We see that in this formula the principal copula has always the sense of implication because the proposition is a secondary one.

By the definition of equality the consequences of the principle of the syllogism may be stated in the following formulas[1]:

$$(a < b) \quad (b = c) < (a < c),$$
$$(a = b) \quad (b < c) < (a < c),$$
$$(a = b) \quad (b = c) < (a = c).$$

The conclusion is an equality only when both premises are equalities.

The preceding formulas can be generalized as follows:

$$(a < b) \quad (b < c) \quad (c < d) < (a < d),$$
$$(a = b) \quad (b = c) \quad (c = d) < (a = d).$$

Here we have the two chief formulas of the *sorites*. Many other combinations may be easily imagined, but we can have an equality for a conclusion only when all the premises are equalities. This statement is of great practical value. In a succession of deductions we must pay close attention to see if the transition from one proposition to the other takes place by means of an equivalence or only of an implication. There is no equivalence between two extreme propositions unless all intermediate deductions are equivalences; in other words, if there is one single implication in the chain, the relation of the two extreme propositions is only that of implication.

7. **Multiplication and Addition.**—The algebra of logic admits of three operations, *logical multiplication*, *logical addition*, and *negation*. The two former are binary operations, that is to say, combinations of two terms having as a consequent a third term which may or may not be different from each of them. The existence of the logical product and logical sum of two terms must necessarily answer the purpose of a

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1 Strictly speaking, these formulas presuppose the laws of multiplication which will be established further on; but it is fitting to cite them here in order to compare them with the principle of the syllogism from which they are derived.

double postulate, for simply to define an entity is not enough for it to exist. The two postulates may be formulated thus:

(Ax. III). Given any two terms, $a$ and $b$, then there is a term $p$ such that

$$p < a,\ p < b,$$

and that for every value of $x$ for which

$$x < a,\ x < b,$$

we have also

$$x < p.$$

(Ax. IV). Given any two terms, $a$ and $b$, then there exists a term $s$ such that

$$a < s,\ b < s,$$

and that, for any value of $x$ for which

$$a < x,\ b < x,$$

we have also

$$s < x.$$

It is easily proved that the terms $p$ and $s$ determined by the given conditions are unique, and accordingly we can define *the* product $ab$ and *the* sum $a + b$ as being respectively the terms $p$ and $s$.

C. I.: 1. The product of two classes is a class $p$ which is contained in each of them and which contains every (other) class contained in each of them;

2. The sum of two classes $a$ and $b$ is a class $s$ which contains each of them and which is contained in every (other) class which contains each of them.

Taking the words "less than" and "greater than" in a metaphorical sense which the analogy of the relation $<$ with the mathematical relation of inequality suggests, it may be said that the product of two classes is the greatest class contained in both, and the sum of two classes is the smallest class which contains both.[1] Consequently the product of two

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[1] According to another analogy DEDEKIND designated the logical sum and product by the same signs as the least common multiple and greatest common divisor (*Was sind und was sollen die Zahlen?* Nos. 8 and 17, 1887. [Cf. English translation entitled *Essays on Number* (Chicago, Open Court Publishing Co. 1901, pp. 46 and 48)] GEORG CANTOR originally gave them the same designation (*Mathematische Annalen*, Vol. XVII, 1880).

classes is the part that is common to each (the class of their common elements) and the sum of two classes is the class of all the elements which belong to at least one of them.

P. I.: 1. The product of two propositions is a proposition which implies each of them and which is implied by every proposition which implies both:

2. The sum of two propositions is the proposition which is implied by each of them and which implies every proposition implied by both.

Therefore we can say that the product of two propositions is their weakest common cause, and that their sum is their strongest common consequence, strong and weak being used in a sense that every proposition which implies another is stronger than the latter and the latter is weaker than the one which implies it. Thus it is easily seen that the product of two propositions consists in their *simultaneous affirmation:* "$a$ and $b$ are true", or simply "$a$ and $b$"; and that their sum consists in their *alternative affirmation*, "either $a$ or $b$ is true", or simply "$a$ or $b$".

*Remark.*—Logical addition thus defined is not disjunctive;[1] that is to say, it does not presuppose that the two summands have no element in common.

**8. Principles of Simplification and Composition.**—The two preceding definitions, or rather the postulates which precede and justify them, yield directly the following formulas:

(1) $$ab < a, \qquad ab < b,$$

(2) $$(x < a)\ (x < b) < (x < ab),$$

(3) $$a < a + b, \qquad b < a + \mathrm{b},$$

(4) $$(a < x)\ (b < x) < (a + b < x).$$

Formulas (1) and (3) bear the name of the *principle of simplification* because by means of them the premises of an

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1 [BOOLE, closely following analogy with ordinary mathematics, premised, as a necessary condition to the definition of "$x + y$", that $x$ and $y$ were mutually exclusive. JEVONS, and practically all mathematical logicians after him, advocated, on various grounds, the definition of "logical addition" in a form which does not necessitate mutual exclusiveness.]