APPENDIX A

Radon transform and propagation of singularities in $\mathbf{R}^n$

In Theorem 5.2 of Chap. 4, we proved the singularity expansion of the Radon transform for an asymptotically hyperbolic metric using the parametrics for the perturbed wave equation. It is also the case for the wave equation in the asymptotically Euclidean space. In this appendix, we state the precise results as well as the relations between the Radon transform, the asymptotic profiles of the wave equation and scattering matrices in a general short-range perturbation regime. The main results are Theorem 1.14, Lemma 1.17, which can be utilized directly in the inverse scattering for the wave equation, and Theorems 6.7, 6.10, which show how the Radon transform is related with the propagation of singularities.

The Radon transform associated with the Euclidean metric is defined by

$$\left(\mathcal{R}_0 f\right)(s,\theta) = \int_{s=x\cdot\theta} f(x)d\Pi_x, \quad s\in\mathbf{R}, \quad \theta\in S^{n-1},$$

$d\Pi_x$ being the measure induced on the hyperplane $\{x\in\mathbf{R}^n;\, s = x\cdot\theta\}$ from the Lebesgue measure dx on $\mathbf{R}^n$. This is rewritten as

$$\left(\mathcal{R}_0 f\right)(s,\theta) = (2\pi)^{(n-1)/2}\int_{-\infty}^{\infty} e^{isk}\widehat{f}(k\theta)dk,$$

where $\widehat{f}$ is the Fourier transform:

$$\widehat{f}(\xi) = (2\pi)^{-n/2}\int_{\mathbf{R}^n} e^{-ix\cdot\xi}f(x)dx.$$

Let us consider the Riemannian metric on $\mathbf{R}^n$ satisfying the following condition:

$$|\partial_x^\alpha(g_{ij}(x)-\delta_{ij})| \leq C_\alpha(1+|x|)^{-1-\epsilon_0-|\alpha|}, \quad \forall\alpha, \tag{0.1}$$

where $\epsilon_0 > 0$ is a constant. In Chap. 2, §7, we have already constructed a generalized Fourier transformation $\mathcal{F}^{(\pm)}$ for Δ_g. As in Chap. 2, §7, we construct $\mathcal{F}_\pm$ from $\mathcal{F}^{(\pm)}$, and define the modified Radon transform $\mathcal{R}_\pm$ by

$$\mathcal{R}_\pm f(s,\theta) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isk}(\mathcal{F}_\pm f)(k,\theta)dk.$$

For the Euclidean Laplacian in $\mathbf{R}^n$ this turns out to be

$$\mathcal{R}_\pm = \left(\mp\partial_s + 0\right)^{\frac{n-1}{2}}\mathcal{R}_0.$$

The main issue of this chapter is the *singular support theorem* for $\mathcal{R}_\pm$. We construct $\varphi(x,\theta)\in C^\infty(\mathbf{R}^n\times S^{n-1})$ such that

$$|\partial_\theta^\alpha\partial_x^\beta(\varphi(x,\theta) - x\cdot\theta)| \leq C_{\alpha\beta}(1+|x|)^{-|\beta|-\epsilon_0},$$

and it solves the eikonal equation

$$g^{ij}(x)(\partial_i\varphi(x,\theta))(\partial_j\varphi(x,\theta)) = 1, \quad \partial_i = \partial/\partial x_i,$$

in an appropriate region in $\mathbf{R}^n$. We put $\Sigma(s,\theta) = \{x \in \mathbf{R}^n; s = \varphi(x,\theta)\}$, which describes a wave front of a plane wave solution to the wave equation $\partial_t^2 u = \Delta_g u$. Then by observing the propagation of singularities, we obtain the following theorem: Let $\mathcal{R}_+(s,\theta,x)$ be the distribution kernel of $\mathcal{R}_+$. Then if we fix $s > 0$ large enough, we have the following singularity expansion:

$$\mathcal{R}_+(s,\theta,x) \sim \sum_{j=0}^{\infty}(s-\varphi(x,\theta))_-^{-\frac{n+1}{2}+j} r_j(x,\theta).$$

Let $\Sigma(s)$ be the envelope of the family of hypersurfaces $\{\Sigma(s,\theta)\ ;\ \theta \in S^{n-1}\}$, which describes a spherical wave front. We then show that f (satisfying a suitable condition on the wave front set) is piecewise smooth near $\Sigma(\sigma)$ with interface $\Sigma(\sigma)$ if and only if $\big(\mathcal{R}_+f\big)(s)$ is piecewise smooth near $\{s = \sigma\}$ with interface $s = \sigma$. Moreover we also obtain the singularity expansion of $\mathcal{R}_+f$ in terms of spherical wave solution to the eikonal equation.

1. Fourier and Radon transforms for perturbed metric

1.1. Spectral properties. The Laplace-Beltrami operator Δ_g is symmetric in $L^2(\mathbf{R}^n;\sqrt{g(x)}dx)$. To avoid the denstity $\sqrt{g(x)}$, we apply a unitary transformation : $u \to ug(x)^{1/4}$, and consider the differential operator

$$H = -g(x)^{1/4}\Delta_g g(x)^{-1/4} = -\sum_{i,j=1}^{n} a_{ij}(x)\partial_i\partial_j + \sum_{i=1}^{n} b_i(x)\partial_i + c(x)$$

on $L^2(\mathbf{R}^n;dx)$. Note that $a_{ij}(x) = g^{ij}(x)$ and $a_{ij}(x) - \delta_{ij}, b_i(x), c(x)$ satisfy

$$|\partial_x^\alpha a(x)| \leq C_\alpha(1+|x|)^{-|\alpha|-1-\epsilon_0}, \quad \forall\alpha.$$

We put

$$H_0 = -\Delta = -\sum_{i=1}^{n}(\partial/\partial x_i)^2, \quad V = H - H_0,$$

$$R_0(z) = (H_0 - z)^{-1}, \quad R(z) = (H-z)^{-1}.$$

Theorem 1.1. *(1)* $\sigma(H) = \sigma_{ac}(H) = [0,\infty)$.
(2) $\sigma_p(H) = \sigma_{sc}(H) = \emptyset$.
(3) For any $\lambda > 0$ *and* $f, g \in \mathcal{B}$, *there exists a limit*

$$\lim_{\epsilon\to 0}\big(R(\lambda \pm i\epsilon)f, g\big) =: \big(R(\lambda \pm i0)f, g\big).$$

(4) For any $0 < a < b < \infty$, *there exists a constant* $C > 0$ *such that*

$$\|R(\lambda \pm i0)f\|_{\mathcal{B}^*} \leq C\|f\|_{\mathcal{B}}, \quad a < \lambda < b.$$

(5) For any f, $g \in \mathcal{B}$, $(R(\lambda \pm i0)f, g)$ *is a continuous function of* $\lambda > 0$.

The proof is omitted. The limiting absorption principle in weighted L^2 spaces was proved in, e.g., [**58**], and in $\mathcal{B} - \mathcal{B}^*$ spaces by Agmon and Agmon-Hörmander [**55**], and [**71**].

1.2. Generalized Fourier transform. Let us recall the notation in Chap. 2, §7. For $k \in \mathbf{R} \setminus \{0\}$ and $f \in \mathcal{B}$, we define

$$\big(\mathcal{F}^0(k)f\big)(\omega) = (2\pi)^{-n/2} \int_{\mathbf{R}^n} e^{-ik\omega\cdot x} f(x)dx.$$

It has the following properties

$$\mathcal{F}^0(k) \in \mathbf{B}(\mathcal{B}; L^2(S^{n-1})),$$

$$\mathcal{F}^0(-k) = J\mathcal{F}^0(k), \tag{1.1}$$

J being the anti-podal operator defined by

$$\big(J\psi\big)(\omega) = \psi(-\omega). \tag{1.2}$$

We put

$$\widehat{\mathcal{H}}_{>0} = L^2((0,\infty); L^2(S^{n-1}); k^{n-1}dk),$$
$$\widehat{\mathcal{H}}_{<0} = L^2((-\infty,0); L^2(S^{n-1}); |k|^{n-1}dk).$$

Then the operator $(\mathcal{F}^0 f)(k) := \mathcal{F}^0(k)f$ is uniquely extended to a unitary operator from $L^2(\mathbf{R}^n)$ to $\widehat{\mathcal{H}}_{>0}$. It is also extended to a unitary operator from $L^2(\mathbf{R}^n)$ to $\widehat{\mathcal{H}}_{<0}$. With these properties in mind, we define the generalized Fourier transform $\mathcal{F}^{(\pm)}(k)$ by the following formula:

$$\mathcal{F}^{(\pm)}(k) = \mathcal{F}^0(k)\big(1 - VR((k \pm i0)^2)\big).$$

Note that $(k+i0)^2 = k^2 + i0$ for $k > 0$ and $(k+i0)^2 = k^2 - i0$ for $k < 0$. By (1.2) we have

$$\mathcal{F}^{(+)}(-k) = J\mathcal{F}^{(-)}(k). \tag{1.3}$$

By Theorem 2.7.11, $\mathcal{F}^{(\pm)}$ is uniquely extended to a unitary operator from $L^2(\mathbf{R}^n)$ to $\widehat{\mathcal{H}}_{>0}$ and diagonalizes H, and $\mathcal{F}^{(\pm)}$ is also unitary from $L^2(\mathbf{R}^n)$ to $\widehat{\mathcal{H}}_{<0}$.

Remark. One can also prove that $\big(\mathcal{F}^{(\pm)}f\big)(k,\theta)$ is smooth with respect to k and θ. In fact, let $\varphi(\lambda) \in C_0^\infty((0,\infty))$, $f(x) \in C_0^\infty(\mathbf{R}^n)$ and put $g(\xi) = \big(\mathcal{F}^{(\pm)}(k)\varphi(L)f\big)(\omega)$ with $k = |\xi|$, $\omega = \xi/|\xi|$. Then $g(\xi) \in C^\infty(\mathbf{R}^n)$. For the case of the Schrödinger operator $-\Delta + V$ where V is a real-valued potential, we have proven this property in [**59**] by using a parametrics at infinity of the time evlolution equation. One can repeat the same argument by using the geometrical optics solutions to be constructed in §3 of this chapter.

The following theorem is proved in the same way as in [**132**].

Theorem 1.2. *For $k \in \mathbf{R} \setminus \{0\}$ and $f \in \mathcal{B}$*

$$R((k+i0)^2)f(x) \simeq C_0(k) r^{-(n-1)/2} e^{ikr} \Big(\mathcal{F}^{(+)}(k)f\Big)(\omega),$$

where $r = |x|$, $\omega = x/r$, and

$$C_0(k) = \sqrt{\frac{\pi}{2}}(-ik+0)^{(n-3)/2}.$$

Here $f \simeq g$ means that

$$\lim_{R\to\infty} \frac{1}{R} \int_{|x|<R} |f(x) - g(x)|^2 dx = 0.$$

1.3. Wave operators and scattering matrix. The wave operator $W_\pm$ for the Schrödinger equation is defined by the following strong limit in $L^2(\mathbf{R}^n)$:

$$W_\pm = \mathrm{s} - \lim_{t\to\pm\infty} e^{itH}e^{-itH_0}.$$

It is well-known that this limit exists and regarding $\mathcal{F}^0$ and $\mathcal{F}^{(\pm)}$ as unitary from $L^2(\mathbf{R}^n)$ to $\widehat{\mathcal{H}}_{>0}$, we have the following relation

$$W_\pm = \big(\mathcal{F}^{(\pm)}\big)^*\mathcal{F}^0. \tag{1.4}$$

The wave operator for the wave equation is usually defined by the energy norm. We can also employ the following equivalent operator

$$\mathrm{s} - \lim_{t\to\pm\infty} e^{it\sqrt{H}}e^{-it\sqrt{H_0}} = W_\pm = \big(\mathcal{F}^{(\pm)}\big)^*\mathcal{F}^0. \tag{1.5}$$

The point is that the limit in the left-hand side exists, and coincides with the wave operator for the Schrödinger equation. This fact, called the invariance principle, is known to hold in a broad situations (see e.g. [**80**], p. 579). The equality (1.5) can of course be proved directly by using $\mathcal{F}^{(\pm)}$ (see e.g. [**102**]).

As a by-product, one can show that the solution $u(t)$ of the wave equation

$$\begin{cases} \partial_t^2 u = -Hu, \\ u(0) = f, \quad \partial_t u(0) = -i\sqrt{H}f \end{cases}$$

behaves as follows

$$\|u(t) - e^{-it\sqrt{H_0}}f_\pm\|_{L^2} \to 0 \quad \text{as} \quad t \to \pm\infty,$$

where $f_\pm = \big(\mathcal{F}^0\big)^*\mathcal{F}^{(\pm)}f$. Therefore $\mathcal{F}^{(\pm)}$ represents the far field behavior of waves. The same fact can be proven for the Schrödinger equation.

Definition 1.3. Regarding $\mathcal{F}^0$ and $\mathcal{F}^{(\pm)}$ as unitary from $L^2(\mathbf{R}^n)$ to $\widehat{\mathcal{H}}_{>0}$, we define the scattering operator S, its Fourier transform $\widehat{S}$, and the physical S-matrix $\widehat{S}_{phy}(k)$ by

$$S = \big(W_+\big)^*W_-, \quad \widehat{S} = \big(\mathcal{F}^0\big)^*S\mathcal{F}^0 = \mathcal{F}^{(+)}\big(\mathcal{F}^{(-)}\big)^*.$$

$$\widehat{S}_{phy}(k) = I - \pi i k^{n-2}\mathcal{F}^{(+)}(k)V\mathcal{F}^0(k)^*, \quad k > 0.$$

Lemma 1.4. *$\widehat{S}_{phy}(k)$ is unitary on $L^2(S^{n-1})$ for any $k > 0$, and*

$$\big(\widehat{S}f\big)(k) = \widehat{S}_{phy}(k)f(k), \quad \forall f \in \widehat{\mathcal{H}}_{>0}, \quad \text{a.e. } k > 0,$$

$$\mathcal{F}^{(+)}(k) = \widehat{S}_{phy}(k)\mathcal{F}^{(-)}(k), \quad \forall k > 0.$$

Definition 1.5. For $k > 0$, we define the geometric scattering matrix $\widehat{S}_{geo}(k)$ by

$$\widehat{S}_{geo}(k) = \widehat{S}_{phy}(k)J.$$

The following theorem is proved in the same way as in [**132**], (see also [**60**], [**62**]).

Theorem 1.6. *Let $k > 0$, and put*

$$\mathcal{N}(k) = \{u \in \mathcal{B}^*; (H - k^2)u = 0\}.$$

(1) We have

$$\mathcal{N}(k) = \mathcal{F}^{(\pm)}(k)^*\big(L^2(S^{n-1})\big).$$

(2) For any $u \in \mathcal{N}(k)$ there exist $\varphi_\pm \in L^2(S^{n-1})$ such that

$$u(x) \simeq \frac{e^{i(kr-(n-1)\pi/4)}}{r^{(n-1)/2}}\varphi_+(\omega) + \frac{e^{-i(kr-(n-1)\pi/4)}}{r^{(n-1)/2}}\varphi_-(\omega), \tag{1.6}$$

where $r = |x|$, $\omega = x/r$.
(3) For any $\varphi_- \in L^2(S^{n-1})$, there exist a unique $u \in \mathcal{N}(k)$ and $\varphi_+ \in L^2(S^{n-1})$ such that the expansion (1.6) holds. Moreover they are related as follows :

$$\varphi_+ = \widehat{S}_{geo}(k)\varphi_-.$$

1.4. Modified Radon transform. It is convenient to change the definition of the generalized Fourier transform slightly. For $k \in \mathbf{R} \setminus \{0\}$, we define

$$\mathcal{F}_\pm(k) = \frac{1}{\sqrt{2}}(\mp ik + 0)^{(n-1)/2}\mathcal{F}^{(\pm)}(k),$$

$$\mathcal{F}_0(k) = \frac{1}{\sqrt{2}}(-ik + 0)^{(n-1)/2}\mathcal{F}^{0}(k),$$

and put $\big(\mathcal{F}_\pm f\big)(k) = \mathcal{F}_\pm(k)f$, $\big(\mathcal{F}_0 f\big)(k) = \mathcal{F}_0(k)f$. Note that by (1.3)

$$\mathcal{F}_+(-k) = J\mathcal{F}_-(k). \tag{1.7}$$

Theorem 1.7. *(1) $\mathcal{F}_\pm : L^2(\mathbf{R}^n) \to L^2(\mathbf{R}; L^2(S^{n-1}); dk)$ is an isometry. Moreover we have*

$$\left(\mathcal{F}_\pm H f\right)(k) = k^2 \left(\mathcal{F}_\pm f\right)(k).$$

(2) For $k > 0$, we have

$$\mathcal{F}_+(k) = (-i)^{n-1}\widehat{S}_{phy}(k)J\mathcal{F}_+(-k).$$

Consequently, the range of $\mathcal{F}_\pm$ has the following characterization:

$$g \in \operatorname{Ran}\mathcal{F}_+ \iff (-i)^{n-1}\widehat{S}_{phy}(k)Jg(-k) = g(k),\ k > 0,$$

$$g \in \operatorname{Ran}\mathcal{F}_- \iff (-i)^{n-1}\widehat{S}_{phy}(k)g(k) = Jg(-k),\ k > 0.$$

(3) Let r_+ (r_-) be the projection onto $\widehat{\mathcal{H}}_{>0}$ ($\widehat{\mathcal{H}}_{<0}$). Then we have

$$W_+ = 2\big(\mathcal{F}_+\big)^* r_+\mathcal{F}_0, \quad W_- = 2\big(\mathcal{F}_+\big)^* r_-\mathcal{F}_0, \tag{1.8}$$

$$W_+ = 2(-i)^{n-1}\big(\mathcal{F}_-\big)^* r_-\mathcal{F}_0, \quad W_- = 2i^{n-1}\big(\mathcal{F}_-\big)^* r_+\mathcal{F}_0. \tag{1.9}$$

Proof. Theorem 2.7.11 proves (1). Lemma 1.4 and (1.3) imply $\widehat{S}_{phy}(k)J\mathcal{F}^{(+)}(-k) = \mathcal{F}^{(+)}(k)$ for $k > 0$, which proves (2). The formula (1.4) proves (1.8) for W_+. For $f, g \in \mathcal{B}$, we have by using (1.3) and (1.4) for W_-

$$\begin{aligned}
\big(W_- f, g\big) &= (\mathcal{F}^0 f, \mathcal{F}^{(-)} g) \\
&= \int_0^\infty (\mathcal{F}^0(k)f, \mathcal{F}^{(-)}(k)g)k^{n-1}dk \\
&= \int_{-\infty}^0 (J\mathcal{F}^0(k)f, J\mathcal{F}^{(+)}(k)g)|k|^{n-1}dk \\
&= \int_{-\infty}^0 ((-ik+0)^{(n-1)/2}\mathcal{F}^0(k)f, (-ik+0)^{(n-1)/2}\mathcal{F}^{(+)}(k)g)dk \\
&= 2\big((\mathcal{F}_+)^* r_-\mathcal{F}_0 f, g\big).
\end{aligned}$$

This proves (1.8) for W_-. By a similar computation using

$$(\mp ik+0)^\alpha = e^{\mp \mathrm{sgn}(k)\alpha\pi i/2}|k|^\alpha, \quad \mathrm{sgn}(k) = k/|k|,$$

we have

$$\begin{aligned}
\big(W_+f,g\big) &= \int_0^\infty (\mathcal{F}^0(k)f, \mathcal{F}^{(+)}(k)g)k^{n-1}dk \\
&= \int_{-\infty}^0 (J\mathcal{F}^0(k)f, J\mathcal{F}^{(-)}(k)g)|k|^{n-1}dk \\
&= \int_{-\infty}^0 ((ik+0)^{(n-1)/2}\mathcal{F}^0(k)f, (ik+0)^{(n-1)/2}\mathcal{F}^{(-)}(k)g)dk \\
&= (-i)^{n-1}\int_{-\infty}^0 ((-ik+0)^{(n-1)/2}\mathcal{F}^0(k)f, (ik+0)^{(n-1)/2}\mathcal{F}^{(-)}(k)g)dk \\
&= 2(-i)^{n-1}\big((\mathcal{F}_-)^* r_- \mathcal{F}_0 f, g\big),
\end{aligned}$$

which proves (1.9) for W_+. Finally by (1.4)

$$\begin{aligned}
\big(W_-f,g\big) &= \int_0^\infty (\mathcal{F}^0(k)f, \mathcal{F}^{(-)}(k)g)k^{n-1}dk \\
&= \int_0^\infty ((ik+0)^{(n-1)/2}\mathcal{F}^0(k)f, (ik+0)^{(n-1)/2}\mathcal{F}^{(-)}(k)g)dk \\
&= i^{n-1}\int_0^\infty ((-ik+0)^{(n-1)/2}\mathcal{F}^0(k)f, (ik+0)^{(n-1)/2}\mathcal{F}^{(-)}(k)g)dk \\
&= 2i^{n-1}\big((\mathcal{F}_-)^* r_+ \mathcal{F}_0 f, g\big),
\end{aligned}$$

which proves (1.9) for W_-. □

As a consequence of Theorem 1.7 (2), we have

$$g \in \mathrm{Ran}\,\mathcal{F}_0 \Longleftrightarrow g(-k,-\omega) = i^{n-1}g(k,\omega), \quad k>0.$$

The projection onto the range of $\mathcal{F}_0$ is written as follows.

Lemma 1.8. *We define the operator $\widetilde{J}$ by $(\widetilde{J}f)(k,\omega) = f(-k,-\omega)$. Then*

$$\mathcal{F}_0(\mathcal{F}_0)^* = \frac{1}{2} + \frac{1}{2}\big((-i)^{n-1}r_+ + i^{n-1}r_-\big)\widetilde{J}.$$

Proof. We put $\big(U_0f\big)(k,\omega) = \frac{1}{\sqrt{2}}|k|^{(n-1)/2}\widehat{f}(k\omega)$. Then U_0 is an isometry from $L^2(\mathbf{R}^n)$ to $L^2(\mathbf{R}; L^2(S^{n-1}); dk)$ and

$$g \in \mathrm{Ran}\,U_0 \Longleftrightarrow g = \widetilde{J}g.$$

Since $U_0(U_0)^*$ is the projection onto the range of U_0, we have

$$U_0(U_0)^* = \frac{1}{2}(1+\widetilde{J}).$$

Let $h = \zeta^{1/2}r_+ + \overline{\zeta}^{1/2}r_-$, $\zeta = e^{-(n-1)\pi i/2}$. Then we have $\mathcal{F}_0 = hU_0$, hence

$$\mathcal{F}_0(\mathcal{F}_0)^* = hU_0(U_0)^*h^*.$$

As can be checked easily

$$\widetilde{J}r_\pm = r_\mp\widetilde{J}. \tag{1.10}$$

Using these formulas we obtain the lemma by a direct computation. □

Corollary 1.9.

$$\mathcal{F}_+ = r_+\mathcal{F}_0(W_+)^* + r_-\mathcal{F}_0(W_-)^*, \tag{1.11}$$

$$\mathcal{F}_- = i^{n-1}r_+\mathcal{F}_0(W_-)^* + (-i)^{n-1}r_-\mathcal{F}_0(W_+)^*. \tag{1.12}$$

Proof. By (1.8), $\mathcal{F}_0\big(W_\pm\big)^* = 2\mathcal{F}_0(\mathcal{F}_0)^*r_\pm\mathcal{F}_+$. By Lemma 1.8 and (1.10) we have

$$r_\pm\mathcal{F}_0(\mathcal{F}_0)^*r_\pm = \frac{1}{2}r_\pm.$$

This proves (1.11). By (1.9), we have $\mathcal{F}_0(W_+)^* = 2i^{n-1}\mathcal{F}_0(\mathcal{F}_0)^*r_-\mathcal{F}_-$, and $\mathcal{F}_0(W_-)^* = 2(-i)^{n-1}\mathcal{F}_0(\mathcal{F}_0)^*r_+\mathcal{F}_-$. Therefore

$$r_-\mathcal{F}_0(W_+)^* = i^{n-1}r_-\mathcal{F}_-, \quad r_+\mathcal{F}_0(W_-)^* = (-i)^{n-1}r_+\mathcal{F}_-.$$

Hence (1.12) follows. □

Definition 1.10. The modified Radon transform $\mathcal{R}_\pm$ is defined by

$$\left(\mathcal{R}_\pm f\right)(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iks}\left(\mathcal{F}_\pm f\right)(k)dk.$$

By (1.7) and Theorem 1.7, we have

Theorem 1.11. *$\mathcal{R}_\pm : L^2(\mathbf{R}^n) \to L^2(\mathbf{R}; L^2(S^{n-1}); dk)$ is an isometry and*

$$\left(\mathcal{R}_\pm Hf\right)(s) = -\partial_s^2\left(\mathcal{R}_\pm f\right)(s).$$

Moreover

$$\left(\mathcal{R}_+ f\right)(-s) = J\left(\mathcal{R}_- f\right)(s).$$

Definition 1.12. For an open interval $I \subset \mathbf{R}$, let $\widehat{H}^m(I)$ be the set of functions $\phi(s,\omega)$ satisfying

$$\sum_{0\le j\le m}\int_I \left\|\partial_s^j\phi(s,\cdot)\right\|^2_{L^2(S^{n-1})}ds < \infty.$$

If $I = \mathbf{R}$, we simply write $\widehat{H}^m$, in which case m can be any real number by passing to the Fourier transformation.

Lemma 1.13. *For any $m \ge 0$ we have*

$$f \in H^m \Longleftrightarrow \mathcal{R}_\pm f \in \widehat{H}^m.$$

Proof. A direct consequence of Theorem 1.11. □

1.5. Asymptotic profiles of solutions to the wave equation. The following theroem is proved in the same way as Theorem 2.8.9.

Theorem 1.14. *For $x \in \mathbf{R}^n$, we write $r = |x|, \omega = x/r$. Then for $f \in L^2(\mathbf{R}^n)$, we have as $t \to \infty$*

$$\left\|\left(\cos(t\sqrt{H})f\right)(x) - \frac{r^{-(n-1)/2}}{\sqrt{2}}\left(\mathcal{R}_+ f\right)(r-t,\omega)\right\| \to 0,$$

$$\left\|\left(\sin(t\sqrt{H})f\right)(x) - \frac{i\,r^{-(n-1)/2}}{\sqrt{2}}\left(h\big(-i\frac{\partial}{\partial s}\big)\mathcal{R}_+ f\right)(r-t,\omega)\right\| \to 0,$$

where $\|\cdot\|$ is the $L^2(\mathbf{R}^n)$-norm, and $h(k) = 1\ (k>0),\ h(k) = -1\ (k<0)$.

1.6. Relation between scattering operators. The scattering operator is also defined by the Radon transform, namely

Definition 1.15. $\mathcal{S}_R = \mathcal{R}_+ (\mathcal{R}_-)^*$.

The following lemma follows easily from Theorem 1.11 and Lemma 1.13.

Lemma 1.16. *(1) $\mathcal{S}_R$ is a partial isometry with initial set Ran$\left(\mathcal{R}_-\right)$ and final set Ran$\left(\mathcal{R}_+\right)$.*
(2) $\partial_s^2 \mathcal{S}_R = \mathcal{S}_R \partial_s^2$.
(3) $\mathcal{S}_R \widehat{H}^m \subset \widehat{H}^m$, $\forall m \geq 0$.

The relation to the scattering operator S in Definition 1.3 is as follows.

Lemma 1.17. *Let $\mathcal{F}_1$ be the 1-dimensional Fourier transform, $r_\pm$ the projection in Theorem 1.7 (3) and $\widetilde{J}$ as in Lemma 1.8. Then we have*

$$\mathcal{F}_1 \mathcal{S}_R (\mathcal{F}_1)^* = (-i)^{n-1} r_+ \mathcal{F}_0 S (\mathcal{F}_0)^* r_+ + i^{n-1} r_- \mathcal{F}_0 S^* (\mathcal{F}_0)^* r_- + \frac{1}{2}\widetilde{J}.$$

Proof. Since $\mathcal{F}_1 \mathcal{S}_R (\mathcal{F}_1)^* = \mathcal{F}_+ (\mathcal{F}_-)^*$, the lemma follows from Corollary 1.9. □

2. Asymptotic solutions

2.1. Geometrical optics. In this section we construct an asymptotic solution to the equation

$$-\Delta_g \left(e^{ik\varphi} a\right) = k^2 e^{ik\varphi} a,$$

$k \in \mathbf{R}$ being a large parameter. We put $a = \sum_{j=0}^N k^{-j} a_j$. Then we have

$$\begin{aligned} e^{-ik\varphi}(-\Delta_g - k^2) e^{ik\varphi} a = &\ k^2 \left[g^{\alpha\beta}(\partial_\alpha \varphi)(\partial_\beta \varphi) - 1\right] a - ikTa - \Delta_g a \\ = &\ k^2 \left[g^{\alpha\beta}(\partial_\alpha \varphi)(\partial_\beta \varphi) - 1\right] a - ikTa_0 \\ &- i \sum_{j=0}^{N-1} k^{-j}(Ta_{j+1} - i\Delta_g a_j) - ik^{-N}\Delta_g a_N, \end{aligned} \tag{2.1}$$

where T is the following differential operator

$$T = 2g^{\alpha\beta}(\partial_\alpha \varphi)\partial_\beta + \Delta_g \varphi.$$

We define the Hamiltonian $h(x,p)$ by

$$h(x,p) = \frac{1}{2} g^{ij}(x) p_i p_j.$$

Our aim is to constrcut a real function $\varphi(x,\theta) \in C^\infty(\mathbf{R}^n \times S^{n-1})$ which behaves like $x \cdot \theta + O(|x|^{-\epsilon_0})$ as $|x| \to \infty$, and solves the eikonal equation $h(x, \nabla_x \varphi) = 1/2$ in the region $\{x \cdot \theta + |x_\perp|/\epsilon > R\}$, where $x_\perp = x - (x \cdot \theta)\theta$, and R, $1/\epsilon$ are sufficiently large constants. We shall parametrize the bicharacteristics by the asymptotic data at infinity.

We fix $\theta \in S^{n-1}$ arbitrarily. We seek a solution $x(t), p(t)$ of the Hamilton-Jacobi equation

$$\frac{dx}{dt} = \frac{\partial h}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial h}{\partial x}, \tag{2.2}$$

having the following asymptotics:

$$x(t) = t\theta + y + O(t^{-\epsilon_0}), \quad p(t) = \theta + O(t^{-1-\epsilon_0}), \quad (t \to \infty)$$

for some $y \in \mathbf{R}^n$. A simple calculation shows that $x(t)$ satisfies the following integral equation

$$x(t) = t\theta + y + \int_t^\infty (s-t)\frac{d^2x(s)}{ds^2}ds.$$

Since Hamilton's equation (2.2) coincides with the equation of geodesic, we have

$$\frac{d^2x^k}{dt^2} = -\Gamma^k_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} = -\Gamma^k_{ij}g^{i\alpha}g^{j\beta}p_\alpha p_\beta,$$

Γ^k_{ij} being Christoffel's symbol. In view of these formulas, we put

$$\begin{aligned} z(t) &= x(t) - t\theta - y, \\ A^k(t,s,y,\theta;z,p) &= (t-s)\Gamma^k_{ij}(s\theta+y+z)g^{i\alpha}(s\theta+y+z)g^{j\beta}(s\theta+y+z)p_\alpha p_\beta, \\ B^k(s,y,\theta;z,p) &= \frac{1}{2}\frac{\partial g^{ij}}{\partial x^k}(s\theta+y+z)p_ip_j, \\ A &= (A^1,\cdots,A^n), \quad B = (B^1,\cdots,B^n), \end{aligned}$$

and consider the integral equation

$$(2.3) \qquad \begin{cases} z(t) = \displaystyle\int_t^\infty A(t,s,y,\theta;z(s),p(s))ds, \\ p(t) = \theta + \displaystyle\int_t^\infty B(s,y,\theta;z(s),p(s))ds. \end{cases}$$

We fix a sufficiently small $\epsilon > 0$. For a sufficiently large $R > 0$, let $\Omega_{R,\epsilon}(\theta)$ be the region defined by

$$\Omega_{R,\epsilon}(\theta) = \{(t,y,z)\,;\, t + |y|/\epsilon > R,\ y\cdot\theta = 0,\ |z| < 3\}.$$

Then taking R large enough we have by a simple computation

$$(2.4) \qquad |t\theta + y + z| \geq C(|t| + |y| + R), \quad \forall (t,y,z) \in \Omega_{R,\epsilon}(\theta),$$

where the constant C is independent of $(t,y,z) \in \Omega_{R,\epsilon}(\theta)$ and $R > 0$. We put

$$X(t) = (z(t), p(t)),$$

and define the non-linear map $\mathcal{L}(X)$ by

$$\mathcal{L}(X)(t,y,\theta) = \left(\int_t^\infty A(t,s,y,\theta;z(s),p(s))ds, \int_t^\infty B(s,y,\theta;z(s),p(s))ds\right).$$

We parametrize y in the following way. Take vectors $e_1(\theta),\cdots,e_{n-1}(\theta)$ so that $e_1(\theta),\cdots,e_{n-1}(\theta)$ and θ form an orthonormal basis of $\mathbf{R}^n$. Then if $y\cdot\theta = 0$, y is written as $y = \sum_{i=1}^{n-1} y_ie_i(\theta)$. This $(y_1,\cdots,y_{n-1})$ gives the desired parametrization. Note that $e_1(\theta),\cdots,e_{n-1}(\theta)$ can be chosen to be smooth with respect to $\theta \in S^{n-1}$ (at least locally). We put

$$|X|_\infty = \sup_{(t,y,z)\in\Omega_{R,\epsilon}(\theta)} |X(t)|.$$

Lemma 2.1. *Suppose $|X|_\infty < 2$, $|\widetilde{X}|_\infty < 2$. Then the following inequalities hold:*

$$\left|\partial_t^m\partial_y^\alpha\mathcal{L}(X)(t,y,\theta)\right| \leq C_{m\alpha}(|t|+|y|+R)^{-\epsilon_0-m-|\alpha|}, \quad \forall m,\alpha,$$

$$\left|\mathcal{L}(X)(t,y,\theta) - \mathcal{L}(\widetilde{X})(t,y,\theta)\right| \leq C(|t|+|y|+R)^{-\epsilon_0}|X-\widetilde{X}|_\infty.$$

Proof. This is a direct consequence of (2.4) and the estimate $\partial_x^\alpha \Gamma_{ij}^k(x) = O(|x|^{-2-\epsilon_0-|\alpha|})$, which follows from (0.1). □

We now put $X_0 = (0,\theta)$ and take $R > 0$ large enough. Then by Lemma 2.1 and the standard method of iteration, there exists a unique solution $X(t,y,\theta)$ of the integral equation

$$X = X_0 + \mathcal{L}(X)$$

in the region $\{t + |y|/\epsilon > R,\ y\cdot\theta = 0\}$ satisfying

$$\left|\partial_t^m \partial_y^\alpha (X(t,y,\theta) - X_0)\right| \leq C_{m\alpha}(|t| + |y| + R)^{-\epsilon_0 - m - |\alpha|}, \quad \forall m,\ \alpha.$$

Returning back to the equation (2.2), we have proven the following lemma.

Lemma 2.2. *Take $\theta \in S^{n-1}$ arbitrarily and $R > 0$ large enough. Then there exists a unique solution $x(t,y,\theta), p(t,y,\theta)$ of the equation (2.2) such that in the region $\{t + |y|/\epsilon > R,\ y\cdot\theta = 0\}$ it satisfies*

$$\left|\partial_t^m \partial_y^\alpha (x(t,y,\theta) - t\theta - y)\right| \leq C_{m\alpha}(|t| + |y| + R)^{-\epsilon_0 - m - |\alpha|}, \quad \forall m,\ \alpha,$$

$$\left|\partial_t^m \partial_y^\alpha (p(t,y,\theta) - \theta)\right| \leq C_{m\alpha}(|t| + |y| + R)^{-1-\epsilon_0 - m - |\alpha|}, \quad \forall m,\ \alpha.$$

Proof. By differentiating the integral equation (2.3), we have

$$\frac{dx^k}{dt} = \theta^k + \int_t^\infty \Gamma_{ij}^k g^{i\alpha} g^{j\beta} p_\alpha p_\beta ds,$$

$$\frac{dp_k}{dt} = -\frac{1}{2}\frac{\partial g^{\alpha\beta}}{\partial x^k} p_\alpha p_\beta = -\frac{\partial h}{\partial x^k}. \tag{2.5}$$

Therefore we have to show that

$$g^{k\alpha} p_\alpha = \theta^k + \int_t^\infty \Gamma_{ij}^k g^{i\alpha} g^{j\beta} p_\alpha p_\beta ds.$$

Since both sides tend to θ^k as $t \to \infty$, we have only to show that their time derivatives coincide. By (2.5), the formula to show is

$$\frac{\partial g^{k\alpha}}{\partial x^i} g^{i\beta} - \frac{1}{2} g^{ki} \frac{\partial g^{\alpha\beta}}{\partial x^i} = -\Gamma_{ij}^k g^{i\alpha} g^{j\beta},$$

which follows from a direct computation and the formula

$$\frac{\partial g^{ij}}{\partial x^m} = -g^{ik}\left(\frac{\partial g_{kr}}{\partial x^m}\right) g^{rj}.$$

The estimates of $x(t), p(t)$ are easy to derive. □

Lemma 2.3. *As a 2-form on the region $\{(t,y)\,;\, t + |y|/\epsilon > R,\ y\cdot\theta = 0\}$, we have*

$$\sum_{i=1}^n dp_i(t,y,\theta) \wedge dx^i(t,y,\theta) = 0.$$

Proof. Without loss of generality we assume $\theta = (0,\cdots,0,1)$ and put $y = (u^1,\cdots,u^{n-1},0)$, $t = u^n$. Then we have

$$\sum_i dp_i \wedge dx^i = \sum_{j<k} [p,x]_{jk} du^j \wedge du^k,$$

$$[p,x]_{jk} = \frac{\partial p}{\partial u^j}\cdot\frac{\partial x}{\partial u^k} - \frac{\partial p}{\partial u^k}\cdot\frac{\partial x}{\partial u^j}.$$

Noting that

$$\frac{\partial}{\partial t}\left(\frac{\partial p}{\partial u^j}\cdot\frac{\partial x}{\partial u^k}\right) = -\frac{\partial^2 h}{\partial x^i \partial x^m}\frac{\partial x^m}{\partial u^j}\frac{\partial x^i}{\partial u^k} + \frac{\partial^2 h}{\partial p_i \partial p_m}\frac{\partial p_i}{\partial u^k}\frac{\partial p_m}{\partial u^j}$$

is symmetric with respect to j and k, we have

$$\frac{\partial}{\partial t}[p, x]_{jk} = 0.$$

By Lemma 2.2, $[p, x]_{jk} \to 0$ as $t \to \infty$. Hence $[p, x]_{jk} = 0$, which proves the lemma. □

For $x \in \mathbf{R}^n$, we put $x_\perp = x - (x\cdot\theta)\theta$ and define the region $\triangle_{R,\epsilon}(\theta)$ by

$$\triangle_{R,\epsilon}(\theta) = \{x \in \mathbf{R}^n\,;\, x\cdot\theta + |x_\perp|/\epsilon > R\}.$$

In the coordinates with basis $\theta, e_1(\theta), \cdots, e_{n-1}(\theta)$, the differential of the map $(t, y) \to x(t, y, \theta)$ is $I + O(R^{-\epsilon_0})$. Therefore the following lemma holds.

Lemma 2.4. *For large $R > 0$, the map $(t, y) \to x(t, y, \theta)$ is a diffeomorphism and its image includes $\triangle_{2R,\epsilon}(\theta)$.*

Let $t = t(x, \theta), y = y(x, \theta)$ be the inverse of the map : $(t, y) \to x(t, y, \theta)$. We put $p(x, \theta) = p(t(x, \theta), y(x, \theta), \theta)$ for the sake of simplicity. Lemma 2.3 implies $d(\sum_j p_j(x, \theta)dx^j) = 0$, which shows

$$\frac{\partial p_j(x,\theta)}{\partial x^i} = \frac{\partial p_i(x,\theta)}{\partial x^j}. \tag{2.6}$$

We put

$$f(x,\theta) = p(x,\theta) - \theta = \int_t^\infty \frac{\partial h}{\partial x}\big(x(s,y,\theta), p(s,y,\theta)\big)ds\Big|_{t=t(x,\theta),y=y(x.\theta)},$$

and define $\Psi(x, \theta)$ by

$$\Psi(x,\theta) = x\cdot\theta - \int_0^\infty f(x + t\theta, \theta)\cdot\theta dt.$$

Lemma 2.5. *On $\triangle_{2R,\epsilon}(\theta)$, we have*

$$\nabla_x\Psi(x,\theta) = p(x,\theta), \tag{2.7}$$

$$h(x, \nabla_x\Psi(x,\theta)) = 1/2, \tag{2.8}$$

$$|\partial_x^\alpha(\Psi(x,\theta) - x\cdot\theta)| \le C_\alpha(1 + |x|)^{-\epsilon_0-|\alpha|}, \quad \forall\alpha. \tag{2.9}$$

$$\Psi(x,\theta) = t(x,\theta). \tag{2.10}$$

Proof. Letting $f = (f_1, \cdots, f_n)$, we have $\dfrac{\partial f_j}{\partial x^i}(x,\theta) = \dfrac{\partial f_i}{\partial x^j}(x,\theta)$ by (2.6). We then have

$$\begin{aligned}
\frac{\partial \Psi}{\partial x^i} &= \theta_i - \int_0^\infty \sum_j \frac{\partial f_j}{\partial x^i}(x + t\theta, \theta)\theta_j dt \\
&= \theta_i - \int_0^\infty \sum_j \frac{\partial f_i}{\partial x^j}(x + t\theta, \theta)\theta_j dt \\
&= \theta_i - \int_0^\infty \frac{d}{dt} f_i(x + t\theta, \theta) dt \\
&= \theta_i + f_i(x,\theta) \\
&= p_i(x,\theta),
\end{aligned}$$

which proves (2.7). Since $x(t), p(t)$ solve the equation (3.2), $h(x(t), p(t))$ is a constant. Letting $t \to \infty$, this constant is seen to be equal to $1/2$, which together with (2.7) proves (2.8). The estimate (2.9) follows from Lemma 2.1. By (2.7), we have

$$\frac{\partial \Psi}{\partial t} = (\partial_i \Psi)\,\frac{\partial x^i}{\partial t} = g^{ij}\,(\partial_i \Psi)\,(\partial_j \Psi) = 1.$$

Therefore $\Psi = t + t_0(y,\theta)$ for some $t_0(y,\theta)$. However by Lemma 3.2, $x(t,y,\theta)\cdot\theta = t + O(t^{-\epsilon_0})$, which implies $t_0(y,\theta) = \Psi - x\cdot\theta + O(t^{-\epsilon_0}) = O(t^{-\epsilon_0})$. Therefore $t_0(y,\theta) = 0$, which proves (2.10). □

The equality (2.6) yields the following corollary.

Corollary 2.6. *For any smooth function $f(x)$ on $\mathbf{R}^n$, we have*

$$\frac{\partial}{\partial t} f(x(t,y,\theta))\Big|_{t=t(x,\theta),y=y(x,\theta)} = g^{ij}(x)\frac{\partial \Psi(x,\theta)}{\partial x^j}\frac{\partial f(x)}{\partial x^i}.$$

By the above construction, $\Psi(x,\theta)$ is actually a function on the fibered space $\{(\theta,x)\ ;\ \theta \in S^{n-1}, x \in \Delta_{2R,\epsilon}(\theta)\}$ and satisfies

$$|\partial_\theta^\alpha \partial_x^\beta (\Psi(x,\theta) - x\cdot\theta)| \le C_{\alpha\beta}(1+|x|)^{-\epsilon_0-|\beta|}, \quad \forall\alpha,\ \beta.$$

Definition 2.7. We take $\chi_\infty(t) \in C^\infty(\mathbf{R})$ and $\chi(t) \in C^\infty(\mathbf{R})$ such that $\chi_\infty(t) = 1,\ (t > 3R),\ \ \chi_\infty(t) = 0,\ (t < 2R),\ \ \chi(t) = 1,\ (t > -1+2\epsilon)$, $\chi(t) = 0, (t < -1+\epsilon)$, where $1/R$ and $\epsilon > 0$ are sufficiently small constants. We define

$$\varphi(x,\theta) = x\cdot\theta + \chi_\infty(|x|)\chi(\widehat{x}\cdot\theta)\Big(\Psi(x,\theta) - x\cdot\theta\Big),$$

$$\varphi_\pm(x,\xi) = \pm|\xi|\,\varphi(x,\pm\widehat{\xi}), \quad \widehat{\xi} = \xi/|\xi|.$$

The following lemma is a direct consequence of the above definition.

Theorem 2.8. *(1) $\varphi_\pm(x,\xi) \in C^\infty(\mathbf{R}^n \times (\mathbf{R}^n \setminus \{0\}))$ and*

$$\Big|\partial_\xi^\alpha \partial_x^\beta \Big(\varphi_\pm(x,\xi) - x\cdot\xi\Big)\Big| \le C_{\alpha\beta}|\xi|^{1-|\alpha|}(1+|x|)^{-\epsilon_0-|\beta|}.$$

(2) If $|x| > 3R$ and $\pm\widehat{x}\cdot\widehat{\xi} > -1+2\epsilon$, it satisfies the eikonal equation

$$h(x, \nabla_x\varphi_\pm(x,\xi)) = |\xi|^2/2.$$

(3) $\varphi_-(x,\xi) = -\varphi_+(x,-\xi)$.

2.2. Asymptotic solutions. We employ the above $\varphi(x,\theta)$ as φ in (2.1). Letting

$$a_0(x,\theta) = \exp\left(\int_t^{\infty} \frac{1}{2}(\Delta_g\varphi)(x(s,y,\theta),\theta)ds\right)\Big|_{t=t(x,\theta),y=y(x,\theta)}, \tag{2.11}$$

and using Corollary 2.6, we have

$$Ta_0(x,\theta) = 0 \quad \text{for} \quad |x| > 3R, \quad \widehat{x}\cdot\theta > -1+2\epsilon.$$

By Theorem 2.8 (1), $a_0(x,\theta)$ satisfies

$$|\partial_\theta^\alpha\partial_x^\beta\,(a_0(x,\theta)-1)\,| \leq C_{\alpha\beta}(1+|x|)^{-|\beta|-\epsilon_0}.$$

We integrate the higer order transport equation

$$Ta_j - i\Delta_g a_{j-1} = 0, \quad j \geq 1$$

in a similar manner, and obtain

$$|\partial_\theta^\alpha\partial_x^\beta\,a_j(x,\theta)| \leq C_{\alpha\beta}(1+|x|)^{-j-|\beta|-\epsilon_0}.$$

Let $\chi(t), \chi_\epsilon(t) \in C^\infty(\mathbf{R})$ be such that $\chi(t) = 1$ $(t > 4)$, $\chi(t) = 0$ $(t < 3)$, $\chi_\epsilon(t) = 1$ $(t > -1+3\epsilon)$, $\chi_\epsilon(t) = 0$ $(t < -1+2\epsilon)$. We put

$$a(x,k,\theta) = g(x)^{1/4}\chi_\epsilon(\widehat{x}\cdot\theta)\sum_{j=0}^{\infty} k^{-j}a_j(x,\theta)\chi(\epsilon_j|x|)\chi(\epsilon_j|k|). \tag{2.12}$$

By a suitable choice of the sequence $\epsilon_0 > \epsilon_1 > \cdots \to 0$, this series converges and defines a smooth function. We finally define

$$a_\pm(x,\xi) = a(x,\pm|\xi|,\pm\widehat{\xi}\,).$$

The following lemma holds.

Lemma 2.9. *(1) On $\mathbf{R}^n\times\mathbf{R}^n$, $a_\pm(x,\xi)$ satisfies*

$$|\partial_\xi^\alpha\partial_x^\beta\,a_\pm(x,\xi)| \leq C_{\alpha\beta}(1+|\xi|)^{-|\alpha|}(1+|x|)^{-|\beta|}.$$

(2) Let $g_\pm(x,\xi) = e^{-i\varphi_\pm(x,\xi)}(L-|\xi|^2)e^{i\varphi_\pm(x,\xi)}a_\pm(x,\xi)$. Then it satisfies

$$\left|\partial_\xi^\alpha\partial_x^\beta\,g_\pm(x,\xi)\right| \leq C_{\alpha\beta N}(1+|\xi|)^{-N}(1+|x|)^{-N}$$

for any $N > 0$ in the region $|x| > 4R$, $\pm\widehat{x}\cdot\widehat{\xi} > -1+3\epsilon$.

3. Fourier integral operators and functional calculus

3.1. Product formula for FIO. Lets us recall the theory of FIO's. Since we need precise product formulas, we employ the computation by [**86**], [**87**]. For $m \in \mathbf{R}$, let S^m be the class of symbols defined by

$$S^m \ni p(x,\xi) \Longleftrightarrow \left|\partial_\xi^\alpha\partial_x^\beta\,p(x,\xi)\right| \leq C_{\alpha\beta}(1+|\xi|)^{m-|\alpha|}, \quad \forall\alpha,\ \beta.$$

The phase function $\varphi(x,\xi) \in C^\infty(\mathbf{R}^n\times\mathbf{R}^n)$ is assumed to be real-valued and satisfy the following conditions (3.1) $\sim$ (3.4) for a sufficiently small constant $\delta_0 > 0$:

$$\varphi(x,\xi) - x\cdot\xi \in S^1, \tag{3.1}$$

$$\left|\nabla_\xi\,(\varphi(x,\xi) - x\cdot\xi)\,\right| < \delta_0, \tag{3.2}$$

$$\left|\nabla_x\,(\varphi(x,\xi) - x\cdot\xi)\,\right| < \delta_0(1+|\xi|), \tag{3.3}$$

$$\left|\frac{\partial^2}{\partial x \partial \xi}\varphi(x,\xi) - I\right| < \delta_0. \tag{3.4}$$

We define FIO's $I_{\varphi,a},\ I_{\varphi^*,a}$ by

$$I_{\varphi,a}u(x) = (2\pi)^{-n} \iint_{\mathbf{R}^n \times \mathbf{R}^n} e^{i(\varphi(x,\xi)-y\cdot\xi)} a(x,\xi)u(y)dyd\xi,$$

$$I_{\varphi^*,a}u(x) = (2\pi)^{-n} \iint_{\mathbf{R}^n \times \mathbf{R}^n} e^{i(x\cdot\xi-\varphi(y,\xi))} a(y,\xi)u(y)dyd\xi.$$

We put $D_x = -i\partial_x$ and define the ψDO $p(x, D_x)$ with symbol $p(x,\xi)$ by

$$p(x,D_x)u(x) = (2\pi)^{-n} \iint_{\mathbf{R}^n \times \mathbf{R}^n} e^{i(x-y)\cdot\xi} p(x,\xi)u(y)dyd\xi.$$

Using the conditions (3.1) $\sim$ (3.4) we can prove the following lemma.

Lemma 3.1. *(1) The map $\mathbf{R}^n \ni \xi \to \eta = \nabla_x\varphi(x,\xi) \in \mathbf{R}^n$ is a global diffeomorphism on $\mathbf{R}^n$. Letting its inverse by $\xi(x,\eta)$, we have*

$$\xi(x,\eta) - \eta \in S^1,$$

$$C^{-1}(1+|\eta|) \le 1 + |\xi| \le C(1+|\eta|).$$

(2) The map $\mathbf{R}^n \ni x \to y = \nabla_\xi\varphi(x,\xi)$ is a global diffeomorphism on $\mathbf{R}^n$. Letting $x(y,\xi)$ be its inverse, we have

$$x(y,\xi) - y \in S^0,$$

$$C^{-1}(1+|y|) \le 1 + |x| \le C(1+|y|).$$

In the following Theorem 3.2, all symbols $c(x,\xi)$ belong to $S^{s_1+s_2}$ and have the following asymptoic expansion:

$$c(x,\xi) \sim \sum_{j=1}^{\infty} c_j(x,\xi), \quad c_j(x,\xi) \in S^{s_1+s_2-j}. \tag{3.5}$$

Theorem 3.2. *Let $a \in S^{s_1},\ b \in S^{s_2}$. Then we have the following formulas.*

$$\begin{cases} I_{\varphi,a}I_{\varphi^*,b} = c(x,D_x), \\ c(x,\eta) \sim a(x,\xi)b(x,\xi)\det\left(\dfrac{\partial^2}{\partial x\partial\xi}\varphi(x,\xi)\right)^{-1}\Bigg|_{\xi=\xi(x,\eta)} + \cdots, \end{cases} \tag{3.6}$$

where $\xi(x,\eta)$ is the inverse map of $\eta = \nabla_x\varphi(x,\xi)$,

$$\begin{cases} I_{\varphi^*,a}I_{\varphi,b} = c(x,D_x), \\ c(y,\xi) \sim a(x,\xi)b(x,\xi)\det\left(\dfrac{\partial^2}{\partial x\partial\xi}\varphi(x,\xi)\right)^{-1}\Bigg|_{x=x(y,\xi)} + \cdots, \end{cases} \tag{3.7}$$

where $x(y,\xi)$ is the inverse map of $y = \nabla_\xi\varphi(x,\xi)$,

$$\begin{cases} I_{\varphi,a}b(x,D_x) = I_{\varphi,c}, \\ c(x,\xi) \sim a(x,\xi)b(\nabla_\xi\varphi(x,\xi),\xi) + \cdots, \end{cases} \tag{3.8}$$

$$\text{(3.9)} \qquad \begin{cases} a(x, D_x)I_{\varphi,b} = I_{\varphi,c}, \\ c(x,\xi) \sim a(x, \nabla_x\varphi(x,\xi))b(x,\xi) + \cdots . \end{cases}$$

For the proof, see [**86**], Theorems 2.1 $\sim$ 2.4. We need the following explicit form of the asymptotic expansion (3.5) later. We put

$$\widetilde{\nabla}_\xi\varphi(x,\xi,\eta) = \int_0^1 (\nabla_\xi\varphi)(x, t\xi + (1-t)\eta)dt,$$

$$\widetilde{\nabla}_x\varphi(x,y,\xi) = \int_0^1 (\nabla_x\varphi)(tx + (1-t)y, \xi)dt.$$

Then $c(x,\xi)$ in (3.8) has the following asymptotic expansion:

$$\text{(3.10)} \qquad c(x,\eta) \sim \sum_\alpha \frac{1}{\alpha!}\partial_\xi^\alpha \left\{ a(x,\xi)(D_x^\alpha b)(\widetilde{\nabla}_\xi\varphi(x,\xi,\eta),\eta)\right\}\Big|_{\xi=\eta},$$

and $c(x,\xi)$ in (3.9) has the following asymptotic expansion:

$$\text{(3.11)} \qquad c(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!}D_y^\alpha \left\{ (\partial_\xi^\alpha a)(x, \widetilde{\nabla}_x\varphi(x,y,\xi))b(y,\xi)\right\}\Big|_{y=x},$$

(see [**86**], (2.41), (2.57)).

3.2. Functional calculus. In Chap. 3, §2, we have introduced the almost analytic extension $F(z)$ of $f(t)$. By the construction procedure, we see that $\partial_t F(t+is)$ is an almost analytic extension of $f'(t)$. Let

$$\text{(3.12)} \qquad X = (1+|x|^2)^{1/2}, \quad \Lambda = (1+|D_x|^2)^{1/2}.$$

Lemma 3.3. *Let $f(t) \in C_0^\infty(\mathbf{R})$. Then we have for any $N > 0$*

$$\text{(3.13)} \qquad f(H) = f(H_0) + \sum_{n=1}^N p_n(x, D_x)f^{(n)}(H_0) + R_N,$$

where $p_n(x,D_x) = \sum_{|\alpha|\le\mu(n)} a_\alpha^{(n)}(x)D_x^\alpha$ such that $|\partial_x^\beta a_\alpha^{(n)}(x)| \le C_{\alpha\beta}(1+|x|)^{-|\beta|-1-\epsilon_0}$, and R_N satisfies

$$\text{(3.14)} \qquad X^N\Lambda^N R_N \Lambda^N X^N \in \mathbf{B}(L^2(\mathbf{R}^n)).$$

Proof. We first prove the lemma with the property (3.14) replaced by

$$\text{(3.15)} \qquad X^N R_N X^N \in \mathbf{B}(L^2(\mathbf{R}^n)).$$

We prove the case $N = 1$. By the resolvent equation, we have

$$\begin{aligned} (z-H)^{-1} - (z-H_0)^{-1} &= (z-H)^{-1}V(z-H_0)^{-1} \\ &= V(z-H)^{-1}(z-H_0)^{-1} + [(z-H)^{-1}, V](z-H_0)^{-1} \\ &= V(z-H_0)^{-2} + K(z), \end{aligned}$$

$$\begin{aligned} K(z) &= V(z-H)^{-1}V(z-H_0)^{-2} \\ &\quad + (z-H)^{-1}[H,V](z-H)^{-1}(z-H_0)^{-1}. \end{aligned}$$

Therefore by virtue of Lemma 3.2.1

$$\text{(3.16)} \qquad \begin{aligned} f(H) - f(H_0) &= V\frac{1}{2\pi i}\int_{\mathbf{C}} \overline{\partial_z}F(z)(z-H_0)^{-2}dzd\overline{z} \\ &\quad + \frac{1}{2\pi i}\int_{\mathbf{C}} \overline{\partial_z}F(z)K(z)dzd\overline{z}. \end{aligned}$$

Since $\partial_t F(t+is)$ is an almost analytic extension of $f'(t)$, we have by integration by parts

$$f'(H_0) = \frac{1}{2\pi i}\int_{\mathbf{C}} \overline{\partial_z}\partial_t F(z)(z-H_0)^{-1}dzd\overline{z}$$
$$= \frac{1}{2\pi i}\int_{\mathbf{C}} \overline{\partial_z}F(z)(z-H_0)^{-2}dzd\overline{z}.$$

Therefore the 1st term of the right-hand side of (3.16) is equal to $Vf'(H_0)$. If P_j is a differential operator of order $j=1,2$ with bounded coefficients, we have by passing to the spectral decomposition

$$\|P_j(z-H)^{-1}\| \leq C\,|\mathrm{Im}\, z|^{-1}(1+|z|)^{j/2}.$$

We then have

$$\|XK(z)X\| \leq C|\mathrm{Im}\, z|^{-p}(1+|z|)^p,$$

for some $p \geq 2$. Since $F(z)$ satisfies $|\overline{\partial_z}F(z)| \leq C|\mathrm{Im}\, z|^p(1+|z|)^{s-p-1}$ for any $s<0$, the remainder term has the desired estimate (3.15). The proof for $N \geq 2$ is similar.

Now for $f \in C_0^\infty(\mathbf{R})$ we take $\chi \in C_0^\infty(\mathbf{R})$ such that $\chi(t)=1$ on $\mathrm{supp}\,\chi$. We multiply (3.13) by the expansion

$$\chi(H) = \chi(H_0) + \sum_{j=1}^N \chi^{(j)}(H_0)q_j(x,D_x) + (\widetilde{R}_N)^*,$$

with $q_j(x,D_x)$ and $\widetilde{R}_N$ having the above mentioned properties. We then have

$$f(H_0)\chi(H) = f(H_0) + f(H_0)(\widetilde{R}_N)^*.$$

Since $\widetilde{R}_N$ satisfies (3.15), one can prove that $f(H_0)(\widetilde{R}_N)^*$ satisfies (3.14). One can deal with $p_n(x,D_x)f^{(n)}(H_0)\chi(H)$ and $R_N\chi(H)$ in a similar manner. □

4. Parametrices and regularizers

We construct parametrices for the wave equation in the form of a FIO using $\varphi_\pm$ and $a_\pm$ in §2. Recall that $\varphi_\pm, a_\pm$ contain cut-off functions. Here we need another cut-off function which restricts x and ξ in a smaller region. Let R and ϵ be as in Definition 2.7. Take $\chi_\infty(t),\ \chi(t) \in C^\infty(\mathbf{R})$ such that $\chi_\infty(t)=1$ $(t>10R)$, $\chi_\infty(t)=0$ $(t<9R)$, $\chi(t)=1$ $(t>-1+5\epsilon)$, $\chi(t)=0$ $(t<-1+4\epsilon)$, and put

$$\chi_\pm(x,\xi) = \chi_\infty(|x|)\chi_\infty(|\xi|)\chi(\pm\widehat{x}\cdot\widehat{\xi}). \tag{4.1}$$

Definition 4.1. Let $\varphi_\pm$, $a_\pm$ be as in Theorem 2.8 and Lemma 2.9, and $\chi_\pm$ as in (4.1). We define a FIO $U_\pm(t)$ by

$$U_\pm(t) = I_{\varphi_\pm,a_\pm}e^{-it\sqrt{H_0}}I_{\varphi_\pm^*,\chi_\pm}.$$

In the following, $\|\cdot\|$ denotes either the operator norm $\|T\|_{\mathbf{B}(L^2(\mathbf{R}^n))}$ of a bounded operator T on $L^2(\mathbf{R}^n)$ or the L^2-norm $\|u\|_{L^2(\mathbf{R}^n)}$ of a vector $u \in L^2(\mathbf{R}^n)$. There will be no fear of confusion. We put

$$G_+(t) = \frac{d}{dt}\left(e^{it\sqrt{H}}U_+(t)\right).$$

Let X and Λ be as in (3.12).

Lemma 4.2. *For any $N>0$, there exists a constant $C_N>0$ such that*

$$\|\Lambda^N G_+(t)\Lambda^N X^N\| \leq C_N(1+t)^{-N}, \quad t>0.$$

Proof. We have

$$G_+(t) = e^{it\sqrt{H}}\left(i\sqrt{H}U_+(t) + \frac{d}{dt}U_+(t)\right).$$

We decompose this operator into two parts and make use of the tools in §3.

Low energy part. First we deal with the low energy part. We take $\chi_0(t) \in C^\infty(\mathbf{R})$ such that $\chi_0(t) = 1$ $(t < 1)$, $\chi_0(t) = 0$ $(t > 2)$ and consider $\Lambda^N e^{it\sqrt{H}}\sqrt{H}\chi_0(H)U_+(t)$. Noting that

$$\Lambda^N e^{it\sqrt{H}}\sqrt{H}\chi_0(H)U_+(t) = \Lambda^N(1+H)^{-N/2}e^{it\sqrt{H}}(1+H)^{N/2}\sqrt{H}\chi_0(H)U_+(t),$$

we have only to show

$$\|\chi_0(H)U_+(t)\Lambda^N X^N\| \le C_N(1+t)^{-N}, \quad \forall t, N > 0. \tag{4.2}$$

We decompose $\chi_0(H)U_+(t)$ into two parts:

$$\chi_0(H)U_+(t) = \chi_0(H)I_{\varphi_+,a_+}\cdot e^{-it\sqrt{H_0}}I_{\varphi_+^*,\chi_+}. \tag{4.3}$$

Proposition 4.3. $\chi_0(H)I_{\varphi_+,a_+}\Lambda^N X^N \in \mathbf{B}(L^2(\mathbf{R}^n)), \quad \forall N > 0.$

Proof. Lemma 3.3 entails the asymptotic expansion

$$\chi_0(H) = \chi_0(H_0) + \sum_{n=1}^{N} p_n(x, D_x) + R_N, \tag{4.4}$$

$$p_n(x,\xi) = 0 \quad \text{for} \quad |\xi| > 2, \quad X^N\Lambda^N R_N\Lambda^N X^N \in \mathbf{B}(L^2(\mathbf{R}^n)). \tag{4.5}$$

By the construction of $a_+(x,\xi)$ in §2 (see (2.12)), $|\xi| \ge 1/\epsilon_0$ and $|x| \ge 1/\epsilon_0$ on $\operatorname{supp} a_+(x,\xi)$. Therefore in the expression

$$\iint e^{-ix\cdot\eta}\chi_0(|\eta|^2)e^{i\varphi_+(x,\xi)}a_+(x,\xi)(1+|\xi|^2)^{N/2}(1-\Delta_\xi)^{N/2}\widehat{f}(\xi)d\xi dx, \tag{4.6}$$

which is the Fourier transform of $\chi_0(L_0)I_{\varphi_+,a_+}\Lambda^N X^N f$, the phase has the following estimate

$$\left|\nabla_x\big(x\cdot\eta - \varphi_+(x,\xi)\big)\right| \ge C(1+|\xi|), \quad C > 0.$$

Using the differential operator

$$P = i\left|\eta - \nabla_x\varphi_+(x,\xi)\right|^{-2}\big(\eta - \nabla_x\varphi_+(x,\xi)\big)\cdot\nabla_x,$$

and integration by parts, we can then rewrite (4.6) as

$$\iint e^{-i(x\cdot\eta-\varphi_+(x,\xi))}\chi_0(|\eta|^2)\big(P^*\big)^{2N}a_+(x,\xi)(1+|\xi|^2)^{N/2}(1-\Delta_\xi)^{N/2}\widehat{f}(\xi)d\xi dx.$$

Since $|\big(P^*\big)^{2N}a_+(x,\xi)| \le C_N(1+|x|)^{-2N}(1+|\xi|)^{-2N}$, by integrating by parts with respect to ξ, the proposition is proved if $\chi_0(H)$ is replaced by $\chi_0(H_0)$. By (4.5) one can prove the same result if $\chi_0(H_0)$ is replaced by $p_n(x,D_x)$ or R_N. This proves the above proposition. □

By (4.3) and Proposition 4.3, the proof of (4.2) is reduced to the following Proposition.

Proposition 4.4.

$$\|X^{-N}\Lambda^{-N}e^{-it\sqrt{H_0}}I_{\varphi_+^*,\chi_+}\Lambda^N X^N\| \le C_N(1+t)^{-N}, \quad \forall t, N > 0.$$

Proof. We estimate the phase function of

$$e^{-it\sqrt{H_0}} I_{\varphi_+^*,\chi_+} f = (2\pi)^{-n} \iint e^{i(x\cdot\xi - t|\xi| - \varphi_+(y,\xi))} \chi_+(y,\xi) f(y) dy d\xi.$$

First we have

$$|\nabla_\xi (t|\xi| + \varphi_+(y,\xi))| \geq |t\widehat{\xi} + y| - C|y|^{-\epsilon_0}.$$

Here the localization with respect to the directions of y and ξ plays an important role. Since $\widehat{\xi}\cdot\widehat{y} > -1 + 4\epsilon$ on $\operatorname{supp}\chi_+(y,\xi)$, we have

$$\begin{aligned} |t\widehat{\xi} + y|^2 &= t^2 + 2t|y|\widehat{\xi}\cdot\widehat{y} + |y|^2 \\ &\geq t^2 - 2t|y|(1-4\epsilon) + |y|^2 \\ &\geq 4\epsilon(t^2 + |y|^2). \end{aligned}$$

By choosing R large enough, we have

$$|\nabla_\xi (t|\xi| + \varphi_+(y,\xi))| \geq C(t + |y|) \tag{4.7}$$

with a constant $C > 0$ independent of y and $t > 0$. Integration by parts then proves the proposition. □

High energy part. Next we consider $i\sqrt{H}(1-\chi_0(H))U_+(t) + \dfrac{d}{dt}U_+(t)$. By the definition of g_+ in Lemma 2.9, we have

$$HI_{\varphi_+,a_+} - I_{\varphi_+,a_+}H_0 = I_{\varphi_+,g_+}, \tag{4.8}$$

which implies

$$I_{\varphi_+,a_+}(H_0 - z)^{-1} - (H-z)^{-1}I_{\varphi_+,a_+} = (H-z)^{-1}I_{\varphi_+,g_+}(H_0-z)^{-1}.$$

We put $f(t) = t^{-1/2}(1-\chi_0(t))$ and let $F(z)$ be its almost analytic extension. Then we have by virtue of Lemma 4.3

$$f(H)I_{\varphi_+,a_+} - I_{\varphi_+,a_+}f(H_0) = B, \tag{4.9}$$

$$B = \frac{1}{2\pi i}\int_{\mathbf{C}} \overline{\partial_z} F(z)(H-z)^{-1} I_{\varphi_+,g_+}(H_0-z)^{-1} dz d\overline{z}.$$

Using this formula, we then have

$$\begin{aligned} \sqrt{H}(1-\chi_0(H))I_{\varphi_+,a_+} &= f(H)HI_{\varphi_+,a_+} \\ &= f(H)I_{\varphi_+,a_+}H_0 + f(H)I_{\varphi_+.g_+} \\ &= I_{\varphi_+,a_+}f(H_0)H_0 + BH_0 + f(H)I_{\varphi_+,g_+}, \end{aligned}$$

where we have used (4.8), (4.9) in the first and second lines. Therefore we have

$$\begin{aligned} &i\sqrt{H}(1-\chi_0(H))U_+(t) + \frac{d}{dt}U_+(t) \\ &= iBH_0 e^{-it\sqrt{H_0}} I_{\varphi_+^*,\chi_+} + if(H)I_{\varphi_+,g_+}e^{-it\sqrt{H_0}}I_{\varphi_+^*,\chi_+} \\ &\quad - iI_{\varphi_+,a_+}\sqrt{H_0}\chi_0(H_0)e^{-it\sqrt{H_0}}I_{\varphi^*,\chi_+}. \end{aligned} \tag{4.10}$$

The third term of the right-hand side vanishes, since $\chi_0(|\xi|^2)\chi_+(y,\xi) = 0$. Let us consider the second term. Taking notice of the relation

$$\Lambda^N e^{it\sqrt{H}} f(H) = \Lambda^N(1+H)^{-N/2}\cdot e^{it\sqrt{H}}\cdot f(H)(1+H)^{N/2}\Lambda^{-N}\cdot\Lambda^N,$$

we have only to show the following

Proposition 4.5.

$$\|\Lambda^N I_{\varphi_+,g_+} e^{-it\sqrt{H_0}} I_{\varphi_+^*,\chi_+} \Lambda^N X^N\| \leq C_N(1+t)^{-N}, \ \forall t, N > 0.$$

Proof. We choose $\psi_1(t), \psi_2(t) \in C^\infty(\mathbf{R})$ such that $\psi_1(t)+\psi_2(t) = 1$ $(t \in \mathbf{R})$, $\psi_1(t) = 1$ $(t < -1+3\epsilon)$, $\psi_1(t) = 0$ $(t > -1+7\epsilon/2)$, and put

$$J_k(t)f = (2\pi)^{-n} \iint e^{i(\varphi_+(x,\xi)-t|\xi|-\varphi_+(y,\xi))} \psi_k(\widehat{x}\cdot\widehat{\xi}) g_+(x,\xi)\chi_+(y,\xi)f(y)dyd\xi.$$

Then $I_{\varphi_+,g_+} e^{-it\sqrt{L_0}} I_{\varphi_+^*,\chi_+} = J_1(t)+J_2(t)$. Note that $\widehat{x}\cdot\widehat{\xi} > -1+3\epsilon$ on the support of $\psi_2(\widehat{x}\cdot\widehat{\xi})$, on which region $g_+(x,\xi)$ decays rapidly in x and ξ by Lemma 3.9. Using (4.7) and integrating by parts, we then have

$$\|\Lambda^N J_2(t)\Lambda^N X^N\| \leq C_N(1+t)^{-N}, \quad \forall t, N > 0.$$

We next show that on the support of the integrand of $J_1(t)$

$$|\nabla_\xi(\varphi_+(x,\xi) - t|\xi| - \varphi_+(y,\xi))| \geq C(t+|x|+|y|) \tag{4.11}$$

for a constant $C > 0$. Once this is proved, one can prove

$$\|\Lambda^N J_1(t)\Lambda^N X^N\| \leq C_N(1+t)^{-N}, \quad \forall t, N > 0$$

by integration by parts. To prove (4.11), we put

$$D_+ = \{y \in \mathbf{R}^n\,;\, \widehat{y}\cdot\widehat{\xi} > -1+4\epsilon\}, \quad D_- = \{x \in \mathbf{R}^n\,;\, \widehat{x}\cdot\widehat{\xi} < -1+7\epsilon/2\}.$$

Then there exists $0 < c_0 < 1$ such that

$$y\cdot x \leq c_0|y||x| \quad \text{if} \quad y \in D_+, \quad x \in D_-.$$

We also see that $y + t\widehat{\xi} \in D_+$ if $y \in D_+$, $t \geq 0$. Therefore

$$|y+t\widehat{\xi}-x|^2 \geq (1-c_0)(|y+t\widehat{\xi}|^2+|x|^2).$$

In the proof of Proposition 5.4, we have already seen that $|y+t\widehat{\xi}| \geq C(t+|y|)$ for some $C > 0$. This proves (4.11). □

It remains to consider the first term of the right-hand side of (4.10).

Proposition 4.6.

$$\|\Lambda^N BH_0 e^{-it\sqrt{H_0}} I_{\varphi_+^*,\chi_+}\Lambda^N X^N\| \leq C_N(1+t)^{-N}, \quad \forall t, N > 0.$$

Proof. We rewrite $BH_0e^{-it\sqrt{H_0}}I_{\varphi_+^*,\chi_+}$ as

$$\frac{1}{2\pi i}\int_{\mathbf{C}} \left(\overline{\partial_z}F(z)\right)|\operatorname{Im} z|^{-m}(1+|z|)^{m-1}\cdot|\operatorname{Im} z|(H-z)^{-1}$$
$$\cdot I_{\varphi_+,g_+}\cdot\left(\frac{|\operatorname{Im} z|}{1+|z|}\right)^{m-1}(H_0-z)^{-1}L_0e^{-it\sqrt{H_0}}I_{\varphi_+^*,\chi_+}dzd\overline{z},$$

m being an arbitrily chosen integer. By the property of almost analytic extension, $\left(\overline{\partial_z}F(z)\right)|\operatorname{Im} z|^{-m}(1+|z|)^{m-1}$ is integrable, and $\||\operatorname{Im} z|(H-z)^{-1}\|$ is uniformly bounded on $\mathbf{C}$. We show that by taking m large enough, one can deal with $|\operatorname{Im} z|^{m-1}(1+|z|)^{-m+1}(H_0-z)^{-1}L_0$ like a ψDO with smooth symbol whose operator norm is uniformly bounded in z. To show this, we have only to prove

$$\left(\frac{|\operatorname{Im} z|}{1+|z|}\right)^{|\alpha|+1}|\partial_\xi^\alpha(|\xi|^2-z)^{-1}| \leq C(1+|z|)^{-1}, \tag{4.12}$$

where C is a constant independent of $\xi \in \mathbf{R}$ and $z \in \mathbf{C} \setminus \mathbf{R}$. In fact, one can show by induction that

$$\partial_\xi^\alpha (|\xi|^2 - z)^{-1} = \sum_{n=1}^{|\alpha|} \frac{P_n(\xi)}{(|\xi|^2 - z)^{n+1}},$$

where $P_n(\xi)$ is a polynomial of order n. Using the inequality $|\xi| \leq C(1+|z|+||\xi|^2 - z|)$, we have $|P_n(\xi)| \leq C((1+|z|)^n + ||\xi|^2 - z|^n)$, which implies

$$|\partial_\xi^\alpha (|\xi|^2 - z)^{-1}| \leq C \sum_{n=1}^{|\alpha|+1} \frac{(1+|z|)^{n-1}}{||\xi|^2 - z|^n}.$$

This proves (4.12). Then by the same computation as in the proof of Proposition 4.5, we can prove the desired estimate. □

The proof of Lemma 4.2 is now completed. □

Lemma 4.7. *For any $f \in L^2(\mathbf{R}^n)$ we have in the sense of $L^2(\mathbf{R}^n)$*

$$U_\pm(t) f = e^{-it\sqrt{H_0}} I_{\varphi_\pm^*, \chi_\pm} f + o(1), \quad t \to \pm\infty.$$

Proof. We have only to prove that

$$I_{\varphi_\pm, a_\pm} e^{-it\sqrt{H_0}} g = e^{-it\sqrt{H_0}} g + o(1), \quad \text{as} \quad t \pm \infty$$

for g satisfying $\widehat{g}(\xi) = \chi_\infty(\xi)\widehat{g}(\xi) \in C_0^\infty(\mathbf{R}^n)$. We prove the case as $t \to \infty$. Take $\chi_0(t), \chi_1(t) \in C^\infty(\mathbf{R})$ such that $\chi_0(t) + \chi_1(t) = 1$ ($t \in \mathbf{R}$), $\chi_0(t) = 1$ ($t < 1/3$), $\chi_0(t) = 0$ ($t > 2/3$). Then we have

$$\chi_0\Big(\frac{|x|}{t}\Big) I_{\varphi_+, a_+} e^{-it\sqrt{H_0}} g = (2\pi)^{-n/2} \int_{\mathbf{R}^n} e^{i(\varphi_+(x,\xi) - t|\xi|)} \chi_0\Big(\frac{|x|}{t}\Big) a_+(x,\xi) \widehat{g}(\xi) d\xi.$$

Since $\nabla_\xi(\varphi_+(x,\xi) - t|\xi|) = x - t\widehat{\xi} + O(|x|^{-\epsilon_0})$, we have

$$|\nabla_\xi(\varphi_+(x,\xi) - t|\xi|)| \geq Ct$$

for some constant $C > 0$ on the support of the integrand. By integration by parts, we then have

$$\|\chi_0\Big(\frac{|x|}{t}\Big) I_{\varphi_+, a_+} e^{-it\sqrt{H_0}} g\| \leq C_N t^{-N}, \quad \forall N, t > 0.$$

We rewrite $\chi_1(\frac{|x|}{t}) I_{\varphi_+, a_+} e^{-it\sqrt{H_0}} g$ as above. Since $a_+(x,\xi) = \chi(\epsilon_0|\xi|)\chi_\epsilon(\widehat{x} \cdot \widehat{\xi}) + O(|x|^{-\epsilon_0})$ (see (2.12)), and the integral over the region $\{\widehat{x} \cdot \xi < 0\}$ disappears (which is proven by the same method of integration by parts), we have

$$\chi_1\Big(\frac{|x|}{t}\Big) I_{\varphi_+, a_+} e^{-it\sqrt{H_0}} g = (2\pi)^{-n/2} \int_{\mathbf{R}^n} e^{i(\varphi_+(x,\xi) - t|\xi|)} \chi_1\Big(\frac{|x|}{t}\Big) \chi(\epsilon_0|\xi|) \widehat{g}(\xi) d\xi + o(1).$$

In (4.1), we take R large enough so that $\chi_\infty(|\xi|) = \chi_\infty(|\xi|)\chi(\epsilon_0|\xi|)$. Then we have $\chi(\epsilon_0|\xi|)\widehat{g}(\xi) = \chi_\infty(|\xi|)\widehat{g}(\xi) = \widehat{g}(\xi)$. Therefore

$$\begin{aligned} \chi_1\Big(\frac{|x|}{t}\Big) I_{\varphi_+, a_+} e^{-it\sqrt{H_0}} g &= \chi_1\Big(\frac{|x|}{t}\Big) e^{-it\sqrt{H_0}} g + o(1) \\ &= e^{-it\sqrt{H_0}} g + o(1), \end{aligned}$$

which proves the lemma □

Let $\widehat{H}^m$ be the Sobolev space in Definition 1.12.

Definition 4.8. (1) An operator R is called a *regularizer of order* N if it satisfies

$$R \in \bigcap_{m=-\infty}^{\infty} \mathbf{B}(H^m\,;\,H^{m+N}) \quad \text{or} \quad R \in \bigcap_{m=-\infty}^{\infty} \mathbf{B}(H^m\,;\,\widehat{H}^{m+N}).$$

If N can be taken arbitarily large, R is simply called a regularizer.
(2) A ψDO P_+ (P_-) is called an *approximate outgoing (incoming) projection* if its symbol $p_+(x,\xi)$ ($p_-(x,\xi)$) has the form

$$p_\pm(y,\xi) = \chi_\pm(x,\xi)\Big|_{x=x_\pm(y,\xi)},$$

where $\chi_\pm(x,\xi)$ is specified in (4.1), and $x_\pm(y,\xi)$ is the inverse function of $y = \nabla_\xi \varphi_\pm(x,\xi)$.

Let $W_\pm$ be the wave operator defined in Subsection 1.3.

Theorem 4.9. *For any $N > 0$, there exist an approximate outgoing (incoming) projection P_+ (P_-) and a regularizer of order N, which is denotede by $R^N_\pm$, such that*

$$W_\pm P_\pm = I_{\varphi_\pm,a_\pm} P_\pm + R^N_\pm.$$

Proof. We consider W_+. Lemmas 4.2 and 4.7 imply

$$W_+ I_{\varphi_+^*,\chi_+} = I_{\varphi_+,a_+} I_{\varphi_+^*,\chi_+} + \int_0^\infty G_+(t)dt, \tag{4.13}$$

the 2nd term of the right-hand side being a regularizer. In the following we use the abbreviation

$$b\Big|_{x_+(y,\xi)} = b(x,\xi)\Big|_{x=x_+(y,\xi)}.$$

We now put $b_0(x,\xi) = \det\left(\partial^2\varphi_+/\partial x\partial\xi\right)\Big|_{x_+(y,\xi)}$, and let

$$I_{\varphi_+^*,\chi_+} I_{\varphi_+,b_0} = c_+(x,D_x).$$

Then we have modulo a regularizer

$$W_+ c_+(x,D_x) \equiv I_{\varphi_+,a_+} c_+(x,D_x).$$

By virtue of (4.7), $c_+(x,\xi)$ has an asymptotic expansion

$$c_+(y,\xi) \sim \chi_+\Big|_{x_+(y,\xi)} + c_1(y,\xi) + \cdots, \quad c_1 \in S^{-1}.$$

Let $\widetilde{\chi}_+(x,\xi)$ be a function similar to $\chi_+(x,\xi)$ such that $\chi_+(x,\xi) = 1$ on $\operatorname{supp}\widetilde{\chi}_+(x,\xi)$. Namely, we slightly shrink the support of χ_+. Let $q_1 \in S^{-1}$ and Q_1 be a ψDO with symbol $\widetilde{\chi}\Big|_{x_+(y,\xi)} + q_1(y,\xi)$. Then the symbol of $c_+(x,D_x)Q_1$ has an asymptotic expansion

$$\begin{aligned} &\chi_+\Big|_{x_+(y,\xi)}\widetilde{\chi}_+\Big|_{x_+(y,\xi)} + \chi_+\Big|_{x_+(y,\xi)} q_1 + c_1\widetilde{\chi}_+\Big|_{x_+(y,\xi)} \\ &+ \sum_{|\alpha|=1} \partial_\xi^\alpha \chi_+\Big|_{x_+(y,\xi)} \cdot D_y^\alpha \widetilde{\chi}_+\Big|_{x_+(y,\xi)} \quad \text{mod} \quad S^{-2}. \end{aligned}$$

We choose q_1 as follows:

$$q_1 = -\frac{1}{\chi_+\Big|_{x_+(y,\xi)}} \left(c_1\widetilde{\chi}_+\Big|_{x_+(y,\xi)} + \sum_{|\alpha|=1} \partial_\xi^\alpha \chi_+\Big|_{x_+(y,\xi)} \cdot D_y^\alpha \widetilde{\chi}_+\Big|_{x_+(y,\xi)} \right).$$

Since $\chi_+ = 1$ on $\operatorname{supp}\widetilde{\chi}_+$, $q_1(y,\xi)$ is smooth and

$$I_{\varphi_+^*,\chi_+} I_{\varphi_+,c_+} Q_1 = \widetilde{c}_+(x, D_x),$$

$$\widetilde{c}_+(y,\xi) \sim \chi_+\Big|_{x_+(y,\xi)} + c_2(y,\xi) + \cdots, \quad c_2 \in S^{-2}.$$

Repeating this procedure, we complete the proof the theorem. □

5. Propagation of singularities

5.1. Singularity expansions I. We show how $\mathcal{R}_+$ describes the singularities of solutions to the wave equation. We start with the following lemma, which can be proved easily by integration by parts.

Lemma 5.1. *The integral operator defined by*

$$(Af)(s,\omega) = \int_{-\infty}^{\infty}\int_{\mathbf{R}^n} e^{ik(s-\omega\cdot y)} a(s,\omega,k,y) f(y) dk dy$$

($s \in \mathbf{R}^1$, $\omega \in S^{n-1}$) is a regularizer if there exist constants $\nu \in \mathbf{R}$ and $C_0 > 0$ such that

$$\left|\partial_s^\alpha \partial_k^\beta \partial_y^\gamma\, a(s,\omega,k,y)\right| \le C_{\alpha\beta\gamma}(1+|k|)^{\nu-\beta}, \quad \forall \alpha,\beta,\gamma, \tag{5.1}$$

$$|s - \omega\cdot y| \ge C_0(1 + |s| + |y|) \tag{5.2}$$

on the support of $a(s,\omega,k,y)$.

By Corollary 1.9, we have the following expression:

$$\begin{aligned}\big(\mathcal{R}_+ f\big)(s) = &\frac{1}{\sqrt{2\pi}}\int_0^\infty e^{iks}\left(\mathcal{F}_0(W_+)^* f\right)(k)dk \\ &+ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^0 e^{iks}\left(\mathcal{F}_0(W_-)^* f\right)(k)dk.\end{aligned} \tag{5.3}$$

We take $\chi_R(s) \in C^\infty(\mathbf{R})$ such that $\chi_R(s) = 0$ $(s < 15R)$, $\chi_R(s) = 1$ $(s > 20R)$, and study the singularity of $\chi_R(s)\mathcal{R}_+ f(s)$ with respect to s.

Lemma 5.2. *We take $N > 0$ large enough. Then there exist approximate outgoing, incoming projections P_+, P_- such that*

$$\chi_R(s)\int_0^\infty e^{iks}\mathcal{F}_0(k)(W_+)^* dk \equiv \chi_R(s)\int_0^\infty e^{iks}\mathcal{F}_0(k)P_+^* I_{\varphi_+^*,\overline{a_+}} dk, \tag{5.4}$$

$$\chi_R(s)\int_{-\infty}^0 e^{iks}\mathcal{F}_0(k)(W_-)^* dk \equiv \chi_R(s)\int_{-\infty}^0 e^{iks}\mathcal{F}_0(k)P_-^* I_{\varphi_-^*,\overline{a_-}} dk \tag{5.5}$$

modulo regularizers of order N.

Proof. We compute the first term of the right-hand side of (5.3). Let $\chi_\infty(t)$ and $\chi(t)$ be as in (4.1). Modulo a regularizer, we can insert $\chi_\infty(|D_x|)$ between $\mathcal{F}_0(k)$ and $(W_+)^*$. Let Q_0 and Q_∞ be defined by

$$Q_0 f(x) = (2\pi)^{-n}\iint e^{i(x-y)\cdot\xi}\big(1 - \chi_\infty(|x_+(y,\xi)|)\big)\chi_\infty(|\xi|)f(y)dyd\xi,$$

$$Q_\infty f(x) = (2\pi)^{-n}\iint e^{i(x-y)\cdot\xi}\chi_\infty(|x_+(y,\xi)|)\chi_\infty(|\xi|)f(y)dyd\xi,$$

where $x_\pm(y,\xi)$ is the inverse function of $y = \nabla_\xi \varphi_\pm(x,\xi)$. Then we have

$$\chi_R(s)\int_0^\infty e^{iks}\mathcal{F}_0(k)Q_0 f\,dk = \int_0^\infty\int_{\mathbf{R}^n} e^{ik(s-\omega\cdot y)}a(s,\omega,k,y)f(y)dydk, \tag{5.6}$$

$$a(s,\omega,k,y) = \frac{\chi_R(s)}{\sqrt{2}(2\pi)^{n/2}}(-ik+0)^{(n-1)/2}(1-\chi_\infty(|x_+(y,k\omega)|))\chi_\infty(k).$$

Since $|y| \le 11R$ on the support of $a(s,\omega,k,y)$, the condition (5.2) is satisfied. Moreover by differentiating $y = \nabla_\xi\varphi_+(x,\xi)$, we have

$$\left|\partial_k^m\partial_y^\gamma\, x_+(y,k\omega)\right| \le C_{m\gamma}(1+|k|)^{-m}, \quad \forall m\ge 1, \quad \forall\gamma,$$

from which one can show that the condition (5.1) is also satisfied. Hence by Lemma 5.1, (5.6) is a regularizer.

Therefore we have only to consider

$$\chi_R(s)\int_0^\infty e^{iks}\big(\mathcal{F}_0 Q_\infty (W_+)^* f\big)(k)dk. \tag{5.7}$$

We put $\chi_-(t) = 1-\chi(t)$ and let Q_- be defined by

$$Q_-f(x) = (2\pi)^{-n}\iint e^{i(x-y)\cdot\xi}\chi_\infty(|x_+(y,\xi)|)\chi_\infty(|\xi|)\chi_-\big(\frac{x_+(y,\xi)}{|x_+(y,k\omega)|}\cdot\frac{\xi}{|\xi|}\big)f(y)dyd\xi.$$

Then the operator (5.7) is split into two parts:

$$\chi_R(s)\int_0^\infty e^{iks}\big(\mathcal{F}_0 P_+^*(W_+)^*f\big)(k)dk + \chi_R(s)\int_0^\infty e^{iks}\big(\mathcal{F}_0 Q_-(W_+)^*f\big)(k)dk.$$

The second term is rewritten as, up to a constant,

$$\chi_R(s)\int_0^\infty\int_{\mathbf{R}^n} e^{ik(s-\omega\cdot y)}\chi_-\big(\frac{x_+(y,k\omega)}{|x_+(y,k\omega)|}\cdot\frac{k\omega}{|k\omega|}\big)\cdots dkdy,$$

which is a regularizer by virtue of Lemma 5.1, since $s > 15R$ and $\omega\cdot y \le -|y|/2$ on the support of the integrand. By Theorem 4.9,

$$P_+^*(W_+)^* \equiv P_+^* I_{\varphi_+^*,\overline{a_+}}$$

modulo a regularizer of order N. We have thus proved (5.4).

Next we consider the second term of the right-hand side of (5.3). We repeat the same arguments as above with $x_+(y,\xi)$ replaced by $x_-(y,\xi)$ and $\int_0^\infty\cdots dk$ by $\int_{-\infty}^0\cdots dk$. Let $\chi_+(t) = 1-\chi(-t)$ and Q_+ be defeined by

$$Q_+f(x) = (2\pi)^{-n}\iint e^{i(x-y)\cdot\xi}\chi_\infty(|x_-(y,\xi)|)\chi_\infty(|\xi|)\chi_+\big(\frac{x_-(y,\xi)}{|x_-(y,k\omega)|}\cdot\frac{\xi}{|\xi|}\big)f(y)dyd\xi.$$

Then as above, we are led to consider

$$\chi_R(s)\int_{-\infty}^0 e^{iks}\big(\mathcal{F}_0 P_-^*(W_-)^*f\big)(k)dk + \chi_R(s)\int_{-\infty}^0 e^{iks}\big(\mathcal{F}_0 Q_+(W_-)^*f\big)(k)dk$$

modulo a regularizer. Since $k<0$ this time, we have

$$\chi_+\big(\frac{x_-(y,k\omega)}{|x_-(y,k\omega)|}\cdot\frac{k\omega}{|k\omega|}\big) = \chi_+\big(-\frac{x_-(y,k\omega)}{|x_-(y,k\omega)|}\cdot\omega\big),$$

on which support, we have $\omega\cdot y \le -|y|/2$. Therefore the second term is a regularizer. Again using Theorem 4.9, we have

$$P_-^*(W_-)^* \equiv P_-^* I_{\varphi_-^*,\overline{a_-}}$$

modulo a regularizer of order N. We have thus derived (5.5) □

Let $(s)_-^\alpha$ be the homogeneous distribution defined in Chap.4, §5.

Lemma 5.3. *Let $\chi_\infty(k)$ be as in (4.1), and put*

$$D_j(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iks}\left(-ik+0\right)^{\frac{n-1}{2}-j}\chi_\infty(|k|)dk. \tag{5.8}$$

Then we have

$$D_j(s) = \big(s\big)_-^{-\frac{n+1}{2}+j} + \Psi_0(s),$$

where $\Psi_0(s)$ is a polynomially bounded smooth function on $\mathbf{R}$.

Proof. Letting $\psi_0(t)$ be the Fourier transform of $1-\chi_\infty(|k|)$, we have

$$D_j(s) = \big(s\big)_-^{-\frac{n+1}{2}+j} - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\big(s+t\big)_-^{-\frac{n+1}{2}+j}\psi_0(t)dt,$$

from which the lemma follows immediately. □

In the following we use the notation $\sim$ in the same meaning as in (3.5). Namely

$$c(x,\xi) \sim \sum_{j=0}^{\infty}|\xi|^{-j}c_j(x,\widehat{\xi})$$

if and only if

$$\Big|\partial_\xi^\alpha\partial_x^\beta\big(c(x,\xi) - \sum_{j=0}^{N-1}|\xi|^{-j}c_j(x,\widehat{\xi})\big)\Big| \le C_{\alpha\beta N}|\xi|^{-N-|\alpha|}, \quad |\xi|>1$$

holds for any α,β and N. Note that this asymptotic expansion can be differentiated term by term with respect to x and ξ.

By Theorem 3.2, we have for some $b_\pm(x,\xi)\in S^0$,

$$I_{\varphi_\pm,a_\pm}P_\pm = I_{\varphi_\pm,b_\pm}. \tag{5.9}$$

Lemma 5.4. *There exist $b_j(x,\theta)$ $(j=0,1,2,\cdots)$ such that $b_\pm(x,\xi)$ have the following asymptotic expansions as $|\xi|\to\infty$:*

$$b_\pm(x,\xi) \sim \sum_{j=0}^{\infty}(\pm|\xi|)^{-j}b_j(x,\pm\widehat{\xi}), \tag{5.10}$$

$$b_0(x,\theta) = g(x)^{1/4}a_0(x,\theta)\chi_\infty(|x|)\chi(\widehat{x}\cdot\theta), \tag{5.11}$$

where $a_0(x,\theta)$ is given in (2.11) and χ_∞, χ are given in (4.1).

Granting this lemma for the moment, we state the main theorem of this section.

Theorem 5.5. *Let $\mathcal{R}_+(s,\theta,x)$ be the distribution kernel of $\mathcal{R}_+$. Then there exist $s_0>0$ such that for any $N>(n+1)/2$, the followlwing expansion holds for $s>s_0$:*

$$\mathcal{R}_+(s,\theta,x) = \sum_{j=0}^{N-1}(s-\varphi(x,\theta))_-^{-\frac{n+1}{2}+j}r_j(x,\theta) + r^{(N)}(s,\theta,x),$$

where $(s_0,\infty)\ni s\to r^{(N)}(s,\theta,x)\in\mathcal{D}'(S^{n-1}\times\mathbf{R}^n)$ is in $C^{\mu(N)}$, $\mu(N)$ is the greatest integer $\le N-(n+1)/2$, $\varphi(x,\theta)$ is given by Definition 2.7, and

$$r_j(x,\theta) = 2^{-1/2}(2\pi)^{(1-n)/2}i^{-j}\overline{b_j(x,\theta)}, \tag{5.12}$$

$b_j(x,\theta)$ *being given in Lemma 5.5.*

Proof. First let us note that

$$\varphi_-(x,k\theta) = k\varphi_+(x,\theta) \quad \text{for} \quad k<0, \tag{5.13}$$

$$b_-(x,k\theta) \sim \sum_{j=0}^{\infty} k^{-j} b_j(x,\theta) \quad \text{as} \quad k \to -\infty. \tag{5.14}$$

In fact by Theorem 2.8 (3) we have for $k<0$

$$\varphi_-(x,k\theta) = -\varphi_+(x,-k\theta) = -\varphi_+(x,|k|\theta) = -|k|\varphi_+(x,\theta) = k\varphi_+(x,\theta),$$

which proves (5.13). By (5.10) we have as $k \to -\infty$

$$b_-(x,k\theta) \sim \sum_{j=0}^{\infty} (-|k|)^{-j} b_j\big(x, -\frac{k\theta}{|k\theta|}\big) = \sum_{j=0}^{\infty} k^{-j} b_j(x,\theta)$$

which proves (5.14).

Take $f \in C_0^\infty(\mathbf{R}^n)$. Since $\varphi_+(x,\theta) = \varphi(x,\theta)$ by Definition 2.7, using (5.10) we have as $k \to \infty$

$$\begin{aligned} &\mathcal{F}_0(k)\big(I_{\varphi_+,b_+}\big)^* f \\ = \;& \frac{1}{\sqrt{2}(2\pi)^{n/2}}\big(-ik+0\big)^{(n-1)/2} \int_{\mathbf{R}^n} e^{-i\varphi_+(x,k\theta)}\overline{b_+(x,k\theta)} f(x)dx \\ \sim \;& \frac{1}{\sqrt{2}(2\pi)^{n/2}} \sum_{j=0}^{\infty} \int_{\mathbf{R}^n} e^{-ik\varphi(x,\theta)}\big(-ik+0\big)^{\frac{n-1}{2}-j} \chi_\infty(k) i^{-j} \overline{b_j(x,\theta)} f(x)dx, \end{aligned}$$

where $\chi_\infty(k)$ is as in (4.1). Here we have used the fact that

$$(-ik+0)^\alpha(-ik)^m = (-ik+0)^{\alpha+m} \quad \text{if} \quad 0 \neq k \in \mathbf{R}, \quad \alpha \in \mathbf{R}, \quad m \in \mathbf{Z}.$$

By (5.13) and (5.14), we have as $k \to -\infty$

$$\begin{aligned} &\mathcal{F}_0(k)\big(I_{\varphi_-,b_-}\big)^* f \\ = \;& \frac{1}{\sqrt{2}(2\pi)^{n/2}}\big(-ik+0\big)^{(n-1)/2} \int_{\mathbf{R}^n} e^{-i\varphi_-(x,k\theta)}\overline{b_-(x,k\theta)} f(x)dx \\ \sim \;& \frac{e^{-(n-1)\pi i/4}}{\sqrt{2}(2\pi)^{n/2}} \sum_{j=0}^{\infty} \int_{\mathbf{R}^n} e^{-ik\varphi(x,\theta)}\big(-ik+0\big)^{\frac{n-1}{2}-j} \chi_\infty(k) i^{-j} \overline{b_j(x,\theta)} f(x)dx. \end{aligned}$$

Using (5.3), Lemma 5.2 and (5.9), we have

$$\begin{aligned} \chi_R(s)\mathcal{R}_+ f(s) \equiv \;& \frac{\chi_R(s)}{\sqrt{2\pi}} \int_0^\infty e^{iks}\mathcal{F}_0(k)\left(I_{\varphi_+,b_+}\right)^* f dk \\ &+ \frac{\chi_R(s)}{\sqrt{2\pi}} \int_{-\infty}^0 e^{iks}\mathcal{F}_0(k)\left(I_{\varphi_-,b_-}\right)^* f dk \end{aligned}$$

modulo a regularizer of order N. We replace $\mathcal{F}_0(k)\left(I_{\varphi_\pm,b_\pm}\right)^*$ by the above asymptotic expansion to obtain

$$\begin{aligned} \chi_R(s)\mathcal{R}_+ f(s) \equiv \;& \frac{\chi_R(s)}{\sqrt{2}(2\pi)^{(n+1)/2}} \sum_{j=0}^{N} \int_{-\infty}^{\infty} \int_{\mathbf{R}^n} e^{ik(s-\varphi(x,\theta))} \\ &\cdot (-ik+0)^{\frac{n-1}{2}-j} \chi_\infty(k) i^{-j} \overline{b_j(x,\theta)} f(x)dxdk \end{aligned}$$

modulo a term sufficiently regular in s. Performing the integral in k and using Lemma 5.3, we have

$$\chi_R(s)\mathcal{R}_+f(s) \equiv \frac{\chi_R(s)}{\sqrt{2}(2\pi)^{(n+1)/2}}\sum_{j=0}^{N}\int_{\mathbf{R}^n}\big(s-\varphi(x,\theta)\big)_-^{-\frac{n+1}{2}+j} i^{-j}\overline{b_j(x,\theta)}f(x)dx,$$

modulo a term sufficiently regular in s, which proves the asymptotic expansion of $\mathcal{R}_+(s,\theta,x)$. □

It remains to prove Lemma 5.4. Let $\big(\nabla_\xi\varphi_\pm\big)^{-1}(x,\xi)$ the inverse of the map $: x \to \nabla_\xi\varphi_\pm(x,\xi)$. Then by (4.1), the symbol $p_\pm(x,\xi)$ of $P_\pm$ is written as

$$p_\pm(x,\xi) = \chi_\pm \circ \big(\nabla_\xi\varphi_\pm\big)^{-1}(x,\xi). \tag{5.15}$$

Now in view of (3.10), we have

$$b_\pm(x,\eta) \sim \sum_\alpha \frac{1}{\alpha!}\,\partial_\xi^\alpha\left\{a_\pm(x,\xi)\big(D_x^\alpha p_\pm\big)\big(\widetilde{\nabla}_\xi\varphi_\pm(x,\xi,\eta),\eta\big)\right\}\Big|_{\xi=\eta}. \tag{5.16}$$

Each term of the right-hand side consists of a sum of functions homogeneous in η. We rearrange them as

$$b_\pm(x,\eta) \sim \sum_{j=0}^{\infty}|\eta|^{-j}b_\pm^{(j)}(x,\widehat{\eta}), \tag{5.17}$$

and compare (5.16) and (5.17) to obtain

$$b_\pm^{(0)}(x,\theta) = g(x)^{1/4}\chi_\epsilon(\pm x\cdot\theta)a_0(x,\pm\theta)p_\pm\big(\widetilde{\nabla}_\xi\varphi_\pm(x,\xi,\eta),\eta\big)\Big|_{\xi=\eta=\theta},$$

where we have used (2.12). Since

$$\widetilde{\nabla}_\xi\varphi_\pm(x,\xi,\eta)\Big|_{\xi=\eta} = \big(\nabla_\xi\varphi_\pm\big)(x,\eta),$$

we have by (5.15)

$$p_\pm\big(\widetilde{\nabla}_\xi\varphi_\pm(x,\xi,\eta),\eta\big)\Big|_{\xi=\eta=\theta} = \chi_\pm(x,\theta),$$

which proves (5.11).

To prove (5.10), we make the following definition. Two functions $f_+(x,\xi)$ and $f_-(x,\xi)$ are said to be *compatible* if there exist $f_j(x,\theta)$ $(j=0,1,2,\cdots)$ such that $f_\pm(x,\xi)$ have the following asymptotic expansion as $|\xi|\to\infty$:

$$f_\pm(x,\xi) \sim \sum_{j=0}^{\infty}\big(\pm|\xi|\big)^{-j}f_j(x,\pm\widehat{\xi}).$$

Lemma 5.6. *(1) If $f_+(x,\xi)$ and $f_-(x,\xi)$ are compatible, so are $\partial_\xi^\alpha f_+(x,\xi)$ and $\partial_\xi^\alpha f_-(x,\xi)$.*
(2) If $f_+(x,\xi)$ and $f_-(x,\xi)$ as well as $g_+(x,\xi)$ and $g_-(x,\xi)$ are compatible, so are $f_+(x,\xi)g_+(x,\xi)$ and $f_-(x,\xi)g_-(x,\xi)$.
(3) $\partial_\xi^\beta\big(D_x^\alpha p_+\big)\big(\widetilde{\nabla}_\xi\varphi_+(x,\xi,\eta),\eta\big)\Big|_{\xi=\eta}$ and $\partial_\xi^\beta\big(D_x^\alpha p_-\big)\big(\widetilde{\nabla}_\xi\varphi_-(x,\xi,\eta),\eta\big)\Big|_{\xi=\eta}$ are compatible.

Proof. The assertions follow from a direct computation. In order to prove (1), we let $\partial_i = \partial/\partial\xi_i$ and take notice of

$$\partial_i f_+(x,\xi) \sim \sum_{m=0}^{\infty} |\xi|^{-m-1}\Big\{ - m\widehat{\xi}_i f_m(x,\widehat{\xi}\,) + \sum_{j=1}^{n} \Big(\partial_j f_m\Big)(x,\widehat{\xi}\,)(\delta_{ij} - \widehat{\xi}_i\widehat{\xi}_j)\Big\},$$

$$\partial_i f_-(x,\xi) \sim \sum_{m=0}^{\infty} (-|\xi|)^{-m-1}\Big\{ m\widehat{\xi}_i f_m(x,-\widehat{\xi}\,) + \sum_{j=1}^{n} \Big(\partial_j f_m\Big)(x,-\widehat{\xi}\,)(\delta_{ij} - \widehat{\xi}_i\widehat{\xi}_j)\Big\}.$$

The assertion (2) is obvious. To show (3), note that by Definition 3.7

$$\partial_i\varphi_-(x,\xi) = -\widehat{\xi}_i\varphi(x,-\widehat{\xi}) + \sum_{j=1}^{n}\left(\frac{\partial\varphi}{\partial\xi_j}\right)(x,-\widehat{\xi})(\delta_{ij} - \widehat{\xi}_i\widehat{\xi}_j) = \left(\partial_i\varphi_+\right)(x,-\xi).$$

Since $\nabla_\xi\varphi_\pm$ are homogeneous of degree 0, this means that $\nabla_\xi\varphi_+(x,\xi)$ and $\nabla_\xi\varphi_-(x,\xi)$ are compatible. Since $D_x^\alpha p_+(x,\xi)$ and $D_x^\alpha p_-(x,\xi)$ are compatible, one can prove (3) inductively. □

By Lemma 5.6 and (5.16), $b_\pm(x,\xi)$ are compatible. This proves Lemma 5.4.

5.2. Recovering partial regularities near infinity. Let us rewrite Theorem 5.6 in the operator form. Let $D_j(s)$ and $r_j(x,\theta)$ be as in (5.8) and (5.12), rspectively. We put

$$\left(\mathcal{R}_+^{(j)} f\right)(s,\theta) = \int_{\mathbf{R}^n} D_j(s - \varphi(x,\theta)) r_j(x,\theta) f(x)dx.$$

Lemma 5.7. *(1) For any $j,m \geq 0$, we have $\mathcal{R}_+^{(j)} \in \mathbf{B}(H^m;\widehat{H}^{j+m})$.*
(2) Let $\chi_R(s)$ be as in Lemma 5.2. Then for any N

$$\chi_R(s)\mathcal{R}_+ \equiv \chi_R(s)\sum_{j=0}^{N-1}\mathcal{R}_+^{(j)}$$

modulo a regularizer of order N.

Proof. To prove the assertion (1), we have only to note that the operator

$$\int_{\mathbf{R}^n} e^{-i\varphi(x,\xi)}\overline{r_j(x,\xi/|\xi|)}\chi_\infty(|\xi|)f(x)dx$$

is L^2-bounded. The assertion (2) has been proven in Theorem 5.5. □

The purpose of this section is to prove Lemma 1.13 in a localized form. Let us recall that the stationary phase method shows the scattered waves propagate to infinity along the directions close to $\widehat{\xi} = \pm\widehat{x}$. With this in mind, we prepare the following notion.

Definition 5.8. For a constant $0 < \delta < 1$, let $S(\delta)$ be the set of symbols $p(x,\xi) \in S^0$ such that $\operatorname{supp} p \subset \{(x,\xi)\,;\, |\widehat{x}\cdot\widehat{\xi}\,| < \delta\}$. We say that $f \in L^2(\mathbf{R}^n)$ is *regular in non-scattering region* if there exists $0 < \delta < 1$ such that $p(x,D_x)f \in H^\infty(\mathbf{R}^n)$, $\forall p(x,\xi) \in S(\delta)$.

If f is regular in non-scattering region, its wave front set, denoted by WF (f), satisfies WF $(f) \cap \{|\widehat{x}\cdot\widehat{\xi}| < \delta\} = \emptyset$. As an example, let $B_R = \{x \in \mathbf{R}^n; |x| < R\}$. If $f \in H^\infty(B_R)$ and $f(x) = 0$ for $|x| > R$, by the stationary phase method, $f(x)$ is shown to be regular in non-scattering region (see Lemma 6.8). The necessity of this notion will be made clear in the proof of Lemma 5.9. We put $\widehat{H}^m(s>\sigma)=\widehat{H}^m(I_\sigma)$ and $H^m(|x|>\rho)=H^m(B_\rho^c)$, where $I_\sigma=(\sigma,\infty)$ and $B_\rho^c=\{x\in\mathbf{R}^n;|x|>\rho\}$.

Lemma 5.9. *There exist constants $\rho>\sigma>0$ such that the following assertion holds: If $f\in L^2(\mathbf{R}^n)$ is regular in non-scattering region and $\mathcal{R}_+^{(0)}f\in\widehat{H}^m(s>\sigma)$ for some $m\geq 0$, then $f\in H^m(|x|>\rho)$. Moreover ρ can be chosen arbitrarily close to σ.*

Proof. The proof is complicated and is split into several parts. Let $\chi(s)\in C^\infty(\mathbf{R})$ be such that $\chi(s)=1$ $(s>\sigma+2)$, $\chi(s)=0$ $(s<\sigma+1)$, where $\sigma>0$ will be determined later. We put

$$u(s,\theta)=\chi(s)\int_{-\infty}^{\infty}\int_{\mathbf{R}^n}e^{ik(s-\varphi(x,\theta))}(-ik+0)^{\frac{n-1}{2}}\chi_\infty(|k|)r_0(x,\theta)f(x)dxdk,$$

and assume that $u\in\widehat{H}^m$. We take $\psi_0(t),\psi_\infty(t)\in C^\infty(\mathbf{R})$ such that $\psi_0(t)+\psi_\infty(t)=1$ $(t\in\mathbf{R})$, $\psi_\infty(t)=1$ $(t>2)$, $\psi_\infty(t)=0$ $(t<1)$, and $c_0(t),c_1(t)\in C^\infty(\mathbf{R})$ such that $c_0(t)+c_1(t)=1$ $(t\in\mathbf{R})$, $c_1(t)=1$ $(|t|>\delta/2)$, $c_1(t)=0$ $(|t|<\delta/4)$, where δ is the constant appearing in the assumption of regularity in non-scattering region for f. We split $f(x)$ into 3 parts :

$$f(x)=\psi_\infty(|x|)c_1(\widehat{x}\cdot\theta)f(x)+\psi_0(|x|)f(x)+\psi_\infty(|x|)c_0(\widehat{x}\cdot\theta)f(x).$$

1st Step. We put

$$u_1(s,\theta)=\chi(s)\int_{-\infty}^{\infty}\int_{\mathbf{R}^n}e^{ik(s-\varphi(x,\theta))}(-ik+0)^{\frac{n-1}{2}}\chi_\infty(|k|)\cdot r_0(x,\theta)\psi_\infty(|x|)c_0(\widehat{x}\cdot\theta)f(x)dxdk,$$

and show that $u_1\in\widehat{H}^\infty$. This is proved if we show

$$v_1(x):=(2\pi)^{-n}\iint_{\mathbf{R}^n\times\mathbf{R}^n}e^{i(x\cdot\xi-\varphi(y,\xi))}\chi_\infty(|\xi|)r_0(y,\pm\widehat{\xi})\psi_\infty(|y|)c_0(\pm\widehat{y}\cdot\widehat{\xi})f(y)dyd\xi$$

is in H^∞. In view of (3.6), we have

$$w_1:=I_{\varphi,1}v_1=Pf,$$

where, modulo a regularizer, P is a ψDO whose symbol is supported in the region $\{|\widehat{x}\cdot\widehat{\xi}|<\delta\}$. Therefore $w_1\in H^\infty$, since f is regular in non-scattering region. Computing $I_{\varphi^*,1}w_1$ and using (3.7), we then have

$$(1+P_1+P_2+\cdots)v_1=g,$$

where $P_i\in S^{-i}$ and $g\in H^\infty$. By multiplying suitable ψDO's, we have $v_1\in H^\infty$.

2nd Step. Next we consider

$$\chi(s)\int_{-\infty}^{\infty}\int_{\mathbf{R}^n}e^{ik(s-\varphi(x,\theta))}(-ik+0)^{\frac{n-1}{2}}\chi_\infty(|k|)r_0(x,\theta)\cdot\big[\psi_\infty(|x|)c_1(\widehat{x}\cdot\theta)+\psi_0(|x|)\big]f(x)dxdk. \tag{5.18}$$

Let $\widetilde{\chi}(s)\in C^\infty(\mathbf{R})$ be such that $\widetilde{\chi}(s)=1$ $(s>\sigma)$, $\widetilde{\chi}(s)=0$ $(s<\sigma-1)$. By integration by parts, the operator

$$\chi(s)\iint e^{ik(s-\varphi(x,\theta))}\,(1-\widetilde{\chi}(\varphi(x,\theta)))\cdots dxdk$$

is a regularizer. In fact, since $\varphi(x,\theta)<\sigma$, we have $|s-\varphi(x,\theta)|\geq C(s+|x|)$ for a constant $C>0$ thanks to the factor $\psi_\infty(|x|)c_1(\widehat{x}\cdot\theta)+\psi_0(|x|)$.

We are thus led to consider

$$\begin{aligned} u_2(s,\theta) =& \chi(s)\int_{-\infty}^{\infty}\int_{\mathbf{R}^n} e^{ik(s-\varphi(x,\theta))}(-ik+0)^{\frac{n-1}{2}}\chi_\infty(|k|) \\ & \cdot \widetilde{\chi}(\varphi(x,\theta))r_0(x,\theta)\big[\psi_\infty(|x|)c_1(\widehat{x}\cdot\theta)+\psi_0(|x|)\big]f(x)dxdk, \end{aligned}$$

which belongs to $\widehat{H}^m$. Here we choose σ large enough so as to be able to apply Lemma 2.4, and make the change of variables $x \to (t,y) = (t(x,\theta), y(x,\theta))$. Since $t(x,\theta) = \varphi(x,\theta)$ by virtue of Lemma 2.5, the above integral is rewritten as

$$\frac{1}{2\pi}\chi(s)\iint e^{ik(s-t)}q_0(t,k,y,\theta)\widetilde{f}(t,y,\theta)dkdtdy =: v_2(s,\theta), \tag{5.19}$$

$$\begin{aligned} q_0(t,k,y,\theta) =& 2\pi(-ik+0)^{\frac{n-1}{2}}\chi_\infty(|k|) \\ & \cdot \widetilde{\chi}(t)r_0(x,\theta)\left[\psi_\infty(|x|)c_1(\widehat{x}\cdot\theta)+\psi_0(|x|)\right]J(t,y,\theta), \end{aligned} \tag{5.20}$$

$J(t,y,\theta)$ being the Jacobian of the map : $x \to (t,y)$, and in the expression of q_0, x should be read as $x(t,y,\theta)$, $\widetilde{f}(t,y,\theta) = f(x)$. This reduces the problem to the 1-dimensional ψDO calculus.

Let Q_0 be the 1-dimensional ψDO with symbol $\overline{q_0(t,k,y,\theta)}$, where y,θ are regarded as parameters. Then (5.19) reads

$$\int \chi(s)\left(Q_0^*\widetilde{f}(\cdot,y,\theta)\right)(s)dy = v_2(s,\theta),$$

where $v_2 \in \widehat{H}^m$. By ψDO calculus, we have modulo $\widehat{H}^{m+1}$

$$\int \chi(s)\left(Q_0^*\widetilde{f}(\cdot,y,\theta)\right)(s)dy \equiv \int \left(P_0^*\widetilde{f}(\cdot,y,\theta)\right)(s)dy \in \widehat{H}^m, \tag{5.21}$$

where the symbol of P_0 is the product of $\chi(t)$ and $q_0(t,k,y,\theta)$, namely, it is obtained with $\widetilde{\chi}(t)$ replaced by $\chi(t)$ in (5.20). Passing to the Fourier transformation with respect to s in (5.21), we get

$$\begin{aligned} &\iint e^{-ikt}(-ik+0)^{\frac{n-1}{2}}\chi_\infty(|k|)\chi(t)r_0(x,\theta) \\ &\cdot\left[\psi_\infty(|x|)c_1(\widehat{x}\cdot\theta)+\psi_0(|x|)\right]J(t,y,\theta)\widetilde{f}(t,y,\theta)dtdy =: w(k,\theta), \end{aligned}$$

where $w(k,\theta)$ satisfies

$$\int (1+|k|)^{2m}\|w(k,\cdot)\|^2_{L^2(S^{n-1})}dk < \infty.$$

Transforming back to the original variable x, we get

$$\begin{aligned} &(-ik+0)^{\frac{n-1}{2}}\chi_\infty(|k|)\int e^{-ik\varphi(x,\theta)}\chi(\varphi(x,\theta))r_0(x,\theta) \\ &\cdot\left[\psi_\infty(|x|)c_1(\widehat{x}\cdot\theta)+\psi_0(|x|)\right]f(x)dx = w(k,\theta). \end{aligned} \tag{5.22}$$

We try to regard (5.22) as a FIO putting $\xi = k\theta$. Here we must note that the term $\chi(\varphi(x,\theta))$ behaves like

$$|\partial_\theta^\alpha\chi(\varphi(x,\theta))| \le C_\alpha(1+|x|)^{|\alpha|},$$

which seems to cause a trouble in defining a suitable class of symbols. However thanks to the locaization factor $\psi_\infty(|x|)c_1(\widehat{x}\cdot\theta)+\psi_0(|x|)$, the amplitude $b(x,\theta)$ of (5.22) has the estimate

$$|\partial_\theta^\alpha\partial_x^\beta b(x,\theta)| \le C_{\alpha\beta}(1+|x|)^{-|\beta|}.$$

In fact, by the estimate (2.9), on the support of $\chi'(\varphi(x,\theta))$, $|x\cdot\theta|$ is bounded. Due to the locaization factor $\psi_\infty(|x|)c_1(\widehat{x}\cdot\theta)+\psi_0(|x|)$, if $|x\cdot\theta|$ is bounded so is x. Therefore, the derivatives of $\chi(\varphi(x,\theta))$ does no harm to our analysis. This is the reason why we have introduced the notion of regularity in non-scattering region.

3rd Step. We consider (5.22) separately in the region $k>0$ and $k<0$. For $\pm k>0$, we put $k=\pm|\xi|$ and $\theta=\pm\widehat{\xi}$. Then we can rewrite (5.22) as

$$\int e^{-i\varphi_\pm(x,\xi)}p_\pm(x,\xi)f(x)dx = g_\pm(\xi),$$

where $p_\pm(x,\xi)\in S^0$ has its support in the region $\pm\widehat{x}\cdot\widehat{\xi}>\delta/3$ and $g_\pm(\xi)$ satisfies $(1+|\xi|)^m g_\pm(\xi)\in L^2(\mathbf{R}^n)$. We now mulitiply $e^{i\varphi_\pm(x,\xi)}$ and integrate in ξ. Then we have by FIO calculus

$$q_\pm(x,D_x)\chi(|x|)f\in H^m,$$

where $q_\pm(x,\xi)\in S^0$, $q_\pm(x,\xi)=1$ for $\pm\widehat{x}\cdot\widehat{\xi}>\delta$ and $|x|>1$, $q_\pm(x,\xi)=0$ for $\pm\widehat{x}\cdot\widehat{\xi}<\delta/5$ and $|x|>1$, and $\chi(t)\in C^\infty(\mathbf{R})$ such that $\chi(t)=1\ \ (t>\sigma+2)$, $\chi(t)=0\ \ (t<\sigma+1)$. Taking into account that f is regular in non-scattering region, we finally prove that $f\in H^m(|x|>\rho)$ for $\rho=s+2$. By examining the proof, we see that ρ can be chosen arbitrarily close to σ. □

Theorem 5.10. *There exist $\rho>\sigma>0$ such that if f is regular in non-scattering region and $\mathcal{R}_+f\in\widehat{H}^m(s>\sigma)$ for some $m\geq 1$, then $f\in H^m(|x|>\rho)$. Moreover ρ can be chosen arbitrarily close to σ.*

Proof. If $\mathcal{R}_+f\in\widehat{H}^1(s>\sigma)$, we have $\mathcal{R}_+^{(0)}f\in\widehat{H}^1(s>\sigma)$ by Lemma 5.6 (1). Therefore the case $m=1$ follows from Lemma 5.9. Let us assume the theorem when $m=k-1$. Then if $\mathcal{R}_+f\in\widehat{H}^k(s>\sigma)$, we have $f\in H^{k-1}(|x|>\rho)$. Therefore if $j\geq 1$, we have $\mathcal{R}_+^{(j)}f\in\widehat{H}^k(s>\sigma)$, which implies that $\mathcal{R}_+^{(0)}f\in\widehat{H}^k(s>\sigma)$. By Lemma 5.9, we have $f\in H^k(|x|>\rho)$, which completes the proof. □

6. Singular support theorem

6.1. Envelope. Let us first recall the classical notion of envelope. Let U and Ω be open sets in $\mathbf{R}^n$ and $\mathbf{R}^{n-1}$, respectively. Suppose a real-valued function $\phi(x,\omega)\in C^\infty(U\times\Omega)$ satisfies

$$\det\left(\nabla_x\phi,\frac{\partial}{\partial\omega_1}\nabla_x\phi,\cdots,\frac{\partial}{\partial\omega_{n-1}}\nabla_x\phi\right)\neq 0,\quad x\in U,\quad \omega\in\Omega, \tag{6.1}$$

$$\det\left(\frac{\partial^2\phi}{\partial\omega_i\partial\omega_j}\right)_{1\leq i,j\leq n-1}\neq 0,\quad x\in U,\quad \omega\in\Omega. \tag{6.2}$$

Given an interval $I\subset\mathbf{R}$, we consider a family of surfaces

$$\Sigma(s,\omega)=\{x\in U\ ;\ \phi(x,\omega)=s\},\quad s\in I,\quad \omega\in\Omega.$$

Assume that for $x\in U$ there exists a unique solution $\omega(x)$ to the system of equations

$$\frac{\partial\phi}{\partial\omega_1}(x,\omega)=\cdots=\frac{\partial\phi}{\partial\omega_{n-1}}(x,\omega)=0. \tag{6.3}$$

Then the envelope $\Sigma(s)$ of $\{\Sigma(s,\omega)\}_{\omega\in\Omega}$ is defined by

$$\Sigma(s)=\{x\in U\ ;\ \phi(x,\omega(x))=s\}.$$

We put $y = (s, \omega)$ and $f(x, y) = \big(f_1(x, y), \cdots, f_n(x, y)\big)$, where

$$f_i(x, y) = \partial\phi(x, \omega)/\partial\omega_i, \quad (1 \le i \le n-1), \quad f_n(x, y) = \phi(x, \omega) - s.$$

Then the equation for the envelope and the conditions (6.1), (6.2) are rewritten as

$$f(x, y) = 0, \quad \det\left(\frac{\partial f}{\partial x}\right) \neq 0, \quad \det\left(\frac{\partial f}{\partial y}\right) \neq 0.$$

Hence by the implicit function theorem the map : $U \ni x \to y(x) = (s(x), \omega(x)) \in I \times \Omega$ is a diffeomorphism. Let $X(s, \omega)$ be its inverse.

Lemma 6.1. *Let $g_{ij}(x)dx^i dx^j$ be a Riemannian metric on U and put $h(x, \xi) = \frac{1}{2}g^{ij}(x)\xi_i\xi_j$. Assume that $\phi(x, \omega)$ satisfies the eikonal equation*

$$h(x, \nabla_x\phi(x, \omega)) = 1/2, \quad x \in U, \quad \omega \in \Omega. \tag{6.4}$$

(1) We put $\Phi(x) = \phi(x, \omega(x))$. Then $\Phi(x)$ also satisfies the eikonal equation

$$h(x, \nabla_x\Phi(x)) = 1/2, \quad x \in X.$$

(2) Let $P(s, \omega) = (\nabla_x\Phi)(X(s, \omega))$. Then we have for $s \in I$ and $\omega \in \Omega$,

$$\begin{cases} \dfrac{\partial}{\partial s}X(s, \omega) = \left(\dfrac{\partial h}{\partial \xi}\right)(X(s, \omega), P(s, \omega)), \\ \dfrac{\partial}{\partial s}P(s, \omega) = -\left(\dfrac{\partial h}{\partial x}\right)(X(s, \omega), P(s, \omega)). \end{cases} \tag{6.5}$$

Proof. By virtue of (6.3), we have

$$\nabla_x\Phi(x) = (\nabla_x\phi)(x, \omega(x)), \tag{6.6}$$

which implies (1). We let $k(x, \omega) = (\nabla_x\phi)(x, \omega)$ and differentiate (6.4) by ω_j to have

$$\left(\frac{\partial h}{\partial \xi}\right)(x, k(x, \omega)) \cdot \frac{\partial k}{\partial \omega_j}(x, \omega) = 0, \quad 1 \le j \le n-1.$$

Using (6.6), we have $P(s, \omega) = k(X(s, \omega), \omega)$, hence

$$\left(\frac{\partial k}{\partial \omega_j}\right)(P(s, \omega), \omega) \cdot \left(\frac{\partial h}{\partial \xi}\right)(X(s, \omega), P(s, \omega)) = 0, \quad 1 \le j \le n-1. \tag{6.7}$$

On the other hand, we have by differentiating $\big(\partial\phi/\partial\omega_j\big)(X(s, \omega), \omega) = 0$ by s

$$\left(\frac{\partial k}{\partial \omega_j}\right)(X(s, \omega), \omega) \cdot \frac{\partial X}{\partial s}(s, \omega) = 0, \quad 1 \le j \le n-1. \tag{6.8}$$

By (6.1), $\partial k/\partial\omega_1, \cdots, \partial k/\partial\omega_{n-1}$ are linearly independent. Therefore by (6.7) and (6.8) we have

$$\frac{\partial X}{\partial s}(s, \omega) = \lambda(s, \omega)\left(\frac{\partial h}{\partial \xi}\right)(X(s, \omega), P(s, \omega))$$

for some scalar function $\lambda(s, \omega)$. Differentiating $s = \phi(X(s, \omega), \omega)$ with respect to s, we then have

$$1 = k \cdot \frac{\partial X}{\partial s} = \lambda k \cdot \left(\frac{\partial h}{\partial \xi}\right)(X, k) = 2\lambda h(X, P) = \lambda.$$

Finally by differentiating $P_i(s,\omega) = (\partial\phi/\partial x_i)\,(X(s,\omega),\omega)$ we have

$$\begin{aligned}\frac{\partial}{\partial s}P_i(s,\omega) &= \sum_j \left(\frac{\partial^2\phi}{\partial x_i\partial x_j}\right)(X(s,\omega),\omega)\frac{\partial X_j}{\partial s}(s,\omega)\\ &= \left(\frac{\partial k}{\partial x_i}\right)(X(s,\omega),\omega)\cdot\left(\frac{\partial h}{\partial \xi}\right)(X(s,\omega),P(s,\omega))\\ &= -\left(\frac{\partial h}{\partial x_i}\right)(X(s,\omega),P(s,\omega)),\end{aligned}$$

since by differentiating $h(x,k(x,\omega)) = 1/2$, we get

$$\left(\frac{\partial h}{\partial x_i}\right)(x,k(x,\omega)) + \left(\frac{\partial h}{\partial \xi}\right)(x,k(x,\omega))\cdot\frac{\partial k}{\partial x_i}(x,\omega) = 0. \quad \Box$$

Let us note that by (6.6), $\Sigma(s,\omega)$ is tangent to $\Sigma(s)$ at $X(s,\omega)$.

We now put

$$\Sigma^{(\pm)}(s,\theta) = \{x\in\mathbf{R}^n;\varphi_\pm(x,\theta)=s\},$$

and construct the envelope of $\{\Sigma^{(\pm)}(s,\theta)\}_{\theta\in S^{n-1}}$. Since $\varphi_+(x,\theta) = -\varphi_-(x,-\theta)$ by Theorem 2.8 (3), we have

$$\Sigma^{(+)}(s,\theta) = \Sigma^{(-)}(-s,-\theta).$$

Therefore we have only to consider $\varphi_+(x,\theta)=\varphi(x,\theta)$. For $\varphi(x,\theta)$, the assumptions (6.1), (6.2) are satisfied on the region $\{|x|>r_0\}\times S^{n-1}$, where $r_0>0$ is chosen largre enogh. We consider the equation

$$\nabla_\theta\varphi(x,\theta)=0,\quad x\cdot\theta>0, \tag{6.9}$$

∇_θ being the gradient on S^{n-1}, which corresponds to (6.3). If $\varphi(x,\theta)=x\cdot\theta$, the solution is unique and given by $\theta=\widehat{x}$. Since $\partial_x^\alpha(\varphi(x,\theta)-x\cdot\theta) = O(|x|^{-|\alpha|-\epsilon_0})$, we see that (6.9) has a unique solution $\theta(x) = \widehat{x}+O(|x|^{-\epsilon_0})$. Let $s(x)=\varphi(x,\theta(x))$ and $X(s,\theta)$ be the inverse of the map : $x\to(s(x),\theta(x))$. We summarize the properties of these diffeomorphisms in the following theorem. We put $\Sigma(s,\theta)=\Sigma^{(+)}(s,\theta)$.

Theorem 6.2. *There exist $r_0>0$ and $s_0>0$ for which the following assertions hold.*
(1) For any $x\in\mathbf{R}^n$ such that $|x|>r_0$, there exists a unique $\theta(x)\in S^{n-1}$ satsifying $\big(\nabla_\theta\varphi\big)(x,\theta(x))=0$ and $\theta(x)\cdot x>0$. We define

$$\Phi(x)=\varphi(x,\theta(x))\quad\text{for}\quad |x|>r_0,$$

and extend it smoothly for $|x|\le r_0$ so that $\Phi(x)$ is monotone increasing with respect to $|x|$. Then $\Phi(x)\sim|x|$ as $|x|\to\infty$ and satisfies the eikonal equation

$$g^{ij}(x)(\partial_i\Phi(x))(\partial_j\Phi(x))=1\quad\text{for}\quad|x|>r_0.$$

(2) For any $s>s_0$, the set

$$\Sigma(s)=\{x\in\mathbf{R}^n;\Phi(x)=s\}$$

is a strictly convex compact hypersurface.
(3) For any $s>s_0$ and $x\in\Sigma(s)$, $\Sigma(s)$ is tangent to $\Sigma(s,\theta(x))$ at x. Moreover $\theta(x)$ is a unique point θ in S^{n-1} such that $\Sigma(s)$ is tangent to $\Sigma(s,\theta)$ at x. We also have for $|x|>r_0$

$$\max_{\theta\in S^{n-1}}\varphi(x,\theta)=\Phi(x), \tag{6.10}$$

and the maximum is attained if and only if $\theta = \theta(x)$.
(4) For any $s > s_0$ *and* $\theta \in S^{n-1}$, *there exists a unique* $X(s,\theta) \in \Sigma(s)$ *such that* $\Sigma(s,\theta)$ *is tangent to* $\Sigma(s)$ *at* $X(s,\theta)$. *We also have for any* $\theta \in S^{n-1}$

$$\max_{x\in\Sigma(s)} \varphi(x,\theta) = s = \Phi(X(s,\theta)), \tag{6.11}$$

and the maximum is attained if and only if $x = X(s,\theta)$.
(5) For any $s > s_0$, *the map*

$$S^{n-1} \ni \theta \to X(s,\theta) \in \Sigma(s)$$

is a diffeomorphism and its inverse is given by

$$\Sigma(s) \ni x \to \theta(x) \in S^{n-1}.$$

(6) The map

$$X : (s_0,\infty) \times S^{n-1} \ni (s,\theta) \to X(s,\theta) \in \mathbf{R}^n$$

is a diffeomorphism whose image contains the region $\{x\,;\,|x| > r_0\}$. *The inverse of this map is*

$$X^{-1} : x \to \big(\Phi(x), \theta(x)\big).$$

It has the following estimates ($\widehat{x} = x/|x|$)

$$|\partial_x^\alpha(\Phi(x) - |x|)| \leq C_\alpha(1+|x|)^{-\epsilon_0-|\alpha|}, \quad \forall\alpha, \tag{6.12}$$

$$|\partial_x^\alpha(\theta(x) - \widehat{x})| \leq C_\alpha(1+|x|)^{-1-\epsilon_0-|\alpha|}, \quad \forall\alpha. \tag{6.13}$$

(7) The diffeomorphism X^{-1} *gives the geodesic polar coordinates in a neighborhood of infinity, and in this coordinate system the Riemannian metric* $G = g_{ij}(x)dx^i dx^j$ *takes the following form*

$$X^*G = (ds)^2 + \sum_{i,j=1}^{n-1} h_{ij}(s,\theta)d\theta^i d\theta^j.$$

Proof. As is noted above $\varphi(x,\theta) = x\cdot\theta$ for the Euclidean metric, hence $\theta(x) = \widehat{x}$, $\Phi(x) = |x|$, and the theorem is obvious. The assertion (1) follows from Lemma 6.1. Since $\Sigma(s)$ is a slight perturbation of sphere, (2) follows. The first part of the assertion (3) is obvious. We shall prove (6.10). If $\varphi(x,\theta)$ attains its maximum at θ, $(\nabla_\theta\varphi)(x,\theta) = 0$ holds. This equation has two solutions $\widetilde{\theta}_\pm$ such that $\pm x\cdot\widetilde{\theta}_\pm > 0$. The Hessian matrix of $\varphi(x,\theta)$ at $\widetilde{\theta}_+$ ($\widetilde{\theta}_-$) is negative (positive) definite. Hence the maximum is attained at $\widetilde{\theta}_+$, furthermore, $\widetilde{\theta}_+ = \theta(x)$. The first part of (4) is obvious. At the point x where $\varphi(x,\theta)$ attains its maximum on $\Sigma(s)$, $\nabla_x\Phi(x)$ and $\nabla_x\varphi(x,\theta)$ are propotional. This is just the point on which two surfaces $\Sigma(s)$ and $\Sigma(s,\theta)$ are tangent each other, hence (6.11) holds. The mapping properties in (5) and (6) are clear. From the equation $\nabla_\theta\varphi(x,\theta) = 0$, we get $\nabla_\theta\widehat{x}\cdot\theta = O(|x|^{-1-\epsilon_0})$, from which (6.13) follows. The estimate (6.12) then follows from Theorem 2.8 (1). Let us prove (7). By the equation (6.5), $X(s,\theta)$ is a geodesic. Hence $(s(x),\theta(x))$ are geodesic polar coordinates. We put $\overline{x}^i = \theta_i(x)$ $(1 \leq i \leq n-1)$, $\overline{x}^n = \Phi(x)$. Then the associated Riemannian metric $\overline{g}_{ij}$ is computed as follows :

$$\overline{g}^{nn} = g^{ij}\frac{\partial\overline{x}^n}{\partial x^i}\frac{\partial\overline{x}^n}{\partial x^j} = g^{ij}\left(\partial_i\Phi\right)\left(\partial_j\Phi\right) = 1,$$

$$\overline{g}^{nk} = g^{ij}\frac{\partial\overline{x}^n}{\partial x^i}\frac{\partial\overline{x}^k}{\partial x^j} = g^{ij}\left(\partial_i\Phi\right)\left(\partial_j\theta_k\right) = 0,$$

for $1 \leq k \leq n-1$. Here we have used the equation (7.5) and

$$0 = \frac{\partial \theta_k}{\partial s} = \frac{\partial \theta_k}{\partial x^m}\frac{\partial X^m}{\partial s} = (\partial_m \theta_k)\, g^{im} P_i = 0.$$

This proves (7). □

Corollary 6.3. *For large $|x|$, we have $\varphi(x,\theta) \leq \Phi(x)$, and the equality holds if and only if $\theta = \theta(x)$, equivalently, $x = X(s,\theta)$ for some $s > s_0$.*

6.2. Singularity expansions II. Our next aim is to compute an asymptotic expansion around $s = \sigma$ of the integral (coupling of distribution and test function, actually)

$$\int_{\mathbf{R}^n} (s - \varphi(x,\theta))_-^\alpha (\sigma - \Phi(x))_+^\beta f(x)dx, \quad f \in C_0^\infty(\mathbf{R}^n). \tag{6.14}$$

For any $\theta \in S^{n-1}$, we have constructed a bicharacteristic $x(t,y,\theta)$, $p(t,y,\theta)$ having the properties in Lemma 2.2. We use the variables t, y to calculate (6.14), which is possible by virtue of Lemma 2.4. In perfoming the computation below it will be helpful to recall that for the Euclidean metric $\sum_{i=1}^n (dx^i)^2$, $x(t,y,\theta) = t\theta + y$, $\theta \cdot y = 0$, $\varphi(x,\theta) = x \cdot \theta$ and $\Phi(x) = |x|$.

Let $\widetilde{\Phi}(t,y,\theta) = \Phi(x(t,y,\theta))$. Then since $t = \varphi(x,\theta)$ by Lemma 2.5 we have by Corollary 6.3

$$\widetilde{\Phi}(t,y,\theta) - t = \Phi(x) - \varphi(x,\theta) \geq 0,$$

and for a fixed t the last equality holds only at one point, which we denote by $y(t,\theta)$. At $y(t,\theta)$ the surface $t = \Phi(x)$ is tangent to the surafce $t = \varphi(x,\theta)$. Therefore $(t, y(t,\theta))$ is the coordiante of $X(t,\theta)$ given in Theorem 6.2 (4). By the Taylor expansion with respect to y we have

$$\widetilde{\Phi}(t,y,\theta) - t = \frac{1}{2}\langle A(y - y(t,\theta)), y - y(t,\theta)\rangle + O(|y - y(t,\theta)|^3),$$

as $y \to y(t,\theta)$, where

$$A = A(t,\theta) = \left(\frac{\partial^2 \widetilde{\Phi}}{\partial y_i \partial y_j}(t, y(t,\theta), \theta)\right)$$

is a positive definite matrix and $\langle\ ,\ \rangle$ is the Euclidean inner product of $\mathbf{R}^{n-1}$. By the Morse lemma, one can find a function $z = z(t,y,\theta)$ defined in a neighborhood of $y(t,\theta)$ such that

$$\Phi(x) = \widetilde{\Phi}(t,y,\theta) = t + \frac{1}{2}\langle A(t,\theta)z, z\rangle,$$

and $z = y - y(t,\theta) + O(|y - y(t,\theta)|^2)$. We now make a new change of variables: $x \to (t,z)$ and put $\widetilde{f}(t,z,\theta) = f(x)$. We denote by

$$J_P(t,z,\theta) = |\det\left(\partial x/\partial(t,z)\right)|$$

the associated Jacobian. (Here the subscript P menas that we are using the plane wave like characteristic surface $t = \varphi(x,\theta)$). Then we have

$$\begin{aligned} &\int \left(s - \varphi(x,\theta)\right)_-^\alpha \left(\sigma - \Phi(x)\right)_+^\beta f(x)dx \\ &= \iint \left(s - t\right)_-^\alpha \left(\sigma - t - \frac{1}{2}\langle A(t,\theta)z, z\rangle\right)_+^\beta \widetilde{f}(t,z,\theta) J_P(t,z,\theta) dtdz. \end{aligned} \tag{6.15}$$

We say that $g(s,\theta)$ admits the asymptotic expansion

$$g(s,\theta) \sim \sum_{k=0}^{\infty}(\sigma - s)_+^{\lambda+k} g_k(\theta), \quad g_k \in C^\infty(S^{n-1})$$

around $s = \sigma$, if there exists $\epsilon_0 > 0$ with the following property. For any $N > 0$, there exist $G_N(s,\theta), H_N(s,\theta) \in C^\infty(\mathbf{R}; L^2(S^{n-1}))$ such that

$$g(s,\theta) = \sum_{k=0}^{N-1}(\sigma - s)_+^{\lambda+k} g_k(\theta) + (\sigma - s)_+^{\lambda+N} G_N(s,\theta) + H_N(s,\theta)$$

holds for $|s-\sigma| < \epsilon_0$. Similarly, we say that $f(x)$ admits the asymptotic expansion

$$f(x) \sim \sum_{k=0}^{\infty}(\sigma - \Phi(x))_+^{\lambda+k} f_k(\theta), \quad f_k(\theta) \in C^\infty(\Sigma(\sigma))$$

around $\Phi(x) = \sigma$, where $\Sigma(\sigma) = \{\sigma = \Phi(x)\}$ and θ denotes the local coordinate on $\Sigma(\sigma)$, if there exists $\epsilon_0 > 0$ with the following property. For any $N > 0$, there exist $G_N(x), H_N(x) \in C^\infty(\mathbf{R}^n)$ such that

$$f(x) = \sum_{k=0}^{N-1}(\sigma - \Phi(x))_+^{\lambda+k} f_k(\theta) + (\sigma - \Phi(x))_+^{\lambda+N} G_N(x) + H_N(x)$$

holds when $|\Phi(x) - \sigma| < \epsilon_0$.

Lemma 6.4. *Let $g(t,z) \in C_0^\infty(\mathbf{R} \times \mathbf{R}^{n-1})$, and $\sigma > 0$ be a sufficiently large constant. Then if $\beta > -1$, we have the following asymptotic expansion around $s = \sigma$*

$$\begin{aligned} &\iint (s-t)_-^{\alpha}\left(\sigma - t - \frac{1}{2}\langle A(t,\theta)z, z\rangle\right)_+^{\beta} g(t,z)\,dtdz \\ &\sim \sum_{k=0}^{\infty}(\sigma - s)_+^{\alpha+\beta+\frac{n+1}{2}+k}\left(P_k^{(\alpha,\beta)} g\right)(\sigma, 0), \end{aligned} \tag{6.16}$$

where $P_k^{(\alpha,\beta)}$ is a differential operator having the following form

$$P_k^{(\alpha,\beta)} = \sum_{\substack{m+|\gamma|/2 \le k,\\ |\gamma| = \text{even}}} C_{km\gamma}(\alpha,\beta) p_{km\gamma}(\sigma,\theta)\partial_t^m \partial_z^\gamma. \tag{6.17}$$

If $|\gamma| = m = k = 0$, we have

$$C_{000}(\alpha,\beta) p_{000}(\sigma,\theta) = (2\pi)^{\frac{n-1}{2}} \det A(\sigma,\theta)^{-1/2}. \tag{6.18}$$

Proof. First let us note that the left-hand side of (6.16) vanishes if $s > \sigma$. For $s < \sigma$, we put $\epsilon = \sigma - s$, $s - t = \epsilon\rho$, $z = \sqrt{2\epsilon(1+\rho)}A(t,\theta)^{-1/2}w$ and

$$\begin{aligned} g_\epsilon(\rho, w) = g\big(&\sigma - \epsilon(1+\rho), \sqrt{2\epsilon(1+\rho)}A(\sigma - \epsilon(1+\rho),\theta)^{-1/2}w\big) \\ &\cdot \det A(\sigma - \epsilon(1+\rho),\theta)^{-1/2}. \end{aligned}$$

Note that since $\sigma \geq t + \frac{1}{2}\langle Az, z\rangle \geq t$, we have $\sigma - t = \epsilon(1+\rho) \geq 0$. Then the left-hand side of (6.16) is rewritten as

$$2^{\frac{n-1}{2}} \frac{\Gamma(\beta + \frac{n+1}{2})}{\Gamma(\beta+1)} \epsilon^{\alpha+\beta+\frac{n+1}{2}} \times \int_{-1}^{0} \int_{|w|<1} (\rho)_-^{\alpha} (1+\rho)_+^{\beta+\frac{n-1}{2}} (1-|w|^2)^{\beta}\, g_\epsilon(\rho, w)\, d\rho\, dw. \tag{6.19}$$

Since $A(t,\theta)$ is a positive definite matrix and smooth in t, so is $A(t,\theta)^{-1/2}$. This follows from the well-known Dunford-Taylor integral of bounded operators (see e.g. p. 44 of [**80**]). We put $\delta = \sqrt{\epsilon(1+\rho)}$ and expand $g_\epsilon(\rho, w)$ into a Taylor series with respect to δ to see that each term of the expansion consists of the product of a function of σ, θ and

$$\delta^{2p+|\gamma|} w^{\gamma} \left(\partial_s^m \partial_z^{\gamma} g\right)(\sigma, 0), \quad m \leq p. \tag{6.20}$$

In fact, we first expand $g\big(\sigma - \delta^2, \delta y\big)$ to obtain terms like $\delta^{2m+|\gamma|} y^{\gamma} (\partial_s^m \partial_z^{\gamma} g)(\sigma, 0)$, and next expand $y = \sqrt{2} A(\sigma - \delta^2, \theta)^{-1/2} w$ and $\det A(\sigma - \delta^2, \theta)^{-1/2}$ to have (6.20). We replace $g_\epsilon(\rho, w)$ in (6.19) by this asymptotic expansion. If $|\gamma|$ is odd, $\int (1 - |w|^2)^{\beta} w^{\gamma} dw = 0$. Therefore, letting $k = p + |\gamma|/2$ and rearranging the terms, we obtain (6.16). To compute (6.18), we have only to use (5.1) and the formula

$$\int_{|w|<1} (1 - |w|^2)^{\beta} dw = \pi^{\frac{n-1}{2}} \frac{\Gamma(\beta+1)}{\Gamma(\beta + \frac{n+1}{2})}.$$

Here we have assumed $\beta > -1$ to guarantee the convergence of the integral □

Lemma 6.5. *Let $\sigma > 0$ be sufficiently large, and assume that $\beta > -1$. Then for any $f(x) \in C_0^{\infty}(\mathbf{R}^n)$, we have the following asymptotic expansion around $s = \sigma$:*

$$\int \big(s - \varphi(x,\theta)\big)_-^{\alpha} \big(\sigma - \Phi(x)\big)_+^{\beta} f(x) dx \sim \sum_{k=0}^{\infty} \big(\sigma - s\big)_+^{\alpha+\beta+\frac{n+1}{2}+k} g_k^{(\alpha,\beta)}(\sigma, \theta). \tag{6.21}$$

Each term of the expansion (6.21) is represented by a differential operator $M_k^{(\alpha,\beta)}$ on $\mathbf{R} \times S^{n-1}$ in the following way:

$$g_k^{(\alpha,\beta)}(\sigma,\theta) = \Big(M_k^{(\alpha,\beta)} f \circ X\Big)(\sigma, \theta),$$

where $X(s,\theta)$ is the diffeomorphism in Theorem 6.2 (6). In the local coordinates $M_k^{(\alpha,\beta)}$ has the following expression

$$M_k^{(\alpha,\beta)} = \sum_{j+|\gamma|/2 \leq k} C_{kj\gamma}(\alpha,\beta) m_{kj\gamma}(s,\theta) \partial_s^j \partial_\theta^{\gamma}. \tag{6.22}$$

In particular,

$$M_0^{(\alpha,\beta)} = (2\pi)^{\frac{n-1}{2}} \det\left(A(\sigma,\theta)\right)^{-1/2} J_P(\sigma, 0, \theta). \tag{6.23}$$

Proof. We plug (6.15) with (6.16). Let $X : (s,\theta) \to X(s,\theta)$ be the diffeomorphism in Theorem 6.2 (6). In the (t,y) coordinate system employed to derive (6.15), the condition $z = 0$ and $t = \sigma$ means that $y = y(\sigma,\theta)$ and $\varphi(x(\sigma, y, \theta), \theta) = \sigma$, which represents the point $X(\sigma,\theta)$. Therefore each term of the asymptotic expansion (6.21) is a derivative of $f(x)$ evaluated at $x = X(\sigma,\theta)$. Moreover

$$\begin{aligned}\partial_t\Big|_{t=s,y=y(s,\theta)} &= \sum_{i,j=1}^{n} g^{ij}(X(s,\theta))\left(\frac{\partial\varphi}{\partial x_j}\right)(X(s,\theta),\theta)\frac{\partial}{\partial x_i} \\ &= \sum_{i,j=1}^{n} g^{ij}(X(s,\theta))\left(\frac{\partial\Phi}{\partial x_j}\right)(X(s,\theta))\frac{\partial}{\partial x_i},\end{aligned}$$

which is equal to ∂_s in the coordinate system $(s,\theta) = X^{-1}(x)$. Thus we have the asymptotic expansion (6.21). The formulas (6.22) and (6.23) follow from (6.17) and (6.18). □

The first term $M_0^{(\alpha,\beta)}$ is written by geometric quantities. By a simple computation one can show that

$$(\det A(\sigma,\theta))^{-1/2} = |\nabla_x\Phi(x)|^{-(n-1)/2}\left(\det\mathcal{H}_{PS}\left(\frac{\partial x}{\partial y_i},\frac{\partial x}{\partial y_j}\right)\right)^{-1/2}\Big|_{x=X(\sigma,\theta)},$$

$$\mathcal{H}_{PS} = \mathcal{H}_P - \mathcal{H}_S,$$

where $x = x(t,y,\theta)$, $\mathcal{H}_P$ and $\mathcal{H}_S$ are second fundamental forms on $\{\sigma = \varphi(x,\theta)\}$ and $\{\sigma = \Phi(x)\}$ induced from the Euclidean metric, and

$$J_P(\sigma,0,\theta) = \left|G(x)^{-1}\nabla_x\Phi(x)\right|(\det G_S(x))^{1/2}\Big|_{x=X(\sigma,\theta)},$$

where $G(x) = \big(g_{ij}(x)\big)$, and $G_S(x)$ is the matrix of first fundamental form on $\{\sigma = \Phi(x)\}$ induced from the Euclidean metric.

Theorem 6.6. *Let $\sigma > 0$ be sufficiently large and $\lambda > -1/2$. Then for any $f \in C_0^\infty(\mathbf{R}^n)$, we have the following asymptotic expansion around $s = \sigma$*

$$\left(\mathcal{R}_+(\sigma-\Phi(x))_+^\lambda f\right)(s,\theta) \sim \sum_{k=0}^{\infty}(\sigma-s)_+^{\lambda+k} g_k^{(\lambda)}(\sigma,\theta).$$

Proof. This follows from Theorem 5.5 and Lemma 6.5. Note that $(\sigma - \Phi(x))_+^\lambda f \in L^2(\mathbf{R}^n)$ if $\lambda > -1/2$. □

In order to prove the converse of Theorem 6.6, we expand $(\sigma-\Phi(x))_+^\lambda f(x)$ into an asymptotic series $\sum_{k=0}^\infty(\sigma-\Phi(x))_+^{\lambda+k} f_k(x)$ and study the relations between f_k and g_k. We compute in the following way. For $f(x) \in C_0^\infty(\mathbf{R}^n)$, take $\chi(x) \in C_0^\infty(\mathbf{R}^n)$ such that $\chi(x) = 1$ on $\operatorname{supp} f$. Then by Taylor expansion

$$(\sigma-\Phi(x))_+^\lambda f(x) = \sum_{j=0}^{N}(\sigma-\Phi(x))_+^{\lambda+j} f_j^{(\sigma)}\chi(x) + F_N(x),$$

where $f_j^{(\sigma)}$ is a smooth function on $\{\sigma = \Phi(x)\}$ and $F_N(x)$ is a compactly supported $C^{\mu(N)}$-function, where $\mu(N) \to \infty$ as $N \to \infty$. This implies modulo $C^{\mu(N)}$-function

$$\left(\mathcal{R}_+\big((\sigma-\Phi(x))_+^\lambda f(x)\big)\right)(s,\theta) \equiv \sum_{j=0}^{N}\left(\mathcal{R}_+\big((\sigma-\Phi(x))_+^{\lambda+j} f_j^{(\sigma)}\chi(x)\big)\right)(s,\theta),$$

and up to a smooth function the right-hand side is equal to

$$\sum_{i,j}\int (s-\varphi(x))_-^{-\frac{n+1}{2}+i}(\sigma-\Phi(x))_+^{\lambda+j} r_i f_j^{(\sigma)}\,dx$$

near $s=\sigma$, since $\chi(x)\equiv 1$ near $\{\sigma=\Phi(x)\}$. Omitting the cut-off function $\chi(x)$, we express this computation as

$$\left(\mathcal{R}_+\left((\sigma-\Phi(x))_+^{\lambda} f(x)\right)\right)(s,\theta)\sim\sum_{j=0}^{\infty}\left(\mathcal{R}_+\left((\sigma-\Phi(x))_+^{\lambda+j} f_j^{(\sigma)}\right)\right)(s,\theta),$$

which will not give a confusion.

In order to write down the expansion it is convenient to use the diffeomorphism $X(s,\theta)$ in Theorem 6.2 (6). We insert the asymptotic expansion

$$\left((\sigma-\Phi(x))_+^{\lambda} f\circ X\right)(s,\theta)\sim\sum_{k=0}^{\infty}(\sigma-s)_+^{\lambda+k} f_k(\sigma,\theta)$$

into the formula in Theorem 6.6 and obtain

$$\Big(\mathcal{R}_+\Big(\sum_{k=0}^{\infty}(\sigma-\Phi(x))_+^{\lambda+k} f_k^*\Big)\Big)(\tau,\theta)\sim\sum_{k=0}^{\infty}(\sigma-\tau)_+^{\lambda+k} g_k(\lambda,\sigma,\theta),$$

where $f_k^*=f_k\circ X^{-1}$. Note that we fix σ and regard f_k^* as a function on $\{\sigma=\Phi(x)\}$. Let us look at $g_k(\lambda,\sigma,\theta)$ more precisely. Using Theorem 5.5 and Lemma 6.5, we have

$$\begin{aligned}&\Big(\mathcal{R}_+\Big(\sum_{\alpha=0}^{\infty}(\sigma-\Phi)_+^{\lambda+\alpha} f_\alpha^*\Big)\Big)(\tau,\theta)\\ &\sim\sum_{k=0}^{\infty}(\sigma-\tau)_+^{\lambda+k}\sum_{\alpha+\beta+\gamma=k} g_\gamma^{(-\frac{n+1}{2}+\beta,\lambda+\alpha)}(\sigma,\theta),\end{aligned}$$

$$g_\gamma^{(-\frac{n+1}{2}+\beta,\lambda+\alpha)}(\sigma,\theta)=M_\gamma^{(-\frac{n+1}{2}+\beta,\lambda+\alpha)} r_\beta f_\alpha^*\circ X.$$

Therefore we have

$$g_k(\lambda,\sigma,\theta)=\sum_{\alpha=0}^{k}\left(\sum_{\beta+\gamma=k-\alpha} M_\gamma^{(-\frac{n+1}{2}+\beta,\lambda+\alpha)} r_\beta\right) f_\alpha^*\circ X.$$

Hence we have the following formula

$$\begin{aligned}g_k(\lambda,\sigma,\theta)&=P_0^{(k)}(\lambda)f_k(\sigma,\theta)+P_2^{(k-1)}(\lambda)f_{k-1}(\sigma,\theta)\\ &\quad+\cdots+P_{2k}^{(0)}(\lambda)f_0(\sigma,\theta),\end{aligned}\tag{6.24}$$

where $P_{2(k-j)}^{(j)}(\lambda)$ is a differential operator with respect to θ, and $P_0^{(k)}$ is the operator of multiplication by

$$P_0^{(k)}(\sigma,\theta)=(2\pi)^{\frac{n-1}{2}}\det A(\sigma,\theta)^{-1/2} J_P(\sigma,0,\theta) r_0(X(\sigma,\theta),\theta).\tag{6.25}$$

Using (6.25), one can solve (6.24) with respect to f_j to have

$$\begin{aligned}f_k(\lambda,\sigma,\theta)&=Q_0^{(k)}(\lambda)g_k(\sigma,\theta)+Q_2^{(k-1)}(\lambda)g_{k-1}(\sigma,\theta)\\ &\quad+\cdots+Q_{2k}^{(0)}(\lambda)g_0(\sigma,\theta),\end{aligned}\tag{6.26}$$

where $Q^{(j)}_{2(k-j)}(\lambda)$ is a differential operator with respect to θ, and

$$Q^{(k)}_0(\sigma,\theta) = 1/P^{(k)}_0(\sigma,\theta).$$

Theorem 6.7. *Let $\sigma > 0$ be sufficiently large and $\lambda > -1/2$. Given any $g(s,\theta)$ having the following asymptotic expansion around $s = \sigma$*

$$g(s,\theta) \sim \sum_{k=0}^{\infty} (\sigma - s)_+^{\lambda+k} g_k(\theta)$$

with $g_k(\theta) \in C^\infty(S^{n-1})$, there exists $f(x)$ such that around $s = \sigma$

$$(\mathcal{R}_+ f)(s,\theta) \sim \sum_{k=0}^{\infty} (\sigma - s)_+^{\lambda+k} g_k(\theta),$$

and $f(x)$ admits the asymptotic expansion

$$f(x) \sim \sum_{k=0}^{\infty} (\sigma - \Phi(x))_+^{\lambda+k} f_k(\theta) \tag{6.27}$$

around $\Sigma(\sigma)$, θ being the local coordinates on $\Sigma(\sigma)$. Furthermore

$$g_0(\theta) = N(\sigma,\theta) f_0(X(\sigma,\theta)),$$

$N(\sigma,\theta)$ being given by (6.25). This $f(x)$ is unique in the sense that if there exist two such $f^{(1)}(x)$ and $f^{(2)}(x)$, $f^{(1)}(x) - f^{(2)}(x)$ is smooth. In particular, $f^{(1)}(x)$ and $f^{(2)}(x)$ have the asymptotic expansion as in (6.27) with the same $f_k(\theta)$.

Proof. By (6.26), one can construct $f_k(\theta)$. Using Borel's procedure we then construct $f(x)$ having the asymptotic expansion $f(x) \sim \sum_{k=0}^{\infty} (\sigma - \Phi(x))_+^{\lambda+k} f_k(\theta)$. Suppose there exist two such $f^{(1)}$ and $f^{(2)}$. As is seen by the lemma below, $f^{(1)} - f^{(2)}$ is regular in non-scattering region, hence it is in H^∞ by Theorem 5.11. □

Lemma 6.8. *For $\sigma > 0$ large enough, let $u(x) = (\sigma - \Phi(x))_+^{\mu} f(x)$, where $f(x) \in C^\infty(\mathbf{R}^n)$ wwhose support is sufficiently close to $\{\sigma = \Phi(x)\}$, and $\mu > -1/2$. Then $u(x)$ is regular in non scattering region.*

Proof. Let P be the ψDO with symbol $p(x,\xi) \in S^0$ such that for some $0 < \delta < 1$, $\operatorname{supp} p(x,\xi) \subset \{|\widehat{x} \cdot \widehat{\xi}| < \delta\}$. Then by using the polar coordinates (s,θ) in Theorem 6.2 (6),

$$\begin{aligned}\widehat{Pu}(\xi) &= (2\pi)^{-n/2} \int_{\Phi(x)<\sigma} e^{-ix\cdot\xi} \overline{p(x,\xi)} u(x) dx \\ &= \int_0^\sigma \int_{S^{n-1}} e^{-iX(s,\theta)\cdot\xi} (\sigma - s)^\mu p(X(x,\theta),\xi) g(s,\theta) ds d\theta,\end{aligned}$$

with suitable $g(s,\theta) \in C^\infty$. We apply the stationary phase method (as $|\xi| \to \infty$) to the integral on S^{n-1}. Since $X(s,\theta)$ is close to $s\theta$, the critical points are close to $\pm\widehat{\xi}$, on which $p(X(s,\theta),\xi)$ vanishes. Therefore above integral is rapidly decreasing in ξ. □

6.3. Singular support theorem. The following Theorem 6.10 will elucidate how the modified Radon transform describes the propagation of singularities for the wave equation.

Definition 6.9. Assume $\Sigma(t) \subset \{|x| > r_0\}$. A function $f(x) \in L^2(\mathbf{R}^n)$ is said to be piecewise $H^\infty(|x| > r_0)$ with interface $\Sigma(t)$ if there exist $f_1, f_2 \in H^\infty(|x| > r_0)$ such that $f = \big(t - \Phi(x)\big)_+^0 f_1 + \big(t - \Phi(x)\big)_-^0 f_2$ on $|x| > r_0$. Similarly a function $f(s) \in L^2(\mathbf{R}; L^2(S^{n-1}))$ is said to be piecewise $\widehat{H}^\infty(s > s_0)$ with interface $s = t$ ($> s_0$) if there exist $f_1, f_2 \in \widehat{H}^\infty(s > s_0)$ such that $f = (t-s)_+^0 f_1 + (t-s)_-^0 f_2$ for $s > s_0$.

Theorem 6.10. *Pick $r_0, s_0 > 0$ large enough, and let $t >$ max $\{r_0 + 1, s_0 + 1\}$. Assume that $f \in L^2(\mathbf{R}^n)$ is regular in non-scattering region. Then f is piecewise $H^\infty(|x| > r_0)$ with interface $\Sigma(t)$ if and only if $\mathcal{R}_+ f$ is piecewise $\widehat{H}^\infty(s > s_0)$ with interface $s = t$.*

Proof. Suppose f is piecewise $H^\infty(|x| > r_0)$ with interface $\Sigma(t)$. Up to an H^∞-function, f is equal to $(t - \Phi(x))_+^0 \widetilde{f}(x)$ with $\widetilde{f} \in H^\infty(\mathbf{R}^n)$. By Theorem 5.5, $(\mathcal{R}_+ f)(s, \theta)$ is smooth with respect to s if $s \neq t$. By Theorem 6.6, $(\mathcal{R}_+ f)(s, \theta) \sim \sum_{k \geq 0} (t-s)_+^k g_k(\theta)$ around $s = t$. Therefore $\mathcal{R}_+ f$ is piecewise $\widehat{H}^\infty(s > s_0)$ with interface $s = t$.

Conversely, suppose $\mathcal{R}_+ f$ is piecewise $\widehat{H}^\infty(s > s_0)$ with interface $s = t$. Up to an $\widehat{H}^\infty$-function, $(\mathcal{R}_+ f)(s, \theta) \equiv (t-s)_+^0 g(s, \theta)$ with $g \in \widehat{H}^\infty(s > s_0)$. By Theorem 6.7, there exists $\widetilde{f}$ such that $(\mathcal{R}_+ \widetilde{f})(s, \theta) \sim (t-s)_+^0 g(s, \theta)$ around $s = t$. Then $\mathcal{R}_+(f - \widetilde{f}) \in \widehat{H}^\infty(s > s_0)$. By Theorem 5.10, $f - \widetilde{f} \in H^\infty(|x| > r_0)$. This shows that f is piecewise $H^\infty(|x| > r_0)$ with interface $\Sigma(t)$. □

The meaning of Theorem 6.10 in propagation of singularities is as follows. We put $v(t, s) = \big(\mathcal{R}_+ \partial_t u(t)\big)(s)$ for the solution $u(t)$ to the wave equation $\partial_t^2 u = Hu$ with initial data $u(0) = 0$, $\partial_t u(0) = f$. Then $v(t, s)$ solves the 1-dimensional wave equation

$$\begin{cases} (\partial_t^2 - \partial_s^2) v(t, s) = 0, \\ v(0, s) = \big(\mathcal{R}_+ f\big)(s), \quad \partial_t v(0, s) = 0, \end{cases}$$

hence is written as

$$v(t, s) = \frac{1}{2}\Big((\mathcal{R}_+ f)(s + t) + (\mathcal{R}_+ f)(s - t)\Big).$$

If σ is sufficiently large, $t \geq 0$ and f is regular in non-scattering region, we then see that f is piecewise $H^m(|x| > r_0)$ with interface $\Sigma(\sigma)$ if and only if $(\mathcal{R}_+ \partial_t u(t))(s)$ is piecewise $\widehat{H}^m(s > s_0)$ with interface $s = t + \sigma$, which is equivalent to that $\partial_t u(t)$ is piecewise $H^m(|x| > t + r_0)$ with interface $\Sigma(t + \sigma)$.