CHAPTER 3

Manifolds with hyperbolic ends

1. Classification of 2-dimensional hyperbolic manifolds

The hyperbolic manifold is, by definition, a complete Riemannian manifold with all sectional curvatures equal to -1. General hyperbolic manifolds are constructed by the action of discrete groups on the upper-half space. The resulting quotient manifold is either compact, or non-compact but of finte volume, or non-compact with infinite volume. In the latter two cases, the manifold can be split into bounded part and unbounded part, this latter being called the end. To study the general structure of ends is beyond our scope. We briefly look at the 2-dimensional case.

1.1. Möbius transformation. Recall that $\mathbf{C}_+ = \{z = x + iy\,; y > 0\}$ is a 2-dimensional hyperbolic space equipped with the metric

$$ds^2 = \frac{(dx)^2 + (dy)^2}{y^2}. \tag{1.1}$$

Let $\partial\mathbf{C}_+ = \partial\mathbf{H}^2 = \{(x, 0)\,; x \in \mathbf{R}\} \cup \infty = \mathbf{R} \cup \infty$. For a matrix

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{R})$$

the Möbius transformation is defined by

$$\mathbf{C}_+ \ni z \to \gamma \cdot z := \frac{az + b}{cz + d}, \tag{1.2}$$

which is an isometry on $\mathbf{H}^2$. Since γ and $-\gamma$ define the same action, one usually identifies them and considers the factor group:

$$PSL(2, \mathbf{R}) := SL(2, \mathbf{R})/\{\pm I\}.$$

The non-trivial Möbius transformations γ are classified into 3 categories :

$$\begin{aligned} \textit{elliptic} &\Longleftrightarrow \text{there is only one fixed point in } \mathbf{C}_+ \\ &\Longleftrightarrow |\mathrm{tr}\,\gamma| < 2, \\ \textit{parabolic} &\Longleftrightarrow \text{there is only one degenerate fixed point on } \partial\mathbf{C}_+ \\ &\Longleftrightarrow |\mathrm{tr}\,\gamma| = 2, \\ \textit{hyperbolic} &\Longleftrightarrow \text{there are two fixed points on } \partial\mathbf{C}_+ \\ &\Longleftrightarrow |\mathrm{tr}\,\gamma| > 2. \end{aligned}$$

1.2. Fuchsian group. Let Γ be a discrete subgroup of $SL(2, \mathbf{R})$, which is usually called a *Fuchsian* group. As a short introduction to the theory of Fuchsian groups, we refer [**81**]. Let $\mathcal{M} = \Gamma\backslash\mathbf{H}^2$ be the fundamental domain by the action (1.2). Γ is said to be *geometrically finite* if $\mathcal{M}$ is chosen to be a finite-sided convex polygon. The sides are then geodesics of $\mathbf{H}^2$. The geometric finiteness is equivalent to that Γ is finitely generated.

1.3. Examples. As a simple example, consider the cyclic group Γ which generates the action $z \to z+1$. This is parabolic with fixed point ∞. The associated fundamental domain is $\mathcal{M} = (-1/2, 1/2] \times (0,\infty)$, with which one can endow the metric (1.1). It has two infinities : $(-1/2,1/2] \times \{0\}$ and ∞. The part $(-1/2, 1/2] \times (0,1)$ has an infinite volume. Let us call it *regular infinity* in this note. The part $(-1/2,1/2] \times (1,\infty)$ has a finite volume, and is called *cusp*. The sides $x = \pm 1/2$ are geodesics.

Another simple example is the cyclic group generated by the hyperbolic action $z \to \lambda z$, $\lambda > 1$. The sides of the fundamental domain $\mathcal{M} = \{1 \leq |z| \leq \lambda\}$ are semi-circles orthogonal to $\{y = 0\}$, which are geodesics. The quotient manifold is diffeomorphic to $S^1 \times (-\infty,\infty)$. It is parametrized by (t,r), where $t \in \mathbf{R}/\log\lambda\mathbf{Z}$ and r is the signed distance from the segment $\{(0,t)\,;1 \leq t \leq \lambda\}$. The metric is then written as

$$ds^2 = (dr)^2 + \cosh^2 r\,(dt)^2. \tag{1.3}$$

The part $x > 0$ (or $x < 0$) of $\mathcal{M}$ is called *funnel*. Letting $y = 2e^{-r}$, one can rewrite (1.3) as

$$ds^2 = \Big(\frac{dy}{y}\Big)^2 + \Big(\frac{1}{y} + \frac{y}{4}\Big)^2 (dt)^2.$$

This means that the funnel can be regarded as a perturbation of the regular infinity.

1.4. Classification. The set of limit points of a Fuchsian group Γ, denoted by $\Lambda(\Gamma)$, is defined as follows : $w \in \Lambda(\Gamma)$ if there exist $z_0 \in \mathbf{C}_+$ and distinct $\gamma_n \in \Gamma$, $n = 1,2,\cdots$, such that $\gamma_n \cdot z_0 \to w$. Since Γ acts discontinuously on $\mathbf{C}_+$, $\Lambda(\Gamma) \subset \partial\mathbf{H}^2$. There are only 3 possibilities.

- (*Elementary*) : $\Lambda(\Gamma)$ is a finite set.
- (*The 1st kind*) : $\Lambda(\Gamma) = \partial\mathbf{H}$.
- (*The 2nd kind*) : $\Lambda(\Gamma)$ is a perfect (i.e. every point is an accumulation point), nowhere dense set of $\partial\mathbf{H}$.

If $\Lambda(\Gamma)$ is a finite set, Γ is said to be *elementary*. Any elementary group is either cyclic or is conjugate in $PSL(2,\mathbf{R})$ to a group generated by $\gamma \cdot z = \lambda z$, $(\lambda > 1)$, and $\gamma' \cdot z = -1/z$.

For non-elementary case, we have the following theorem.

Theorem 1.1. *Let $\mathcal{M} = \Gamma\backslash\mathbf{H}^2$ be a non-elementary geometrically finite hyperbolic manifold. Then there exists a compact subset $\mathcal{K}$ such that $\mathcal{M} \setminus \mathcal{K}$ is a finite disjoint union of cusps and funnels.*

For the proof of this theorem, see [**21**], p. 27, Theorem 2.13.

One more explanation is necessary about Theorem 1.1. Let Γ be a Fuchsian group. For a point $z_0 \in \overline{\mathbf{C}_+}$, we put

$$\Gamma_{z_0} = \{\gamma \in \Gamma\,; \gamma \cdot z_0 = z_0\}.$$

If $\Gamma_{z_0} \neq \{1\}$, z_0 is called a fixed point of Γ. A fixed point in $\mathbf{C}_+$ is called an elliptic fixed point. Let $\mathcal{M}_{sing}$ be the set of elliptic fixed points of Γ. By a suitable choice of local coordinates, $\mathcal{M} = \Gamma\backslash\mathbf{H^2}$ becomes a Riemann surface, moreover by introducing the metric $y^{-2}\big((dx)^2 + (dy)^2\big)$, $\mathcal{M} \setminus \mathcal{M}_{sing}$ is a hyperbolic manifold. However, this metric is singular around the points from $\mathcal{M}_{sing}$. In this case, there exists a neighborhood U of $z_0 \in \mathcal{M}_{sing}$ such that $U = \Gamma_{z_0}\backslash B$, where B is a ball in $\mathbf{H}^2$. Then $\mathcal{M}$ turns out to be an *orbifold*. Theorem 1.1 also holds for the orbifold case. However, in this note, we do not enter into the orbifold structure in detail. The case $\Gamma = SL(2,\mathbf{Z})$ will be explained in §5.

2. Model space

By the above classification, it is natural to consider the manifold whose ends are asymptotically equal to either $\mathcal{M}_{reg} = M \times (0,1)$, or $\mathcal{M}_{cusp} = M \times (1,\infty)$, where M is a compact manifold, and the metrics of $\mathcal{M}_{reg}$ and $\mathcal{M}_{cusp}$ have the form

$$ds^2 = \frac{(dy)^2 + h(x,dx)}{y^2}, \tag{2.1}$$

where $h(x,dx) = \sum_{i,j=1}^{n-1} h_{ij}(x)dx^i dx^j$ is the metric on M, x being local coordinates on M. Let Δ_M be the Laplace-Beltrami operator on M, $0 = \lambda_0 < \lambda_1 \leq \cdots$ the eigenvalues, and $\varphi_m(x)$, $m = 0,1,2,\cdots$, the associated complete orthonormal system of eigenvectors of $-\Delta_M$. We define for $\phi \in L^2(M)$

$$P_m\phi = (\phi,\varphi_m)_{L^2(M)}\,\varphi_m, \tag{2.2}$$

$$\Pi_m\phi = (\phi,\varphi_m)_{L^2(M)}. \tag{2.3}$$

We now let $\mathcal{M} = M \times (0,\infty)$ equipped with the metric (2.1). The Laplace-Beltrami operator on $\mathcal{M}$ is $y^2(\partial_y^2 + \Delta_M) - (n-2)y\partial_y$. We put

$$H_{free} = -y^2(\partial_y^2 + \Delta_M) + (n-2)y\partial_y - \frac{(n-1)^2}{4} = -\Delta_{\mathcal{M}} - \frac{(n-1)^2}{4}. \tag{2.4}$$

Here we need to explain the change of usage of suffix. In Chapters 1 and 2, we used the subscript 0 to denote *unperturbed* operators. However, in the sequel, we use the suffix *free* for that purpose. The suffix 0 will be used to distinguish the case in which the eigenvalue $\lambda_0 = 0$ is involved.

Spectral properties of H_{free} can be studied in essentially the same way as in Chap. 2. We have only to replace the space $L^2(\mathbf{R}^{n-1})$ by $L^2(M)$ and the Fourier transform by the eigenfunction expansion associated with $-\Delta_M$. The expansion coefficient of $f(x,y)$ is denoted by

$$\widehat{f}_m(y) = (f(\cdot,y),\varphi_m)_{L^2(M)} = \big(\Pi_m\, f\big)(y). \tag{2.5}$$

For $f \in C_0^\infty(\mathcal{M})$, we have

$$\big(\Pi_m H_{free} f\big)(y) = L_{free}(\sqrt{\lambda_m})\widehat{f}_m(y),$$

where $L_{free}(\zeta)$ is defined by Chap. 1. (3.7). As in Corollary 1.3.10, for $\lambda_m \neq 0$, the Green operator of $L_{free}(\sqrt{\lambda_m}) - \lambda \mp i\epsilon$ is

$$\big(L_{free}(\sqrt{\lambda_m}) - \lambda \mp i\epsilon)\big)^{-1} = G_{free}(\sqrt{\lambda_m}, \mp i\sqrt{\lambda \pm i\epsilon}),$$

where $G_{free}(\zeta,\nu)$ is defined by Definition 1.3.5. The Fourier transformation associated with $L_{free}(\sqrt{\lambda_m})$ is given in Chap.1, (3.22):

$$\left(F_{free,m}\psi\right)(k) = \frac{\big(2k\sinh(k\pi)\big)^{1/2}}{\pi}\int_0^\infty y^{(n-1)/2}K_{ik}(\sqrt{\lambda_m}\,y)\psi(y)\frac{dy}{y^n}. \tag{2.6}$$

Letting $\zeta = \sqrt{\lambda_m}$ in Theorem 1.3.13, we obtain the following theorem.

Theorem 2.1. *Let* $\lambda_m \neq 0$.
(1) $F_{free,m}$ *is a unitary operator from* $L^2((0,\infty); dy/y^n)$ *onto* $L^2((0,\infty); dk)$.
(2) For $\psi \in D(L_{free}(\sqrt{\lambda_m}))$

$$(F_{free,m}L_{free}(\sqrt{\lambda_m})\psi)(k) = k^2(F_{free,m}\psi)(k).$$

(3) For $\psi \in L^2((0,\infty); dy/y^n)$ *the inversion formula holds :*

$$\begin{aligned}\psi &= \big(F_{free,m}\big)^* F_{free,m}\psi \\ &= y^{(n-1)/2}\int_0^\infty \frac{(2k\sinh(k\pi))^{1/2}}{\pi}K_{ik}(\sqrt{\lambda_m}\,y)(F_{free,m}\psi)(k)dk.\end{aligned}$$

We consider the case $\lambda_m = 0$, i.e. $m = 0$:

$$L_{free}(0) = -y^2\partial_y^2 + (n-2)y\partial_y - \frac{(n-1)^2}{4}.$$

Since this is Euler's operator, we have

$$(L_{free}(0) - \lambda \mp i\epsilon))^{-1} = G_{free,0}(\mp i\sqrt{\lambda \pm i\epsilon}),$$

$$G_{free,0}(\nu)\psi(y) = \int_0^\infty G_{free,0}(y,y';\nu)\psi(y')\frac{dy'}{(y')^n}, \tag{2.7}$$

$$G_{free,0}(y,y',\nu) = \frac{1}{2\nu}\begin{cases} y^{\frac{n-1}{2}+\nu}(y')^{\frac{n-1}{2}-\nu}, & 0 < y < y', \\ y^{\frac{n-1}{2}-\nu}(y')^{\frac{n-1}{2}+\nu}, & 0 < y' < y. \end{cases} \tag{2.8}$$

In the same way as in Lemma 1.3.8, we can prove

$$\|G_{free,0}(\nu)\psi\|_{\mathcal{B}^*} \leq \frac{C}{|\nu|}\|\psi\|_{\mathcal{B}},$$

where the constant C is independent of ν. The Fourier transform $F_{free,0}$ associated with $L_{free}(0)$ has 2 components:

$$F_{free,0} = \big(F^{(+)}_{free,0}, F^{(-)}_{free,0}\big), \tag{2.9}$$

$$\big(F^{(\pm)}_{free,0}\psi\big)(k) = \frac{1}{\sqrt{2\pi}}\int_0^\infty y^{\frac{n-1}{2}\pm ik}\psi(y)\frac{dy}{y^n}. \tag{2.10}$$

Let us check this fact. By (2.7), we have for $\psi \in C_0^\infty((0,\infty))$

$$G_{free,0}(\mp ik)\psi(y) \sim \pm\frac{i}{k}\sqrt{\frac{\pi}{2}}\begin{cases} y^{\frac{n-1}{2}\mp ik}F^{(\pm)}_{free,0}(k)\psi, & y \to 0, \\ y^{\frac{n-1}{2}\pm ik}F^{(\mp)}_{free,0}(k)\psi, & y \to \infty. \end{cases}$$

On the other hand, we have

$$\begin{aligned} & \frac{1}{2\pi i}\big(G_{free,0}(-ik) - G_{free,0}(ik)\big)\psi \\ = & \frac{1}{4\pi k}\int_0^\infty (yy')^{\frac{n-1}{2}}\left\{\big(\frac{y'}{y}\big)^{ik} + \big(\frac{y}{y'}\big)^{ik}\right\}\psi(y')\frac{dy'}{(y')^n} \\ = & \frac{1}{2k\sqrt{2\pi}}\left(y^{\frac{n-1}{2}-ik}F^{(+)}_{free,0}(k)\psi + y^{\frac{n-1}{2}+ik}F^{(-)}_{free,0}(k)\psi\right). \end{aligned}$$

Hence we have

$$\frac{1}{2\pi i}\left(\big[G_{free,0}(-ik) - G_{free,0}(ik)\big]\psi, \psi\right) = \frac{1}{2k}\left|(F_{free,0}\psi)(k)\right|^2.$$

Integrating this equality and arguing as in Chap. 1, §3, we obtain the following Theorem 2.2. Alternatively, one can use the fact that

$$(F_{free,0}\psi)(k) = \left(\widetilde{\psi}(-k), \widetilde{\psi}(k)\right),$$

where $\widetilde{\psi}$ is the Fourier transform of $U\psi(t) = e^{-(n-1)t/2}\psi(e^t)$. In fact, U is unitary from $L^2((0,\infty); dy/y^n)$ to $L^2(\mathbf{R}; dt)$, and we have

$$U\left(-y^2\partial_y^2 + (n-2)y\partial_y - \frac{(n-1)^2}{4}\right)U^* = -\partial_t^2. \tag{2.11}$$

Theorem 2.2. *(1)* $F_{free,0} : L^2((0,\infty); dy/y^n) \to (L^2((0,\infty); dk))^2$ *is unitary.*
(2) For $f \in D(L_{free,0}(0))$,

$$(F_{free,0}L_{free,0}(0)f)(k) = k^2(F_{free,0}f)(k).$$

(3) For $f \in L^2((0,\infty); dy/y^n)$, *the inversion formula holds:*

$$\begin{aligned} f &= (F_{free,0})^* F_{free,0} f \\ &= \frac{1}{\sqrt{2\pi}}\int_0^\infty y^{(n-1)/2}\left(y^{-ik}F^{(+)}_{free,0}(k)f + y^{ik}F^{(-)}_{free,0}(k)f\right)dk. \end{aligned}$$

We now return to the operator H_{free} whose resolvent is written as

$$(H_{free} - \lambda \mp i0)^{-1}f = \sum_{m=0}^{\infty}\varphi_m(x)\left(G_{free}(\sqrt{\lambda_m}, \mp i\sqrt{\lambda})\widehat{f}_m\right)(y). \tag{2.12}$$

Here $G_{free}(\sqrt{\lambda_0}, \mp i\sqrt{\lambda}) = G_{free,0}(\mp i\sqrt{\lambda})$. Repeating the proof of Lemma 1.4.1, we can show the following lemma.

Lemma 2.3. $H_{free}\Big|_{C_0^\infty(\Omega)}$ *is essentially self-adjoint.*

Recall that the generalized Fourier transform is derived from the asymptotic behavior of the resolvent at infinity. For $M \times (0,\infty)$, there are two infinities ; $y = 0$ and $y = \infty$, the former corresponding to the regular infinity, the latter to the cusp. We put the suffix *reg* or c for the Fourier transforms associated with regular infinity or cusp.

Definition 2.4. Let $\mathcal{D}(M \times (0,\infty))$ be the set of functions $f(x,y) \in C^\infty(M \times (0,\infty))$ such that $\widehat{f}_m \in C_0^\infty((0,\infty))$, moreover $\widehat{f}_m = 0$ except for a finite number of m. We put

$$\mathbf{h} = L^2(M) \oplus \mathbf{C}, \quad \widehat{\mathcal{H}} = L^2((0,\infty); \mathbf{h}; dk),$$

$$\mathcal{F}^{(\pm)}_{free} = \left(\mathcal{F}^{(\pm)}_{reg,free}, \mathcal{F}^{(\pm)}_{c,free}\right),$$

and define on $\mathcal{D}(M \times (0,\infty))$

$$\mathcal{F}^{(\pm)}_{reg,free} = \sum_{m=0}^{\infty} C^{(\pm)}_m(k)P_m \otimes F^{(\pm)}_{free,m}, \tag{2.13}$$

$$F^{(\pm)}_{free,m} = \begin{cases} F_{free,m} & (\lambda_m \neq 0) \\ F^{(\pm)}_{free,0} & (\lambda_m = 0), \end{cases} \tag{2.14}$$

(2.15)
$$C_m^{(\pm)}(k) = \begin{cases} \left(\dfrac{\sqrt{\lambda_m}}{2}\right)^{\mp ik} & (\lambda_m \neq 0) \\ \dfrac{\pm i}{k\omega_\pm(k)}\sqrt{\dfrac{\pi}{2}} & (\lambda_m = 0), \end{cases}$$

(2.16)
$$\mathcal{F}^{(\pm)}_{c,free} = P_0 \otimes F^{(\mp)}_{free,0}.$$

We define $\mathcal{B}, \mathcal{B}^*$, and $L^{2,s}$ by putting $\mathbf{h} = L^2(M) \oplus \mathbf{C}$ in Chap. 1, §2. Note that, geometrically, $\mathcal{B}$ corresponds to the diadic decomposition with respect to the geodesic distance, and $\mathcal{B}^*$ to the integral mean over the geodesic ball. Let

$$R_{free}(z) = (H_{free} - z)^{-1}.$$

Then Theorem 2.1.3 remains valid for H_{free} if $\mathcal{X}^s$ is replaced by $L^{2,s}$.

Theorem 2.5. *(1)* $\sigma(H_{free}) = [0,\infty)$.
(2) $\sigma_p(H_{free}) = \emptyset$.
(3) For $\lambda > 0$ *and* $f, g \in \mathcal{B}$*, the following weak limit exists*

$$\lim_{\epsilon\to 0}(R_{free}(\lambda \pm i\epsilon)f, g) =: (R_{free}(\lambda \pm i0)f, g).$$

Moreover

$$\|R_{free}(\lambda \pm i0)f\|_{\mathcal{B}^*} \leq C\|f\|_{\mathcal{B}},$$

where the constant C *does not depend on* λ *if* λ *varies over a compact set in* $(0,\infty)$.
(4) Letting $\mathcal{F}^{(\pm)}_{free}(k)f = (\mathcal{F}^{(\pm)}_{free}f)(k)$ *for* $f \in \mathcal{D}(M \times (0,\infty))$*, we have*

$$\|\mathcal{F}^{(\pm)}_{free}(k)f\|_{\mathbf{h}} \leq C\|f\|_{\mathcal{B}},$$

where the constant C *does not depend on* k *if* k *varies over a compact set in* $(0,\infty)$.
(5) $\mathcal{F}^{(\pm)}_{free}$ *is uniquely extended to a unitary operator from* $L^2(M\times(0,\infty); \sqrt{g_M}dxdy/y^n)$ *to* $\widehat{\mathcal{H}}$*. Moreover if* $f \in D(H_{free})$

$$(\mathcal{F}^{(\pm)}_{free}H_{free}f)(k) = k^2(\mathcal{F}^{(\pm)}_{free}f)(k).$$

Proof. The assertions (1), (2) follow from Lemma 1.3.2. Note that $L_{free}(0)$ should be treated separately, however, it is easy by (2.11). The proof of (3) is almost the same as Theorem 2.2.3 (2), (3), the term $L_{free}(0)$ requires a small change, though. In the next section, we shall give the proof for the more general case (see Theorem 3.8). Applying Stone's formulas for each $L_{free}(\sqrt{\lambda_m})$, we have

$$\frac{1}{2\pi i}([R_{free}(\lambda + i0) - R_{free}(\lambda - i0)]f, f) = \|\mathcal{F}^{(\pm)}_{free}(k)f\|^2,$$

which implies (4). Since each $\mathcal{F}_{free,m}$ is unitary, (5) follows. □

The relation of $\mathcal{F}^{(\pm)}_{free}$ and the asymptotic behavior of the resolvent is as follows.

Theorem 2.6. *For* $k > 0$ *and* $f \in \mathcal{B}$*, we have*

(2.17)
$$\lim_{R\to\infty}\frac{1}{\log R}\int_{1/R<y<1}\|R_{free}(k^2 \pm i0)f - v^{(\pm)}_{reg}\|^2_{L^2(M)}\frac{dy}{y^n} = 0,$$

$$v^{(\pm)}_{reg} = \omega_\pm(k)\, y^{(n-1)/2\mp ik}\mathcal{F}^{(\pm)}_{reg,free}(k)f,$$

(2.18)
$$\lim_{R\to\infty}\frac{1}{\log R}\int_{1<y<R}\|R_{free}(k^2 \pm i0)f - v^{(\pm)}_{c}\|^2_{L^2(M)}\frac{dy}{y^n} = 0,$$

$$v_c^{(\pm)} = \omega_\pm^{(c)}(k)\, y^{(n-1)/2\pm ik} \mathcal{F}^{(\pm)}_{c,free}(k) f.$$

Here $\omega_\pm(k)$ is defined by Chap. 1 (4.15), and

$$\omega_\pm^{(c)}(k) = \pm\frac{i}{k}\sqrt{\frac{\pi}{2}}.$$

Proof. By Theorem 2.5(3) and (4), we have only to prove the theorem for $f \in \mathcal{D}(M \times (0,\infty))$. Assume that $f = 0$ for $y < \epsilon$ and $y > 1/\epsilon$. Then if $y < \epsilon$, we have by (2.12), (2.8) and Chap.1 Definition 3.5

$$\begin{aligned} & R_{free}(k^2 \pm i0) f \\ = & \pm \frac{i}{k}\sqrt{\frac{\pi}{2}}\frac{1}{\sqrt{|M|}} y^{(n-1)/2\mp ik} F^{(\mp)}_{free,0}(k)\widehat{f}_0 \\ & + \frac{\pi}{\big(2k\sinh(k\pi)\big)^{1/2}} \sum_{m\geq 1} \varphi_m(x) y^{(n-1)/2} I_{\mp ik}(\sqrt{\lambda_m} y) F_{free,m}(k)\widehat{f}_m. \end{aligned}$$

Using Definition 2.4 and Chap. 1 (3.5), we obtain (2.17).

For $y > 1/\epsilon$, we have by using Chap. 1 (3.10)

$$\begin{aligned} & \|R_{free}(k^2 \pm i0) f - \frac{1}{\sqrt{|M|}} G_{free,0}(\mp ik)\widehat{f}_0\|^2_{L^2(M)} \\ \leq \ & C y^{n-2} \sum_{m\geq 1} \Big(\int_0^\infty |\widehat{f}_m(y)| \frac{dy}{y^{(n+2)/2}} \Big)^2, \end{aligned}$$

which proves (2.18). □

3. Manifolds with hyperbolic ends

3.1. The formula of Helffer-Sjöstrand. We prepare a useful tool from functional analysis introduced by Helffer-Sjöstrand [**49**]. Let $\sigma \in \mathbf{R}$, and suppose $f(t) \in C^\infty(\mathbf{R})$ satisfies

$$|f^{(k)}(t)| \leq C_k (1+|t|)^{\sigma-k}, \quad \forall k, \quad \forall t \in \mathbf{R}. \tag{3.1}$$

Then there exists $F(z) \in C^\infty(\mathbf{C})$ such that

$$\begin{cases} F(t) = f(t), \quad t \in \mathbf{R}, \\ |F(z)| \leq C(1+|z|)^\sigma, \\ |\overline{\partial_z} F(z)| \leq C_n |\mathrm{Im}\, z|^n (1+|z|)^{\sigma-n-1}, \quad \forall n, \\ \mathrm{supp}\, F(z) \subset \{|\mathrm{Im}\, z| \leq 2 + 2|\mathrm{Re}\, z|\}. \end{cases} \tag{3.2}$$

Here $\overline{\partial_z} = \frac{1}{2}(\partial_x + i\partial_y)$. This function F is called an *almost analytic extension* of f. If $f \in C_0^\infty(\mathbf{R})$, we can construct $F(z) \in C_0^\infty(\mathbf{C})$.

Let us explain the idea of the proof. For $z \in \mathbf{C}$, let $\langle z\rangle = (1+|z|^2)^{1/2}$. Take $\chi(y) \in C_0^\infty(\mathbf{R})$ such that $\chi(y) = 1$ ($|y| < 1$), $\chi(y) = 0$ ($|y| > 2$), and put

$$F(z) = \sum_{n=0}^{N-1} \frac{i^n}{n!} f^{(n)}(x) y^n \chi\Big(\frac{y}{\langle x\rangle}\Big).$$

Then we have

$$2\overline{\partial_z}F(z) = \frac{i^{N-1}}{(N-1)!}f^{(N)}(x)y^{N-1}\chi\Big(\frac{y}{\langle x\rangle}\Big) + \sum_{n=0}^{N-1}\frac{i^n}{n!}f^{(n)}(x)y^n\chi'\Big(\frac{y}{\langle x\rangle}\Big)\Big(\frac{i}{\langle x\rangle} - \frac{xy}{\langle x\rangle^3}\Big).$$

On the support of the first term of the right-hand side, $|y| \leq 2\langle x\rangle$. Hence for $1 \leq n \leq N-1$, it is dominated by $C\langle x\rangle^{\sigma-N}|y|^{N-1} \leq C|y|^n\langle z\rangle^{\sigma-n-1}$. On the support of the 2nd term, $\langle x\rangle \leq |y| \leq 2\langle x\rangle$. Hence, it is dominated by

$$C\sum_{n=0}^{N-1}\frac{1}{n!}\langle x\rangle^{\sigma-n-1}|y|^n\Big|\chi'\Big(\frac{y}{\langle x\rangle}\Big)\Big| \leq C\langle x\rangle^{\sigma-1}\exp\frac{|y|}{\langle x\rangle} \leq C_n|y|^n\langle z\rangle^{\sigma-n-1}.$$

Hence, $|\overline{\partial_z}F(z)| \leq C_n|\mathrm{Im}\,z|^n(1+|z|)^{\sigma-n-1}$ holds for $1 \leq n \leq N-1$. By the similar computation, one can show $|F(z)| \leq C(1+|z|)^\sigma$. For the general construction of $F(z)$, see e.g. [**62**] p. 363.

Lemma 3.1. *Let $f(t)$ and $F(z)$ be as above. Suppose $\sigma < 0$. Then for any self-adjoint operator A, the following formula holds*

$$f(A) = \frac{1}{2\pi i}\int_{\mathbf{C}}\overline{\partial_z}F(z)(z-A)^{-1}dzd\overline{z}.$$

Proof. For $\lambda \in \mathbf{R}$, we have by the generalized Cauchy formula

$$F(\lambda) = \frac{1}{2\pi i}\int_{|z|=R}\frac{F(z)}{z-\lambda}dz + \frac{1}{2\pi i}\int_{|z|<R}\frac{\overline{\partial_z}F(z)}{z-\lambda}dzd\overline{z}.$$

Letting $R \to \infty$, we have

$$F(\lambda) = \frac{1}{2\pi i}\int_{\mathbf{C}}\frac{\overline{\partial_z}F(z)}{z-\lambda}dzd\overline{z},$$

where the integral is absolutely convergent. Let $E(\lambda)$ be the spectral decomposition of A. Then we have

$$\begin{aligned} f(A) &= \int_{-\infty}^{\infty}f(\lambda)dE(\lambda) \\ &= \frac{1}{2\pi i}\int_{-\infty}^{\infty}\int_{\mathbf{C}}\frac{\overline{\partial_z}F(z)}{z-\lambda}dzd\overline{z}dE(\lambda) \\ &= \frac{1}{2\pi i}\int_{\mathbf{C}}\overline{\partial_z}F(z)(z-A)^{-1}dzd\overline{z}. \quad \Box \end{aligned}$$

Let us mention here useful formulas to compute the commutator of functions of self-adjont operators. For two operators P, A, we put

$$\mathrm{ad}_0(P, A) = P,$$

$$\mathrm{ad}_n(P, A) = [\mathrm{ad}_{n-1}(P, A), A], \quad \forall n \geq 1.$$

If A is self-adjoint and $f(s)$ satisfies $|f^{(k)}(s)| \leq C_k(1+|s|)^{\sigma-k}$, $\forall k \geq 0$, we have

$$[P, f(A)] = \sum_{k=1}^{n-1}\frac{(-1)^{k-1}}{k!}\mathrm{ad}_k(P, A)f^{(k)}(A) + R_{n,l}, \tag{3.3}$$

$$R_{n,l} = \frac{1}{2\pi i}\int_{\mathbf{C}} \overline{\partial_z}F(z)(A-z)^{-1}\mathrm{ad}_n(P,A)(A-z)^{-n}dzd\overline{z}. \tag{3.4}$$

$$[P, f(A)] = \sum_{k=1}^{n-1}\frac{1}{k!}f^{(k)}(A)\mathrm{ad}_k(P,A) + R_{n,r}, \tag{3.5}$$

$$R_{n,r} = \frac{(-1)^{(n+1}}{2\pi i}\int_{\mathbf{C}} \overline{\partial_z}F(z)(A-z)^{-n}\mathrm{ad}_n(P,A)(A-z)^{-1}dzd\overline{z}. \tag{3.6}$$

Here, $F(z)$ is an almost analytic extension of f, and we assume that

$$\|(A-z)^{-n}\mathrm{ad}_n(P,A)(A-z)^{-1}\| \leq C|\mathrm{Im}\, z|^{-n-1}\langle z\rangle^{\mu(n)},$$

$$\sigma - n + \mu(n) < 0,$$

in order to guarantee the convergence of the integrals (3.4), (3.6). Formal derivation of (3.3), (3.5) is rather easy. However, rigorous derivation requires examination of the domain of $\mathrm{ad}_n(P,A)$. When P and A are differential operators, this domain question boils down to the regularity estimate for $(A-z)^{-1}$.

3.2. Assumptions on ends. Now we consider an n-dimensional connected Riemannian manifold $\mathcal{M}$, which is written as a union of open sets:

$$\mathcal{M} = \mathcal{K} \cup \mathcal{M}_1 \cup \cdots \cup \mathcal{M}_N.$$

We assume that

(A-1) *$\overline{\mathcal{K}}$ is compact.*

(A-2) $\mathcal{M}_p \cap \mathcal{M}_q = \emptyset, \quad p \neq q.$

(A-3) *Each $\mathcal{M}_p$, $p = 1, \cdots, N$, is diffeomorphic either to $\mathcal{M}_{reg} = M_p \times (0,1)$ or to $\mathcal{M}_c = M_p \times (1,\infty)$,* M_p being a compact Riemannian manifold of dimension *$n-1$, which is allowed to be different for each p.*

(A-4) *On each $\mathcal{M}_p$, the Riemannian metric ds^2 has the following form*

$$ds^2 = y^{-2}\left((dy)^2 + h_p(x,dx) + A_p(x,y,dx,dy)\right), \tag{3.7}$$

$$A_p(x,y,dx,dy) = \sum_{i,j=1}^{n-1} a_{p,ij}(x,y)dx^i dx^j + 2\sum_{i=1}^{n-1} a_{p,in}(x,y)dx^i dy + a_{p,nn}(x,y)(dy)^2,$$

where $h_p(x,dx) = \sum_{i,j=1}^{n-1} h_{p,ij}(x)dx^i dx^j$ is a positive definite metric on M_p, and $a_{p,ij}(x,y), 1 \leq i,j \leq n$, satisfies the following condition

$$|\widetilde{D}_x^\alpha D_y^\beta\, a(x,y)| \leq C_{\alpha\beta}(1+|\log y|)^{-\min(|\alpha|+\beta,1)-1-\epsilon}, \quad \forall \alpha, \beta \tag{3.8}$$

for some $\epsilon > 0$. Here $D_y = y\partial_y$, $\widetilde{D}_x = \tilde{y}(y)\partial_x$, $\tilde{y}(y) \in C^\infty((0,\infty))$ such that $\tilde{y}(y) = y$ for $y > 2$ and $\tilde{y}(y) = 1$ for $0 < y < 1$.

Following Example 1.3, we call $\mathcal{M}_p = M_p \times (0,1)$ a *regular end* and $\mathcal{M}_p = M_p \times (1,\infty)$ a *cusp*.

Let us note that the above model in particular contains $\mathbf{H}^n$. In fact, we take $\mathcal{K} = B_2(0,1)$, and $\mathcal{M}_1 = \mathbf{H}^n \setminus B_{\log 2}(0,1)$, where $B_r(0,1)$ is the geodesic ball of radius r centered at $(0,1)$. Using geodesic polar coordinates, $\mathcal{M}_1$ is isometric to $S^{n-1} \times (\log 2, \infty)$ equipped with the metric $(dr)^2 + \sinh^2 r(d\theta)^2$. Taking $e^r = 2/y$, we see that $\mathcal{M}_1 = \mathcal{M}_{reg} = S^{n-1} \times (0,1)$ equiped with the metric $y^{-2}\Big((dy)^2 + (d\theta)^2 + (y^4/16 - y^2/2)(d\theta)^2\Big)$.

The 2nd important remark is that, if $\mathcal{M}_p$ is equal to $\mathcal{M}_{reg}$, one can assume that the above metric (3.7) takes the form

$$ds^2 = y^{-2}\Big((dy)^2 + h_p(x,dx) + \sum_{i,j=1}^{n-1} a_{p,ij}(x,y)dx^i dx^j\Big) \tag{3.9}$$

and each $a_{p,ij}(x,y)$ satisfies the condition (3.8). This can be proved in the same way as in Theorem 4.1.6 to be given in Chap. 4. Therefore in the following we consider the metric of the form (3.9) for such ends.

Let Δ_g be the Laplace-Beltrami operator on $\mathcal{M}$. As has been discussed in Chap. 2, §2, we pass to the gauge transformation

$$-\Delta_g - \frac{(n-1)^2}{4} \to H =: -\rho^{1/4}\Delta_g\rho^{-1/4} - \frac{(n-1)^2}{4}, \tag{3.10}$$

where $\rho \in C^\infty(\mathcal{M})$ is a positive function such that on each end $\mathcal{M}_p$

$$\rho = \det g^{(p)}/\det g^{(p)}_{free}, \tag{3.11}$$

$g^{(p)}_{free}$ and $g^{(p)}$ being the unperturbed and perturbed metrics

$$g^{(p)}_{free} = y^{-2}\left((dy)^2 + h_p(x,dx)\right), \tag{3.12}$$

$$g^{(p)} = y^{-2}\left((dy)^2 + h_p(x,dx) + A_p(x,y;dx,dy)\right) \tag{3.13}$$

satisfying the above assumptions. Then H is written as

$$H = -\Delta_g + L_2 - \frac{(n-1)^2}{4}, \tag{3.14}$$

L_2 being a 2nd order differential operator on $\mathcal{M}$, and satisfies the following conditions.

(A-5) *H is formally self-adjoint. Namely,*

$$(H\varphi,\psi) = (\varphi, H\psi), \quad \forall\varphi,\psi \in C_0^\infty(\mathcal{M}),$$

where (,) is the inner product of $L^2(\mathcal{M})$, i,e,

$$(\varphi,\psi) = \int_{\mathcal{M}} \varphi\overline{\psi}d\mathcal{M},$$

$d\mathcal{M}$ being the measure which coincides with the unperturbed metric on each $\mathcal{M}_p$.
(A-6) *L_2 is short-range on each $\mathcal{M}_p$ $(1 \le p \le N)$. Namely, if L_2 is represented as*

$$L_1 = \sum_{|\alpha|\le 2} a_\alpha(x,y)D^\alpha, \quad D = (D_x, D_y) = (y\partial_x, y\partial_y),$$

there exists a constant $\epsilon > 0$ such that

$$|\widetilde{D}_x^\beta D_y^k a_\alpha(x,y)| \le C_{\beta,k}(1 + |\log y|)^{-|\beta|-k-1-\epsilon}, \quad \forall\beta, \quad \forall k.$$

We use the following partition of unity. Fix $x_0 \in \mathcal{K}$ arbitrarily, and pick $\chi_0 \in C_0^\infty(\mathcal{M})$, such that

$$\chi_0(x) = \begin{cases} 1, & \text{dist}\,(x,x_0) < R, \\ 0, & \text{dist}\,(x,x_0) > R+1, \end{cases}$$

where $\mathrm{dist}(x, x_0)$ is the distance between x and x_0. Taking R large enough, we define $\chi_j \in C^\infty(\mathcal{M})\, j = 1, \ldots, N$, such that

$$\chi_j(x) = \begin{cases} 1 - \chi_0(x), & x \in \mathcal{M}_j, \\ 0, & x \notin \mathcal{M}_j. \end{cases}$$

Then we have

$$\begin{cases} \sum_{j=0}^{N} \chi_j = 1, \\ \mathrm{supp}\, \chi_j \subset \mathcal{M}_j, \quad 1 \le j \le N, \\ \chi_0 = 1 \quad \text{on} \quad \mathcal{K}. \end{cases} \tag{3.15}$$

For $1 \le j \le N$, we construct $\widetilde{\chi}_j \in C^\infty(\mathcal{M})$ such that

$$\mathrm{supp}\, \widetilde{\chi}_j \subset \mathcal{M}_j, \quad \widetilde{\chi}_j = 1 \quad \text{on} \quad \mathrm{supp}\, \chi_j.$$

Theorem 3.2. *(1)* $H\big|_{C_0^\infty(\mathcal{M})}$ *is essentially self-adjoint.*
(2) $\sigma_e(H) = [0, \infty)$.

Proof. To prove assertion (1), we first observe that Theorem 2.1.3(4) and (6) remain valid for H, if we substitute the spaces $\mathcal{X}^s$ with

$$L^{2,s} = \{U \in L^2_{loc} : \int_{\mathcal{M}} \left(1 + \log^2 \left(d(x, x_0)\right)\right)^s |u(x)|^2 < \infty\}.$$

Using this analog of Theorem 2.1.3 (4), assertion (1) is proven in the same way as in Theorem 2.1.4.

To show (2), we derive a formula for the resolvent by using the partition of unity (3.15). Recall that $\mathcal{M}_j$ is diffeomorphic to $M_j \times (0, 1)$ or $M_j \times (1, \infty)$. Let $H_{free(j)}$ be defined by (2.4) with M replaced by M_j, and put

$$R(z) = (H - z)^{-1}, \quad R_{free(j)}(z) = (H_{free(j)} - z)^{-1}. \tag{3.16}$$

Note that we are using the suffix *free(j)* to specify unperturbed operators with respect to the model space $M_j \times (0, \infty)$. Since

$$(H-z)\chi_j R_{free(j)}(z)\widetilde{\chi}_j = \chi_j + \chi_j(H - H_{free(j)})R_{free(j)}(z)\widetilde{\chi}_j + [H, \chi_j]R_{free(j)}(z)\widetilde{\chi}_j,$$

we have

$$\chi_j R_{free(j)}(z)\widetilde{\chi}_j = R(z)\chi_j + R(z)A_j(z)\widetilde{\chi}_j,$$
$$A_j(z) = [H, \chi_j]R_{free(j)}(z) + \chi_j(H - H_{free(j)})\widetilde{\chi}_j R_{free(j)}(z).$$

Letting

$$A(z) = \sum_{j=1}^{N} A_j(z)\widetilde{\chi}_j, \tag{3.17}$$

we then have

$$R(z) = \sum_{j=1}^{N} \chi_j R_{free(j)}(z)\widetilde{\chi}_j + R(z)(\chi_0 - A(z)).$$

By the assumption (A-4), $R(z)(\chi_0 - A(z))$ is compact. Indeed, for $z \notin \mathbf{R}$, $A_j(z)$ is bounded from $W^{2,2}(\mathcal{M})$ to $L^{2,s}$ with $0 < s < 1 + \epsilon$. Since $R(z)$ is locally smoothening, this implies the desired compactness if one considers the adjoint $(A(z)^* - \chi_0)R(z)^*$.

To prove (2), we first show $(-\infty, 0) \subset \sigma_d(H)$. It is sufficient to prove that $f(H)$ is compact for any $f \in C_0^\infty((-\infty, 0))$. Let F be an almost analytic extension of f. Then, by Lemma 3.1, we have

$$f(H) = \sum_{j=1}^{N} \chi_j f(H_{free(j)})\widetilde{\chi}_j - K,$$

$$K = \frac{1}{2\pi i}\int_{\mathbf{C}} \overline{\partial_z} F(z) R(z) \left(\chi_0 - A(z)\right) dz d\overline{z}.$$

Note that K is compact, since $|\partial_{\overline{z}} F(z)| \leq C_l (1 + |z|)^{-l}$, for all $l > 0$, and so is $R(z)(\chi_0 - A(z))$. Since $\sigma(H_{free(j)}) = [0, \infty)$, we have $f(H_{free(j)}) = 0$. Therefore $f(H)$ is compact, which proves $\sigma_e(H) \subset [0, \infty)$. The converse inclusion relation is proven by Weyl's method of singular sequence as in Lemma 1.3.12. □

3.3. Limiting absorption principle.

Lemma 3.3. *Let $f(x) \in L^1(0, \infty; dx)$ and put*

$$u(x) = \int_x^\infty f(t)dt.$$

Then for $s > 1/2$

$$\int_0^\infty x^{2(s-1)} |u(x)|^2 dx \leq \frac{4}{(2s-1)^2}\int_0^\infty x^{2s}|f(x)|^2 dx.$$

Proof. We use the following inequality of Hardy : For $p > 1,\ g(x) \in L^1(0, \infty)$, we put

$$F(x) = \int_x^\infty g(t)dt.$$

Then we have

$$\int_0^\infty |F(x)|^p dx \leq p^p \int_0^\infty |xg(x)|^p dx$$

([**46**], p. 244). Letting $\epsilon = 2s - 1 > 0,\ y = x^\epsilon$ for $u(x)$ in the Lemma, we have

$$(2s-1)\int_0^\infty x^{2(s-1)}|u(x)|^2 dx = \int_0^\infty |u(y^{1/\epsilon})|^2 dy,$$

$$u(y^{1/\epsilon}) = \frac{1}{\epsilon}\int_y^\infty f(z^{1/\epsilon}) z^{1/\epsilon - 1} dz.$$

By Hardy's inequality, with $g(z) = \frac{1}{\epsilon} f(z^{1/\epsilon}) z(1-\epsilon)/\epsilon$ and $p = 2$,

$$\begin{aligned}\int_0^\infty |u(y^{1/\epsilon})|^2 dy &\leq \frac{4}{\epsilon^2}\int_0^\infty |f(y^{1/\epsilon})|^2 y^{2/\epsilon} dy \\ &= \frac{4}{\epsilon}\int_0^\infty |f(x)|^2 x^{2s} dx,\end{aligned}$$

which implies the Lemma. □

On each end $\mathcal{M}_j$ of $\mathcal{M}$, the spaces $L^{2,s},\ \mathcal{B},\ \mathcal{B}^*$ are defined in the same way as before with $\mathbf{h} = L^2(M_j)$. Using the above partition of unity χ_j, we put

$$\|u\|_s = \|\chi_0 u\|_{L^2} + \sum_{j=1}^{N} \|\chi_j u\|_s,$$

$$\|u\|_{\mathcal{B}} = \|\chi_0 u\|_{L^2} + \sum_{j=1}^{N} \|\chi_j u\|_{\mathcal{B}},$$

$$\|u\|_{\mathcal{B}^*} = \|\chi_0 u\|_{L^2} + \sum_{j=1}^{N} \|\chi_j u\|_{\mathcal{B}^*},$$

where $\|\chi_j u\|_s$ is defined by

$$\|\chi_j u\|_s = \left(\int_0^\infty (1+|\log y|)^{2s} \|\chi_j u(y)\|^2_{L^2(M_j)} \frac{dy}{y^n}\right)^{1/2},$$

and $\|\chi_j u\|_{\mathcal{B}}$, $\|\chi_j u\|_{\mathcal{B}^*}$ are defined similarly.

Let us note that many a-priori estimates and preliminary results which are proven in Chapter 2 for $\mathbf{H}^n$ may be straightforwardly generalized for $\mathcal{M}$. For example, Theorem 2.1.3 remains valid if we use $L^{2,s}$ instead of $\mathcal{X}^s$. Similarly, Theorem 2.2.10 can be extended to the case in which $(H-\lambda)u = 0$ in one of the regular ends $M_p \times (0, y_0)$ $(0 < y_0 < 1)$. Analogous extensions are true for Lemmas 2.2.4 $\sim$ 2.2.8 and so on.

Lemma 3.4. *Suppose all $\mathcal{M}_j$ $(1 \leq j \leq N)$ have a cusp. If $u \in \mathcal{B}^*$ satisfies $(H-\lambda)u = 0$ for some $\lambda > 0$ and, on each $\mathcal{M}_j$,*

$$\lim_{R\to\infty} \frac{1}{\log R} \int_2^R \|u(y)\|^2_{L^2(M_j)} \frac{dy}{y^n} = 0,$$

then $u \in L^{2,s}$, $\forall s > 0$. Moreover, for any $s > 0$ and any compact interval $I \subset (0,\infty)$, there exists a constant $C_s > 0$ such that

$$\|u\|_s \leq C_s \|u\|_{\mathcal{B}^*}, \quad \forall \lambda \in I. \tag{3.18}$$

Proof. For simplicity's sake, we assume that $N = 1$. Letting $U = \chi_1 u$ and $M = M_1$, we have for $\epsilon > 0$ given in the assumption (A-4)

$$\begin{cases} \Big(-y^2(\partial_y^2 + \Delta_M) + (n-2)y\partial_y - \dfrac{(n-1)^2}{4} - \lambda\Big)U = F, \\ U \in \mathcal{B}^*, \quad F \in L^{2,(1+\epsilon)/2}. \end{cases} \tag{3.19}$$

In fact, F consists of U and its 1st and 2nd order derivatives, which, by Theorem 2.1.3, are in $L^{2,-(1+\epsilon)/2}$, multiplied by functions decaying like $(1+|\log y|)^{-1-\epsilon}$, $\epsilon > 0$. Therefore, F is in $L^{2,(1+\epsilon)/2}$.

We apply the boot-strap arguments. In view of Lemma 2.2.6, letting $\mathbf{h} = L^2(M)$ and Δ_M the Laplace-Beltrami operator on M, we have

$$\int_0^\infty y^2 \|\sqrt{-\Delta_M} U\|^2_{\mathbf{h}} \frac{dy}{y^n} \leq C\left(\|U\|^2_{\mathcal{B}^*} + \|F\|^2_{\mathcal{B}}\right). \tag{3.20}$$

Let P_0 be the projection associated with the 0 eigenvalue of Δ_M, and put

$$U_0 = P_0 U, \quad U' = U - P_0 U.$$

Then we have by (3.20)

$$\|U'\|_s \leq C_s(\|U\|_{\mathcal{B}^*} + \|F\|_{\mathcal{B}}), \quad \forall s > 0.$$

Since U' satisfies the equation

$$(H_0 - \lambda)U' = F' \in L^{2,(1+\epsilon)/2},$$

we have, by Theorem 2.1.3 (6), that

$$U',\ D_iU',\ D_iD_jU' \in L^{2,(1+\epsilon)/2}. \tag{3.21}$$

Letting

$$t = \log y, \quad u_0(t) = e^{-(n-1)t/2}U_0(e^t), \quad f_0(t) = e^{-(n-1)t/2}F_0(e^t),$$

we see that $u_0(t)$ satisfies

$$\begin{cases} (-\partial_t^2 - \lambda)u_0 = f_0, \\ \lim_{R\to\infty} \dfrac{1}{R}\displaystyle\int_2^R |u_0(t)|^2 dt = 0, \\ (1+t)^{(1+\epsilon)/2} f_0(t) \in L^2((2,\infty); dt). \end{cases} \tag{3.22}$$

Recall that the Green function of the 1-dimensional Helmholtz equation

$$\Big(-\frac{d^2}{dt^2} - z\Big)u = f, \quad \operatorname{Im} z \geq 0$$

is given by $\dfrac{i}{2\sqrt{z}}e^{i\sqrt{z}|t-s|}$. Hence u_0 is represented as

$$\begin{aligned} u_0(t) &= \frac{i}{2\sqrt{\lambda}}\int_0^\infty e^{i\sqrt{\lambda}|t-s|}f_0(s)ds + C_+e^{i\sqrt{\lambda}t} + C_-e^{-i\sqrt{\lambda}t} \\ &= \frac{i}{2\sqrt{\lambda}}\int_0^t e^{i\sqrt{\lambda}(t-s)}f_0(s)ds + \frac{i}{2\sqrt{\lambda}}\int_t^\infty e^{i\sqrt{\lambda}(s-t)}f_0(s)ds \\ &\quad + C_+e^{i\sqrt{\lambda}t} + C_-e^{-i\sqrt{\lambda}t}. \end{aligned}$$

Since $f_0(t) \in L^1((0,\infty)); dt)$, we have

$$u_0(t) \sim \Big(C_+ + \frac{i}{2\sqrt{\lambda}}\int_0^\infty e^{-i\sqrt{\lambda}s}f_0(s)ds\Big)e^{i\sqrt{\lambda}t} + C_-e^{-i\sqrt{\lambda}t}, \quad t \to \infty,$$

$$u_0(t) \sim C_+e^{i\sqrt{\lambda}t} + \Big(C_- + \frac{i}{2\sqrt{\lambda}}\int_0^\infty e^{i\sqrt{\lambda}s}f_0(s)ds\Big)e^{-i\sqrt{\lambda}t}, \quad t \to -\infty.$$

They imply, by (3.22),

$$C_+ = 0 = -\frac{i}{2\sqrt{\lambda}}\int_0^\infty e^{-i\sqrt{\lambda}s}f_0(s)ds,$$

$$C_- = 0 = -\frac{i}{2\sqrt{\lambda}}\int_0^\infty e^{i\sqrt{\lambda}s}f_0(s)ds.$$

We then have

$$u_0(t) = \frac{i}{2\sqrt{\lambda}}\left(e^{-i\sqrt{\lambda}t}\int_t^\infty e^{i\sqrt{\lambda}s}f_0(s)ds - e^{i\sqrt{\lambda}t}\int_t^\infty e^{-i\sqrt{\lambda}s}f_0(s)ds\right).$$

Using Lemma 3.3, we then have

$$(1+t)^{(-1+\epsilon)/2}u_0,\ (1+t)^{(-1+\epsilon)/2}\frac{d}{dt}u_0 \in L^2((0,\infty); dt). \tag{3.23}$$

Then by (3.22), we also have

$$(1+t)^{(-1+\epsilon)/2}\frac{d^2}{dt^2}u_0 \in L^2((0,\infty); dt). \tag{3.24}$$

By (3.21), (3.23) and (3.24), we have $U, D_iU, D_iD_jU \in L^{2,(-1+\epsilon)/2}$. Hence we have $F \in L^{2,(1+2\epsilon)/2}$.

We return to the equation (3.19), and apply the same arguments as above. Then we have $U, D_iU, D_iD_jU \in L^{2,(-1+2\epsilon)/2}$, hence $F \in L^{2,(1+3\epsilon)/2}$. We repeat these procedures to obtain $U \in L^{2,(-1+N\epsilon)/2}$, $\forall N > 0$ and the inequality (3.18). $\square$

Theorem 3.5. *(1) If one of $\mathcal{M}_j$ has a regular infinity, $\sigma_p(H) \cap (0, \infty) = \emptyset$. (2) If all of $\mathcal{M}_j$ have a cusp, then $\sigma_p(H) \cap (0, \infty)$ is discrete with finite multiplicities, whose possible accumulation points are 0 and ∞.*

Proof. We shall prove (1). Let u be the eigenvector of H with eigenvalue $\lambda \in (0, \infty)$. Applying Theorem 2.2.10 on $\mathcal{M}_j$ having a regular infinity, we see that u vanishes in a neighborhood of infinity of $\mathcal{M}_j$. By the unique continuation theorem, u vanishes identically on $\mathcal{M}$.

To prove (2) assume that there exist an infinite number of eigenvlaues (counting multiplicities) in a compact interval $I \subset (0, \infty)$. Let $u_n, n = 1, 2, \cdots$, be the associated orthonormal system of eigenvectors. Choose $x_0 \in \mathcal{K}$ arbitrarily, and let χ_R be such that $\chi(x) = 1$ for $\mathrm{dist}\,(x, x_0) < R$, $\chi(x) = 0$ for $\mathrm{dist}\,(x, x_0) > R$. By (3.18), for any $\epsilon > 0$, there exists $R > 0$ independent of n such that $\|(1 - \chi_R)u_n\|_{L^2} < \epsilon$ and $\|\chi_R u_n\|_{L^2} \geq 1 - 2\sqrt{\epsilon}$. Using Rellich's theorem, one can choose a subsequence of $\{\chi_R u_n\}_{n\geq 1}$ which converges in L^2,

$$\chi_R u_n \to u, \quad \|u\|_{L^2} \geq 1 - 2\sqrt{\epsilon}.$$

Thus, for sufficiently large n, m,

$$\begin{aligned}
&(u_n, u_m) = (\chi_R u_n, \, \chi_R u_m) + ((1 - \chi_R)u_n, \, \chi_R u_m) \\
&+ (\chi_R u_n, \, (1 - \chi_R)u_m) + ((1 - \chi_R)u_n, \, (1 - \chi_R)u_m) \\
&\geq (1 - 2\sqrt{\epsilon})^2 - 3\epsilon > 0, \quad \text{if } \epsilon < \frac{1}{16}.
\end{aligned}$$

This is a contradiction to $(u_n, u_m) = 0$. $\square$

Theorem 3.6. *Suppose $\lambda > 0$, and $u \in \mathcal{B}^*$ satisfies $(H - \lambda)u = 0$. Furthermore, assume that, when $\mathcal{M}_j$ has a regular infinity,*

$$\lim_{R\to\infty} \frac{1}{\log R} \int_{1/R}^{1/2} \|u(\cdot, y)\|^2_{L^2(M_j)} \frac{dy}{y^n} = 0,$$

and when $\mathcal{M}_j$ has a cusp,

$$\lim_{R\to\infty} \frac{1}{\log R} \int_{2}^{R} \|u(\cdot, y)\|^2_{L^2(M_j)} \frac{dy}{y^n} = 0.$$

Then:
(1) If one of $\mathcal{M}_j$ has a regular infinity, then $u = 0$.
(2) If all of $\mathcal{M}_j$ have a cusp, then $u \in L^{2,s}$, $\forall s > 0$.

Proof. Applying Theorem 2.2.10 to $\mathcal{M}_j$ with regular infinity, we see that u vanishes on an open set of $\mathcal{M}_j$, hence $u = 0$ by the unique continuation theorem. The assertion (2) follows from Lemma 3.4. $\square$

As in Chap. 2, §2, we put

$$\sigma_\pm(\lambda) = \frac{n-1}{2} \mp i\sqrt{\lambda}.$$

We say that a solution $u \in \mathcal{B}^*$ of the equation

$$(H - \lambda)u = f \in \mathcal{B}$$

satisfies the outgoing radiation condition, when $\mathcal{M}_j$ has a regular infinity , if

$$\lim_{R\to\infty}\frac{1}{\log R}\int_{1/R}^{1/2}\|(D_y-\sigma_+(\lambda))u(\cdot,y)\|^2_{L^2(M_j)}\frac{dy}{y^n}=0, \tag{3.25}$$

and when $\mathcal{M}_j$ has a cusp

$$\lim_{R\to\infty}\frac{1}{\log R}\int_{2}^{R}\|(D_y-\sigma_-(\lambda))u(\cdot,y)\|^2_{L^2(M_j)}\frac{dy}{y^n}=0. \tag{3.26}$$

The incoming radiation condition is defined similarly by exchanging $\sigma_+(\lambda)$ and $\sigma_-(\lambda)$.

Let us remark that, compared to the case of $\mathbf{H}^n$ (see Chap. 2, (2.20)), the condition (3.26) seems to be confusing. Due to the presence of 0-eigenvalue of Δ_M, there exist generalized eigenfunctions for H_{free} which behave like $y^{(n-1)/2\pm i\sqrt{\lambda}}$ as $y\to\infty$. To distinguish these two functions, we need (3.26).

Theorem 3.7. *Let $\lambda>0$ and suppose $u\in\mathcal{B}^*$ satisfies $(H-\lambda)u=0$ and the outgoing or incoming radiation condition. Then:*
(1) If one of $\mathcal{M}_j$ has a regular infinity, then $u=0$.
(2) If all $\mathcal{M}_j$ have a cusp, then $u\in L^{2,s}$, $\forall s>0$.

Proof. We assume that the ends $\mathcal{M}_1,\cdots,\mathcal{M}_\mu$ have regular infinities, and $\mathcal{M}_{\mu+1},\cdots,\mathcal{M}_N$ have cusps. Recall that for $1\le j\le\mu$, $\mathcal{M}_j$ is diffeomorphic to $M_j\times(0,1)$, and for $\mu+1\le j\le N$, $\mathcal{M}_j$ is diffeomorphic to $M_j\times(1,\infty)$. Let $\{\chi_j\}_{j=0}^N$ be a smooth partition of unity such that $\sum_{j=0}^N\chi_j=1$ on $\mathcal{M}$, and $\operatorname{supp}\chi_j\subset\mathcal{M}_j$ for $1\le j\le N$. We shall assume that for $1\le j\le\mu$,

$$\chi_j(y)=\begin{cases}1, & (y<1/2),\\ 0, & (y>3/4),\end{cases}$$

and for $\mu+1\le j\le N$,

$$\chi_j(y)=\begin{cases}0, & (y<3/2),\\ 1, & (y>2).\end{cases}$$

We take $\rho(t)\in C_0^\infty(\mathbf{R})$ such that $\rho(t)=\rho(-t)$ and

$$\rho(t)=\begin{cases}c, & |t|<1,\\ 0, & |t|>2,\end{cases}$$

where c is a positive constant such that

$$\int_{-\infty}^0\rho(t)dt=\int_0^\infty\rho(t)dt=1.$$

We put

$$\varphi(t)=\int_{-\infty}^t\rho(s)ds,\quad \psi(t)=\int_t^\infty\rho(s)ds,$$

and

$$\varphi_R(y)=\varphi\left(\frac{\log y}{\log R}\right),\quad \psi_R(y)=\psi\left(\frac{\log y}{\log R}\right).$$

Then we have

$$\chi_j(y)\varphi_R(y)\in C_0^\infty(\mathcal{M}_j)\quad\text{for}\quad 1\le j\le\mu,$$
$$\chi_j(y)\psi_R(y)\in C_0^\infty(\mathcal{M}_j)\quad\text{for}\quad \mu+1\le j\le N.$$

Moreover,

$$\lim_{R\to\infty}\varphi_R(y)=\varphi(0)=1,\quad \lim_{R\to\infty}\psi_R(y)=\psi(0)=1. \tag{3.27}$$

Since $(H-\lambda)u=0$, we have

$$0=((H-\lambda)u,\chi_j\varphi_R u)=(u,[H,\chi_j\varphi_R]u).$$

Therefore, we have

$$\begin{gathered}(u,[H,\chi_j]\varphi_R u)+(u,\chi_j[H,\varphi_R]u)=0,\\ (u,[H,\chi_j]\psi_R u)+(u,\chi_j[H,\psi_R]u)=0,\\ (u,[H,\chi_0]u)=0.\end{gathered}$$

We add them, and let $R\to\infty$. Then by (3.27)

$$\sum_{j=1}^{\mu}(u,[H,\chi_j]\varphi_R u)+\sum_{j=\mu+1}^{N}(u,[H,\chi_j]\psi_R u)+(u,[H,\chi_0]u)\to\sum_{j=0}^{N}(u,[H,\chi_j]u)=0.$$

Therefore, as $R\to\infty$,

$$\sum_{j=1}^{\mu}(u,\chi_j[H,\varphi_R]u)+\sum_{j=\mu+1}^{N}(u,\chi_j[H,\psi_R]u)\to 0. \tag{3.28}$$

We put

$$V_j=H-\Big(-D_y^2+(n-1)D_y-y^2\Delta_{M_j}-\frac{(n-1)^2}{4}\Big).$$

Then we have, for $1\le j\le\mu$,

$$\begin{aligned}[H,\varphi_R]&=[-D_y^2+(n-1)D_y,\varphi_R]+[V_j,\varphi_R]\\ &=-\frac{2}{\log R}\rho\left(\frac{\log y}{\log R}\right)(D_y-\frac{n-1}{2})+\frac{1}{\log R}L_{j,R}.\end{aligned} \tag{3.29}$$

Here $L_{j,R}$ is a 1st order differential operator

$$L_{j,R}=a_{j,R}(x,y)D_y+b_{j,R}(x,y)D_x+c_{j,R}, \tag{3.30}$$

whose coefficients satisfy, due to (3.8),

$$|a_{j,R}(x,y)|+|b_{j,R}(x,y)|+|c_{j,R}(x,y)|\le C(1+|\log y|)^{-1-\epsilon}, \tag{3.31}$$

where the constant C is independent of $R>1$. Similarly, we have, for $\mu+1\le j\le N$,

$$\begin{aligned}[H,\psi_R]&=[-D_y^2+(n-1)D_y,\psi_R]+[V_j,\varphi_R]\\ &=\frac{2}{\log R}\rho\left(\frac{\log y}{\log R}\right)(D_y-\frac{n-1}{2})+\frac{1}{\log R}L_{j,R},\end{aligned} \tag{3.32}$$

where $L_{j,R}$ is a 1st order differential operator having the same property as above. In view of (3.28), we then have

$$\begin{aligned}&-\sum_{j=1}^{\mu}\frac{2}{\log R}(\chi_j\rho\left(\frac{\log y}{\log R}\right)(D_y-\frac{n-1}{2})u,u)\\ &+\sum_{j=\mu+1}^{N}\frac{2}{\log R}(\chi_j\rho\left(\frac{\log y}{\log R}\right)(D_y-\frac{n-1}{2})u,u)\\ &+\sum_{j=1}^{N}\frac{1}{\log R}(\chi_j L_{j,R}u,u)\to 0.\end{aligned} \tag{3.33}$$

We consider the case when u satisfies the outgoing radiation condition. Then we have, by (3.33),

$$\sum_{j=1}^{N}\frac{2i\sqrt{\lambda}}{\log R}\,(\chi_j\rho(\frac{\log y}{\log R})u,u)\to 0, \tag{3.34}$$

since one can replace $(D_y-(n-1)/2)$ by $-i\sqrt{\lambda}$ for $1\le j\le\mu$, by $i\sqrt{\lambda}$ for $\mu+1\le j\le N$, and $(\chi_j L_{j,R}u,u)/\log R\to 0$. This shows that, for $1\le j\le N$,

$$\frac{1}{\log R}\int_0^\infty \chi_j(y)\rho(\frac{\log y}{\log R})\|u(\cdot,y)\|^2_{L^2(M_j)}\frac{dy}{y^n}\to 0.$$

Thus, u satisfies conditions of Theorem 3.6, providing the desired result.

The case in which u satisfies the incoming radiation condition is proved similarly. □

These preparations are sufficient to prove the limiting absorption principle for H as in Chap. 2, §2.

Theorem 3.8. *For $\lambda\in\sigma_e(H)\setminus\sigma_p(H)$, there exists a limit*

$$\lim_{\epsilon\to 0}R(\lambda\pm i\epsilon)\equiv R(\lambda\pm i0)\in\mathbf{B}(\mathcal{B};\mathcal{B}^*)$$

in the weak $*$ sense. Moreover, for any compact interval $I\subset\sigma_e(H)\setminus\sigma_p(H)$, there exists a constant $C>0$ such that

$$\|R(\lambda\pm i0)f\|_{\mathcal{B}^*}\le C\|f\|_{\mathcal{B}},\quad \lambda\in I.$$

For $f\in\mathcal{B}$, we put $u=R(\lambda\pm i0)f$. Then u is a unique solution to the equation $(H-\lambda)u=f$ satisfying the outgoing (for the case $+$) or incoming (for the case $-$) radiation condition. For $f,g\in\mathcal{B}$, $(R(\lambda\pm i0)f,g)$ is continuous with respect to $\lambda\in\sigma_e(H)\setminus\sigma_p(H)$.

In order to prove Theorem 3.8, recall that Lemmas 2.2.4 $\sim$ 2.2.9 also hold for $M_j\times(0,\infty)$ with $\mathbf{h}$ replaced by $L^2(M_j)$. Let χ_j be the partition of unity (3.15), and put $u=R(\lambda+i\epsilon)f,\ u_j=\chi_j u$. Then, with ϵ defined by (3.8),

$$\|u_j\|_{\mathcal{B}^*}\le C_s\left(\|f\|_{\mathcal{B}}+\|u\|_{-s}\right),\quad 1/2<s<(1+\epsilon)/2, \tag{3.35}$$

where C_s is independent of $\lambda\in I$. Indeed, we first observe that

$$(H-\lambda-i\epsilon)u_j=\chi_j f+[H,\chi_j]u.$$

By Theorem 2.1.3 (6),

$$\|D_i u_j\|_{-s},\|D_iD_l u_j\|_{-s}\le C_s\left(\|f\|_{\mathcal{B}}+\|u_j\|_{-s}\right),$$

and as $[H,\chi_j]$, $\left[H_{free(j)},\chi_j\right]$ are compactly supported, we also have

$$\|\,[H,\,\chi_j]\,u\|_{\mathcal{B}},\ \|\left[H_{free(j)},\,\chi_j\right]u\|_{\mathcal{B}}\le C_s\left(\|f\|_{\mathcal{B}}+\|u_j\|_{-s}\right).$$

At last, rewriting the equation for u_j as

$$(H_{free(j)}-\lambda-i\epsilon)u_j=\chi_j f+[H_{free(j)},\chi_j]u+\chi_j Vu,$$

and using (3.8), we obtain (3.35) by Lemma 2.2.9. Summing up (3.35), we obtain

$$\|u\|_{\mathcal{B}^*}\le C_s\left(\|f\|_{\mathcal{B}}+\|u\|_{-s}\right),\quad 1/2<s<(1+\epsilon)/2,$$

Once we have derived this estimate, the remaining arguments are essentially the same as those in Chap. 2. Namely, arguing in the same way as in Lemma 2.2.13, we can prove the following lemma.

Lemma 3.9. *Take $s > 1/2$ sufficiently close to $1/2$. Let I be any compact interval in $(0,\infty)\setminus\sigma_p(H)$, and put $J = \{\lambda \pm i\epsilon; \lambda \in I, 0 < \epsilon < 1\}$.*
(1) There exists a constant $C_s > 0$ such that

$$\sup_{z\in J}\|R(z)f\|_{-s} \leq C_s\|f\|_{\mathcal{B}}.$$

(2) For any $f \in \mathcal{B}$ and $\lambda \in (0,\infty)\setminus\sigma_p(H)$, the strong limit $\lim_{\epsilon\to 0} R(\lambda \pm i0)f$ exists in $L^{2,-s}$.
(3) $R(\lambda \pm i0)f$ is an $L^{2,-s}$-valued continuous function of $\lambda \in (0,\infty)\setminus\sigma_p(H)$.

Since $L^{2,s}$ $(s > 1/2)$ is dense in $\mathcal{B}$, Theorem 3.8 follows from Lemma 3.9 and (3.35). □

3.4. Fourier transform associated with H**.** One can apply the abstract theory in Chap. 2, §4 to H after suitable modifications. However, we shall give here a direct approach to the spectral representation for H.

Let $H_{free(j)}$ be as above and χ_j as in (3.15). We put

$$\widetilde{V}_j = H - H_{free(j)} \quad \text{on} \quad \mathcal{M}_j. \tag{3.36}$$

This is symmetric, since so are H and $H_{free(j)}$ on $C_0^\infty(\mathcal{M}_j)$. Using

$$(H_{free(j)} - \lambda)\chi_j R(\lambda \pm i0) = \chi_j + \Big([H_{free(j)}, \chi_j] - \chi_j\widetilde{V}_j\Big) R(\lambda \pm i0), \tag{3.37}$$

we have

$$\begin{aligned}\chi_j R(\lambda \pm i0) = &R_{free(j)}(\lambda \pm i0)\chi_j \\ &+ R_{free(j)}(\lambda \pm i0)\Big([H_{free(j)}, \chi_j] - \chi_j\widetilde{V}_j\Big) R(\lambda \pm i0).\end{aligned} \tag{3.38}$$

This formula suggests how the generalized Fourier transform is constructed by the perturbation method.

3.4.1. *Definition of $\mathcal{F}^{(\pm)}_{free(j)}(k)$.* Let $0 = \lambda_{j,0} < \lambda_{j,1} \leq \lambda_{j,2} \leq \cdots$ be the eigenvalues of the Laplace-Beltrami operator on M_j and $|M_j|^{-1/2} = \varphi_{j,0}, \varphi_{j,1}, \varphi_{j,2}, \cdots$ the associated orthonormal eigenvectors, where $|M_j|$ is the volume of M_j. We define, for $\phi \in L^2(M_j)$,

$$P_{j,m}\phi = (\phi, \varphi_{j,m})_{L^2(M_j)}\varphi_{j,m}, \tag{3.39}$$

$$\Pi_{j,m}\phi = (\phi, \varphi_{j,m})_{L^2(M_j)}. \tag{3.40}$$

Assume that for $1 \leq j \leq \mu$, $\mathcal{M}_j$ has a regular infinity, and for $\mu + 1 \leq j \leq N$, $\mathcal{M}_j$ has a cusp.

(i) For $1 \leq j \leq \mu$ (the case of regular infinity), we define

$$\mathcal{F}^{(\pm)}_{free(j)}(k) = \sum_{m=0}^{\infty} C^{(\pm)}_{j,m}(k)\, P_{j,m} \otimes F^{(\pm)}_{free(j),m}(k), \tag{3.41}$$

where $F^{(\pm)}_{free(j),m}$ is defined by (2.6), (2.10), (2.14) with M replaced by M_j, and $C^{(\pm)}_{j,m}(k)$ is the constant in (2.15) with λ_m replaced by $\lambda_{j,m}$, i.e.

$$C^{(\pm)}_{j,m}(k) = \begin{cases} \left(\dfrac{\sqrt{\lambda_{j,m}}}{2}\right)^{\mp ik}, & (\lambda_{j,m} \neq 0), \\ \dfrac{\pm i}{k\omega_\pm(k)}\sqrt{\dfrac{\pi}{2}}, & (\lambda_{j,m} = 0). \end{cases} \tag{3.42}$$

Thus, in this case, $F^{(\pm)}_{free(j)}(k) = F^{(\pm)}_{reg,free(j)}(k)$, see (2.13).

(ii) For $\mu + 1 \leq j \leq N$ (the case of cusp), we define

$$\mathcal{F}^{(\pm)}_{free(j)}(k) = P_{j,0} \otimes F^{(\mp)}_{free(j),0}(k). \tag{3.43}$$

Thus, in this case, $F^{(\pm)}_{free(j)}(k) = F^{(\pm)}_{c,free(j)}(k)$, see (2.16).

3.4.2. *Definition of $\mathcal{F}^{(\pm)}(k)$.* For $1 \leq j \leq N$, we define

$$\mathcal{F}^{(\pm)}_j(k) = \mathcal{F}^{(\pm)}_{free(j)}(k) Q_j(k^2 \pm i0), \tag{3.44}$$

$$Q_j(z) = \chi_j + \left([H_{free(j)}, \chi_j] - \chi_j \widetilde{V}_j\right) R(z) = (H_{free(j)} - z)\chi_j R(z). \tag{3.45}$$

Finally, we define the Fourier transform associated with H by

$$\mathcal{F}^{(\pm)}(k) = \left(\mathcal{F}^{(\pm)}_1(k), \cdots, \mathcal{F}^{(\pm)}_N(k)\right). \tag{3.46}$$

3.4.3. *Asymptotic expansion of the resolvent.* For $f, g \in \mathcal{B}^*$ on $\mathcal{M}$, by $f \simeq g$ we mean that on each end the following expansion

$$\lim_{R\to\infty} \frac{1}{\log R} \int_{1/R}^{R} \rho_j(y) \|f(y) - g(y)\|^2_{L^2(M_j)} \frac{dy}{y^n} = 0$$

holds, where $\rho_j(y) = 1$ $(y < 1)$, $\rho_j(y) = 0$ $(y > 1)$ when $\mathcal{M}_j$ has a regular infinity, and $\rho_j(y) = 0$ $(y < 1)$, $\rho_j(y) = 1$ $(y > 1)$ when $\mathcal{M}_j$ has a cusp. Applying Theorem 2.6 on each end, we get the following theorem.

Theorem 3.10. *Let $f \in \mathcal{B}$, $k^2 \in \sigma_e(H) \setminus \sigma_p(H)$, and χ_j the partition of unity from (3.15). Then we have*

$$\begin{aligned} R(k^2 \pm i0)f \simeq \omega_\pm(k) \sum_{j=1}^{\mu} \chi_j y^{(n-1)/2 \mp ik} \mathcal{F}^{(\pm)}_j(k) f \\ + \omega^{(c)}_\pm(k) \sum_{j=\mu+1}^{N} \chi_j y^{(n-1)/2 \pm ik} \mathcal{F}^{(\pm)}_j(k) f. \end{aligned}$$

We put

$$\mathbf{h}_\infty = \left(\mathop{\oplus}_{j=1}^{\mu} L^2(M_j)\right) \oplus \left(\mathop{\oplus}_{j=\mu+1}^{N} P_{j,0} L^2(M_j)\right), \tag{3.47}$$

As a matter of fact,

$$P_{j,0} L^2(M_j) = \mathbf{C}\,\varphi_{j,0} = \{c\,\varphi_{j,0}\,;\, c \in \mathbf{C}\}, \quad \varphi_{j,0} = |M_j|^{-1.2},$$

equipped with the inner product

$$(c_1\varphi_{j,0}, c_2\varphi_{j,0})_{\mathbf{C}_j} = c_1\overline{c_2}. \tag{3.48}$$

For $\phi, \psi \in \mathbf{h}_\infty$ we define the inner product by

$$(\phi, \psi)_{\mathbf{h}_\infty} = \sum_{j=1}^{\mu} (\phi_j, \psi_j)_{L^2(M_j)} + \sum_{j=\mu+1}^{N} (\phi_j, \psi_j)_{\mathbf{C}_j}. \tag{3.49}$$

We then have the following lemma.

Lemma 3.11. *For $f, g \in \mathcal{B}$ and $k^2 \in \sigma_e(H) \setminus \sigma_p(H)$,*

$$\frac{k}{\pi i}\left(\left[R(k^2+i0) - R(k^2-i0)\right] f, g\right) = \left(\mathcal{F}^{(\pm)}(k)f, \mathcal{F}^{(\pm)}(k)g\right)_{\mathbf{h}_\infty}.$$

Proof. Take $\chi \in C_0^\infty(\mathbf{R})$ such that $\chi(t) = 1$ $(|t| < 1)$, $\chi(t) = 0$ $(|t| > 2)$. Let $\chi_R \in C_0^\infty(\mathcal{M})$ be such that $\chi_R = 1$ on a neighborhood of $\mathcal{K}$, $\chi_R = \chi(\log y / \log R)$ on each $\mathcal{M}_j$, where $R > 0$ is a large parameter. Let χ_j be the partition of unity from (3.15). Putting $u = R(k^2 + i0)f$, $v = R(k^2+i0)g$, we have

$$(\chi_R u, Hv) - (Hu, \chi_R v) = ([H, \chi_R]u, v) = \sum_{j=1}^{N} (\chi_j [H, \chi_R] u, v),$$

since $\chi_R = 1$ on a neighborhood of $\operatorname{supp} \chi_0$. Next we take $\widetilde{\chi}_j \in C^\infty(\mathcal{M}_j)$ such that $\operatorname{supp} \widetilde{\chi}_j \subset \mathcal{M}_j$ and $\widetilde{\chi}_j = 1$ on $\operatorname{supp} \chi_j$. Then, by Theorems 3.8, 2.1.3 (5) and (3.8), we have, as $R \to \infty$,

$$\begin{aligned}(\chi_R u, Hv) - (Hu, \chi_R v) &= \sum_{j=1}^{N} (\chi_j [H, \chi_R] \widetilde{\chi}_j u, v) \\ &= \sum_{j=1}^{N} (\chi_j [H_{free(j)}, \chi_R] \widetilde{\chi}_j u, v) + o(1).\end{aligned}$$

On each end, we have

$$\begin{aligned}\left[-y^2 \partial_y^2 + (n-2) y \partial_y, \chi_R \right] &= -\frac{2}{\log R} \chi'\Big(\frac{\log y}{\log R}\Big)\Big(D_y - \frac{n-1}{2}\Big) \\ &\quad - \left(\frac{1}{\log R}\right)^2 \chi''\Big(\frac{\log y}{\log R}\Big).\end{aligned}$$

Therefore,

$$(\chi_j [H_{free(j)}, \chi_R] \widetilde{\chi}_j u, v) = -\frac{2}{\log R}\left(\chi_j \chi'\Big(\frac{\log y}{\log R}\Big)\Big(D_y - \frac{n-1}{2}\Big) u, v\right) + o(1).$$

Since, by Theorem 3.8, u satisfies the outgoing radiation condition, for $1 \leq j \leq \mu$, one can replace $(D_y - (n-1)/2)u$ by $-iku$. Hence,

$$\begin{aligned}(\chi_j [H_{free(j)}, \chi_R] \widetilde{\chi}_j u, v) &= \frac{2ik}{\log R}\left(\chi_j \chi'\Big(\frac{\log y}{\log R}\Big) u, v\right) + o(1) \\ &= \frac{2ik}{\log R} \cdot \frac{\pi}{2k^2}\left(\chi_j \chi'\Big(\frac{\log y}{\log R}\Big) y^{n-1} \mathcal{F}_j^{(+)}(k) f, \mathcal{F}_j^{(+)}(k) g\right) + o(1) \\ &= \frac{\pi i}{k}\left(\mathcal{F}_j^{(+)}(k) f, \mathcal{F}_j^{(+)}(k) g\right)_{L^2(M_j)} + o(1),\end{aligned}$$

where we have used Theorem 3.10 in the 2nd line, and

$$\frac{1}{\log R}\int_{-\infty}^{0} \chi'\Big(\frac{\log y}{\log R}\Big) \frac{dy}{y} = 1.$$

For $\mu+1 \leq j \leq N$, one replaces $(D_y - (n-1)/2)u$ by iku, and uses

$$\frac{1}{\log R}\int_0^\infty \chi'\Big(\frac{\log y}{\log R}\Big)\frac{dy}{y} = -1$$

to obtain

$$\begin{aligned}(\chi_j[H_{free(j)}, \chi_R]\widetilde{\chi}_j u, v) &= -\frac{2ik}{\log R}\left(\chi_j\chi'\Big(\frac{\log y}{\log R}\Big)u, v\right) + o(1)\\ &= \frac{\pi i}{k}\big(\mathcal{F}_j^{(+)}(k)f, \mathcal{F}_j^{(+)}(k)g\big)_{\mathbf{C}_j} + o(1).\end{aligned}$$

Using

$$\begin{aligned}(\chi_R u, Hv) - (Hu, \chi_R v) &\to (u, g) - (f, v)\\ &= (R(k^2+i0)f, g) - (f, R(k^2+i0)g),\end{aligned}$$

we complete the proof of the lemma. □

We put

$$\widehat{\mathcal{H}} = L^2((0,\infty); \mathbf{h}_\infty; dk).$$

Theorem 3.12. *We define* $\big(\mathcal{F}^{(\pm)}f\big)(k) = \mathcal{F}^{(\pm)}(k)f$ *for* $f \in \mathcal{B}$. *Then* $\mathcal{F}^{(\pm)}$ *is uniquely extended to a bounded operator from* $L^2(\mathcal{M})$ *to* $\widehat{\mathcal{H}}$ *with the following properties.*

(1) Ran $\mathcal{F}^{(\pm)} = \widehat{\mathcal{H}}$.
(2) $\|f\| = \|\mathcal{F}^{(\pm)}f\|$ *for* $f \in \mathcal{H}_{ac}(H)$.
(3) $\mathcal{F}^{(\pm)}f = 0$ *for* $f \in \mathcal{H}_p(H)$.
(4) $\left(\mathcal{F}^{(\pm)}Hf\right)(k) = k^2\left(\mathcal{F}^{(\pm)}f\right)(k)$ *for* $f \in D(H)$.
(5) $\mathcal{F}^{(\pm)}(k)^* \in \mathbf{B}(\mathbf{h}_\infty; \mathcal{B}^*)$ *and* $(H-k^2)\mathcal{F}^{(\pm)}(k)^* = 0$ *for* $k^2 \in (0,\infty)\setminus\sigma_p(H)$.
(6) For $f \in \mathcal{H}_{ac}(H)$, *the inversion formula holds:*

$$f = \left(\mathcal{F}^{(\pm)}\right)^*\mathcal{F}^{(\pm)}f = \sum_{j=1}^N\int_0^\infty \mathcal{F}_j^{(\pm)}(k)^*\left(\mathcal{F}_j^{(\pm)}f\right)(k)dk.$$

Remark The meaning of the integral in (6) is as follows. Let $(0,\infty)\setminus\sigma_p(H) = \cup_{i=1}^\infty I_i$, $I_i = (a_i, b_i)$ being non-overlapping connected open interval. For $g(k) \in \widehat{\mathcal{H}}$, we have by (5)

$$\int_{\sqrt{a_i}+\epsilon}^{\sqrt{b_i}-\epsilon}\mathcal{F}_j^{(\pm)}(k)^*g(k)dk \in \mathcal{B}^*.$$

As a matter of fact, it belongs to $L^2(\mathcal{M})$, and

$$\lim_{\epsilon\to 0}\int_{\sqrt{a_i}+\epsilon}^{\sqrt{b_i}-\epsilon}\mathcal{F}_j^{(\pm)}(k)^*g(k)dk \in L^2(\mathcal{M})$$

in the sense of strong convergence in $L^2(\mathcal{M})$. Denoting this limit by

$$\int_{\sqrt{I_i}}\mathcal{F}_j^{(\pm)}(k)^*g(k)dk,$$

we define

$$\int_0^\infty \mathcal{F}_j^{(\pm)}(k)^*g(k)dk = \sum_{i=1}^\infty\int_{\sqrt{I_i}}\mathcal{F}_j^{(\pm)}(k)^*g(k)dk.$$

Proof. Let $E(\lambda)$ be the spectral decomposition for H. Since the interval (a_i, b_i) does not contain eigenvalues of H, we have by Lemma 3.11 and Stone's formula

$$\frac{1}{2\pi i}\int_{a_i+\epsilon}^{b_i-\epsilon}\left([R(\lambda+i0)-R(\lambda-i0)]f,f\right)d\lambda=\int_{\sqrt{a_i+\epsilon}}^{\sqrt{b_i-\epsilon}}\|\mathcal{F}^{(\pm)}(k)f\|^2dk,$$

for $f \in \mathcal{B}$. When $\epsilon \to 0$, the left-hand side converges to $(E((a_i, b_i))f, f)$. Therefore, so does the right-hand side and

$$(E(I_i)f,f)=\int_{\sqrt{I_i}}\|\mathcal{F}^{(\pm)}(k)f\|^2dk.$$

Since the end points of $(a_i.b_i)$ are eigenvalues, we have adding these formulas

$$\big(E((0,\infty)\setminus\cup_{\lambda_n\in\sigma_p(H)}\{\lambda_n\})f,f\big)=\int_0^\infty\|\mathcal{F}^{(\pm)}(k)f\|^2dk.$$

Let $P_{ac}(H)$ be the projection onto the absolutely continuous subspace for H. Then

$$E((0,\infty)\setminus\cup_{\lambda_n\in\sigma_p(H)}\{\lambda_n\})=P_{ac}(H).$$

Therefore, we have

$$(P_{ac}(H)f,f)=\int_0^\infty\|\mathcal{F}^{(\pm)}(k)f\|^2dk,$$

which proves (2), (3).

Let $f \in C_0^\infty(\mathcal{M})$. By (3.44), (3.45) and Theorem 2.1 (2), we have

$$\begin{aligned}\mathcal{F}_j^{(\pm)}(k)(H-k^2)f&=\mathcal{F}_{free(j)}^{(\pm)}(k)Q_j(k^2\pm i0)(H-k^2)f\\&=\mathcal{F}_{free(j)}^{(\pm)}(k)(H_{free(j)}-k^2)\chi_jf=0.\end{aligned}$$

To prove (4) for $f \in D(H)$, we have only to approximate it by a sequence in $C_0^\infty(\mathcal{M})$.

Theorem 3.8 and Lemma 3.11 imply that $\mathcal{F}^{(\pm)}(k) \in \mathbf{B}(\mathcal{B}; \mathbf{h}_\infty)$. Therefore, $\mathcal{F}^{(\pm)}(k)^* \in \mathbf{B}(\mathbf{h}_\infty; \mathcal{B}^*)$. This and (4) yield (5).

To prove (1). we have only to show that $\operatorname{Ran}\mathcal{F}^{(\pm)}$ is dense in $\widehat{\mathcal{H}}$, since $\operatorname{Ran}\mathcal{F}^{(\pm)}$ is closed by (2), (3). The idea is the same as the case of Lemma 1.3.19. For the sake of notational simplicity, we assume that there are only 2 ends, $\mathcal{M}_1$ with regular infinity and $\mathcal{M}_2$ with cusp. Suppose

$$(\varphi_1(k),\varphi_2(k)\varphi_{2,0})\in\mathbf{h}_\infty=L^2((0,\infty);L^2(M_1);dk)\times L^2((0,\infty);\mathbf{C};dk),$$

where $\varphi_{2,0} = |M_2|^{-1/2}$ is the eigenfunction of Δ_{M_2} associated with zero eigenvalue, is orthogonal to $\operatorname{Ran}\mathcal{F}^{(+)}$. Let $\{e_1, e_2, \cdots\}$ be a complete orthnormal system of $L^2(M_1)$, and put

$$\varphi_{1,n}(k)=(\varphi_1(k),e_n)_{L^2(M_1)}.$$

Let $\mathcal{L}(\psi)$ be the set of Lebesgue points of $\psi \in L^1_{loc}((0,\infty))$ introduced in the proof of Lemma 1.3.19. We take

$$\ell\in\Big(\cap_{n=1}^\infty\mathcal{L}(\varphi_{1,n})\Big)\cap\Big(\mathcal{L}(\|\varphi_1(k)\|^2_{L^2(M_1)})\Big)\cap\Big(\mathcal{L}(\varphi_2)\Big)\cap\Big(\mathcal{L}(|\varphi_2|^2)\Big).$$

Let $\{\chi_j\}_{j=0}^2$ be the partition of unity from (3.15). We fix m arbitrarily, and put

$$u_\ell=\omega_+(\ell)\chi_1(y)y^{(n-1)/2-i\ell}\alpha e_m+\omega_+^{(c)}(\ell)\chi_j(y)y^{(n-1)/2+i\ell}\beta\varphi_{2,0},$$

α, β being arbitraily chosen constants. We further put

$$(H-\ell^2)u_\ell = g_\ell.$$

Then, as can be checked easily, $g_\ell \in L^{2,(1+\epsilon)/2}$, and by Theorems 3.8 and 3.10, u_ℓ is written as $u_\ell = R(\ell^2+i0)g_\ell$. Moreover, etting $\mathcal{F}^{(+)}(k)g_\ell = (C_1(k), C_2(k)\varphi_{2,0})$, we see that $(C_1(k), C_2(k)\varphi_{2,0})$ is an $L^2(M_1)\times \mathbf{C}$-valued continuous function of $k>0$, satisfying

$$(C_1(\ell), e_n) = \delta_{mn}\alpha, \qquad C_2(\ell) = \beta. \tag{3.50}$$

By our assumption, $(\varphi_1(k), \varphi_2(k)\varphi_{2,0})$ is orthogonal to $\mathcal{F}^{(+)}(k)E_H(I)g_\ell$, I being any interval of $(0,\infty)$. Hence,

$$\int_I \Big((\varphi_1(k), C_1(k))_{L^2(M_1)} + \varphi_2(k)\overline{C_2(k)} \Big)\, dk = 0$$

for any interval $I \subset (0,\infty)$. By the same arguments as in the proof of Lemma 1.3.19, we then have

$$\frac{1}{2\epsilon}\int_{\ell-\epsilon}^{\ell+\epsilon} \varphi_2(k)\overline{C_2(k)}dk \to \varphi_2(\ell)\overline{\beta}.$$

The 1st term is computed as

$$\begin{aligned}\frac{1}{2\epsilon}\int_{\ell-\epsilon}^{\ell+\epsilon} (\varphi_1(k), C_1(k))_{L^2(M_1)}dk &= \frac{1}{2\epsilon}\int_{\ell-\epsilon}^{\ell+\epsilon} (\varphi_1(k), C_1(k)-C_1(\ell))_{L^2(M_1)}dk \\ &\quad + \frac{1}{2\epsilon}\int_{\ell-\epsilon}^{\ell+\epsilon} (\varphi_1(k), C_1(\ell))_{L^2(M_1)}dk.\end{aligned}$$

By (3.50), $(\varphi_1(k), C_1(\ell))_{L^2(M_1)} = \varphi_{1,m}(k)\overline{\alpha}$, hence

$$\frac{1}{2\epsilon}\int_{\ell-\epsilon}^{\ell+\epsilon} (\varphi_1(k), C_1(\ell))_{L^2(M_1)}dk \to \varphi_{1,m}(\ell)\overline{\alpha}.$$

We also have

$$\begin{aligned}&\left|\frac{1}{2\epsilon}\int_{\ell-\epsilon}^{\ell+\epsilon} (\varphi_1(k), C_1(k)-C_1(\ell))_{L^2(M_1)}dk\right| \\ &\le \left(\frac{1}{2\epsilon}\int_{\ell-\epsilon}^{\ell+\epsilon} \|\varphi_1(k)\|^2_{L^2(M_1)}dk\right)^{1/2} \times \left(\frac{1}{2\epsilon}\int_{\ell-\epsilon}^{\ell+\epsilon} \|C_1(k)-C_1(\ell)\|^2_{L^2(M_1)}dk\right)^{1/2}.\end{aligned}$$

The right-hand side tends to 0, since ℓ is an Lebesgue point of $\|\varphi_1(k)\|^2_{L^2(M_1)}$, and $C_1(k)$ is an $L^2(M_1)$-valued continuous function of $k>0$. We have, therefore, obtained that

$$\varphi_{1,m}(\ell)\overline{\alpha} + \varphi_2(\ell)\overline{\beta} = 0.$$

Since α, β and m are arbitrarily, we have $\varphi_1(\ell)=0$, $\varphi_2(\ell)=0$, which completes the proof of (1). The proof of (6) is the same as Theorem 1.3.13. $\square$

3.5. S **matrix.** As in Chap. 2, we can prove the existence and completeness of time-dependent wave operators and introduce the Radon transform associated with H. We give a breif sketch of the proof later. Here, instead of this time-depedent approach, we construct the S-matrix by using the generalized Fourier transform.

The following theorem is proved in the same way as Theorem 1.4.3 with $\mathcal{F}_0^{(\pm)}(k)$ replaced by $\mathcal{F}^{(\pm)}(k)$, and is a generalization of the modified *Poisson-Herglotz* formula.

Theorem 3.13. *If* $k^2 \notin \sigma_p(H)$, *we have*

$$\mathcal{F}^{(\pm)}(k)\mathcal{B} = \mathbf{h}_\infty,$$

$$\{u \in \mathcal{B}^* \, ; \, (H - k^2)u = 0\} = \mathcal{F}^{(\pm)}(k)^* \mathbf{h}_\infty.$$

We derive an asymptotic expansion of solutions to the Helmholtz equation. Let V_j be the differential operator defined by

$$V_j = [H_{free(j)}, \chi_j] - \chi_j \widetilde{V}_j \quad (1 \leq j \leq N),$$

where $\widetilde{V}_j$ is defined by (3.36). We put

$$J_j(k) = \sum_{m=1}^{\infty} \left(\frac{\sqrt{\lambda_{j,m}}}{2} \right)^{-2ik} P_{j,m} = \left(\frac{\sqrt{-\Delta_{M_j}}}{2} \right)^{-2ik} (I - P_{j,0}), \tag{3.51}$$

where Δ_{M_j} is the Laplace-Beltami operator on M_j and $P_{j,0}$ is the projection onto the zero eigenspace for Δ_{M_j}. For $1 \leq j, l \leq N$, we define $\widehat{S}_{jl}(k) \in \mathbf{B}(L^2(M_l); L^2(M_j))$ by

$$\widehat{S}_{jl}(k) = \begin{cases} \delta_{jl} J_j(k) + \dfrac{\pi i}{k} \mathcal{F}_j^{(+)}(k) \big(V_l\big)^* \left(\mathcal{F}^{(-)}_{free(l)}(k) \right)^*, & 1 \leq j \leq \mu, \\ \dfrac{\pi i}{k} \mathcal{F}_j^{(+)}(k) \big(V_l\big)^* \left(\mathcal{F}^{(-)}_{free(l)}(k) \right)^*, & \mu + 1 \leq j \leq N. \end{cases} \tag{3.52}$$

Theorem 3.14. *For* $\psi = (\psi_1, \cdots, \psi_N) \in \mathbf{h}_\infty$, *the following asymptotic expansion holds:*

$$\begin{aligned} \big(\mathcal{F}^{(-)}(k)\big)^* \psi &= \sum_{j=1}^{N} \big(\mathcal{F}_j^{(-)}(k)\big)^* \psi_j \\ &\simeq \frac{ik}{\pi} \omega_-(k) \sum_{j=1}^{\mu} \chi_j \, y^{(n-1)/2 + ik} \, \psi_j + \frac{ik}{\pi} \omega_-^{(c)}(k) \sum_{j=\mu+1}^{N} \chi_j \, y^{(n-1)/2 - ik} \, \psi_j \\ &\quad - \frac{ik}{\pi} \omega_+(k) \sum_{j=1}^{\mu} \sum_{l=1}^{N} \chi_j \, y^{(n-1)/2 - ik} \, \widehat{S}_{jl}(k) \psi_l \\ &\quad - \frac{ik}{\pi} \omega_+^{(c)}(k) \sum_{j=\mu+1}^{N} \sum_{l=1}^{N} \chi_j \, y^{(n-1)/2 + ik} \, \widehat{S}_{jl}(k) \psi_l. \end{aligned}$$

Proof. First note that by (3.44)

$$\left(\mathcal{F}_j^{(-)}(k) \right)^* = \chi_j \left(\mathcal{F}^{(-)}_{free(j)}(k) \right)^* + R(k^2 + i0) \big(V_j\big)^* \left(\mathcal{F}^{(-)}_{free(j)}(k) \right)^*. \tag{3.53}$$

By (3.41), for $1 \le j \le \mu$,

$$
\begin{aligned}
\left(\mathcal{F}^{(-)}_{free(j)}(k)\right)^* \phi &= \sum_{m=0}^{\infty} \overline{C^{(-)}_{j,m}(k)} \left(F^{(-)}_{free(j),m}(k)\right)^* P_{j,m}\phi \\
&= \overline{C^{(-)}_{j,0}(k)} \frac{1}{\sqrt{2\pi}} y^{(n-1)/2+ik} P_{j,0}\phi \\
&+ \sum_{m=1}^{\infty} \overline{C^{(-)}_{j,m}(k)} \frac{(2k\sinh(k\pi))^{1/2}}{\pi} y^{(n-1)/2} K_{ik}(\sqrt{\lambda_{j,m}}\, y)\, P_{j,m}\phi,
\end{aligned}
$$

and by (3.43), for $\mu + 1 \le j \le N$,

$$
\left(\mathcal{F}^{(-)}_{free(j)}(k)\right)^* \phi = \frac{1}{\sqrt{2\pi}} y^{(n-1)/2-ik}\phi. \tag{3.54}
$$

Since $\mathcal{F}^{(-)}(k)^* \in \mathbf{B}(\mathbf{h}_\infty; \mathcal{B}^*)$, we have only to prove the theorem for $\psi = (\psi_1, \cdots, \psi_N) \in \mathbf{h}_\infty$ such that for $1 \le j \le \mu$, $P_{j,m}\psi_j = 0$ except for a finite number of m. By using Chap. 1, (3.6), (4.15) and (4.18), for $1 \le j \le \mu$, one can show

$$
\begin{aligned}
\left(\mathcal{F}^{(-)}_{free(j)}(k)\right)^* \psi_j &\simeq \frac{ik}{\pi}\omega_-(k) y^{(n-1)/2+ik}\psi_j \\
&\quad - \frac{ik}{\pi}\omega_+(k) y^{(n-1)/2-ik} \sum_{m\ge 1} \left(\frac{\sqrt{\lambda_m}}{2}\right)^{-2ik} P_{j,m}\psi_j.
\end{aligned} \tag{3.55}
$$

We apply (3.54) and (3.55) to the 1st term of the right-hand side of (3.53). To the 2nd term, we apply Theorem 3.10. We then have, for $1 \le j \le \mu$,

$$
\begin{aligned}
\left(\mathcal{F}^{(-)}_j(k)\right)^* \psi_j &\simeq \frac{ik}{\pi}\omega_-(k)\, \chi_j\, y^{(n-1)/2+ik}\, \psi_j \\
&\quad -\frac{ik}{\pi}\omega_+(k) \sum_{l=1}^{\mu} \chi_l\, y^{(n-1)/2-ik}\, \widehat{S}_{lj}(k)\psi_j \\
&\quad -\frac{ik}{\pi}\omega^{(c)}_+(k) \sum_{l=\mu+1}^{N} \chi_l\, y^{(n-1)/2+ik}\, \widehat{S}_{lj}(k)\psi_j.
\end{aligned}
$$

Similary, one can show, for $\mu + 1 \le j \le N$,

$$
\begin{aligned}
\left(\mathcal{F}^{(-)}_j(k)\right)^* \psi_j &\simeq \frac{ik}{\pi}\omega^{(c)}_-(k)\, \chi_j\, y^{(n-1)/2-ik}\, \psi_j \\
&\quad -\frac{ik}{\pi}\omega_+(k) \sum_{l=1}^{\mu} \chi_l\, y^{(n-1)/2-ik}\, \widehat{S}_{lj}(k)\psi_j \\
&\quad -\frac{ik}{\pi}\omega^{(c)}_+(k) \sum_{l=\mu+1}^{N} \chi_l\, y^{(n-1)/2+ik}\, \widehat{S}_{lj}(k)\psi_j.
\end{aligned}
$$

Summing up these two formulas, we obtain the theorem. □

We define an operator-valued $N \times N$ matrix $\widehat{S}(k)$ by

$$
\widehat{S}(k) = \left(\widehat{S}_{jl}(k)\right), \tag{3.56}
$$

and call it *S-matrix*. This should be more properly called the geometric S-matrix in the context of Chap. 2, §6. This is a bounded operator on $\mathbf{h}_\infty$. Similarly to Theorem 2.7.9, we have the following asymptotic expansion.

Theorem 3.15. *(1) For any $u \in \mathcal{B}^*$ satisfying $(H-k^2)u=0$, there exists a unique $\psi^{(\pm)} = (\psi_1^{(\pm)}, \cdots, \psi_N^{(\pm)}) \in \mathbf{h}_\infty$ such that*

$$\begin{aligned} u \simeq \omega_-(k) \sum_{j=1}^{\mu} \chi_j \, y^{(n-1)/2+ik} \, \psi_j^{(-)} + \omega_-^{(c)}(k) \sum_{j=\mu+1}^{N} \chi_j \, y^{(n-1)/2-ik} \, \psi_j^{(-)} \\ - \omega_+(k) \sum_{j=1}^{\mu} \chi_j \, y^{(n-1)/2-ik} \, \psi_j^{(+)} - \omega_+^{(c)}(k) \sum_{j=\mu+1}^{N} \chi_j \, y^{(n-1)/2+ik} \, \psi_j^{(+)}. \end{aligned}$$

(2) For any $\psi^{(-)} \in \mathbf{h}_\infty$, there exists a unique $\psi^{(+)} \in \mathbf{h}_\infty$ and $u \in \mathcal{B}^*$ satisfying $(H-k^2)u=0$, for which the expansion (1) holds. Moreover

$$\psi^{(+)} = \widehat{S}(k)\psi^{(-)}.$$

Proof. By Theorem 3.13, $u \in \mathcal{F}^{(-)}(k)^*\mathbf{h}_\infty$. Using Theorem 3.14, we prove the result. □

Theorem 3.16. *$\widehat{S}(k)$ is unitary on $\mathbf{h}_\infty$.*

Proof. Let $u \in \mathcal{B}^*$ such that $(H-k^2)u = 0$. By Theorem 3.13, $u = \mathcal{F}^{(+)}(k)^*\psi^{(+)}$, $\psi^{(+)} \in \mathbf{h}_\infty$. By similar arguments as in Theorem 3.14, with $\mathcal{F}^{(+)}(k)^*$ instead of $\mathcal{F}^{(-)}(k)^*$, one can show that there exists $\psi^{(-)} \in \mathbf{h}_\infty$ such that the expansion in Theorem 3.15 (1) holds. In particular, $\psi^{(+)} = \widehat{S}(k)\psi^{(-)}$. This means that $\widehat{S}(k)$ is onto.

Thus, we have only to prove that $\widehat{S}(k)$ is isometric. Take $\psi^{(-)} = (\psi_1^{(-)}, \cdots, \psi_N^{(-)}) \in \mathbf{h}_\infty$ such that for $1 \leq j \leq \mu$, $P_{j,m}\psi_j^{(-)} = 0$ except for a finite number of m. We put for $1 \leq j \leq \mu$

$$a_{j,m} = \begin{cases} P_{j,0}\psi_j^{(-)}, & (m=0) \\ \left(\dfrac{\sqrt{\lambda_{j,m}}}{2}\right)^{-ik} \Gamma(1+ik) P_{j,m}\psi_j^{(-)}, & (m \neq 0) \end{cases}$$

$$u_j^{(-)} = \omega_-(k) \, \chi_j \left(y^{(n-1)/2+ik} a_{j,0} + \sum_{m \geq 1} y^{(n-1)/2} I_{ik}(\sqrt{\lambda_{j,m}}\, y) a_{j,m} \right).$$

Then, as $y \to 0$,

$$u_j^{(-)} \simeq \omega_-(k) \, \chi_j(y) \, y^{(n-1)/2+ik} \psi_j^{(-)}.$$

For $\mu+1 \leq j \leq N$, we put

$$u_j^{(-)} = \omega_-^{(c)}(k) \, \chi_j \, y^{(n-1)/2-ik} \psi_j^{(-)}, \tag{3.57}$$

and define

$$\begin{aligned} u^{(-)} &= \sum_{j=1}^{N} u_j^{(-)}, \quad f = (H-k^2)u^{(-)} \in \mathcal{B}, \\ u^{(+)} &= R(k^2+i0)f, \quad u = u^{(+)} - u^{(-)}, \\ \psi^{(+)} &= \mathcal{F}^{(+)}(k)f. \end{aligned}$$

Then, by Theorem 3.10, u and $\psi^{(\pm)}$ give the expansion in Theorem 3.15 (1). Lemma 3.11 implies

$$\begin{aligned}\frac{1}{2k}\|\psi^{(+)}\|^2 &= \frac{1}{2\pi i}\left(R(k^2+i0)f - R(k^2-i0)f, f\right)\\ &= \frac{1}{2\pi i}\left[(f,u^{(-)}) - (u^{(-)},f)\right].\end{aligned}$$

Here we have used the fact that

$$R(k^2-i0)f = u^{(-)},$$

since $u^{(-)}$ is incoming. Now we do the same computation as in Lemma 3.11. Let χ_R be as in the lemma. Then,

$$\begin{aligned}(f,\chi_R u^{(-)}) - (\chi_R u^{(-)}, f) &= ([H,\chi_R]u^{(-)}, u^{(-)})\\ &= \sum_{j=1}^{N}(\chi_j[H_{free(j)},\chi_R]\widetilde{\chi}_j\, u^{(-)}, u^{(-)}) + o(1).\end{aligned}$$

Recall that

$$[H_{free(j)},\chi_R] = -\frac{2}{\log R}\chi'\big(\frac{\log y}{\log R}\big)\big(D_y - \frac{n-1}{2}\big) + O(|\log R|^{-2}).$$

Then, for $1 \leq j \leq \mu$, using the fact that $u^{(-)}$ has the form (3.57), we have

$$\begin{aligned}(\chi_j[H_{free(j)},\chi_R]\widetilde{\chi}_j\, u^{(-)}, u^{(-)}) &= \frac{2ik}{\log R}\left(\chi'\Big(\frac{\log y}{\log R}\Big)u_j^{(-)}, u_j^{(-)}\right) + o(1)\\ &= \frac{2ik}{\log R}|\omega_-(k)|^2\int_0^1 \chi'\Big(\frac{\log y}{\log R}\Big)\frac{dy}{y}\|\psi_j^{(-)}\|^2 + o(1)\\ &= \frac{\pi i}{k}\|\psi_j^{(-)}\|^2 + o(1),\end{aligned}$$

where, at the last step, we use equation (4.18) of Ch. 1.

Similarly, for $\mu+1 \leq j \leq N$,

$$(\chi_j[H_{free(j)},\chi_R]\widetilde{\chi}_j\, u^{(-)}, u^{(-)}) = \frac{\pi i}{k}\|\psi_j^{(-)}\|^2 + o(1).$$

Taking $R \to \infty$, we obtain $\|\psi^{(+)}\| = \|\psi^{(-)}\|$. □

Corollary 3.17. *$\mathcal{F}^{(+)}(k) = \widehat{S}(k)\mathcal{F}^{(-)}(k)$.*

Proof. The above f satisfies $\psi^{(\pm)} = \mathcal{F}^{(\pm)}(k)f$. Since $\psi^{(+)} = \widehat{S}(k)\psi^{(-)}$ and, by (3.58), $\psi^{(-)} = \mathcal{F}^{(-)}(k)f$, the corollary is proved. □

3.6. Wave operators. We briefly look at the temporal asymptotics of $e^{-it\sqrt{H}}f$ for $f \in \mathcal{H}_{ac}(H)$. Let $\{\chi_j\}_{j=0}^N$ be the partition of unity given in Subsection 3.2. We can then show that

$$\|\chi_0 e^{-it\sqrt{H}}f\| \to 0, \quad \text{as} \quad t \to \pm\infty. \tag{3.58}$$

In fact, by approximating f, we have only to consider the case that $f \in D(H) \cap \mathcal{H}_{ac}(H)$. In this case, we have $\chi_0 e^{-it\sqrt{H}}f = \chi_0(H+i)^{-1}e^{-it\sqrt{H}}(H+i)f$. Since

$(H+i)f \in \mathcal{H}_{ac}(H)$, we have $\chi_0 e^{-itH}(H+i)f \to 0$ weakly as $t \to \pm\infty$. As also $\chi_0(H+i)^{-1}$ is compact, this proves (3.58). It then implies

$$\|e^{-it\sqrt{H}}f - \sum_{j=1}^{N} \chi_j e^{-it\sqrt{H}}f\| \to 0, \quad \text{as} \quad t \to \pm\infty.$$

Consider the behavior of $\chi_j e^{-it\sqrt{H}}f$ on the end $\mathcal{M}_j$. Suppose $\mathcal{M}_j$ is a regular end. Then the argument in Chapter 2 Subsection 8.3 works well without any essential change, and one can show that, as $t \to \infty$,

$$\left\| \chi_j e^{-it\sqrt{H}}f - \chi_j \frac{y^{(n-1)/2}}{\sqrt{\pi}} \int_0^\infty e^{ik(-\log y - t)} \left(\mathcal{F}_j^{(+)} f\right)(k) dk \right\| \to 0.$$

Similarly, for $g \in L^2(\mathcal{M}_j)$,

$$\left\| \chi_j e^{-it\sqrt{H_{free(j)}}}g - \chi_j \frac{y^{(n-1)/2}}{\sqrt{\pi}} \int_0^\infty e^{ik(-\log y - t)} \left(\mathcal{F}_{free(j)}^{(+)} g\right)(k) dk \right\| \to 0.$$

Taking $g = \left(\mathcal{F}_{free(j)}^{(+)}\right)^* \mathcal{F}_j^{(+)} f$, these two limits imply

$$\chi_j e^{-it\sqrt{H}}f \sim \chi_j e^{-it\sqrt{H_{free(j)}}} \left(\mathcal{F}_{free(j)}^{(+)}\right)^* \mathcal{F}_j^{(+)} f.$$

We can prove similar formulae when $\mathcal{M}_j$ is a cusp. This means that, in the long-run, the waves disappear from compact parts of the manifold, and, on each end, they behave like free waves.

Similarly, we can prove

$$\mathrm{s} - \lim_{t \to \infty} e^{it\sqrt{H}} \chi_j e^{-it\sqrt{H_{free(j)}}} = \left(\mathcal{F}_j^{(+)}\right)^* \mathcal{F}_{free(j)}^{(+)},$$

and, therefore, there exist the wave operators,

$$W_\pm = \mathrm{s} - \lim_{t \to \pm\infty} \sum_{j=1}^{N} e^{it\sqrt{H}} \chi_j e^{-it\sqrt{H_{free(j)}}} = \sum_{j=1}^{N} \left(\mathcal{F}_j^{(+)}\right)^* \mathcal{F}_{free(j)}^{(+)}. \tag{3.59}$$

Since $\mathcal{F}_{free(j)}^{(+)}$ are unitary, it follows from Theorem 3.12, that and $W_\pm$ are complete:

$$\mathrm{Ran}\, W_\pm = \mathcal{H}_{ac}(H).$$

As in Chap. 2, §8, we construct $\mathcal{F}_\pm$ from $\mathcal{F}^{(\pm)}$, and define the Radon transform by the formula

$$\left(\mathcal{R}_\pm f\right)(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iks} \left(\mathcal{F}_\pm f\right)(k) dk.$$

Then Theorem 2.8.9 also holds on $\mathcal{M}$.

4. Cusps and generalized eigenfunctions

In the following two sections, we consider the case in which $\mathcal{M}$ has only cusps as infinity. We use the same notation as in the previous section, and for the sake of simplicity assume that $\mathcal{M}$ has only one cusp and the manifold at infinity M satisfies $|M| = 1$. In this section z denotes a point in $\mathcal{M}$. Moreover, we assume:

(C-1) *The end $\mathcal{M}_1$ is identified with $M \times (1, \infty)$ and the metric of $\mathcal{M}$ is*

$$ds^2 = \sum_{i,j=1}^{n} g_{ij}(z) dz^i dz^j = \frac{(dy)^2 + h(x, dx)}{y^2} \quad \textit{on} \quad \mathcal{M}_1, \tag{4.1}$$

where we typically use local coordinates $z = (x, y)$, $x = (x_1, \dots, x_{n-1})$ being local coordinates on M.

4.1. A remark on the S-matrix. In Theorem 3.15, we have proven that for $k > 0$ such that $k^2 \notin \sigma_p(H)$ and $u \in \mathcal{B}^*$ satisfying $(H - k^2)u = 0$, there exist unique constant functions $\psi^{(\pm)} \in P_0 L^2(M)$ such that

$$u \simeq \omega_-^{(c)}(k) y^{(n-1)/2 - ik} \psi^{(-)} - \omega_+^{(c)}(k) y^{(n-1)/2+ik} \psi^{(+)}, \quad \omega_\pm^{(c)}(k) = \pm \frac{i}{k}\sqrt{\frac{\pi}{2}}. \tag{4.2}$$

$w_\pm^{(c)}(k)$ has natural extension to $k < 0$. Then taking $u(k) = \overline{u(-k)}$, we obtain, for $k < 0$, a solution to $(H - k^2)u = 0$ which also satisfies (4.2). With this in mind, we change the notion of the S-matrix as follows. Let

$$\mathcal{N}(k) = \Big\{ u \in \mathcal{B}^* \, ; \, \Big(-\Delta_g - \frac{(n-1)^2}{4} - k^2 \Big) u = 0 \Big\}.$$

Then, for any $0 \neq k \in \mathbf{R}$, such that $k^2 \notin \sigma_p(H)$, $\dim \mathcal{N}(k) = 1$, and one can choose a basis $u(z, k) \in \mathcal{N}(k)$ satisfying

$$u \simeq y^{(n-1)/2 - ik} + \widehat{S}(k) y^{(n-1)/2+ik}, \tag{4.3}$$

$\widehat{S}(k)$ being a complex number of modulus 1. Traditionally, we put

$$\mathcal{S}(s) = \widehat{S}(k), \quad s = (n-1)/2 - ik, \tag{4.4}$$

and call it the S-matrix.

4.2. Eisenstein series. We put

$$\sqrt{\sigma_p(H)} = \{ \zeta \in \mathbf{C} \, ; \, \zeta^2 \in \sigma_p(H) \}.$$

Let $\chi \in C^\infty((0, \infty))$ be such that $\chi(y) = 0$ for $y < 2$, $\chi(y) = 1$ for $y > 3$. We define for $k > 0$ and $\epsilon > 0$

$$\varphi(z, k - i\epsilon) = \chi(y)\, y^{\frac{n-1}{2} + i(k - i\epsilon)} - R((k - i\epsilon)^2)\, [H, \chi]\, y^{\frac{n-1}{2} + i(k-i\epsilon)}. \tag{4.5}$$

Due to (C-1), $\operatorname{supp}([H, \chi]) \subset M \times (2, 3)$ and this function φ satisfies

$$(H - (k - i\epsilon)^2) \varphi(z, k - i\epsilon) = 0.$$

By the reasoning to be explained in the next section, this function is called an *Eisenstein series*. As a function of $k - i\epsilon$, this is meromorphic in the lower-half plane and has poles at $\sqrt{\sigma_p(H)} \cap \mathbf{C}_-$. Note that in the standard notation, we put $s = (n-1)/2 + i(k - i\epsilon)$ and regard φ as a meromorphic function on $\{ s \in \mathbf{C} \, ; \, \operatorname{Re} s > (n-1)/2 \}$. By the limiting absorption principle, letting $\epsilon \to 0$, $\varphi(z, k - i\epsilon)$ is continuously extended to $\mathbf{R} \setminus \sqrt{\sigma_p(H)}$.

Using the definitions (3.44), (2.16), (2.9), and (3.45) with $\widetilde{V} = 0$, we have, for $k \in (0, \infty) \setminus \sqrt{\sigma_p(H)}$,

$$\mathcal{F}^{(+)}(k) f = \frac{1}{\sqrt{2\pi}} \int_{\mathcal{M}} \overline{\varphi(z, k)} f(z) \, d\mathcal{M}.$$

Hence, by Theorem 3.12 we have the following theorem.

Theorem 4.1. *$\mathcal{F}^{(+)}$ maps $\mathcal{H}_{ac}(H)$ onto $L^2((0,\infty)\,;\,P_0(L^2(M))\,;\,dk)$. For any $f \in L^2(\mathcal{M})$, the inversion formula holds:*

$$f(z) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \varphi(z,k)\widetilde{f}(k)dk + \sum_i (f,\psi_i)\psi_i,$$

$$\widetilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{\mathcal{M}} \overline{\varphi(z,k)} f(z) d\mathcal{M},$$

where ψ_i is a normalized eigenvector of H.

4.3. Theory of quadratic forms. Let us recall the theory of quadratic forms associated with self-adjoint extensions of symmetric operators. For the details, see e.g. [**80**] p. 322 or [**62**], p. 38. Let D be a dense subspace of a Hilbert space $\mathcal{H}$. A hermitian quadratic form $a(\cdot,\cdot)$ with domain D is a mapping : $D \times D \to \mathbf{C}$ satisfying

$$a(\lambda u + \mu v, w) = \lambda a(u,w) + \mu a(v,w), \quad \lambda,\mu \in \mathbf{C}, \quad u,v,w \in D$$

$$\overline{a(u,v)} = a(v,u), \quad u,v \in D.$$

A hermitian quadratic form $a(\cdot,\cdot)$ is said to be positive definite if there exists a constant $C > 0$, such that

$$a(u,u) \geq C\|u\|^2, \quad u \in D.$$

In this case $a(\cdot,\cdot)$ defines an inner product on D. If D is complete with respect to the norm $\|u\|_a = \sqrt{a(u,u)}$, $a(\cdot,\cdot)$ is said to be a closed form. We say that $a(\cdot,\cdot)$ is closable if, for any sequence $u_n \in D$ such that $\|u_n\| \to 0, \|u_n - u_m\|_a \to 0$, we have $\|u_n\|_a \to 0$. For a closable form $a(\cdot,\cdot)$, we define a subspace $\widetilde{D}$ by

$$u \in \widetilde{D} \Longleftrightarrow \exists u_n \in D \text{ s.t.} \|u_n - u\| \to 0, \|u_n - u_m\|_a \to 0.$$

For $u,v \in \widetilde{D}$, there exist $u_n, v_n \in D$ such that $u_n \to u$, $v_n \to v$, $\|u_n - u_m\|_a \to 0$, $\|v_n - v_m\|_a \to 0$. Then, the quadratic form, defined by

$$\widetilde{a}(u,v) = \lim_{m,n\to\infty} a(u_m, v_n)$$

can be shown to be positive defnite and closed and is called the closed extension of $a(\cdot,\cdot)$. Then the following theorem holds.

Theorem 4.2. *Let $a(\cdot,\cdot)$ be a positive definite closed form with domain D. Then there exists a unique self-adjoint operator A such that $D(A) \subset D$ and*

$$a(u,v) = (Au,v), \quad u \in D(A), \quad v \in D.$$

Moreover $D = D(A^{1/2})$.

A quadratic form $a(\cdot,\cdot)$ with domain D is said to be bounded from below if there exists a constant $C_0 \geq 0$ such that

$$a(u,u) \geq -C_0\|u\|^2, \quad \forall u \in D.$$

In this case the quadratic form $b(\cdot,\cdot)$ defined by

$$b(u,v) = a(u,v) + (C_0+1)(u,v)$$

is positive definite. $a(\cdot,\cdot)$ is said to be closable if so is $b(\cdot,\cdot)$. Let $\widetilde{b}(\cdot,\cdot)$ be the closed extension of $b(\cdot,\cdot)$. By Theorem 4.2, there exists a unique self-adjoint operator B such that $D(B) \subset \widetilde{D}$ and

$$\widetilde{b}(u,v) = (Bu,v), \quad u \in D(B), \quad v \in \widetilde{D}.$$

Letting

$$\widetilde{a}(u,v) = \widetilde{b}(u,v) - (C_0+1)(u,v),$$
$$A = B - (C_0+1),$$

we have $D(A) = D(B) \subset \widetilde{D}$, and

$$A \geq -C_0, \quad \widetilde{a}(u,v) = (Au,v), \quad u \in D(A), \quad v \in \widetilde{D}.$$

We call A the self-adjoint operator associated with $a(\cdot,\cdot)$.

4.4. 0-mode boundary value problem. We show that the Eisenstein series $\varphi(z,k)$ is meromorphically extended to **C** with respect to k. Following the arguments of [**26**], we consider the boundary value problem as below.

Recall that $\mathcal{M}$ is assumed to be

$$\mathcal{M} = \mathcal{K} \cup \mathcal{M}_1, \quad \mathcal{M}_1 = M \times (1,\infty), \quad |M| = 1, \tag{4.6}$$

where $\overline{\mathcal{K}}$ is compact. We can assume that

$$\mathcal{K} \cap \big(M \times (2,\infty)\big) = \emptyset.$$

Take $a > 3$, and put

$$\mathcal{M}^a_{int} = \mathcal{K} \cup \big(M \times (1,a)\big), \quad \mathcal{M}^a_{ext} = M \times (a,\infty), \quad \Gamma^a = M \times \{a\}.$$

Using the projections P_0 and P' on $L^2(M)$,

$$(P_0\psi)(x) = \int_M \psi(x')dM_{x'}, \quad P' = 1 - P_0,$$

we define the following Hilbert space:

$$\mathcal{H} = L^2(\mathcal{M}^a_{int}) \oplus (P' \otimes I^a_y)L^2(\mathcal{M}^a_{ext}) \subset L^2(\mathcal{M}),$$

with $\pi : L^2(\mathcal{M}) \to \mathcal{H}$ being the associated orthogonal projection. Here, for any $b > 0$, I^b_y is the cut-off projector, in the y-coordinate, onto $y > b$. To define the Sobolev spaces $H^m(\mathcal{M})$, we use representation (4.6) of $\mathcal{M}$. Namely, if U_l, $l = 1,\ldots,L$, is a coordinate covering of M, we use, as a coordinate covering of $\mathcal{M}$,

$$\mathcal{M} = \bigcup_{l=1}^{L+P} \mathcal{U}_l,$$

where $\mathcal{U}_l = U_l \times (1,\infty)$, $l = 1,\ldots,L$; $\{\mathcal{U}_l\}_{l=L+1}^{L+P}$ being a coordinate covering of $\mathcal{M}^2_{int}$. Using the corresponding decomposition of unity,

$$1 = \sum_{l=1}^{L+P} \Psi_l(z), \quad \mathrm{supp}(\Psi_l) \subset \mathcal{U}_l,$$

where we assume, for $y > 2$, $\Psi_l(x,y) = \psi_l(x)$, $\mathrm{supp}(\psi_l) \subset U_l$, $l = 1,\ldots,L$, we define

$$\|f\|^2_{H^m(\mathcal{M})} = \sum_{l=1}^{L+P} \|\Psi_l f\|^2_{H^m(\mathcal{U}_l)}.$$

Here $H^m(\mathcal{U}_l)$, $l = L+1,\ldots,L+P$, are usual Sobolev spaces, while

$$\|\Psi_l f\|^2_{H^m(\mathcal{U}_l)} = \sum_{|\alpha|\leq m} \int_1^\infty \|D^\alpha\left(\Psi_l f\right)\|^2_{L^2(M)} \frac{dy}{y^n}, \quad l = 1,\ldots,L,$$

where $D_i = y\partial_i$, $i = 1,\ldots,n$.

Note that, if $m = 1$, $\|f\|_{H^1}$ is equivalent to the classical invariant definition of H^1 on a Riemannian manifold,

$$\|f\|^2_{H^1(\mathcal{M})} \sim \|f\|^2_{L^2(\mathcal{M})} + \int_{\mathcal{M}} |df|^2_g d\mathcal{M} = \|f\|^2_{L^2(\mathcal{M})} + \int_{\mathcal{M}} g^{ij}\, \partial_i f\, \overline{\partial_j f}\, \sqrt{g}\, dz. \tag{4.7}$$

Next we define

$$\mathcal{H}^m := \pi H^m(\mathcal{M}), \quad m \geq 1.$$

Note that, with I_M being identity on M and $b > 1$, $\left(I_M \otimes I^b_y\right) f \in H^m(M \times (b, \infty))$ iff

$$\sum_j \int_b^\infty y^{2m} \left[(1 + \lambda_j^2)^m |\hat{f}_j(y)|^2 + |\partial_y^m \hat{f}_j(y)|^2\right] \frac{dy}{y^n} < \infty.$$

Here $f(x, y) = \sum_{j=0} \hat{f}_j(y)\phi_j(x)$, for $y > b$. Thus,

$$(P' \otimes I^b_y)\mathcal{H}^m \to H^m(M \times (b, \infty)), \quad b > 1.$$

Also, if $u \in \mathcal{H}^m$, then $\partial_y^j(P' \otimes I^b_y)u$, $0 \leq j \leq m - 1$, is continuous across Γ^a, $a > b$.

We define a quadratic form $l(\cdot, \cdot)$ with domain $\mathcal{H}^1$ by

$$\begin{aligned} l(u, u) = {} & (du, du)_{L^2(\mathcal{M}^a_{int})} + \|u\|^2_{L^2(\mathcal{M}^a_{int})} \\ & + (du, du)_{L^2(\mathcal{M}^a_{ext})} + \|u\|^2_{L^2(\mathcal{M}^a_{ext})}, \end{aligned}$$

see (4.7). Then $l(\cdot, \cdot)$ is a positive definite closed form on $\mathcal{H}^1$, and $\sqrt{l(\cdot, \cdot)}$ is equivalent to the $\mathcal{H}^1$-norm. Hence, by Theorem 4.2, there exists a unique self-adjoint operator L such that $L \geq 1$, $D(L^{1/2}) = \mathcal{H}^1$ and

$$l(u, v) = (Lu, v), \quad \forall u \in D(L), \quad \forall v \in \mathcal{H}^1.$$

We introduce the set D_L by

$$D_L = \{u \in \mathcal{H}^2\, ;\, (\partial_y(P_0 \otimes I_y)u)\, (a - 0) = 0\}. \tag{4.8}$$

Here, for $w \in H^1(M \times (a, a+1))$ or $w \in H^1(M \times (a - 1, a))$, $w(a \pm 0)$ is defined by

$$w(a \pm 0) = \lim_{\epsilon \to 0} w(\cdot, a \pm \epsilon).$$

Lemma 4.3. *(1) L has compact resolvent.*
(2) $D(L) = D_L$.
(3) If $\zeta \notin \sigma(L)$, for any $f \in \mathcal{H}$ and $\lambda \in \mathbf{C}$, there exists a unique solution $u \in D_L$ of the following boundary value problem

$$\begin{cases} \left(-\Delta_g - \dfrac{(n-1)^2}{4} + 1 - \zeta\right)u = f \quad \text{in} \quad \mathcal{M}^a_{int}, \\ \left(-\Delta_g - \dfrac{(n-1)^2}{4} + 1 - \zeta\right)(P' \otimes 1)u = f \quad \text{in} \quad \mathcal{M}^a_{ext}, \\ \left(\partial_y(P_0 \otimes I_y)u\right)(a - 0) = \lambda. \end{cases} \tag{4.9}$$

The solution $u = u(z, \zeta, \lambda)$ is analytic with respect to λ and meromorphic on $\mathbf{C}$ with respect to ζ with possible poles at $\sigma(L)$.

Proof. By (4.1), if $y > 1$, the inverse to g_{ij} is, For $y > 1$, the metric takes the form

$$(g_{ij}) = \begin{pmatrix} h_{ij}(x)/y^2 & 0 \\ 0 & 1/y^2 \end{pmatrix}.$$

Therefore, its inverse is

$$\left(g^{ij}\right) = \begin{pmatrix} y^2 h^{ij}(x) & 0 \\ 0 & y^2 \end{pmatrix}.$$

To show the compactness of the resolvent, we have only to show that if $\{u_j\}$ is a bounded sequence in $\mathcal{H}^1$, it contains a subsequence convergent in $\mathcal{H}$. Since P_m is the projection onto the the eigenspace corresponding to m-th eigenvalue λ_m of $-\Delta_M$, we have, for $u \in H^1(\mathcal{M})$ and $R > a$,

$$\begin{aligned} \int_{M\times(R,\infty)} g^{ij}\, \partial_i u\, \overline{\partial_j u}\, d\mathcal{M} &= \int_{M\times(R,\infty)} y^2 \left(|\partial_y u|^2 + h^{ij}\partial_{x_i} u \overline{\partial_{x_j} u}\right) \frac{dM dy}{y^n} \\ &\geq R^2 \sum_{m=0}^{\infty} \lambda_m \int_R^\infty \|P_m u(y)\|^2_{L^2(M)} \frac{dy}{y^n} \\ &\geq \lambda_1 R^2 \int_{M\times(R,\infty)} |(P'\otimes I_y)u|^2 d\mathcal{M}. \end{aligned}$$

By the above inequality, for any $\epsilon > 0$ there exists $R > 1$ such that

$$\sup_j \int_{M\times(R,\infty)} |(P'\otimes I_y)u_j|^2 d\mathcal{M} < \epsilon.$$

On $\mathcal{M}\backslash M\times(R,\infty)$ we apply Rellich's theorem to extract a convergent subsequence. This proves (1).

Any $u \in D(L)$ is written as $u = L^{-1}f$ for some $f \in \mathcal{H}$. It satisfies

$$l(u,v) = (Lu, v) = (f, v), \quad \forall v \in \mathcal{H}^1. \tag{4.10}$$

Taking v from $C_0^\infty(\mathcal{M}^a_{int})$ and $(P'\otimes I_y)C_0^\infty(\mathcal{M}^a_{ext})$, we see that

$$(-\Delta_g - \frac{(n-1)^2}{4} + 1 - \zeta)u = f \quad \text{weakly in} \quad \mathcal{M}^a_{int}, \quad \text{and} \quad \mathcal{M}^a_{ext}.$$

Therefore, $u \in H^2_{loc}(\mathcal{M}^a_{int})$, $(P'\otimes I_y)u \in H^2_{loc}(\mathcal{M}^a_{ext})$. Take $v = \varphi_m(x)\chi(y)$ $(m \geq 1)$, where $\chi \in C_0^\infty((2,\infty))$ and φ_m is the eigenfunction associated with λ_m. Then from (4.10), we see that $(u(\cdot,y),\varphi_m)$ satisfies a 2nd order differential equation on $(2,\infty)$. Therefore, we have that $(P'\otimes I_y)u \in H^2_{loc}(M\times(2,\infty))$. We then have $u \in H^2(\mathcal{M}^a_{int})$ and, by Theorem 2.1.3, $u = (P'\otimes I_y)u \in H^2(\mathcal{M}^a_{ext})$. By taking $v \in (P_0\otimes I_y)C^\infty(M\times(2,a])$ such that $v = 0$ for $y < 3$ in (4.10), and integrating by parts, we have

$$\Big(\big(y^{(n-2}\partial_y(P_0\otimes I_y)u\big)(a-0), v\Big)_{L^2(\Gamma^a)} = 0.$$

Therefore, $\left(\partial_y(P_0\otimes I_y)u\right)(a-0) = 0$. These facts prove $D(L) \subset D_L$.

Take $u \in D_L$ and put $h = (-\Delta_g - (n-1)^2/4 + 1)u$ for $y \neq a$. Then by integration by parts, we have

$$l(u,v) = (h,v)_{\mathcal{H}}, \quad \forall v \in \mathcal{H}^1.$$

Since $l(u,v) = (L^{1/2}u, L^{1/2}v)_{\mathcal{H}}$, we then have

$$\left|(L^{1/2}u, L^{1/2}v)_{\mathcal{H}}\right| \leq C\|v\|_{\mathcal{H}}, \quad \forall v \in \mathcal{H}^1$$

with a constant C independent of $v \in \mathcal{H}^1 = D(L^{1/2})$. This shows that $L^{1/2}u \in D(L^{1/2})$, which proves $D_L \subset D(L)$. In particular, we have proven for $y \neq a$

$$Lu = \big(-\Delta_g - \frac{(n-1)^2}{4} + 1\big)u, \quad u \in D(L).$$

The uniqueness in (3) follows from $\zeta \notin \sigma(L)$. Indeed, if u_1, u_2 be two different solutions, then $u_i - u_2 \in D_L$ would be an eigenfunction of L. To show the existence, we take $\eta(y) \in C^\infty(\mathcal{M}^a_{int})$ such that $\eta(y) = 0$ for $y < 2$, $\eta(a-0) = 0$, $(\partial_y\eta)(y-a) = 1$, and $\eta(y) = 0$ in $\mathcal{M}^a_{ext}$. Let

$$\tilde{f} = \begin{cases} \big(-\Delta_g - \dfrac{(n-1)^2}{4} + 1 - \zeta\big)\eta & \text{in} \quad \mathcal{M}^a_{int}, \\ 0 \quad \text{in} \quad M^a_{ext}, \end{cases}$$

and put

$$u = u(z, \zeta, \lambda) = \lambda\chi(y) + (L-\zeta)^{-1}f - \lambda(L-\zeta)^{-1}\tilde{f}. \tag{4.11}$$

This is analytic with respect to λ and meromorphic with respect to ζ. □

For $0 < \alpha < \beta < \infty$, we put

$$U^{(\pm)}_{\alpha\beta} = \{\zeta \in \mathbf{C}\,;\, \alpha < \mathrm{Re}\,\zeta < \beta, \quad 0 \leq \pm\mathrm{Im}\,\zeta\}.$$

Lemma 4.4. *On $M \times (0,\infty)$, we consider $H_0 = -y^2(\partial_y^2 + \Delta_M) + (n-2)y\partial_y - (n-1)^2/4$, and $R_0(\zeta) = (H_0 - \zeta)^{-1}$. Suppose $f \in C_0^\infty(\mathcal{M})$ satisfies* $\mathrm{supp}\, f \subset \mathcal{M}_1 = M \times (1,\infty)$. *Let $\rho(y) \in C^\infty((0,\infty))$ be such that $\rho(y) = 0$ for $y < 2$, $\rho(y) = 1$ for $y > 3$. Then, for any $0 < \alpha < \beta < \infty$, there exist $\epsilon > 0$, $C > 0$ such that*

$$\Big|\rho(y)\left((P' \otimes I_y)R_0(\zeta)f\right)(x,y)\Big| \leq Ce^{-\epsilon y}, \quad \zeta \in U^{(\pm)}_{\alpha\beta}.$$

Proof. By (2.12),

$$u(x,y) := (P' \otimes I^a_y)R_0(\zeta + i0)f = \sum_{m\geq 1} \varphi_m(x)\left(G_0(\sqrt{\lambda_m}, \nu)\widehat{f}_m\right)(y),$$

with $\nu = -i\sqrt{\zeta}$, where $G_0(\zeta,\nu)$ is defined by Definition 1.3.5. Then we have by Chap. 1, (3.14)

$$\|u(\cdot,y)\|^2_{L^2(M)} = \sum_{m\geq 1} |G_0(\sqrt{\lambda_m},\nu)\widehat{f}_m(y)|^2 \leq Ce^{-\epsilon y}.$$

Note that $\mathrm{supp}\,\widehat{f}_m(y)$ is away from 0, and the singularities of $I_\nu(y), K_\nu(y)$ at $y = 0$ do no harm. Since, for any $q > 0$, $\|\Delta_x^q u(\cdot,y)\|^2$ is estimated in a similar manner, by Sobolev's inequality we have $|u(x,y)|^2 \leq Ce^{-\epsilon y}$. □

4.5. Meromorphic continuation of the Eisenstein series. Here we pass to the traditional parametrization. For a subset $\mathcal{E} \subset \mathbf{R}$, we write

$$\frac{n-1}{2} \pm \sqrt{-\mathcal{E}} = \left\{s \in \mathbf{C}\,;\, s(n-1-s) - \frac{(n-1)^2}{4} \in \mathcal{E}\right\}.$$

Let $A = L - 1 - \frac{(n-1)^2}{4}$, and put

$$\Sigma(A) = \frac{n-1}{2} \pm \sqrt{-\sigma(A)}, \quad \Sigma(H) = \frac{n-1}{2} \pm \sqrt{-\sigma(H)},$$

$$\Sigma_d(H) = \frac{n-1}{2} \pm \sqrt{-\sigma_d(H)}, \quad \Sigma_p(H) = \frac{n-1}{2} \pm \sqrt{-\sigma_p(H)},$$

$$\mathcal{L} = \left\{s \in \mathbf{C}\,;\, \mathrm{Re}\,s = \frac{n-1}{2}\right\}, \quad \mathcal{L}_\pm = \left\{s \in \mathcal{L}\,;\, \pm\mathrm{Im}\,s > 0\right\}.$$

Note that $\Sigma(H) = \mathcal{L} \cup \Sigma_d(H)$, and that $\Sigma(A)$ is a discrete set, since $\sigma(A)$ is discrete by Lemma 4.3.

In view of (4.5), we define for $\{\mathrm{Re}\, s > (n-1)/2\} \setminus \Sigma_p(H)$

$$\begin{aligned} E(z,s) &= \chi(y)\, y^s - \big(-\Delta_g - s(n-1-s)\big)^{-1} [-\Delta_g, \chi(y)]\, y^s \\ &= \varphi(z, k - i\epsilon), \end{aligned}$$

where $s = (n-1)/2 + i(k - i\epsilon)$ ($\epsilon > 0$). By Theorem 3.8, $E(z,s)$ is extended to $\mathcal{L} \setminus \big(\Sigma_p(H) \cup \{(n-1)/2\}\big)$. We take $s = (n-1)/2 + ik \in \mathcal{L} \setminus \big(\Sigma_p(H) \cup \{(n-1)/2\}\big)$. Since $(-\Delta_g - s(n-1-s))^{-1} f$ satisfies outgoing radiation condition,

$$E(z,s) - y^s \sim C y^{n-1-s}.$$

Comparing with (4.3),

$$E(z,s) \simeq y^s + \mathcal{S}(s) y^{n-1-s}, \quad \text{as} \quad y \to \infty.$$

By Lemma 4.3, for $s \notin \Sigma(A)$, there exists a unique solution $v = v(z,s) \in D_L$ of the following boundary value problem

$$\begin{cases} \big(-\Delta_g - s(n-1-s)\big) v(z,s) = 0 \quad \text{in} \quad \mathcal{M}^a_{int}, \\ \big(-\Delta_g - s(n-1-s)\big)(P' \otimes I^1_y) v(z,s) = 0 \quad \text{in} \quad \mathcal{M}^a_{ext}, \\ \big(y\partial_y (P_0 \otimes I^1_y) v\big)(a-0,s) = 1. \end{cases} \tag{4.12}$$

We define

$$\lambda_a(s) = \Big((P_0 \otimes I^1_y) v\Big)(a-0,s). \tag{4.13}$$

By Lemma 4.3 (3), $\lambda_a(s)$ is meromorphic on $\mathbf{C}$ with respect to s with poles in $\Sigma(A)$.

Lemma 4.5. *(1) For $s \in \mathcal{L} \setminus \big(\Sigma(A) \cup \Sigma_p(H) \cup \{(n-1)/2\}\big)$, we have*

$$\lambda_a(s) = \frac{a^s + a^{n-1-s}\mathcal{S}(s)}{s a^s + (n-1-s) a^{n-1-s} \mathcal{S}(s)}, \quad \mathcal{S}(s) = a^{2s-n+1} \frac{1 - s\lambda_a(s)}{(n-1-s)\lambda_a(s) - 1}. \tag{4.14}$$

(2) Letting $v(z,s)$ be the solution to (4.12), we have

$$E(z,s) - \Big(s a(s) - (n-1-s)\mathcal{S}(s) a^{(n-1-s)}\Big)\, v = \begin{cases} y^s + \mathcal{S}(s) y^{n-1-s}, & \text{on} \quad \mathcal{M}^a_{ext}, \\ 0, \quad \text{on} \quad \mathcal{M}^a_{int}. \end{cases}$$

(3) $\mathcal{S}(s)$ and $E(z,s)$ are extended to meromorphic functions on $\mathbf{C}$.

Proof. Lemma 4.4 implies

$$\big|(P' \otimes I^a_y) E(z,s)\big| \leq C e^{-\epsilon y}, \quad \epsilon > 0.$$

Hence, we have

$$(P_0 \otimes I^a_y) E(z,s) \simeq y^s + \mathcal{S}(s) y^{n-1-s}.$$

On the other hand,for $y > 3$,

$$\left(-y^2 \partial_y^2 + (n-2) y \partial_y - s(n-1-s)\right) (P_0 \otimes I^3_y) E(z,s) = 0.$$

Therefore, we have

$$(P_0 \otimes I^3_y) E(z,s) = y^s + \mathcal{S}(s) y^{n-1-s}, \tag{4.15}$$

since any solution of the equation $\left(-y^2\partial_y^2 + (n-2)y\partial_y - s(n-1-s)\right) u(y) = 0$ is written uniquely by a linear combination of y^s and y^{n-1-s}. Let

$$u = \begin{cases} E(z,s) \quad \text{in} \quad \mathcal{M}^a_{int}, \\ (P' \otimes I^a_y) E(z,s) \quad \text{in} \quad \mathcal{M}^a_{ext}. \end{cases}$$

Then $u \in D_L$, and

$$\begin{cases} \big(-\Delta_g - s(n-1-s)\big)u = 0 \quad \text{in} \quad \mathcal{M}^a_{int}, \\ \big(-\Delta_g - s(n-1-s)\big)(P'\otimes 1)u = 0 \quad \text{in} \quad \mathcal{M}^a_{ext}, \\ \big(y\partial_y(P_0 \otimes I_y)u\big)(a-0,s) = sa^s + (n-1-s)\mathcal{S}(s)a^{n-1-s}. \end{cases}$$

Comparing with (4.12), we obtain, by the uniqueness,

$$u = \big(sa^s + (n-1-s)\mathcal{S}(s)a^{n-1-s}\big)v. \tag{4.16}$$

Using (4.15), we obtain (1). The assertions (2) and (3) are direct consequences of Lemma 4.3 (3), (4.16) and the meromorphy of $\lambda_a(s)$. □

Lemma 4.6. $\lambda_a(s) \in \mathbf{R}$ *for* $s \in \mathcal{L} \setminus \Sigma(A)$ **and** $\lambda_a(s) = \lambda_a(\bar{s})$**.**

Proof. Note that if $v \in D_L$, then $\overline{v} \in D_L$, and also that $s(n-1-s) \in \mathbf{R}$ if $s \in \mathcal{L}$. Then, if $v(z,s)$ satisfies (4.12), so does $\overline{v(z,s)}$. By the uniqueness, $v(z,s)$ is then real-valued. This proves that $\lambda_a(s) \in \mathbf{R}$. As, for $s \in \mathcal{L}$, $s(n-1-s) = \bar{s}(n-1-\bar{s})$ it follows from (4.12) that $\lambda_a(s) = \lambda_a(\bar{s})$. □

Theorem 4.7. $\mathcal{S}(s)$ *is holomorphic on* $\operatorname{Re} s = (n-1)/2$.

Proof. Take $s_1 = (n-1)/2 + ik_1$, $0 \neq k_1 \in \mathbf{R}$, and suppose $\lambda_a(s)$ is holomorphic at s_1. It follows from Lemma 4.6 that $\lambda_a(s_1)$ is real. Then $(n-1-s_1)\lambda_a(s_1) - 1 \neq 0$, hence by Lemma 4.5 (1), $\mathcal{S}(s)$ is holomorphic at s_1.

Suppose $\lambda_a(s)$ has a pole at $s_1 = (n-1)/2 + ik_1$, $0 \neq k_1 \in \mathbf{R}$. Then $\kappa_a(s) = 1/\lambda_a(s)$ is holomorphic at s_1, and $\kappa_a(s_1) = 0$. By the formula

$$\mathcal{S}(s) = a^{2s-n+1}\frac{\kappa_a(s) - s}{n-1-s-\kappa_a(s)}, \tag{4.17}$$

$\mathcal{S}(s)$ is holomorphic at s_1.

Suppose $\lambda_a(s)$ is holomorphic at $s_0 = (n-1)/2$. By Lemma 4.5 (1), if $\lambda_a(s_0) \neq 2/(n-1)$, $\mathcal{S}(s)$ is holomorphic at s_0, and $\mathcal{S}(s_0) = -1$. If $\lambda_a(s_0) = 2/(n-1)$, by the Taylor expansion $\lambda_a(s_0 + w) = 2/(n-1) + cw + O(w^2)$. We then have

$$\mathcal{S}(s_0 + w) = -a^{2w}\frac{\Big(c + \Big(\dfrac{2}{n-1}\Big)^2\Big)w + O(w^2)}{\Big(c - \Big(\dfrac{2}{n-1}\Big)^2\Big)w + O(w^2)}.$$

Since $\lambda_a(s) = \lambda_a(\bar{s})$, we have $c = 0$. Therefore, $\mathcal{S}(s)$ is holomorphic at s_0 and $\mathcal{S}(s_0) = 1$.

Suppose $\lambda_a(s)$ has a pole at $s_0 = (n-1)/2$. By (4.17), $S(s)$ is holomorphic at s_0 and $\mathcal{S}(s_0) = -1$. □

Note, since by Theorem 3.16, $\widehat{S}(k)$ is unitary for $k > 0$, $k^2 \notin \sigma_p(H)$, we have $|\mathcal{S}(s)| = 1$ a.e. on $\mathcal{L}$. In particular, due to the proof of Theorem 4.7, $\mathcal{S}((n-1)/2) = \pm 1$.

To prove the holomorphy of $E(z,s)$, we prepare an identity. Let $v(z,s)$ be a solution to (4.12), and put

$$\widetilde{w}(z,s) = \big(sa^s + (n-1-s)a^{n-1-s}\mathcal{S}(s)\big)\, v(z,s),$$

and, for $k \in \mathbf{R}$,

$$w(z,k) = \widetilde{w}\Big(z, \frac{n-1}{2} + ik\Big).$$

It satisfies the equation

$$(L - 1 - s(n-1-s))w = 0, \quad s = \frac{n-1}{2} + ik,$$

and the boundary condition

$$\big((P_0 \otimes I_y^1)\big)w(a-0,k) = a^{(n-1)/2+ik} + a^{(n-1)/2-ik}\mathcal{S}\Big(\frac{n-1}{2} + ik\Big),$$

where we have used the definition of $\lambda_a(s)$ and Lemma 4.5. It also satisfies

$$\begin{aligned}\big(y\partial_y(P_0 \otimes I_y^1)w\big)(a-0,k) =& \Big(\frac{n-1}{2} + ik\Big)a^{(n-1)/2+ik} \\ &+ \Big(\frac{n-1}{2} - ik\Big)a^{(n-1)/2-ik}\mathcal{S}\Big(\frac{n-1}{2} + ik\Big).\end{aligned}$$

Lemma 4.8. *For $k, h \in \mathbf{R}$, the following formula holds:*

$$\begin{aligned}&(w(\cdot,k), w(\cdot,h))_{\mathcal{H}} \\ &= \frac{i}{k-h}\left(a^{i(h-k)}\mathcal{S}\Big(\frac{n-1}{2} + ik\Big)\overline{\mathcal{S}\Big(\frac{n-1}{2} + ih\Big)} - a^{i(k-h)}\right) \\ &- \frac{i}{k+h}\left(a^{i(k+h)}\overline{\mathcal{S}\Big(\frac{n-1}{2} + ih\Big)} - a^{-i(k+h)}\mathcal{S}\Big(\frac{n-1}{2} + ik\Big)\right).\end{aligned} \tag{4.18}$$

Proof. Letting $w_0(y,k) = (P_0 \otimes I_y^1)w\big|_{\mathcal{M}_1}$, we have, by integration by parts and Lemma 4.4,

$$\begin{aligned}&(Lw(k), w(h))_{\mathcal{H}} - (w(k), Lw(h))_{\mathcal{H}} \\ &= \frac{1}{y^{n-2}}\Big(w_0(y,k)(\overline{\partial_y w_0})(y,h) - (\partial_y w_0)(y,k)\overline{w_0(y,h)}\Big)\Big|_{y=a-0}.\end{aligned}$$

Using the equation and the boundary conditions, we have

$$\begin{aligned}&(k^2 - h^2)(w(k), w(h)) \\ &= i(h+k)\left(a^{i(h-k)}\mathcal{S}\Big(\frac{n-1}{2} + ik\Big)\overline{\mathcal{S}\Big(\frac{n-1}{2} + ih\Big)} - a^{i(k-h)}\right) \\ &+ i(h-k)\left(a^{i(k+h)}\overline{\mathcal{S}\Big(\frac{n-1}{2} + ih\Big)} - a^{-i(k+h)}\mathcal{S}\Big(\frac{n-1}{2} + ik\Big)\right),\end{aligned}$$

which proves the lemma. □

Theorem 4.9. *Eisenstein series $E(z,s)$ is holomorphic on* $\mathrm{Re}\, s = (n-1)/2$.

Proof. In view of Lemma 4.5 (2), we have only to show that when $k \to k_0 \in \Sigma(A)$, $\|w(k)\|$ is bounded. We prove this by first letting $h \to k \neq 0$ and $k \to k_0$ in (4.18). Since $\mathcal{S}(s)$ is holomorphic and, by the unitarity, $|\mathcal{S}(s)| = 1$ on $\mathrm{Re}\, s = (n-1)/2$, the 1st term of the right-hand side of (4.18) is bounded in this process. The second term is bounded when $k_0 \neq 0$.

By the note after Theorem 4.7, $\mathcal{S}(s_0) = \pm 1$ for $s_0 = (n-1)/2$. Therefore, the 2nd term of the right-hand side of (4.18) is bounded when $k, h \to k_0$. □

5. $SL(2,\mathbf{Z})\backslash\mathbf{H}^2$ as a Riemann surface

In this section we summarize the basic properties of the quotient manifold by the action of modular group

$$SL(2,\mathbf{Z}) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} ; a,b,c,d \in \mathbf{Z},\ ad-bc=1\right\},$$

where the action $SL(2,\mathbf{Z}) \times \mathbf{C}_+ \ni (\gamma, z) \to \gamma\cdot z \in \mathbf{C}_+$ is defined by (1.2). In the following, I_2 denotes the 2×2 unit matrix.

5.1. Fundamental domain. Let $\mathcal{M} = SL(2,\mathbf{Z})\backslash\mathbf{H}^2$. The fundamental domain $\mathcal{M}^f$ of $\mathcal{M}$ is the following set:

$$\mathcal{M}^f = \{z \in \mathbf{C}_+ \,;\, |z| \geq 1, -1/2 \leq \mathrm{Re}\, z \leq 1/2\},$$

$$\partial M^f = \partial M_1^f \cup \partial M_2^f,$$

$$\partial M_1^f = \Big\{-\frac{1}{2} + iy\,;\, \frac{\sqrt{3}}{2} \leq y < \infty\Big\} \cup \Big\{\frac{1}{2} + iy\,;\, \frac{\sqrt{3}}{2} \leq y < \infty\Big\},$$

$$\partial M_2^f = \Big\{e^{i\varphi}\,;\, \frac{\pi}{3} \leq \varphi \leq \frac{2\pi}{3}\Big\},$$

(see e.g. [**5**] p. 30, [**128**] p. 241). We put

$$\gamma^{(T)} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \gamma^{(I)} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Their actions are

$$\gamma^{(T)}\cdot z = z+1, \quad \gamma^{(I)}\cdot z = -\frac{1}{z}.$$

To get $\mathcal{M}$ from $\mathcal{M}^f$, we glue ∂M_1^f by the action of $\gamma^{(T)}$, i.e. $-\frac{1}{2} + iy \to \frac{1}{2} + iy$, and ∂M_2^f by the action of $\gamma^{(I)}$, i.e. $e^{i\varphi} \to e^{i(\pi-\varphi)}$. We denote this identification by Π, i.e.

$$\mathcal{M} = \mathcal{M}^f/\Pi.$$

The resulting surface $\mathcal{M}$ has two singular points, $p_1 = \Pi(i)$ and $p_2 = \Pi(e^{i\pi/3}) = \Pi(e^{2\pi i/3})$. The nature of these singularities is clarified by the following lemmas (see e.g. [**124**] p. 15, [**128**], p. 247, p. 251). We denote by $\langle\gamma\rangle$ the cyclic group generated by γ.

Lemma 5.1. *$SL(2,\mathbf{Z})$ is generated by $\gamma^{(T)}$ and $\gamma^{(I)}$.*

Lemma 5.2. *For $w \in \mathbf{C}_+$, we put*

$$G_w = \{\gamma \in SL(2,\mathbf{Z})\,;\, \gamma\cdot w = w\}.$$

That $w \in \mathcal{M}^f$ and $G_w \neq \{\pm I_2\}$ occurs only for the following three cases.

(1) $w = i$. In this case $G_w = \left\langle\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right\rangle$.

(2) $w = e^{\pi i/3}$. In this case $G_w = \left\langle\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}\right\rangle$.

(3) $w = e^{2\pi i/3}$. In this case $G_w = \left\langle\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}\right\rangle$.

Note that in the case $w = i$, the order of the group G_w is 2, while in the case $w = e^{\pi i/3}$ and $e^{2\pi i/3}$ (which are identified by $\gamma^{(T)}$ and $\gamma^{(I)}$), the order of the group G_w is 3. As a result, the point p_i has a vicinity $U_i \subset \mathcal{M}$, $i = 1, 2$, which can be represented as $U_1 = \Gamma_1 \backslash B(1/2)$, $U_2 = \Gamma_2 \backslash B(1/2)$, where Γ_1, Γ_2 are the groups of rotations corresponding to G_i and $G_{e^{\pi i/3}}$, and $B(r)$ is the ball of radius $r > 0$ in $\mathbf{C}$ centered at 0. These introduce orbifold structure on $\mathcal{M}$, however, in this note, we do not issue these constructions further.

5.2. Analytic structure. To introduce local coordinates on $\mathcal{M}$, we consider 3 different cases.

1. Let $V_0 = \mathcal{M}^f \setminus \partial\mathcal{M}^f_2$, and $U_0 = \Pi(V_0)$. Define for $p \in U_0$

$$\zeta_0(p) = \varphi_0(z) = e^{2\pi i z}, \quad p = \Pi(z).$$

Then, since two points $-1/2 + iy$, $1/2 + iy$ are identified by the action of $\gamma^{(T)}$, $\zeta_0(p)$ defines analytic coordinates on U_0.

2. Let $V_1 = \mathcal{M}^f \setminus \partial M^f_1$, and $U_1 = \Pi(V_1)$ be a neighborhood of $p_1 = \Pi(i)$. Define for $p \in U_1$

$$\zeta_1(p) = \varphi_1(z) = \left(\frac{z - i}{z + i}\right)^2, \quad \Pi(z) = p.$$

Then, since two points $e^{i\varphi}$, $e^{i(\pi-\varphi)}$, where $\pi/3 \le \varphi < \pi/2$, are identified by the action of $\gamma^{(I)}$, $\zeta_1(p)$ defines analytic coordinates on U_1.

3. Let $V_2 = \mathcal{M}^f \setminus i\mathbf{R}$, and $U_2 = \Pi(V_2)$ be a neighborhood of $p_2 = \Pi(e^{\pi i/3}) = \Pi(e^{2\pi i/3})$. Define for $p \in U_2$

$$\zeta_2(p) = \varphi_2(z) = \begin{cases} \left(\dfrac{z - e^{\pi i/3}}{z - e^{-\pi i/3}}\right)^3, & p = \Pi(z), \quad \operatorname{Re} z > 0, \\ \left(\dfrac{z - e^{2\pi i/3}}{z - e^{-2\pi i/3}}\right)^3, & p = \Pi(z), \quad \operatorname{Re} z < 0. \end{cases}$$

Since two points $-1/2 + iy$, $1/2 + iy$ are identified by the action of $\gamma^{(T)}$, and two points $e^{i\varphi}$, $e^{i(\pi-\varphi)}$, where $\pi/3 \le \varphi < \pi/2$, are identified by the action of $\gamma^{(I)}$, this $\zeta_2(p)$ defines analytic local coordinates on U_2.

To check that φ_1, φ_2 satisfy the desired analytical property, it is convenient to observe that φ_1, φ_2 map the corresponding sectors of the circle $|z| = 1$ onto an interval of a ray emanating from 0.

Since $\zeta_\alpha \circ \zeta_\beta^{-1}$ on $\zeta_\beta(U_\alpha \cap U_\beta)$, $\alpha, \beta = 0, 1, 2$, are analytic, the local coordinate system $\{(U_\alpha, \zeta_\alpha)\}_{\alpha=0}^2$ makes $\mathcal{M}$ a Riemann surface.

5.3. Singularities as a Riemannian manifold. By the metric

$$ds^2 = \frac{(dx)^2 + (dy)^2}{y^2} = -\frac{4dzd\overline{z}}{(z - \overline{z})^2} \quad \text{on} \quad \mathcal{M}^f,$$

$\mathcal{M}$ becomes a hyperbolic space. However, we must pay attention to the points p_1, p_2. By the above local coordinate $\zeta_\alpha(p) = \varphi_\alpha(z)$, $p = \Pi(z)$, $\alpha = 0, 1, 2$, this metric becomes

$$ds^2 = \frac{d\zeta_\alpha d\overline{\zeta_\alpha}}{(\operatorname{Im} z)^2 |\varphi'_\alpha(z)|^2}.$$

Therefore, on the zeros of $\varphi_1'(z)$, i.e. at i, $\varphi_2(z)$, i.e. $e^{\pi i/3}$, $e^{2\pi i/3}$, this Riemannian metric has singularities. In these cases,

$$\zeta_\alpha = \varphi_\alpha(z) = T(z)^{\alpha+1}, \quad T(z) = \frac{z-w}{z-\overline{w}},$$

where $w = i$ for $\alpha = 1$, and $w = e^{\pi i/3}$ and $w = e^{2\pi i/3}$ for $\alpha = 2$. In these cases,

$$z = \frac{w - \overline{w}\zeta^{1/n}}{1-\zeta^{1/n}} = w + (w-\overline{w})\zeta^{1/n} + \cdots .$$

Therefore, $dz/d\zeta = n^{-1}(w-\overline{w})\zeta^{1/n-1} + \cdots$, hence

$$|\varphi_\alpha'(z)|^2 = \left|\frac{dz}{d\zeta}\right|^{-2} = O(|\zeta|^\lambda), \quad \lambda = 2 - \frac{2}{n}. \tag{5.1}$$

Note that $1 \leq \lambda < 2$. The volume element and the Laplace-Beltrami operator are rewritten as

$$\frac{dx\wedge dy}{y^2} = \frac{i}{2y^2}dz\wedge d\overline{z} = \frac{i\big|dz/d\zeta\big|^2}{2(\operatorname{Im} z)^2}d\zeta\wedge d\overline{\zeta}, \tag{5.2}$$

$$y^2\big(\partial_x^2+\partial_y^2\big) = 4(\operatorname{Im} z)^2\frac{\partial^2}{\partial z\partial\overline{z}} = \frac{4(\operatorname{Im} z)^2}{\big|dz/d\zeta\big|^2}\frac{\partial^2}{\partial\zeta\partial\overline{\zeta}}. \tag{5.3}$$

Both of them have singularities at the corresponding w. However, the singularity of the volume element and that of the Laplace-Beltrami operator cancel, since we have, for C^∞-functions f, g supported near w,

$$\int_{\mathcal{M}} y^2\big(\partial_x^2+\partial_y^2\big)f\cdot g\,\frac{dxdy}{y^2} = 2i\int\frac{\partial^2}{\partial\zeta\partial\overline{\zeta}}f\cdot g\,d\zeta d\overline{\zeta}. \tag{5.4}$$

We take small open neighborhoods $\widetilde{U}_i$ of p_i, $i = 1, 2$ such that $\widetilde{U}_1\cap\widetilde{U}_2 = \emptyset$. We construct a partition of unity $\{\chi_\alpha\}_{\alpha=0}^2$ such that $\operatorname{supp}\chi_\alpha\subset\widetilde{U}_\alpha$, $\alpha = 1, 2$, $\operatorname{supp}\chi_0\subset U_0$, and $\sum_{\alpha=0}^3\chi_\alpha = 1$ on $\mathcal{M}$. In addition to the hyperbolic volume element, let

$$dV_E^{(\alpha)} = \frac{i}{2}d\zeta_\alpha\wedge d\overline{\zeta_\alpha}, \tag{5.5}$$

and define a quadratic form $a(u,v)$ by

$$a(u,v) = \sum_{\alpha=0}^3\int\chi_\alpha u\,\overline{v}\;dV_H^{(\alpha)} + \sum_{\alpha=0}^3\int\chi_\alpha\nabla u\cdot\nabla\overline{v}\;dV_E^{(\alpha)}, \tag{5.6}$$

where

$$\nabla = (\partial_t, \partial_s), \quad (\zeta = t + is).$$

We can show that the quadratic form $a(u,v)$ with domain $C_0^\infty(\mathcal{M})$ is closable in $L^2(\mathcal{M}, dv_H)$. Let $\widetilde{a}(u,v)$ be its closed extension, and $\widetilde{H}^1$ the set of u such that $a(u,u) < \infty$ equipped with the inner product (5.6). This is the 1st order Sobolev space on $\mathcal{M}$. By Theorem 4.2, we have a self-adjoint operator A such that $a(u,v) = (Au,v)_{\mathcal{M},g}$ for $u\in D(A)$, $v\in\widetilde{H}^1$. Then $1 - A$ is a self-adjoint realization of the Laplace-Beltrami operator Δ_g.

When we deal with the perturbation problem of Δ_g, we should restrict ourselves to the case that the coefficients of differential of more than one order of the pertubation term vanish around $i, e^{\pi i/3}, e^{2\pi i/3}$. The precise assumption is as follows.

Let $H_0 = -\Delta_g = -y^2(\partial_y^2 + \partial_x^2)$, and V a 2nd order differential operator on $\mathcal{M}$ such that

(M-1) $H = H_0 + V$ is formally self-adjoint.

(M-2) Around $i, e^{\pi i/3}, e^{2\pi i/3}$, V is an operator of multiplication by a bounded real function.

(M-3) Except for the neighborhoods in (M-2), V is a differential operator of the form :

$$V = \sum_{i+j\leq 2} a_{ij}(x,y)(y\partial_x)^i(y\partial_y)^j$$
$$|D^\alpha a_{ij}(x,y)| \leq C_\alpha(1+|\log y|)^{-\min(|\alpha|,1)-1-\epsilon}, \quad \forall\alpha,$$
$$D = (D_x, D_y) = (y\partial_x, y\partial_y).$$

We define a self-adjoint extension of H through the quadratic form discussed in §4. This means that we perturb the hyperbolic metric on $\mathcal{M}$ except for neighborhoods of singular points so that it is asymptotically equal to the original metric at infinity.

Since the measure $dxdy/y^2$ has singularties at $i, e^{\pi/3}, e^{2\pi i/3}$, the following lemma is not obvious.

Lemma 5.3. *For any $R > 1$, let χ_R be the characteristic function of $\mathcal{M}\cap\{y < R\}$. Then $\chi_R(H+i)^{-1}$ is compact in $L^2(\mathcal{M}; dxdy/y^2)$.*

Proof. Assume that $f_n, n = 1, 2, \cdots$, are on the unit sphere of $L^2(\mathcal{M}; dxdy/y^2)$, and let $u_n = (H+i)^{-1}f_n$. By Rellich's theorem, from $\{\chi_R u_n\}_{n\geq 1}$ one can extract a subsequence which converges in L^2 outside small neighborhoods of singular points.

Around $p_1 = i$ and $p_2 = \omega$, we take local coordinate $\zeta = t+is$ as above, and for a suffiently small $r > 0$, let B_r be a disc $\{t^2+s^2 < r^2\}$. Then, if $u \in L^2(\mathcal{M}, \frac{dxdy}{y^2})$ has a support in B_r, we have by (5.2)

$$\int_{B_r}|u|^2 dtds \leq C\int_{B_r}|u|^2 dV_H^{(\alpha)}, \tag{5.7}$$

with a constant $C > 0$. By the Sobolev imbedding $H^s(\mathbf{R}^n) \subset L^p(\mathbf{R}^n)$, where $0 \leq s < n/2$, $p = 2n/(n-2s)$, we have

$$H^1(\mathbf{R}^2) \subset L^p(\mathbf{R}^2), \quad \forall p > 2, \tag{5.8}$$

with continuous inclusion.

We take α, β such that $\alpha^{-1} + \beta^{-1} = 1$, $1 < \alpha < 2/\lambda$, where λ is defined by (5.1). Then, by Hölder's inequality,

$$\int_{B_\delta}|u|^2 dV_H^{(\alpha)} \leq C\int_{B_\delta} r^{-\lambda}|u|^2 dtds \leq C\left(\int_{B_\delta} r^{-\lambda\alpha}dtds\right)^{1/\alpha}\left(\int_{B_\delta}|u|^{2\beta}dtds\right)^{1/\beta}.$$

Since $\lambda\alpha < 2$, the 1st term of the most right-hand side tends to 0 when $\delta \to 0$. To the 2nd term of the most right-hand side we apply (5.8). Then, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\int_{B_\delta}|u|^2 dV_H^{(\alpha)} \leq \epsilon\left(\int_{B_\delta}|u|^2 dV_H^{(\alpha)} + \int_{B_\delta}|\nabla u|^2 dV_E^{(\alpha)}\right), \quad u \in \widetilde{H}^1.$$

Given the bouded sequence $\{u_n\}$ in $\widetilde{H}^1$, the integral of $|u_n|^2$ over B_δ can be made small uniformly in n. Outside B_δ, we use the usual Rellich theorem. This proves the lemma. □

5.4. Spectrum. By the above Lemma 5.3, the results in §3 and §4 also hold for H. Let $R(z) = (H - z)^{-1}$.

Theorem 5.4. *(1)* $\sigma_e(H) = [0,\infty)$.
(2) $\sigma_p(H) \cap (0,\infty)$ *is of finite multiplicity, discrete as a subset in* $\mathbf{R}$*, with possible accumulation points 0 and* ∞*.*
(3) If $\lambda \in (0,\infty) \setminus \sigma_p(H)$, $R(\lambda \pm i0) \in \mathbf{B}(\mathcal{B};\mathcal{B}^*)$.

5.5. Eisenstein series. We return to the case of $H_0 = -y^2(\partial_y^2 + \partial_x^2)$. Let

$$G = SL(2,\mathbf{Z}), \quad G_0 = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} ; n \in \mathbf{Z} \right\},$$

i.e. G_0 is the group of translations by n along the $y-$axis.

Lemma 5.5. *(1) For* $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, g' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in G$,

$$g'g^{-1} \in G_0 \Longleftrightarrow \exists n \in \mathbf{Z} \quad \text{s.t.} \quad a' - a = nc, \ b' - b = nd, \ c' = c, \ d' = d$$

(2) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} * & * \\ c & d \end{pmatrix}$, $(c,d) = 1$, *are the complete representative of* $G_0\backslash G$*. Here* $(c,d) = 1$ *means that* c *and* d *are mutually prime.*

The proof is omitted.

Let us note that for $z = x + iy$

$$\operatorname{Im} g \cdot z = \frac{y}{(cx+d)^2 + c^2y^2}$$

holds. The Eisenstein series is defined by

$$\widetilde{E}(z,s) = \sum_{[g] \in \mathbf{G}_0\backslash\mathbf{G}} (\operatorname{Im} g \cdot z)^s = y^s + \sum_{(c,d)=1} \left(\frac{y}{(cx+d)^2 + c^2y^2} \right)^s. \tag{5.9}$$

We show that it is absolutely convergent for $\operatorname{Re} s > 1$.

Lemma 5.6. *For* $|x| \le 1/2$, $y \ge \sqrt{3}/2$, $cd \ne 0$,

$$\frac{y}{(cx+d)^2 + c^2y^2} \le \frac{2}{\sqrt{3}|cd|}.$$

Proof. Letting $r^2 = x^2 + y^2$, we have

$$(cx+d)^2 + c^2y^2 = r^2 \left(c + \frac{dx}{r^2} \right)^2 + \frac{y^2}{r^2}d^2 \ge \frac{y^2}{r^2}d^2 \ge \frac{3}{4}d^2.$$

This together with the obvious inequality

$$(cx+d)^2 + c^2y^2 \ge c^2y^2$$

proves

$$(cx+d)^2 + c^2y^2 \ge \frac{1}{2}\left(c^2y^2 + \frac{3}{4}d^2 \right) \ge \frac{\sqrt{3}}{2}y|cd|. \quad \square$$

Lemma 5.6 implies the following lemma.

Lemma 5.7. *For* $\operatorname{Re} s > 1$*, the series (5.9) is absolutely convergent and*

$$|\widetilde{E}(z,s) - y^s| \le C_s, \quad \forall z \in \mathcal{M}.$$

Since y^s satisfies on $\mathbf{H}^2$,

$$-\Delta(y^s) - s(1-s)y^s = 0,$$

due to $g \in SL(2,\mathbf{Z})$ being an isometry on $\mathbf{H}^2$,

$$-\Delta\left(\operatorname{Im} g \cdot z)^s\right) - s(1-s)\left(\operatorname{Im} g \cdot z\right)^s = 0.$$

In addition, $\left(\operatorname{Im} g_0 \cdot z\right)^s = \operatorname{Im} z = y$ for $g_0 \in G_0$. Therefore, by Lemma 5.5 (2), $\widetilde{E}(z,s)$ satisfies

$$-\Delta\widetilde{E}(z,s) - s(1-s)\widetilde{E}(z,s) = 0, \quad \text{on } \mathcal{M}.$$

By Lemma 5.7, $\widetilde{E}(z,s) - y^s \in L^\infty(\mathcal{M}) \subset L^2(\mathcal{M})$, in view of $\mathcal{M}$ having finite measure, $L^\infty(\mathcal{M}) \subset L^2(\mathcal{M})$. Therefore, for $\operatorname{Re} s > 1$

$$\widetilde{E}(z,s) = \chi(y)y^s - R_0(s(1-s))\big([H_0,\chi]y^s\big).$$

Here $R_0(\zeta) = (H_0 - \zeta)^{-1}$, and $\chi(y) \in C^\infty((0,\infty))$ such that $\chi(y) = 0$ for $y < 2$, $\chi(y) = 1$ for $y > 3$. This coincides with the Eisenstein series (4.5) introduced in §4. By using properties of number theoretic functions and Poisson's summation formula, the S-matrix is computed as follows (see e.g. [**70**], p. 61).

Theorem 5.8. *For the case of* $H_0 = -y^2(\partial_y^2 + \partial_x^2)$, *we have*

$$\mathcal{S}(s) = \sqrt{\pi}\,\frac{\Gamma(s-1/2)\,\zeta(2s-1)}{\Gamma(s)\,\zeta(2s)},$$

where $\zeta(s)$ *is Riemann's zeta function.*

Remark 5.9. For 3-dimensions, one can define a similar surface by using the Picard group

$$SL(2,\mathbf{Z}+i\mathbf{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} ; a,b,c,d \in \mathbf{Z}+i\mathbf{Z},\ ad-bc=1 \right\},$$

where the action is defined by quarternios. The quotient space $SL(2,\mathbf{Z}+i\mathbf{Z})\backslash\mathbf{H}^3$ is also an orbifold. See [**33**].