Introduction

0.1. Fourier analysis on manifolds. The Fourier transform on $L^2(\mathbf{R}^n)$ and its inversion formula are well-known :

$$\widehat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbf{R}^n} e^{-ix\cdot\xi} f(x) dx, \tag{0.1}$$

$$f(x) = (2\pi)^{-n/2} \int_{\mathbf{R}^n} e^{ix\cdot\xi} \widehat{f}(\xi) d\xi. \tag{0.2}$$

Since $-\Delta e^{ix\cdot\xi} = |\xi|^2 e^{ix\cdot\xi}$, $e^{ix\cdot\xi}$ is an eigenfunction of $-\Delta$. Therefore (0.1) and (0.2) illustrate the expansion of arbitrary functions in terms of eigenfunctions (more appropriately *generalized eigenfunctions* since they do not belong to $L^2(\mathbf{R}^n)$) of the Laplacian.

There are two directions of development of the above fact. One is quantum mechanics, where the Schrödinger operator $H = -\Delta + V(x)$ is the most basic tool to decribe the physical system of atoms or molecules. If H has the continuous spectrum, it is known that there exists a system of generalized eigenfunctions of H which play the same role as $e^{ix\cdot\xi}$. Moreover, by using these generalized eigenfunctions one can define an operator called the scattering matrix or the S-matrix, which is the fundamental object to study the physical properties of quantum mechanical particles through the scattering experiment.

The other direction is the Fourier transform on manifolds, especially on homogeneous spaces of Lie groups, which is a central theme in the representation theory. Hyperbolic manifolds, one of the deepest sources of classical mathematics, appear also in this context. In particular, hyperbolic quotient manifolds by the action of discrete subgroups of $SL(2,\mathbf{R})$ and the associated S-matrix are important objects in number theory.

0.2. Perturbation of the continuous spectrum. The aim of the perturbation theory of continuous spectrum is, given an operator H_0 whose spectral property is rather easy to understand, to study the spectral properties of $H_0 + V$, where V is the perturbation deforming the operator H_0. When $H = H_0 + V$ has the continuous spectrum, an effective way of studying its spectral properties is to construct a generalized Fourier tranform associated with H. To accomplish this idea, it is necessary that the Fourier transform for H_0 can be constructed easily. For example, it is the case for the Laplacian $-\Delta$ on $\mathbf{R}^n$. If the perturbation term V is an operator on the same Hilbert space as for H_0 and is not so strong, one can construct the Fourier transform associated with $H_0 + V$ by using the technique of functional analysis and partial differential equations.

This is not so easy for operators on hyperbolic manifolds. Even the construction of the Fourier transform associated with the Laplace-Beltrami operator on the hyperbolic space is no longer a trivial work. To construct the Fourier transform on hyperbolic spaces based on the upper half space model or the ball model, one needs deep knowledge of Bessel functions. Under the action of discrete subgroups, the properties of groups will reflect on the structure of manifolds or the construction of generalized eigenfunctions.

0.3. Spectral and scattering theory on hyperbolic manifolds. In the present note, we deal with the spectral theory and the associated forward and inverse problems for Laplace-Beltrami operators on hyperbolic manifolds. Since we are mainly interested in its spectral properties, Selberg's work [**123**] and its developments are beyond our scope. As an approach to the hyperbolic manifolds from the spectral theory, the first important paper is that of Faddeev [**34**]. Lang [**90**] is a detailed exposition of Faddeev's article. There are also works of Roelcke [**117**], Venkov [**130**] and a recent article of Iwaniec [**70**]. The study of spectral theory, in particular, that of continuous spectrum is drastically changed in these 30 years. The article of Lax-Phillips [**92**] has distinguished features, leaning over the analysis of wave equation. The derivation of the analytic continuation of Eisenstein series from that of the resolvent was done by Colin de Verdière [**26**]. Agmon [**1**] used the modern spectral theory for this problem. Hislop [**53**] uses Mourre theoryiwhich is a modern powerful technique to study the continuous spectrum of self-adjoint operators, see e.g. [**62**]) to prove the resolvent estimates for the Laplacian on hyperbolic spaces.

The scattering metric proposed by Melrose [**99**] aims at constructing a general calculus on non-compact manifolds on which the scattering theory is developed. Melrose' theory includes the following model. Let $\mathcal{M}$ be a compact n-dimensional Riemannian manifold with boundary. Assume that near the boundary, $\mathcal{M}$ is diffeomorphic to $M \times (0,1)$, M being a compact $n-1$-dimensional manifold, and introduce the following metric

$$ds^2 = \frac{(dy)^2 + A(x,y,dx,dy)}{y^2}, \quad 0 < y < 1, \quad x \in M,$$

where $A(x,y,dx,dy)$ is a symmetirc covariant tensor such that as $y \to 0$

$$A(x,y,dx,dy) \sim A_0(x,dx) + yA_1(x,dx,dy) + y^2A_2(x,dx,dy) + \cdots, \tag{0.3}$$

A_0 being the Riemannian metric on M. This generalizes the upper half-space model of the hyperbolic space. Spectral structures of the associated Laplace-Beltarmi operator were studied by Mazzeo [**95**] and Mazzeo-Melrose [**96**]. Related inverse problem was studied by Joshi-Sa Barreto [**73**]. In particular, Sa Barreto [**120**] proved that the coincidence of the scattering operators gives rise to an isometry of associated metrics. Here the essential role is played by the boundary control method presented by Belishev [**10**], (see also [**13**], [**11**], [**14**]), which makes it possible to reconstruct a Riemannian manifold from the boundary spectral data of the associated Laplace-Belrami operator.

A feature of Melrose theory is that it proves the analytic continuation of the resolvent of Laplace-Beltrami operator for a broad class of metric so that it enables us to study the resonance, another important subject in spectral and scattering theory ([**45**], [**134**]). We do not deal with the resonance in this note. However, let us mention the recent article of Borthwick [**21**] which studies the inverse problem related to the resonance based on Melrose theory and includes a thorough list of references.

In the case of the Schrödinger operator $-\Delta + V(x)$ on $\mathbf{R}^n$, the behavior of solutions to the Schrödinger equation has a clear difference depending on the decay order of the potential at infinity. If we assume that $V(x) = O(|x|^{-\rho})$, $|x| \to \infty$, the border line is the case $\rho = 1$. This is also true on hyperbolic spaces. The difference occurs in the case $\rho = 1$ of the decay order $d_h^{-\rho}$, where d_h denotes the hyperbolic distance. In (0.3), y corresponds to e^{-d_h}. Hence from the view point of perturbation theory, the theory of scattering metric deals with the case in which the perturbation term is expanded as the power of e^{-d_h}.

0.4. Contents of this note. The purpose of this note is the exposition of the basic knowledge of the generalized Fourier transform on asymptotically hyperbolic manifolds and their applications to inverse scattering problem. We deal with the general short-range perturbation of the metric, namely, we consider the metric which differ from the standard hyperbolic metric with the term decaying like $d_h^{-1-\epsilon}$, where d_h is the hyperbolic distance.

More precisely we shall study an n-dimensional connected Riemannian manifold $\mathcal{M}$, which is written as a union of open sets:

$$\mathcal{M} = \mathcal{K} \cup \mathcal{M}_1 \cup \cdots \cup \mathcal{M}_N.$$

The basic assumptions are as follows:

(A-1) $\overline{\mathcal{K}}$ *is compact.*

(A-2) $\mathcal{M}_i \cap \mathcal{M}_j = \emptyset, \quad i \neq j.$

(A-3) *Each* $\mathcal{M}_i$, $i = 1, \cdots, N$, *is diffeomorphic either to* $\mathcal{M}_0 = M \times (0,1)$ *or to* $\mathcal{M}_\infty = M \times (1, \infty)$, M being a compact Riemannian manifold of dimension $n-1$. *Here the manifold* M *is allowed to be different for each* i.

(A-4) *On each* $\mathcal{M}_i$, *the Riemannian metric* ds^2 *has the following form*

$$ds^2 = y^{-2}\left((dy)^2 + h(x, dx) + A(x, y, dx, dy)\right), \tag{0.4}$$

$$A(x,y,dx,dy) = \sum_{i,j=1}^{n-1} a_{ij}(x,y)dx^i dx^j + 2\sum_{i=1}^{n-1} a_{in}(x,y)dx^i dy + a_{nn}(x,y)(dy)^2,$$

where $h(x,dx) = \sum_{i,j=1}^{n-1} h_{ij}(x)dx^i dx^j$ *is a positive definite metric on* M, *and* $a_{ij}(x,y), 1 \leq i,j \leq n$, *satisfies the following condition*

$$|\widetilde{D}_x^\alpha D_y^\beta\, a(x,y)| \leq C_{\alpha\beta}(1 + |\log y|)^{-\min(|\alpha|+\beta,1)-1-\epsilon_0}, \quad \forall \alpha, \beta \tag{0.5}$$

for some $\epsilon_0 > 0$. *Here* $\widetilde{D}_x = \tilde{y}(y)\partial_x$, $\tilde{y}(y) \in C^\infty((0,\infty))$ *such that* $\tilde{y}(y) = y$ *for* $y > 2$ *and* $\tilde{y}(y) = 1$ *for* $0 < y < 1$.

Of course this metric ds^2 depends on the end $\mathcal{M}_i$, hence should be written as $ds^2 = y^{-2}\big((dy)^2 + h_i(x,dx) + A_i(x,y,dx,dy)\big)$.

Picking up the wave equation, we shall study the following scattering problem. Consider the initial value problem for the wave equation

$$\begin{cases} \partial_t^2 u = \Delta_g u \quad \text{on} \quad \mathcal{M}, \\ u\big|_{t=0} = f, \quad \partial_t u\big|_{t=0} = -i\sqrt{-\Delta_g} f, \end{cases}$$

where f is orthogonal to the point spectral subspace for $-\Delta_g$. Then for any compact set K on $\mathcal{M}$, the solution $u(t)$ behaves as

$$\int_K |u(t)|^2 dV_g \to 0, \quad \text{as} \quad t \to \pm\infty.$$

Namely, the wave disappears from any compact set in $\mathcal{M}$. On each end $\mathcal{M}_j$, it will behave like

$$\|u(t) - u_j^{(\pm)}(t)\| \to 0, \quad \text{as} \quad t \to \pm\infty,$$

where $u_j^{(\pm)}(t)$ is the solution to the free wave equation

$$\begin{cases} \partial_t^2 u_j^{(\pm)} = \Delta_{g_j^0} u_j^{(\pm)}, \quad \text{on} \quad \mathcal{M}_j, \\ u_j^{(\pm)}\big|_{t=0} = f_j^{(\pm)}, \quad \partial_t u_j^{(\pm)}\big|_{t=0} = -i\sqrt{-\Delta_{g_j^0}} f_j^{(\pm)}, \end{cases}$$

$\Delta_{g_j^0}$ being the Laplace-Beltrami operator associated with the metric $y^{-2}\big((dy)^2 + h_j(x,dx)\big)$. The scattering operator $\mathcal{S}$ assigns to the asymptotic data in the remote past that in the remote future:

$$\mathcal{S} : \big(f_1^{(-)}, \cdots, f_N^{(-)}\big) \to \big(f_1^{(+)}, \cdots, f_N^{(+)}\big).$$

The inverse scattering is an attempt to recover the metric of $\mathcal{M}$ from the scattering operator $\mathcal{S}$. To study this problem, we first investigate the spectral properties of the associated Laplace-Beltrami operator $-\Delta_g$. Namely

- Location of the essential spectrum.
- Absence of eigenvalues embedded in the continuous spectrum when one of the ends is regular, i.e. one $\mathcal{M}_i$ is diffeomorphic to $M \times (0,1)$.
- Discreteness of embedded eigenvalues in the continuous spectrum when all the ends are cusps, i.e. all $\mathcal{M}_i$ are diffeomorphic to $M_i \times (1,\infty)$.
- Limiting absorption principle for the resolvent and the absolute continuity of the continuous spectrum.

Our next issue is the forward problem. Namely

- Construction of the generalized Fourier transform associated with $-\Delta_g$.
- Asymptotic completeness of time-dependent wave operators.
- Characterization of the space of scattering solutions to the Helmhotz equation in terms of the generalized Fourier transform.
- Asymptotic expansion of scattering solutions to the Helmholtz equation and the S-matrix.

As a byproduct, we also study

- Representation of the fundamental solution to the wave equation in the upper-half space model.
- Radon transform and the propagation of singularities for the wave equation.

Finally, we shall discuss the inverse problem. Namely

- Identification of the Riemannian metric from the scattering matrix.

We show that two asymptotically hyperbolic manifolds satisfying the above assumptions are isometric, if the metrics coincide on one regular end, and also the S-matrices coincide on that end.

The ingredient of each chapter is as follows.

Chapter 1 Fourier transforms on hyperbolic spaces

We discuss the construction of the Fourier transform associated with the Laplace-Beltrami operator of $\mathbf{H}^n$ as well as its spectral properties. Moreover, we characterize the solution space of the Helmholtz equation in terms of the Fourier transform. We also study the fundamental solution to the wave equation and the Radon transform. We mainly use the estimates of Bessel functions. This chapter is the basis of whole arguments in this note. Main results are Theorems 3.13, 4.2, 4.3, 6.5 and 6.6.

Chapter 2 Perturbation of the metric

This is an exposition of spectral and scattering theory for Laplace-Beltrami operators associated with asympotically hyperbolic metrics on $\mathbf{R}^n_+$ and their scattering matrices. As in Chapter 1, we will discuss the generalized Fourier transform, the asymptotoic expansion of the resolvent, the Helmholtz equation and the Radon transform. This is also an introduction to the classical spectral theory. Main results are Theorems 2.3, 7.1, 7.8, 7.9, 7.10 and 8.9.

Chapter 3 Manifolds with hyperbolic ends

The general hyperbolic manifolds are constructed by the action of discrete groups on $\mathbf{H}^n$. We shall consider simple cases and study the spectral properties of the resulting quotient manifolds. We also discuss the action of $SL(2, \mathbf{Z})$. Main results are Theorems 3.8, 3.12, 3.13 and 3.14.

Chapter 4 Radon transform and propagation of singularities in $\mathbf{H}^n$

The Radon transform describes singularities of solutions to the wave equation. We shall discuss this classical matter in this chapter for the hyperbolic space. The goal is Theorem 5.2 which is a generalization of Theorem 6.6 in Chapter 1.

Chapter 5 Introduction to inverse scattering

Local perturbations of the metric of hyperbolic manifolds are identified from the scattering matrix. We shall prove this fact by using spectral theory. Our goal is Theorem 4.8, which asserts that if the metrics coincide on one regular end of the asymptoticaly hyperbolic manifolds, and also the S-matrices coincide on that end, then two manifolds are isometric.

The method we have given here works not only for asymptotically hyperbolic ends but also for the manifolds on which the spectral representation is established. In particlular, Theorem 4.8 holds for manifolds with asymptotically Euclidean ends, or the mixture of Euclidean and hyperbolic ends.

Chapter 6 Boundary control method

To identify the metric, we reduce the problem to that of the inverse spectral problem on non-compact manifolds with compact boundaries. The crucial role is played by the boundary control method developed by Belishev and Kurylev. This section is devoted to a comprehensive and self-contained exposition of this approach. We shall give a complete proof of the BC-method except for Tataru's theorem on the uniqueness of solutions to non-characteristic Cauchy problem for the wave equation.

Appendix A Radon transform and propagation of singularities in $\mathbf{R}^n$

The relation between the propagation of singularities and the Radon transform is not obvious even for the case of perturbed Euclidean metric. We shall give detailed proof for this subject for the case of general short-range perturbation of the Euclidean metric. Main results are Theorem 1.14, Lemma 1.17 and Theorems 6.7 and 6.10.

Let us remark here that our inverse scattering procedure can be made purely stationary. Namely, in this stationary approach, we first define the scattering matrix by observing the asymptotic behavior of solutions to the Helmholtz equation at infinity (see Theorem 3.15 in Chap. 3), from which we derive all informations necessary for the inverse problem. Therefore, the readers who have basic knowledge about the forward scattering, and are interested only in the inverse scattering can skip Sections 5 and 6 of Chapter 1, whole Chapter 4 and Appendix, since they are of independent interest and not used in the arguments for the inverse problem.

Spectral representations, Radon transforms, S-matrices are mutually related as follows. In the time-dependent picture of scattering, the S-matrix assigns the asymptotic profile in the remote future of the solutions to the wave equation to that in the remote past. The Radon transform describes the asymptotic expansion at infinity of the fundamental solution to the wave equation (see Theorem 8.9 of Chap. 3). Using the Fourier slice theorem, one can define the Radon transform in terms of spectral representations (see Definition 5.3 of Chap. 1). In the study of the symmetric spaces, one is intereted in the characterization of the range of the Radon transform by differential operators. In the perturabation thory of the continuous spectrum, the S-matrix describes the range of the Radon transform. Support theorem of the Radon tranform is also an important subject. Here we are interested in its micro-local properties, in particular, the propagation of singularities. Although this is basically known, it is worthwhile to give the precise statements in the general short-range perturbation regime. These issues are discussed in Chap. 4 and the Appendix.

The main part of our results will be proved under a weaker decay assumption on the metric. More precisely, if we assume instead of (A-4) that in the region $0 < y < y_0$

$$ds^2 = y^{-2}\left((dy)^2 + h(x,dx) + B(x,y,dx)\right), \tag{0.6}$$

$$B(x,y,dx) = \sum_{i,j=1}^{n-1} b_{ij}(x,y)dx^i dx^j,$$

where each $b_{ij}(x,y)$ satisfies

$$|\widetilde{D}_x^\alpha D_y^\beta b(x,y)| \leq C_{\alpha\beta}(1+\rho(x,y))^{-1-\epsilon}, \quad \epsilon > 0, \tag{0.7}$$

$\rho(x,y)$ being the distance of $(x,y) \in \mathcal{M}$ from some fixed point, we can derive the same results as those presented below. In fact, we shall prove that the metric of the form (0.4) satisfying (0.5) is transformed to the metric of the form (0.6) satisfying (0.5) (see Theorem 1.6 in Chapter 4), and once we adopt (0.6), we only use the decay assumption (0.7).

Even if we start from the metric of the form (0.4) satisfying (0.7), the results below, except for Theorem 2.10, Corollary 2.11 in Chapter 2 and Theorems in Chapter 4, also hold. The difference is that the non-existence of eigenvalues embedded in the continuous spectrum may not be true. However, even in this case, one can show that the embedded eigenvalues are discrete with possible accumulation points 0 and ∞ just like Chapter 3, Theorem 3.5.

We have tried to make Chapters 1, 2 and 6 as elementary as possible so that one needs little knowledge to understand the spectral theory and inverse problems. The readers interested in only the inverse problems can skip Chapter 4 and Appendix. If one wants to know the essential step of the limiting absorption principle (resolvent estimates), one should skip Chapter 1 and read subsections 2.3, 2.4 and 2.5 of Chapter 2 first. Although it is written for the upper-half space model, the same idea works for the analysis of ends. We employed the method of integration by parts to prove the limiting absorption principle, which is essentially due to Eidus

[**31**]. This approach is simple and needs no preparatory tool, moreover it is flexible and applicable to various situation. For the other approaches, see e.g. [**33**], [**35**], [**85**], [**104**], [**105**].

To construct the generalized Fourier transform, we compute the asymptotic expansion at infinity of the resolvent. This is a classical idea, and has been frequently used (see e.g. [**118**], or [**44**]). We also utilize the Besov type space introduced by Agmon-Hörmander [**2**] to construct eigenoperators, which, as has been done by Yafaev [**132**], makes it possible to characterize the solution space of the Helmholtz equation by the generalized Fourier transform and to derive the S-matrix from the asymptotic expansion of solutions to the Helmholtz equation.

One can deal with other types of metric by the methods employed here. For example, the asymptotically Euclidean ends can be treated in the same way by utilizing results in Chap. 2, §5, §6 and Appendix A. The inverse scattering from asymptotically (Euclidean) cylindrical ends has been studied in [**67**]. In practical situation, this problem includes that of wave guides. In [**68**] and [**69**], inverse scattering from cusp of asymptotically hyperbolic manifolds or orbifolds is studied. The idea consists in generalizing the notion of S-matrix, which makes it possible to determine all geometrically finite hyperbolic surfaces. One can also consider a mixture of these different types of ends.

There are many unknown problems on spectral properties and inverse scattering for a big variety of other types of ends. We hope that the methods in this paper will be helpful for the future study of these fields.

0.5. Remarks on notation.

- For two Banach spaces X, Y, $\mathbf{B}(X;Y)$ denotes the totality of bounded linear operators from X to Y.
- For a self-adjoint operator A

$$\begin{aligned} \sigma(A) &= \text{the spectrum of } A, \\ \sigma_p(A) &= \text{the set of all eigenvalues of } A, \\ \sigma_{ac}(A) &= \text{the absolutely continuous spectrum of } A, \\ \sigma_d(A) &= \text{the dscrete spectrum of } A, \\ \sigma_e(A) &= \text{the essential spectrum of } A. \end{aligned}$$

- For an open set Ω in a manifold, $C_0^\infty(\Omega)$ is the set of all infinitely differentiable functions with compact support in Ω.
- For a measure $d\mu$ on Ω, $L^2(\Omega; d\mu)$ denotes all functions f such that

$$\|f\| = \left(\int_\Omega |f|^2 d\mu\right)^{1/2} < \infty.$$

- For an open set Ω, $H^m(\Omega)$ is the Sobolev space of order m on Ω, namely the set of all functions f on Ω whose all weak derivatives of order up to m belong to $L^2(\Omega; d\mu)$.
- $H^m_{loc}(\Omega)$ denotes the set of all u such that $u \in H^m(\omega)$ for all relatively compact open set ω in Ω.
- In the inequalities, C's denote various constants. Although these constants may vary from line to line, they are denoted by the same letter C.

- Theorems, Lemmas, etc. are quoted as follows. In each chapter, Theorem $m.n$ means Theorem $m.n$ of § m of that chapter. Theorem $p.m.n$ means Theorem $m.n$ of Chapter p.

Throughout this note, we have assumed the standard knowledge of functional analysis. We have also given a brief explanation for the basic knowledge of the spectrum of self-adjoint operators and partial differential equations when it appears. The reader should consult Kato [**80**], Reed-Simon [**115**], Isozaki [**62**] for details.

0.6. Very short perspective. Let us explain the basic strategy of constructing the Fourier transform in this paper taking $\mathbf{R}^1$ as an example. We regard $H = -d^2/dx^2$ as the Laplacian on the 1-dimensional manifold $\mathbf{R}^1$. The resolvent $R(z) = (H - z)^{-1}$ of H has the following expession:

$$R(z)f(x) = \frac{i}{2\sqrt{z}} \int_{-\infty}^{\infty} e^{i\sqrt{z}|x-y|} f(y)dy, \quad \mathrm{Im}\sqrt{z} > 0.$$

Therefore assuming that $f \in L^1(\mathbf{R}^1)$ and $z \to \lambda > 0$, and letting $x \to \pm\infty$, we have

$$R(\lambda + i0)f(x) \sim i\sqrt{\frac{\pi}{2\lambda}} e^{\pm i\sqrt{\lambda}x} \widehat{f}(\pm\sqrt{\lambda}).$$

Let $E_H(\lambda)$ be the spectral measure for H. Then by Stone's formula, we have for $0 < a < b < \infty$

$$(E_H((a,b))f, f) = \lim_{\epsilon\to 0} \frac{1}{2\pi i} \int_a^b ([R(\lambda + i\epsilon) - R(\lambda - i\epsilon)]f, f)d\lambda.$$

Letting $u = R(\lambda + i0)f$, we have by integration by parts

$$\begin{aligned} ([R(\lambda + i0) - R(\lambda - i0)]f, f) &= (u, f) - (f, u) \\ &= \lim_{R\to\infty} \int_{-R}^{R} \left(u''\overline{u} - u\overline{u''}\right) dx \\ &= \lim_{R\to\infty} [u'\overline{u} - u\overline{u'}]_{-R}^{R} \\ &= \frac{\pi i}{\sqrt{\lambda}} \left(|\widehat{f}(\sqrt{\lambda})|^2 + |\widehat{f}(-\sqrt{\lambda})|^2\right), \end{aligned}$$

which implies

$$\|f\|^2 = \lim_{a\to 0, b\to\infty} (E_H((a,b))f, f) = \int_{-\infty}^{\infty} |\widehat{f}(k)|^2 dk.$$

These calculations suggest that

- The Fourier transform is obtained from the asymptotic expansion at infinity of the Green operator of the Laplacian.
- Parseval's formula is a consequence of Stone's formula and integration by parts.

We should stress that

- The limit $R(\lambda \pm i0)$ of the resolvent $R(\lambda \pm i\epsilon)$ as $\epsilon \downarrow 0$ plays an important role.

The procedure of taking the limit as $\epsilon \downarrow 0$ of $R(\lambda \pm i\epsilon)$ is called the *limiting absorption principle.*

We shall explain these matters on asymptotically hyperbolic spaces.