Chapter 3

Complex random variable $\sum_p -\log(1-\frac{e(-\log p)}{p^\sigma})$ on $(\mathbb{R}^{\mathbb{B}}, \mathbf{P})$

In this chapter, we define the complex random variable in the heading above, whose distribution is the limit distribution in the Bohr-Jessen limit theorem.

3.1 Complex random variables $e(\lambda)$

Definition 3.1 For $\lambda \in \mathbb{R} \setminus \{0\}$, we define

$$\begin{array}{cccc} e(\lambda): & \mathbb{R}^{\mathbb{B}} & \to & \mathbb{C} \\ & \Psi & & \Psi \\ & (x_f)_{f\in\mathbb{B}} & \mapsto & x_{\cos\lambda\cdot} + \sqrt{-1}x_{\sin\lambda\cdot}, \end{array}$$

where $\cos\lambda\cdot$ and $\sin\lambda\cdot$ denote almost periodic functions $t \mapsto \cos\lambda t$ and $t \mapsto \sin\lambda t$, respectively.

Note that $e(\lambda) = \pi_{\cos\lambda\cdot} + \sqrt{-1}\pi_{\sin\lambda\cdot}$.

Claim 3.1 *For $f : \mathbb{C} \to \mathbb{C}$ bounded Borel measurable,*

$$E^{\mathbf{P}}\left[f\left(e(\lambda)\right)\right] = \frac{1}{2\pi}\int_0^{2\pi} f\left(e^{\sqrt{-1}t}\right)dt.$$

Proof. We divide the proof into three steps:
<u>1°</u> For $f \in C_b(\mathbb{C};\mathbb{C})$, i.e., bounded continuous function $f : \mathbb{C} \to \mathbb{C}$,

$$\begin{aligned} E^{\mathbf{P}}\left[f\left(e(\lambda)\right)\right] &= E^{\mathbf{P}}\left[f\left(\pi_{\cos\lambda\cdot} + \sqrt{-1}\pi_{\sin\lambda\cdot}\right)\right] \\ &= \iint_{\mathbb{R}^2} f(a+\sqrt{-1}b)\mathbf{P}\circ\pi^{-1}_{(\cos\lambda\cdot,\sin\lambda\cdot)}(dadb) \\ &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(\cos\lambda t + \sqrt{-1}\sin\lambda t)dt \\ &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f\left(e^{\sqrt{-1}\lambda t}\right)dt \end{aligned}$$

$$
\begin{aligned}
&= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f\big(e^{\sqrt{-1}|\lambda|t}\big)dt \\
&= \lim_{T\to\infty} \frac{1}{2|\lambda|T}\int_{-|\lambda|T}^{|\lambda|T} f\big(e^{\sqrt{-1}\tau}\big)d\tau \\
&= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f\big(e^{\sqrt{-1}t}\big)dt \\
&= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f\big(e^{\sqrt{-1}2\pi t}\big)dt \\
&= \lim_{T\to\infty} \frac{1}{2T}\Big(\int_{-\lfloor T\rfloor}^{\lfloor T\rfloor} f\big(e^{\sqrt{-1}2\pi t}\big)dt + \int_{-T}^{-\lfloor T\rfloor} f\big(e^{\sqrt{-1}2\pi t}\big)dt \\
&\qquad + \int_{\lfloor T\rfloor}^{T} f\big(e^{\sqrt{-1}2\pi t}\big)dt\Big) \\
&= \lim_{T\to\infty} \frac{1}{2T}\Big(\sum_{n=-\lfloor T\rfloor}^{\lfloor T\rfloor-1}\int_{n}^{n+1} f\big(e^{\sqrt{-1}2\pi t}\big)dt + \int_{-T}^{-\lfloor T\rfloor} f\big(e^{\sqrt{-1}2\pi t}\big)dt \\
&\qquad + \int_{\lfloor T\rfloor}^{T} f\big(e^{\sqrt{-1}2\pi t}\big)dt\Big) \\
&= \lim_{T\to\infty} \frac{1}{2T}\Big(\sum_{n=-\lfloor T\rfloor}^{\lfloor T\rfloor-1}\int_{0}^{1} f\big(e^{\sqrt{-1}2\pi (t+n)}\big)dt + \int_{-T}^{-\lfloor T\rfloor} f\big(e^{\sqrt{-1}2\pi t}\big)dt \\
&\qquad + \int_{\lfloor T\rfloor}^{T} f\big(e^{\sqrt{-1}2\pi t}\big)dt\Big) \\
&= \lim_{T\to\infty}\Big(\frac{\lfloor T\rfloor}{T}\int_{0}^{1} f\big(e^{\sqrt{-1}2\pi t}\big)dt + \frac{1}{2T}\int_{-T}^{-\lfloor T\rfloor} f\big(e^{\sqrt{-1}2\pi t}\big)dt \\
&\qquad + \frac{1}{2T}\int_{\lfloor T\rfloor}^{T} f\big(e^{\sqrt{-1}2\pi t}\big)dt\Big) \\
&= \int_{0}^{1} f\big(e^{\sqrt{-1}2\pi t}\big)dt \\
&= \frac{1}{2\pi}\int_{0}^{2\pi} f\big(e^{\sqrt{-1}t}\big)dt.
\end{aligned}
$$

2° Let $U(dt)$ be a uniform distribution on $[0, 2\pi)$, i.e., $U(dt) = \frac{dt}{2\pi}$ $(t \in [0, 2\pi))$, and $R(t) = (\cos t, \sin t)$. Then $\mathbf{P}\circ\pi^{-1}_{(\cos\lambda\cdot,\sin\lambda\cdot)} = U\circ R^{-1}$.

∵ For $^\forall f \in C_{\mathrm{b}}(\mathbb{R}^2;\mathbb{R})$, put $\tilde{f} \in C_{\mathrm{b}}(\mathbb{C};\mathbb{R})$ as $\tilde{f}(z) := f(\operatorname{Re} z, \operatorname{Im} z)$, $z \in \mathbb{C}$. By 1°,

$$
\begin{aligned}
\iint_{\mathbb{R}^2} f(a,b)\mathbf{P}\circ\pi^{-1}_{(\cos\lambda\cdot,\sin\lambda\cdot)}(dadb) &= E^{\mathbf{P}}\big[f(\pi_{\cos\lambda\cdot}, \pi_{\sin\lambda\cdot})\big] \\
&= E^{\mathbf{P}}\big[\tilde{f}(e(\lambda))\big] \\
&= \frac{1}{2\pi}\int_0^{2\pi}\tilde{f}(e^{\sqrt{-1}t})dt
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\int_0^{2\pi} f(\cos t, \sin t)dt \\
&= \iint_{\mathbb{R}^2} f(a,b) U \circ R^{-1}(dadb),
\end{aligned}
$$

which shows the assertion of 2°.

3° For $f : \mathbb{C} \to \mathbb{C}$ bounded Borel measurable, we have by 2° that

$$
\begin{aligned}
E^{\mathbf{P}}\big[f(e(\lambda))\big] &= \iint_{\mathbb{R}^2} f(a+\sqrt{-1}b)\mathbf{P} \circ \pi^{-1}_{(\cos\lambda\cdot, \sin\lambda\cdot)}(dadb) \\
&= \iint_{\mathbb{R}^2} f(a+\sqrt{-1}b) U \circ R^{-1}(dadb) \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(e^{\sqrt{-1}t})dt. \qquad \blacksquare
\end{aligned}
$$

Corollary 3.1 $|e(\lambda)| = 1$ **P**-*a.e.*

Proof. Clearly $\mathbb{C} \ni z \mapsto \mathbf{1}_{\mathbb{R}\setminus\{1\}}(|z|) \in \mathbb{R}$ is bounded Borel measurable. Then, by Claim 3.1,

$$
\mathbf{P}\big(|e(\lambda)| \neq 1\big) = E^{\mathbf{P}}\big[\mathbf{1}_{\mathbb{R}\setminus\{1\}}\big(|e(\lambda)|\big)\big] = \frac{1}{2\pi}\int_0^{2\pi} \mathbf{1}_{\mathbb{R}\setminus\{1\}}\big(|e^{\sqrt{-1}t}|\big)dt = 0. \qquad \blacksquare
$$

Claim 3.2 *If* $\{\lambda_k\}_{k=1}^\infty$ *is AI, then a sequence* $\big\{e(\lambda_k)\big\}_{k=1}^\infty$ *of complex random variables is i.i.d.*

Proof. We divide the proof into three steps:

1° For $r \in \mathbb{N}$, $(a_i, b_i) \in \mathbb{R}^2 \setminus \{(0,0)\}$ $(1 \le i \le r)$ and $n_i \in \mathbb{N}$ $(1 \le i \le r)$,

$$
E^{\mathbf{P}}\Big[\prod_{i=1}^r \big(a_i \pi_{\cos\lambda_i\cdot} + b_i \pi_{\sin\lambda_i\cdot}\big)^{n_i}\Big] = \prod_{i=1}^r E^{\mathbf{P}}\Big[\big(a_i \pi_{\cos\lambda_i\cdot} + b_i \pi_{\sin\lambda_i\cdot}\big)^{n_i}\Big].
$$

$\because$ First

L.H.S. (= the left-hand side)

$$
\begin{aligned}
&= \int\cdots\int_{\mathbb{R}^{2r}} \prod_{i=1}^r \big(a_i\alpha_i + b_i\beta_i\big)^{n_i} \mathbf{P} \circ \pi^{-1}_{(\cos\lambda_1\cdot, \sin\lambda_1\cdot, \ldots, \cos\lambda_r\cdot, \sin\lambda_r\cdot)}(d\alpha_1 d\beta_1 \cdots d\alpha_r d\beta_r) \\
&= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} \prod_{i=1}^r \big(a_i \cos\lambda_i t + b_i \sin\lambda_i t\big)^{n_i} dt.
\end{aligned}
$$

For simplicity, let $\rho_i = \sqrt{a_i^2 + b_i^2} > 0$ and $\theta_i \in [0, 2\pi)$ be such that $a_i = \rho_i \cos\theta_i$, $b_i = \rho_i \sin\theta_i$. Clearly

$$
\begin{aligned}
a_i \cos\lambda_i t + b_i \sin\lambda_i t &= \rho_i\big(\cos\theta_i \cos\lambda_i t + \sin\theta_i \sin\lambda_i t\big) \\
&= \rho_i \cos\big(\lambda_i t - \theta_i\big).
\end{aligned}
$$

Thus

R.H.S. (= the right-hand side) in the above

$$
\begin{aligned}
&= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\prod_{i=1}^{r}\rho_i^{n_i}\cos^{n_i}\big(\lambda_i t-\theta_i\big)dt\\
&= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\Big(\prod_{i=1}^{r}\rho_i^{n_i}\Big)\Big(\prod_{i=1}^{r}\cos^{n_i}\big(\lambda_i t-\theta_i\big)\Big)dt\\
&= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\Big(\prod_{i=1}^{r}\rho_i^{n_i}\Big)\Big(\prod_{i=1}^{r}\frac{1}{2^{n_i}}\sum_{l_i=0}^{n_i}\binom{n_i}{l_i}e^{\sqrt{-1}(2l_i-n_i)(\lambda_i t-\theta_i)}\Big)dt\\
&= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\Big(\prod_{i=1}^{r}\rho_i^{n_i}\Big)\Big(\prod_{i=1}^{r}\frac{1}{2^{n_i}}\Big)\\
&\qquad\times\sum_{\substack{0\le l_1\le n_1,\\ \dots,\\ 0\le l_r\le n_r}}\binom{n_1}{l_1}\cdots\binom{n_r}{l_r}e^{\sqrt{-1}(2l_1-n_1)(\lambda_1 t-\theta_1)}\cdots e^{\sqrt{-1}(2l_r-n_r)(\lambda_r t-\theta_r)}dt\\
&= \Big(\prod_{i=1}^{r}\rho_i^{n_i}\Big)\Big(\prod_{i=1}^{r}\frac{1}{2^{n_i}}\Big)\sum_{\substack{0\le l_1\le n_1,\\ \dots,\\ 0\le l_r\le n_r}}\binom{n_1}{l_1}\cdots\binom{n_r}{l_r}e^{\sqrt{-1}\big((n_1-2l_1)\theta_1+\cdots+(n_r-2l_r)\theta_r\big)}\\
&\qquad\qquad\times\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}e^{\sqrt{-1}\big((2l_1-n_1)\lambda_1+\cdots+(2l_r-n_r)\lambda_r\big)t}dt\\
&= \Big(\prod_{i=1}^{r}\rho_i^{n_i}\Big)\Big(\prod_{i=1}^{r}\frac{1}{2^{n_i}}\Big)\sum_{\substack{0\le l_1\le n_1,\\ \dots,\\ 0\le l_r\le n_r}}\binom{n_1}{l_1}\cdots\binom{n_r}{l_r}e^{\sqrt{-1}\big((n_1-2l_1)\theta_1+\cdots+(n_r-2l_r)\theta_r\big)}\\
&\qquad\qquad\times\mathbf{1}_{(2l_1-n_1)\lambda_1+\cdots+(2l_r-n_r)\lambda_r=0}\\
&= \Big(\prod_{i=1}^{r}\rho_i^{n_i}\Big)\Big(\prod_{i=1}^{r}\frac{1}{2^{n_i}}\Big)\sum_{\substack{0\le l_1\le n_1,\dots,0\le l_r\le n_r;\\ (2l_1-n_1)\lambda_1+\cdots+(2l_r-n_r)\lambda_r=0}}\binom{n_1}{l_1}\cdots\binom{n_r}{l_r}\\
&\qquad\qquad\times e^{\sqrt{-1}\big((n_1-2l_1)\theta_1+\cdots+(n_r-2l_r)\theta_r\big)}\\
&= \Big(\prod_{i=1}^{r}\rho_i^{n_i}\Big)\Big(\prod_{i=1}^{r}\frac{1}{2^{n_i}}\Big)\sum_{\substack{0\le l_1\le n_1,\dots,0\le l_r\le n_r;\\ 2l_1=n_1,\dots,2l_r=n_r}}\binom{n_1}{l_1}\cdots\binom{n_r}{l_r}\\
&= \begin{cases} 0 & \text{if } 1\le {}^{\exists}i\le r \text{ s.t. } n_i\in 2\mathbb{N}-1,\\ \Big(\prod_{i=1}^{r}\rho_i^{n_i}\Big)\Big(\prod_{i=1}^{r}\frac{1}{2^{n_i}}\Big)\binom{n_1}{\frac{n_1}{2}}\cdots\binom{n_r}{\frac{n_r}{2}}=\prod_{i=1}^{r}\Big(\rho_i^{n_i}\frac{1}{2^{n_i}}\binom{n_i}{\frac{n_i}{2}}\Big) & \\ & \text{if } n_i\in 2\mathbb{N}\ (1\le i\le r).\end{cases}
\end{aligned}
$$

In particular, when $r = 1$,

$$E^{\mathbf{P}}\Big[\big(a_1\pi_{\cos\lambda_1\cdot} + b_1\pi_{\sin\lambda_1\cdot}\big)^{n_1}\Big] = \begin{cases} 0 & \text{if } n_1 \in 2\mathbb{N}-1, \\ \rho_1^{n_1}\dfrac{1}{2^{n_1}}\dbinom{n_1}{\frac{n_1}{2}} & \text{if } n_1 \in 2\mathbb{N}. \end{cases}$$

Putting this into the above, we have

$$E^{\mathbf{P}}\Big[\prod_{i=1}^{r}\big(a_i\pi_{\cos\lambda_i\cdot} + b_i\pi_{\sin\lambda_i\cdot}\big)^{n_i}\Big] = \prod_{i=1}^{r} E^{\mathbf{P}}\Big[\big(a_i\pi_{\cos\lambda_i\cdot} + b_i\pi_{\sin\lambda_i\cdot}\big)^{n_i}\Big].$$

<u>2°</u> Fix $^\forall r \geq 1$. For $^\forall\big((a_1, b_1), \ldots, (a_r, b_r)\big) \in \prod_{i=1}^{r}\mathbb{R}^2$, it follows from 1° that

$$\begin{aligned}
&E^{\mathbf{P}}\Big[\prod_{i=1}^{r} e^{\sqrt{-1}(a_i\pi_{\cos\lambda_i\cdot}+b_i\pi_{\sin\lambda_i\cdot})}\Big] \\
&= E^{\mathbf{P}}\Big[\prod_{i=1}^{r}\sum_{n_i=0}^{\infty}\frac{(\sqrt{-1}(a_i\pi_{\cos\lambda_i\cdot} + b_i\pi_{\sin\lambda_i\cdot}))^{n_i}}{n_i!}\Big] \\
&= E^{\mathbf{P}}\Big[\sum_{n_1,\ldots,n_r\geq 0}\prod_{i=1}^{r}\frac{(\sqrt{-1}(a_i\pi_{\cos\lambda_i\cdot} + b_i\pi_{\sin\lambda_i\cdot}))^{n_i}}{n_i!}\Big] \\
&= \sum_{n_1,\ldots,n_r\geq 0} E^{\mathbf{P}}\Big[\prod_{i=1}^{r}\frac{(\sqrt{-1}(a_i\pi_{\cos\lambda_i\cdot} + b_i\pi_{\sin\lambda_i\cdot}))^{n_i}}{n_i!}\Big] \\
&= \sum_{n_1,\ldots,n_r\geq 0}\prod_{i=1}^{r} E^{\mathbf{P}}\Big[\frac{(\sqrt{-1}(a_i\pi_{\cos\lambda_i\cdot} + b_i\pi_{\sin\lambda_i\cdot}))^{n_i}}{n_i!}\Big] \\
&= \prod_{i=1}^{r} E^{\mathbf{P}}\Big[\sum_{n_i=0}^{\infty}\frac{(\sqrt{-1}(a_i\pi_{\cos\lambda_i\cdot} + b_i\pi_{\sin\lambda_i\cdot}))^{n_i}}{n_i!}\Big] \\
&= \prod_{i=1}^{r} E^{\mathbf{P}}\Big[e^{\sqrt{-1}(a_i\pi_{\cos\lambda_i\cdot}+b_i\pi_{\sin\lambda_i\cdot})}\Big].
\end{aligned}$$

This shows the independence of $(\pi_{\cos\lambda_1\cdot}, \pi_{\sin\lambda_1\cdot}), \ldots, (\pi_{\cos\lambda_r\cdot}, \pi_{\sin\lambda_r\cdot})$. Thus $\{e(\lambda_k)\}_{k=1}^{\infty}$ is independent.

<u>3°</u> From 2° in the proof of Claim 3.1, $\mathbf{P} \circ (\pi_{\cos\lambda_k\cdot}, \pi_{\sin\lambda_k\cdot})^{-1} = U \circ R^{-1}$ for $^\forall k$. This tells us that $\{e(\lambda_k)\}_{k=1}^{\infty}$ is identically distributed. ■

3.2 Logarithm functions of a complex variable

Before stating Theorem 3.1 below, we here explain the *logarithm function of a complex variable*.

For $z \in \mathbb{C} \setminus (-\infty, 0]$, we define

$$\log z := \int_1^z \frac{1}{w}dw. \tag{3.1}$$

Since $\mathbb{C} \setminus (-\infty, 0]$ is a simply connected domain of $\mathbb{C}$, this integral is determined only by z in virtue of Cauchy's integral theorem. In other words, this is independent of paths in $\mathbb{C} \setminus (-\infty, 0]$ connecting 1 with z. Clearly $\log(\cdot)$ is holomorphic. Also when $z \in (0, \infty)$, $\log z$ is just a usual logarithm function of a real variable.

Note that $\log z = \log |z| + \sqrt{-1} \arg z$, where $-\pi < \arg z < \pi$. Because

$$\begin{aligned}
\log z &= \int_1^z \frac{1}{w} dw \\
&= \int_1^{|z|} \frac{1}{w} dw + \int_{|z|}^z \frac{1}{w} dw \\
&= \int_1^{|z|} \frac{1}{x} dx + \int_0^{\arg z} \frac{1}{|z| e^{\sqrt{-1}\theta}} |z| e^{\sqrt{-1}\theta} \sqrt{-1} d\theta \\
&= \log |z| + \sqrt{-1} \arg z.
\end{aligned}$$

Thus we have an identity

$$\begin{aligned}
e^{\log z} = e^{\log |z| + \sqrt{-1} \arg z} &= e^{\log |z|} e^{\sqrt{-1} \arg z} \\
&= |z| e^{\sqrt{-1} \arg z} \\
&= z. \qquad (3.2)
\end{aligned}$$

($\log(\cdot)$ is an inverse function of $z = e^w$!) Also note that for $|z| < 1$

$$\log(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^n}{n}. \qquad (3.3)$$

Because, by the bounded convergence theorem,

$$\begin{aligned}
\log(1+z) = \int_1^{1+z} \frac{1}{w} dw &= z \int_0^1 \frac{ds}{1+zs} \\
&= z \int_0^1 \sum_{n=0}^{\infty} (-zs)^n ds \\
&= z \sum_{n=0}^{\infty} \int_0^1 (-zs)^n ds \\
&= z \sum_{n=0}^{\infty} (-z)^n \int_0^1 s^n ds \\
&= \sum_{n=0}^{\infty} (-1)^n z^{n+1} \Big[\frac{s^{n+1}}{n+1} \Big]_0^1 \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n z^{n+1}}{n+1} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^n}{n}.
\end{aligned}$$

Now, let $D \subset \mathbb{C}$ be a simply connected domain and $a \in D$. For a holomorphic function $f : D \to \mathbb{C}$ satisfying $f(w) \neq 0$ ($w \in D$) and $f(a) \in \mathbb{C} \setminus (-\infty, 0]$, we define

$$\log f(z) := \log f(a) + \int_a^z \frac{f'(w)}{f(w)} dw, \quad z \in D. \tag{3.4}$$

Here log of R.H.S. is the one defined by (3.1). Clearly $\log f(\cdot)$ is holomorphic. And it holds that

$$e^{\int_c^z \frac{f'(w)}{f(w)} dw} = \frac{f(z)}{f(c)}, \quad {}^\forall z, {}^\forall c \in D. \tag{3.5}$$

For, from

$$\begin{aligned}\left(e^{\int_c^z \frac{f'(w)}{f(w)} dw}\right)' &= e^{\int_c^z \frac{f'(w)}{f(w)} dw}\left(\int_c^z \frac{f'(w)}{f(w)} dw\right)' \\ &= e^{\int_c^z \frac{f'(w)}{f(w)} dw} \frac{f'(z)}{f(z)},\end{aligned}$$

it follows that

$$\begin{aligned}\left(\frac{e^{\int_c^z \frac{f'(w)}{f(w)} dw}}{f(z)}\right)' &= \frac{\left(e^{\int_c^z \frac{f'(w)}{f(w)} dw}\right)' f(z) - e^{\int_c^z \frac{f'(w)}{f(w)} dw} f'(z)}{f^2(z)} \\ &= \frac{e^{\int_c^z \frac{f'(w)}{f(w)} dw} \frac{f'(z)}{f(z)} f(z) - e^{\int_c^z \frac{f'(w)}{f(w)} dw} f'(z)}{f^2(z)} \\ &= 0, \quad {}^\forall z \in D.\end{aligned}$$

Since D is a domain, ${}^\exists w_0 \in \mathbb{C}$ s.t. $\frac{e^{\int_c^z \frac{f'(w)}{f(w)} dw}}{f(z)} = w_0$ (${}^\forall z \in D$). Putting $z = c$ yields that $w_0 = \frac{1}{f(c)}$, from which (3.5) is obvious. By (3.2) and (3.5), we have

$$\begin{aligned}e^{\log f(z)} = e^{\log f(a) + \int_a^z \frac{f'(w)}{f(w)} dw} &= e^{\log f(a)} e^{\int_a^z \frac{f'(w)}{f(w)} dw} \\ &= f(a) \cdot \frac{f(z)}{f(a)} \\ &= f(z), \quad {}^\forall z \in D.\end{aligned} \tag{3.6}$$

By the uniqueness theorem, such a $\log f(\cdot)$ is unique.

3.3 Complex random variable $\sum_p -\log(1-\frac{e(-\log p)}{p^\sigma})$

Theorem 3.1 *For $\sigma > \frac{1}{2}$,*

$$\left\{-\log\left(1-\frac{e(-\log p)}{p^\sigma}\right)\right\}_{p:prime}$$

is a sequence of independent complex random variables, and

$$E^{\mathbf{P}}\left[-\log\left(1-\frac{e(-\log p)}{p^\sigma}\right)\right] = 0, \tag{3.7}$$

$$E^{\mathbf{P}}\Big[\Big|-\log\Big(1-\frac{e(-\log p)}{p^\sigma}\Big)\Big|^2\Big] \le \Big(\frac{2^\sigma}{2^\sigma-1}\Big)^2 \frac{1}{p^{2\sigma}}. \tag{3.8}$$

Thus

$$\sum_{p:prime} -\log\Big(1-\frac{e(-\log p)}{p^\sigma}\Big)$$

is convergent $\mathbf{P}$*-a.e.*

Proof. We divide the proof into two steps:

$\underline{1^{\circ}}$ Let $\lambda \in \mathbb{R}\setminus\{0\}$ and p prime. Since $\big|\frac{e(\lambda)}{p^\sigma}\big| = \frac{1}{p^\sigma} \in (0,1)$ $\mathbf{P}$-a.e. by Corollary 3.1, $-\log\big(1-\frac{e(\lambda)}{p^\sigma}\big)$ is defined $\mathbf{P}$-a.e. Noting that for $|z|<1$,

$$\begin{aligned}\big|\log(1+z)\big| = |z|\Big|\int_0^1 \frac{ds}{1+zs}\Big| &\le |z|\int_0^1 \frac{ds}{|1+zs|}\\ &\le |z|\int_0^1 \frac{ds}{1-|z|}\\ &\qquad \big[\because\ |1+zs| \ge 1-|z|s \ge 1-|z|\big]\\ &= \frac{|z|}{1-|z|},\end{aligned} \tag{3.9}$$

we have an estimate

$$\Big|-\log\Big(1-\frac{e(\lambda)}{p^\sigma}\Big)\Big| \le \frac{\frac{1}{p^\sigma}}{1-\frac{1}{p^\sigma}} = \frac{1}{p^\sigma-1} \le \frac{2^\sigma}{2^\sigma-1}\frac{1}{p^\sigma}. \tag{3.10}$$

By Claim 3.1 and Corollary 3.1,

$$\begin{aligned}E^{\mathbf{P}}\Big[-\log\Big(1-\frac{e(\lambda)}{p^\sigma}\Big)\Big] &= E^{\mathbf{P}}\Big[-\log\Big(1-\frac{e(\lambda)}{p^\sigma}\mathbf{1}_{[0,1]}(|e(\lambda)|)\Big)\Big]\\ &= \frac{1}{2\pi}\int_0^{2\pi} -\log\Big(1-\frac{e^{\sqrt{-1}t}}{p^\sigma}\mathbf{1}_{[0,1]}(|e^{\sqrt{-1}t}|)\Big)dt\\ &= \frac{1}{2\pi}\int_0^{2\pi} -\log\Big(1-\frac{e^{\sqrt{-1}t}}{p^\sigma}\Big)dt\\ &= \frac{1}{2\pi\sqrt{-1}}\int_0^{2\pi} \frac{-\log\Big(1-\frac{e^{\sqrt{-1}t}}{p^\sigma}\Big)}{e^{\sqrt{-1}t}}\sqrt{-1}e^{\sqrt{-1}t}dt\\ &= \frac{1}{2\pi\sqrt{-1}}\int_{|z|=1} \frac{-\log\Big(1-\frac{z}{p^\sigma}\Big)}{z}dz\\ &= -\log\Big(1-\frac{0}{p^\sigma}\Big) \quad \big[\because\ \text{Cauchy's integral representation}\big]\\ &= 0.\end{aligned} \tag{3.11}$$

$\underline{2^{\circ}}$ (3.7) and (3.8) follow from (3.11) and (3.10) with $\lambda = -\log p$, respectively. By Example 2.1 and Claim 3.2, $\{e(-\log p)\}_{p:\text{prime}}$ is a sequence of i.i.d. complex random variables. Thus $\{-\log\big(1-\frac{e(-\log p)}{p^\sigma}\big)\}_{p:\text{prime}}$ is independent.

Next, by (3.7),

$$E^{\mathbf{P}}\left[\mathrm{Re}\left(-\log\left(1-\frac{e(-\log p)}{p^\sigma}\right)\right)\right]=0,$$
$$E^{\mathbf{P}}\left[\mathrm{Im}\left(-\log\left(1-\frac{e(-\log p)}{p^\sigma}\right)\right)\right]=0;$$

by (3.8),

$$\begin{aligned}
&E^{\mathbf{P}}\left[\left(\mathrm{Re}\left(-\log\left(1-\frac{e(-\log p)}{p^\sigma}\right)\right)\right)^2\right],\ E^{\mathbf{P}}\left[\left(\mathrm{Im}\left(-\log\left(1-\frac{e(-\log p)}{p^\sigma}\right)\right)\right)^2\right]\\
&\le E^{\mathbf{P}}\left[\left(\mathrm{Re}\left(-\log\left(1-\frac{e(-\log p)}{p^\sigma}\right)\right)\right)^2+\left(\mathrm{Im}\left(-\log\left(1-\frac{e(-\log p)}{p^\sigma}\right)\right)\right)^2\right]\\
&=E^{\mathbf{P}}\left[\left|-\log\left(1-\frac{e(-\log p)}{p^\sigma}\right)\right|^2\right]\\
&\le\left(\frac{2^\sigma}{2^\sigma-1}\right)^2\frac{1}{p^{2\sigma}}.
\end{aligned}$$

Since $\sum_{p:\mathrm{prime}}\frac{1}{p^{2\sigma}}<\infty\ \left[\because\ 2\sigma>1\right]$, we can apply the a.s. convergence theorem for independent random variables [cf. Claim A.6] to see that

$$\sum_{p:\mathrm{prime}}\mathrm{Re}\left(-\log\left(1-\frac{e(-\log p)}{p^\sigma}\right)\right),$$
$$\sum_{p:\mathrm{prime}}\mathrm{Im}\left(-\log\left(1-\frac{e(-\log p)}{p^\sigma}\right)\right)$$

are convergent $\mathbf{P}$-a.e. Therefore

$$\sum_{p:\mathrm{prime}}-\log\left(1-\frac{e(-\log p)}{p^\sigma}\right)$$

is also convergent $\mathbf{P}$-a.e. ■

Remark 3.1 The distribution of $\sum_p -\log\left(1-\frac{e(-\log p)}{p^\sigma}\right)$ is just the limit distribution in Bohr-Jessen limit theorem !

3.4 Some properties of the distribution of $\sum_p -\log(1-\frac{e(-\log p)}{p^\sigma})$

We view what is easily seen from concrete calculations.

Claim 3.3 *The distributions of real random variables* $\mathrm{Re}\left(\sum_p -\log\left(1-\frac{e(-\log p)}{p^\sigma}\right)\right)$ *and* $\mathrm{Im}\left(\sum_p -\log\left(1-\frac{e(-\log p)}{p^\sigma}\right)\right)$ *are both continuous. Namely, for* $^\forall\alpha, {}^\forall\beta\in\mathbb{R}$,

$$\mathbf{P}\left(\mathrm{Re}\left(\sum_p -\log\left(1-\frac{e(-\log p)}{p^\sigma}\right)\right)=\alpha\right)=0,$$

$$\mathbf{P}\Big(\mathrm{Im}\Big(\sum_p -\log\big(1-\frac{e(-\log p)}{p^\sigma}\big)\Big)=\beta\Big)=0.$$

Proof. Fix $\sigma > \frac{1}{2}$. From the independence of $\big\{-\log\big(1-\frac{e(-\log p_n)}{p_n^\sigma}\big)\big\}_{n=1}^\infty$, it follows that for $\alpha, \beta \in \mathbb{R}$,

$$\begin{aligned}
&\mathbf{P}\Big(\mathrm{Re}\Big(\sum_{n=1}^\infty -\log\big(1-\frac{e(-\log p_n)}{p_n^\sigma}\big)\Big)=\alpha\Big)\\
&=\mathbf{P}\Big(\sum_{n=1}^\infty \mathrm{Re}\Big(-\log\big(1-\frac{e(-\log p_n)}{p_n^\sigma}\big)\Big)=\alpha\Big)\\
&=\mathbf{P}\Big(\mathrm{Re}\Big(-\log\big(1-\frac{e(-\log p_1)}{p_1^\sigma}\big)\Big)+\sum_{n=2}^\infty \mathrm{Re}\Big(-\log\big(1-\frac{e(-\log p_n)}{p_n^\sigma}\big)\Big)=\alpha\Big)\\
&=\mathbf{P}\Big(\mathrm{Re}\Big(-\log\big(1-\frac{e(-\log p_1)}{p_1^\sigma}\big)\Big)=\alpha-\sum_{n=2}^\infty \mathrm{Re}\Big(-\log\big(1-\frac{e(-\log p_n)}{p_n^\sigma}\big)\Big)\Big)\\
&=E^{\mathbf{P}}\Big[\mathbf{P}\Big(\mathrm{Re}\Big(-\log\big(1-\frac{e(-\log p_1)}{p_1^\sigma}\big)\Big)=\alpha-x\Big)\Big|_{x=\sum_{n=2}^\infty \mathrm{Re}\big(-\log(1-\frac{e(-\log p_n)}{p_n^\sigma})\big)}\Big],
\end{aligned}$$

$$\begin{aligned}
&\mathbf{P}\Big(\mathrm{Im}\Big(\sum_{n=1}^\infty -\log\big(1-\frac{e(-\log p_n)}{p_n^\sigma}\big)\Big)=\beta\Big)\\
&=\mathbf{P}\Big(\sum_{n=1}^\infty \mathrm{Im}\Big(-\log\big(1-\frac{e(-\log p_n)}{p_n^\sigma}\big)\Big)=\beta\Big)\\
&=\mathbf{P}\Big(\mathrm{Im}\Big(-\log\big(1-\frac{e(-\log p_1)}{p_1^\sigma}\big)\Big)+\sum_{n=2}^\infty \mathrm{Im}\Big(-\log\big(1-\frac{e(-\log p_n)}{p_n^\sigma}\big)\Big)=\beta\Big)\\
&=\mathbf{P}\Big(\mathrm{Im}\Big(-\log\big(1-\frac{e(-\log p_1)}{p_1^\sigma}\big)\Big)=\beta-\sum_{n=2}^\infty \mathrm{Im}\Big(-\log\big(1-\frac{e(-\log p_n)}{p_n^\sigma}\big)\Big)\Big)\\
&=E^{\mathbf{P}}\Big[\mathbf{P}\Big(\mathrm{Im}\Big(-\log\big(1-\frac{e(-\log p_1)}{p_1^\sigma}\big)\Big)=\beta-y\Big)\Big|_{y=\sum_{n=2}^\infty \mathrm{Im}\big(-\log(1-\frac{e(-\log p_n)}{p_n^\sigma})\big)}\Big].
\end{aligned}$$

By the lemma below, the distributions of $\mathrm{Re}\big(-\log\big(1-\frac{e(-\log p_1)}{p_1^\sigma}\big)\big)$ and $\mathrm{Im}\big(-\log\big(1-\frac{e(-\log p_1)}{p_1^\sigma}\big)\big)$ are both absolutely continuous. Thus we have the assertion of the claim. ■

Lemma 3.1 *Let $\sigma > \frac{1}{2}$ and p prime. Then*

(i) $\mathbf{P}\Big(\mathrm{Re}\Big(-\log\big(1-\frac{e(-\log p)}{p^\sigma}\big)\Big)\in du\Big)$

$$=\frac{2}{\pi}\mathbf{1}_{\left(\log(1+\frac{1}{p^\sigma})^{-1},\,\log(1-\frac{1}{p^\sigma})^{-1}\right)}(u)\frac{1}{\sqrt{\big((e^u(1+\frac{1}{p^\sigma}))^2-1\big)\big(1-(e^u(1-\frac{1}{p^\sigma}))^2\big)}}du.$$

(ii) *Define* $\psi_{p^\sigma}:\mathbb{R}\to\mathbb{R}$ *by*

$$\psi_{p^\sigma}(t)=\begin{cases}\tan^{-1}\Big(\dfrac{\frac{1}{p^\sigma}-\cos t}{\sin t}\Big)+\tan^{-1}\Big(\dfrac{\cos t}{\sin t}\Big), & t\in\mathbb{R}\setminus\pi\mathbb{Z},\\ 0, & t\in\pi\mathbb{Z}.\end{cases}$$

ψ_{p^σ} *is a smooth function with period* 2π*. Its first derivative test on* $[0,2\pi]$ *is as follows:*

t	0	$\cdots$	t_σ	$\cdots$	$\frac{\pi}{2}$	$\cdots$	π	$\cdots$	$\frac{3}{2}\pi$	$\cdots$	$2\pi-t_\sigma$	$\cdots$	2π
ψ'	$\frac{1}{p^\sigma-1}$	$+$	0	$-$	$-$	$-$	$\frac{-1}{p^\sigma-1}$	$-$	$-$	$-$	0	$+$	$\frac{1}{p^\sigma-1}$
ψ	0	$\nearrow$	$\frac{\pi}{2}-t_\sigma$		$\searrow$		0		$\searrow$		$t_\sigma-\frac{\pi}{2}$	$\nearrow$	0

Here $\psi:=\psi_{p^\sigma}$ *and* $t_\sigma:=\cos^{-1}(\frac{1}{p^\sigma})$*. Thus* $\psi|_{[0,t_\sigma]}:[0,t_\sigma]\to[0,\frac{\pi}{2}-t_\sigma]$ *and* $\psi|_{[2\pi-t_\sigma,2\pi]}:[2\pi-t_\sigma,2\pi]\to[t_\sigma-\frac{\pi}{2},0]$ *are strictly increasing, and* $\psi|_{[t_\sigma,2\pi-t_\sigma]}:[t_\sigma,2\pi-t_\sigma]\to[t_\sigma-\frac{\pi}{2},\frac{\pi}{2}-t_\sigma]$ *is strictly decreasing. Denoting their inverse functions by* $\big(\psi|_{[0,t_\sigma]}\big)^{-1}$, $\big(\psi|_{[2\pi-t_\sigma,2\pi]}\big)^{-1}$ *and* $\big(\psi|_{[t_\sigma,2\pi-t_\sigma]}\big)^{-1}$*, respectively, we have*

$$\begin{aligned}&\mathbf{P}\Big(\mathrm{Im}\Big(-\log\big(1-\frac{e(-\log p)}{p^\sigma}\big)\Big)\in dv\Big)\\ &=\frac{1}{2\pi}\Bigg(\mathbf{1}_{(t_\sigma-\frac{\pi}{2},0)}(v)\Big(\frac{-1}{\psi'\big((\psi|_{[t_\sigma,2\pi-t_\sigma]})^{-1}(v)\big)}+\frac{1}{\psi'\big((\psi|_{[2\pi-t_\sigma,2\pi]})^{-1}(v)\big)}\Big)\\ &\quad+\mathbf{1}_{(0,\frac{\pi}{2}-t_\sigma)}(v)\Big(\frac{-1}{\psi'\big((\psi|_{[t_\sigma,2\pi-t_\sigma]})^{-1}(v)\big)}+\frac{1}{\psi'\big((\psi|_{[0,t_\sigma]})^{-1}(v)\big)}\Big)\Bigg)dv.\end{aligned}$$

Proof. First, for $w\in\mathbb{C}$, $|w|<1$,

$$\begin{aligned}-\log(1-w)&=\int_0^1\frac{w}{1-sw}ds\\ &=\int_0^1\frac{w(1-s\overline{w})}{(1-sw)(1-s\overline{w})}ds\\ &=\int_0^1\frac{w-s|w|^2}{1-s(w+\overline{w})+s^2|w|^2}ds\\ &=\int_0^1\frac{\mathrm{Re}\,w-s|w|^2}{1-2s\,\mathrm{Re}\,w+s^2|w|^2}ds+\sqrt{-1}\int_0^1\frac{\mathrm{Im}\,w}{1-2s\,\mathrm{Re}\,w+s^2|w|^2}ds\\ &=-\frac{1}{2}\int_0^1\big(\log(1-2s\,\mathrm{Re}\,w+s^2|w|^2)\big)'ds\\ &\quad+\sqrt{-1}\int_0^1\frac{\mathrm{Im}\,w}{1-2s\,\mathrm{Re}\,w+s^2|w|^2}ds\\ &=-\frac{1}{2}\log(1-2\,\mathrm{Re}\,w+|w|^2)+\sqrt{-1}\int_0^1\frac{\mathrm{Im}\,w}{1-2s\,\mathrm{Re}\,w+s^2|w|^2}ds.\end{aligned}$$

Fix $\sigma>\frac{1}{2}$ and prime p. From the above,

$$\mathrm{Re}\Big(-\log\big(1-\frac{e^{\sqrt{-1}t}}{p^\sigma}\big)\Big)=-\frac{1}{2}\log\Big(1-\frac{2}{p^\sigma}\cos t+\frac{1}{p^{2\sigma}}\Big),\tag{3.12}$$

$$\begin{aligned}\operatorname{Im}\Big(-\log\big(1-\frac{e^{\sqrt{-1}t}}{p^\sigma}\big)\Big) &= \int_0^1 \frac{\frac{\sin t}{p^\sigma}}{1-\frac{2s}{p^\sigma}\cos t+\frac{s^2}{p^{2\sigma}}}ds \\ &= \frac{1}{p^\sigma}\int_0^1 \frac{\sin t}{1-\frac{2s}{p^\sigma}\cos t+\frac{s^2}{p^{2\sigma}}}ds. \end{aligned} \tag{3.13}$$

(i) By Claim 3.1 and (3.12),

$$\begin{aligned}
&\mathbf{P}\bigg(\operatorname{Re}\Big(-\log\big(1-\frac{e(-\log p)}{p^\sigma}\big)\Big)\in\cdot\bigg)\\
&=\frac{1}{2\pi}\int_0^{2\pi}\mathbf{1}_\cdot\bigg(\operatorname{Re}\Big(-\log\big(1-\frac{e^{\sqrt{-1}t}}{p^\sigma}\big)\Big)\bigg)dt\\
&=\frac{1}{2\pi}\int_0^{2\pi}\mathbf{1}_\cdot\bigg(-\frac{1}{2}\log\Big(1-\frac{2}{p^\sigma}\cos t+\frac{1}{p^{2\sigma}}\Big)\bigg)dt\\
&=\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathbf{1}_\cdot\bigg(-\frac{1}{2}\log\Big(1+\frac{2}{p^\sigma}\cos \tau+\frac{1}{p^{2\sigma}}\Big)\bigg)d\tau\\
&\qquad\big[\because \text{ change of variable: } \tau=t-\pi\big]\\
&=\frac{1}{\pi}\int_0^{\pi}\mathbf{1}_\cdot\bigg(-\frac{1}{2}\log\Big(1+\frac{2}{p^\sigma}\cos \tau+\frac{1}{p^{2\sigma}}\Big)\bigg)d\tau\\
&\qquad\big[\because \tau\mapsto\cos\tau \text{ is even}\big]\\
&=\frac{1}{\pi}\int_{-1}^{1}\mathbf{1}_\cdot\bigg(-\frac{1}{2}\log\Big(1+\frac{2}{p^\sigma}s+\frac{1}{p^{2\sigma}}\Big)\bigg)\frac{ds}{\sqrt{1-s^2}}\\
&\qquad\big[\because \text{ change of variable: } s=\cos\tau\big]\\
&=\frac{1}{\pi}\int_{\log(1+\frac{1}{p^\sigma})^{-1}}^{\log(1-\frac{1}{p^\sigma})^{-1}}\mathbf{1}_\cdot(u)\frac{p^\sigma e^{-2u}}{\sqrt{1-\big(\frac{p^\sigma}{2}(e^{-2u}-1-\frac{1}{p^{2\sigma}})\big)^2}}du\\
&\qquad\big[\because \text{ change of variable: } u=-\tfrac{1}{2}\log(1+\tfrac{2}{p^\sigma}s+\tfrac{1}{p^{2\sigma}})\big]\\
&=\frac{2}{\pi}\int_{\log(1+\frac{1}{p^\sigma})^{-1}}^{\log(1-\frac{1}{p^\sigma})^{-1}}\frac{\mathbf{1}_\cdot(u)}{\sqrt{\big((e^u(1+\frac{1}{p^\sigma}))^2-1\big)\big(1-(e^u(1-\frac{1}{p^\sigma}))^2\big)}}du.
\end{aligned}$$

Here the last equality comes from the following calculation:

$$\begin{aligned}
&\frac{p^\sigma e^{-2u}}{\sqrt{1-\big(\frac{p^\sigma}{2}(e^{-2u}-1-\frac{1}{p^{2\sigma}})\big)^2}}\\
&=\bigg(\frac{p^{2\sigma}e^{-4u}}{\big(1-\frac{p^\sigma}{2}(e^{-2u}-1-\frac{1}{p^{2\sigma}})\big)\big(1+\frac{p^\sigma}{2}(e^{-2u}-1-\frac{1}{p^{2\sigma}})\big)}\bigg)^{1/2}\\
&=\bigg(\frac{4p^{2\sigma}}{2e^{2u}\big(1-\frac{p^\sigma}{2}(e^{-2u}-1-\frac{1}{p^{2\sigma}})\big)2e^{2u}\big(1+\frac{p^\sigma}{2}(e^{-2u}-1-\frac{1}{p^{2\sigma}})\big)}\bigg)^{1/2}\\
&=\bigg(\frac{4p^{2\sigma}}{\big(2e^{2u}-p^\sigma(1-e^{2u}-\frac{e^{2u}}{p^{2\sigma}})\big)\big(2e^{2u}+p^\sigma(1-e^{2u}-\frac{e^{2u}}{p^{2\sigma}})\big)}\bigg)^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&= \Big(\frac{4p^{2\sigma}}{\big(e^{2u}(p^\sigma+p^{-\sigma}+2)-p^\sigma\big)\big(p^\sigma-e^{2u}(p^\sigma+p^{-\sigma}-2)\big)}\Big)^{1/2}\\
&= \Big(\frac{4}{p^{-\sigma}\big(e^{2u}(p^{\sigma/2}+p^{-\sigma/2})^2-p^\sigma\big)p^{-\sigma}\big(p^\sigma-e^{2u}(p^{\sigma/2}-p^{-\sigma/2})^2\big)}\Big)^{1/2}\\
&= \Big(\frac{4}{\big(e^{2u}(1+p^{-\sigma})^2-1\big)\big(1-e^{2u}(1-p^{-\sigma})^2\big)}\Big)^{1/2}\\
&= \frac{2}{\sqrt{\big(e^{2u}(1+p^{-\sigma})^2-1\big)\big(1-e^{2u}(1-p^{-\sigma})^2\big)}}.
\end{aligned}
$$

Thus we have the assertion (i).
(ii) For simplicity, put

$$
\begin{aligned}
\psi_{p^\sigma}(t) &:= \mathrm{Im}\Big(-\log\big(1-\frac{e^{\sqrt{-1}t}}{p^\sigma}\big)\Big)\\
&= \frac{1}{p^\sigma}\int_0^1 \frac{\sin t}{1-\frac{2s}{p^\sigma}\cos t+\frac{s^2}{p^{2\sigma}}}ds \quad \big[\text{cf. } (3.13)\big].
\end{aligned}
$$

Clearly ψ_{p^σ} is smooth. Also it has period 2π and vanishes on $\pi\mathbb{Z}$. Since, for $t\in\mathbb{R}\setminus\pi\mathbb{Z}$,

$$
\begin{aligned}
\frac{d}{ds}\Big(\tan^{-1}\Big(\frac{\frac{s}{p^\sigma}-\cos t}{\sin t}\Big)\Big) &= \frac{\frac{1}{p^\sigma\sin t}}{1+\big(\frac{\frac{s}{p^\sigma}-\cos t}{\sin t}\big)^2}\\
&= \frac{1}{p^\sigma}\frac{\sin t}{\sin^2 t+(\frac{s}{p^\sigma}-\cos t)^2}\\
&= \frac{1}{p^\sigma}\frac{\sin t}{\sin^2 t+\frac{s^2}{p^{2\sigma}}-\frac{2s}{p^\sigma}\cos t+\cos^2 t}\\
&= \frac{1}{p^\sigma}\frac{\sin t}{1-\frac{2s}{p^\sigma}\cos t+\frac{s^2}{p^{2\sigma}}},
\end{aligned}
$$

it follows that

$$
\begin{aligned}
\psi_{p^\sigma}(t) &= \Big[\tan^{-1}\Big(\frac{\frac{s}{p^\sigma}-\cos t}{\sin t}\Big)\Big]_0^1\\
&= \tan^{-1}\Big(\frac{\frac{1}{p^\sigma}-\cos t}{\sin t}\Big)-\tan^{-1}\Big(-\frac{\cos t}{\sin t}\Big)\\
&= \tan^{-1}\Big(\frac{\frac{1}{p^\sigma}-\cos t}{\sin t}\Big)+\tan^{-1}\Big(\frac{\cos t}{\sin t}\Big).
\end{aligned}
$$

Differentiating it in t, we have

$$
\begin{aligned}
\psi'_{p^\sigma}(t) = &\frac{1}{1+\big(\frac{\frac{1}{p^\sigma}-\cos t}{\sin t}\big)^2}\frac{\sin^2 t-(\frac{1}{p^\sigma}-\cos t)\cos t}{\sin^2 t}\\
&+\frac{1}{1+\big(\frac{\cos t}{\sin t}\big)^2}\frac{-\sin^2 t-\cos^2 t}{\sin^2 t}
\end{aligned}
$$

$$= \frac{\sin^2 t + \cos^2 t - \frac{\cos t}{p^\sigma}}{\sin^2 t + \left(\frac{1}{p^\sigma} - \cos t\right)^2} + \frac{-\sin^2 t - \cos^2 t}{\sin^2 t + \cos^2 t}$$
$$= \frac{1 - \frac{\cos t}{p^\sigma}}{\sin^2 t + \frac{1}{p^{2\sigma}} - \frac{2}{p^\sigma}\cos t + \cos^2 t} - 1$$
$$= \frac{1 - \frac{\cos t}{p^\sigma} - 1 + \frac{2}{p^\sigma}\cos t - \frac{1}{p^{2\sigma}}}{1 - \frac{2}{p^\sigma}\cos t + \frac{1}{p^{2\sigma}}}$$
$$= \frac{\frac{1}{p^\sigma}\left(\cos t - \frac{1}{p^\sigma}\right)}{1 - \frac{2}{p^\sigma}\cos t + \frac{1}{p^{2\sigma}}},$$

from which, the first derivative test of ψ_{p^σ} on $[0, 2\pi]$ is given as in (ii).

Now, by Claim 3.1,

$$\begin{aligned}
&\mathbf{P}\left(\mathrm{Im}\left(-\log\left(1 - \frac{e(-\log p)}{p^\sigma}\right)\right) \in \cdot\right) \\
&= \frac{1}{2\pi}\int_0^{2\pi} \mathbf{1}_{\cdot}\left(\mathrm{Im}\left(-\log\left(1 - \frac{e^{\sqrt{-1}t}}{p^\sigma}\right)\right)\right)dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} \mathbf{1}_{\cdot}\big(\psi_{p^\sigma}(t)\big)dt \\
&= \frac{1}{2\pi}\left(\int_0^{t_\sigma} \mathbf{1}_{\cdot}\big(\psi_{p^\sigma}(t)\big)dt + \int_{t_\sigma}^{2\pi - t_\sigma} \mathbf{1}_{\cdot}\big(\psi_{p^\sigma}(t)\big)dt + \int_{2\pi - t_\sigma}^{2\pi} \mathbf{1}_{\cdot}\big(\psi_{p^\sigma}(t)\big)dt\right) \\
&= \frac{1}{2\pi}\left(\int_0^{\frac{\pi}{2} - t_\sigma} \mathbf{1}_{\cdot}(v)\frac{dv}{\psi'_{p^\sigma}\big((\psi_{p^\sigma}|_{[0,t_\sigma]})^{-1}(v)\big)}\right. \\
&\quad + \int_{t_\sigma - \frac{\pi}{2}}^{\frac{\pi}{2} - t_\sigma} \mathbf{1}_{\cdot}(v)\frac{-1}{\psi'_{p^\sigma}\big((\psi_{p^\sigma}|_{[t_\sigma, 2\pi - t_\sigma]})^{-1}(v)\big)}dv \\
&\quad \left. + \int_{t_\sigma - \frac{\pi}{2}}^{0} \mathbf{1}_{\cdot}(v)\frac{dv}{\psi'_{p^\sigma}\big((\psi_{p^\sigma}|_{[2\pi - t_\sigma, 2\pi]})^{-1}(v)\big)}\right).
\end{aligned}$$

This shows the assertion (ii). ■

Corollary 3.2 *For $-\infty < \alpha_1 \le \alpha_2 < \infty$, $-\infty < \beta_1 \le \beta_2 < \infty$, a closed rectangle*

$$E_{\alpha_1,\alpha_2;\beta_1,\beta_2} := \left\{a + \sqrt{-1}b; \alpha_1 \le a \le \alpha_2,\ \beta_1 \le b \le \beta_2\right\} \tag{3.14}$$

is a continuity set of the distribution of $\sum_p -\log\left(1 - \frac{e(-\log p)}{p^\sigma}\right)$. Namely,

$$\mathbf{P}\left(\sum_p -\log\left(1 - \frac{e(-\log p)}{p^\sigma}\right) \in \partial E_{\alpha_1,\alpha_2;\beta_1,\beta_2}\right) = 0.$$

Proof. Since

$$\partial E_{\alpha_1,\alpha_2;\beta_1,\beta_2} \subset \left\{\alpha_1 + \sqrt{-1}b; b \in \mathbb{R}\right\} \cup \left\{\alpha_2 + \sqrt{-1}b; b \in \mathbb{R}\right\}$$

$$\cup \left\{a+\sqrt{-1}\beta_1; a \in \mathbb{R}\right\} \cup \left\{a+\sqrt{-1}\beta_2; a \in \mathbb{R}\right\},$$

we have by Claim 3.3 that

$$\begin{aligned}
&\mathbf{P}\Big(\sum_p -\log\Big(1-\frac{e(-\log p)}{p^\sigma}\Big) \in \partial E_{\alpha_1,\alpha_2;\beta_1,\beta_2}\Big)\\
&\le \mathbf{P}\Big(\mathrm{Re}\Big(\sum_p -\log\big(1-\frac{e(-\log p)}{p^\sigma}\big)\Big) = \alpha_1\Big)\\
&\quad + \mathbf{P}\Big(\mathrm{Re}\Big(\sum_p -\log\big(1-\frac{e(-\log p)}{p^\sigma}\big)\Big) = \alpha_2\Big)\\
&\quad + \mathbf{P}\Big(\mathrm{Im}\Big(\sum_p -\log\big(1-\frac{e(-\log p)}{p^\sigma}\big)\Big) = \beta_1\Big)\\
&\quad + \mathbf{P}\Big(\mathrm{Im}\Big(\sum_p -\log\big(1-\frac{e(-\log p)}{p^\sigma}\big)\Big) = \beta_2\Big)\\
&= 0.
\end{aligned}$$

■

We regard a complex random variable $\sum_p -\log\big(1-\frac{e(-\log p)}{p^\sigma}\big)$ as a 2-dimensional random variable

$$\Big(\mathrm{Re}\Big(\sum_p -\log\big(1-\frac{e(-\log p)}{p^\sigma}\big)\Big), \mathrm{Im}\Big(\sum_p -\log\big(1-\frac{e(-\log p)}{p^\sigma}\big)\Big)\Big)$$

and denote this distribution by $p_\sigma(dxdy)$:

$$\begin{aligned}
p_\sigma(dxdy) = \mathbf{P}\bigg(\Big(&\mathrm{Re}\Big(\sum_p -\log\big(1-\frac{e(-\log p)}{p^\sigma}\big)\Big),\\
&\mathrm{Im}\Big(\sum_p -\log\big(1-\frac{e(-\log p)}{p^\sigma}\big)\Big)\Big) \in dxdy\bigg), \quad x, y \in \mathbb{R}.
\end{aligned}$$

Claim 3.4 $\mathrm{supp}^{\dagger 1}\big(p_\sigma(dxdy)\big) = \begin{cases} \textit{compact set}, & \textit{if } \ 1 < \sigma < \infty, \\ \mathbb{R}^2, & \textit{if } \ \frac{1}{2} < \sigma \le 1. \end{cases}$

Proof of the first assertion of Claim 3.4. In case $1 < \sigma < \infty$, it follows from (3.10) that

$$\begin{aligned}
\Big|\sum_p -\log\Big(1-\frac{e(-\log p)}{p^\sigma}\Big)\Big| &\le \sum_p \Big|-\log\Big(1-\frac{e(-\log p)}{p^\sigma}\Big)\Big|\\
&\le \frac{2^\sigma}{2^\sigma-1}\sum_p \frac{1}{p^\sigma} \quad \mathbf{P}\text{-a.e.},
\end{aligned}$$

[†1] For a probability measure ν on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$,

$$\mathrm{supp}\, \nu := \bigcap \big\{F \subset \mathbb{R}^n; F \text{ is a closed set such that } \nu(F^\complement) = 0\big\}$$

is called the *support* of ν.

or, equivalently

$$p_\sigma\left(\left\{(x,y); \sqrt{x^2+y^2} \le \frac{2^\sigma}{2^\sigma-1}\sum_p \frac{1}{p^\sigma}\right\}\right) = 1,$$

which shows that $\operatorname{supp}\big(p_\sigma(dxdy)\big)$ is compact. ■

For the proof of the second assertion of Claim 3.4, we present the following lemma:

Lemma 3.2 *Let $n \ge 2$ and $s_1, \dots, s_n > 0$, and put $S = \{z \in \mathbb{C}; |z| = 1\}$. Then*

$$\left\{\sum_{k=1}^n s_k z_k;\ z_1, \dots, z_n \in S\right\} = \left\{z; \max_{1\le k\le n}\Big(s_k - \sum_{i\ne k} s_i\Big)^+ \le |z| \le s_1 + \cdots + s_n\right\}.$$

Here, for $\alpha \in \mathbb{R}$, $\alpha^+ := \alpha \vee 0$.

Proof. Since the triangle inequality gives that for $z_1, \dots, z_n \in S$,

$$\begin{aligned}
\left|\sum_{k=1}^n s_k z_k\right| &\le \sum_{k=1}^n s_k |z_k| = \sum_{k=1}^n s_k, \\
\left|\sum_{k=1}^n s_k z_k\right| &= \left|s_j z_j + \sum_{k\in\{1,\dots,n\}\setminus\{j\}} s_k z_k\right| \\
&\ge s_j|z_j| - \left|\sum_{k\in\{1,\dots,n\}\setminus\{j\}} s_k z_k\right| \\
&\ge s_j|z_j| - \sum_{k\in\{1,\dots,n\}\setminus\{j\}} s_k|z_k| \\
&= s_j - \sum_{k\in\{1,\dots,n\}\setminus\{j\}} s_k \quad (j = 1, \dots, n),
\end{aligned}$$

it is seen that

$$\max_{1\le j\le n}\Big(s_j - \sum_{k\in\{1,\dots,n\}\setminus\{j\}} s_k\Big)^+ \le \left|\sum_{k=1}^n s_k z_k\right| \le \sum_{k=1}^n s_k,$$

which shows the inclusion '$\subset$'.

In the following, we show the opposite inclusion '$\supset$' by induction on n.

$\underline{1^\circ}$ Let $n = 2$ and $s_1, s_2 > 0$. In case $s_1 = s_2 =: s$,

$$\begin{aligned}
&s_1 + s_2 = 2s, \\
&\max_{1\le k\le 2}\Big(s_k - \sum_{i\ne k} s_i\Big)^+ = (s_1 - s_2)^+ \vee (s_2 - s_1)^+ = 0.
\end{aligned}$$

Note that for $\psi_1, \psi_2 \in \mathbb{R}$,

$$s_1 e^{\sqrt{-1}(\psi_2+\psi_1)} + s_2 e^{\sqrt{-1}(\psi_2-\psi_1)} = s e^{\sqrt{-1}\psi_2}\big(e^{\sqrt{-1}\psi_1} + e^{-\sqrt{-1}\psi_1}\big)$$

$$= \big(2s\cos\psi_1\big)e^{\sqrt{-1}\psi_2}.$$

Let $z \in \mathbb{C}$, $|z| \le 2s$. If $z = 0$, putting $\psi_1 = \frac{\pi}{2}$ yields that

$$s_1 e^{\sqrt{-1}(\psi_2+\psi_1)} + s_2 e^{\sqrt{-1}(\psi_2-\psi_1)} = \big(2s\cos\tfrac{\pi}{2}\big)e^{\sqrt{-1}\psi_2} = 0 = z;$$

if $z \ne 0$, letting $z = |z|e^{\sqrt{-1}\psi}$ for some $\psi \in \mathbb{R}$ and putting $\psi_1 = \cos^{-1}\big(\frac{|z|}{2s}\big)$ and $\psi_2 = \psi$ yield that

$$\begin{aligned} s_1 e^{\sqrt{-1}(\psi_2+\psi_1)} + s_2 e^{\sqrt{-1}(\psi_2-\psi_1)} = \big(2s\cos\psi_1\big)e^{\sqrt{-1}\psi_2} &= 2s\frac{|z|}{2s}e^{\sqrt{-1}\psi} \\ &= |z|e^{\sqrt{-1}\psi} \\ &= z. \end{aligned}$$

Thus it is seen that

$$\big\{z; |z| \le 2s\big\} \subset \big\{s_1 z_1 + s_2 z_2;\ z_1, z_2 \in S\big\}.$$

In case $s_1 > s_2 > 0$,

$$\max_{1\le k\le 2}\Big(s_k - \sum_{i\ne k} s_i\Big)^+ = (s_1 - s_2)^+ \vee (s_2 - s_1)^+ = s_1 - s_2 > 0.$$

Note that for $\psi_1, \psi_2 \in \mathbb{R}$,

$$\begin{aligned} &s_1 e^{\sqrt{-1}(\psi_2+\psi_1)} + s_2 e^{\sqrt{-1}(\psi_2-\psi_1)} \\ &= e^{\sqrt{-1}\psi_2}\big(s_1 e^{\sqrt{-1}\psi_1} + s_2 e^{-\sqrt{-1}\psi_1}\big) \\ &= e^{\sqrt{-1}\psi_2}\big((s_1+s_2)\cos\psi_1 + \sqrt{-1}(s_1-s_2)\sin\psi_1\big). \end{aligned} \tag{3.15}$$

Let $a \in [s_1 - s_2, s_1 + s_2]$. Since

$$\frac{1 - \big(\frac{s_1-s_2}{a}\big)^2}{\big(\frac{s_1+s_2}{a}\big)^2 - \big(\frac{s_1-s_2}{a}\big)^2} \in [0, 1]$$

by $0 < \frac{s_1-s_2}{a} \le 1 \le \frac{s_1+s_2}{a}$, $\frac{s_1-s_2}{a} < \frac{s_1+s_2}{a}$, take

$$\psi_1 := \cos^{-1}\left(\sqrt{\frac{1 - \big(\frac{s_1-s_2}{a}\big)^2}{\big(\frac{s_1+s_2}{a}\big)^2 - \big(\frac{s_1-s_2}{a}\big)^2}}\right) \in \Big[0, \frac{\pi}{2}\Big].$$

Then

$$\begin{aligned} &\Big(\frac{s_1+s_2}{a}\cos\psi_1\Big)^2 + \Big(-\frac{s_1-s_2}{a}\sin\psi_1\Big)^2 \\ &= \Big(\frac{s_1+s_2}{a}\Big)^2\cos^2\psi_1 + \Big(\frac{s_1-s_2}{a}\Big)^2\sin^2\psi_1 \\ &= \Big(\frac{s_1+s_2}{a}\Big)^2\cos^2\psi_1 + \Big(\frac{s_1-s_2}{a}\Big)^2\big(1-\cos^2\psi_1\big) \end{aligned}$$

$$= \Big(\frac{s_1-s_2}{a}\Big)^2 + \Big(\Big(\frac{s_1+s_2}{a}\Big)^2 - \Big(\frac{s_1-s_2}{a}\Big)^2\Big)\cos^2\psi_1$$
$$= \Big(\frac{s_1-s_2}{a}\Big)^2 + \Big(\Big(\frac{s_1+s_2}{a}\Big)^2 - \Big(\frac{s_1-s_2}{a}\Big)^2\Big)\frac{1-\big(\frac{s_1-s_2}{a}\big)^2}{\big(\frac{s_1+s_2}{a}\big)^2-\big(\frac{s_1-s_2}{a}\big)^2}$$
$$= 1,$$

so

$$^\exists\psi_2 \in \Big[\frac{3}{2}\pi, 2\pi\Big] \text{ s.t. } \begin{cases} \dfrac{s_1+s_2}{a}\cos\psi_1 = \cos\psi_2, \\ -\dfrac{s_1-s_2}{a}\sin\psi_1 = \sin\psi_2. \end{cases}$$

By (3.15),

$$\begin{aligned} &s_1 e^{\sqrt{-1}(\psi_2+\psi_1)} + s_2 e^{\sqrt{-1}(\psi_2-\psi_1)} \\ &= e^{\sqrt{-1}\psi_2}\big((s_1+s_2)\cos\psi_1 + \sqrt{-1}(s_1-s_2)\sin\psi_1\big) \\ &= e^{\sqrt{-1}\psi_2}a\big(\cos\psi_2 - \sqrt{-1}\sin\psi_2\big) \\ &= ae^{\sqrt{-1}\psi_2}e^{-\sqrt{-1}\psi_2} \\ &= a. \end{aligned}$$

For a general $z \in \mathbb{C}$, $s_1 - s_2 \le |z| \le s_1 + s_2$, let $z = |z|e^{\sqrt{-1}\psi}$ ($\psi \in \mathbb{R}$), and from the above take θ_1, θ_2 so that $s_1 e^{\sqrt{-1}\theta_1} + s_2 e^{\sqrt{-1}\theta_2} = |z|$. Then

$$\begin{aligned} s_1 e^{\sqrt{-1}(\theta_1+\psi)} + s_2 e^{\sqrt{-1}(\theta_2+\psi)} &= \big(s_1 e^{\sqrt{-1}\theta_1} + s_2 e^{\sqrt{-1}\theta_2}\big)e^{\sqrt{-1}\psi} \\ &= |z|e^{\sqrt{-1}\psi} \\ &= z. \end{aligned}$$

Hence it is seen that

$$\big\{z;\ s_1 - s_2 \le |z| \le s_1 + s_2\big\} \subset \big\{s_1 z_1 + s_2 z_2;\ z_1, z_2 \in S\big\},$$

so that the opposite inclusion '$\supset$' is valid for $n = 2$.
2° Assume that the opposite inclusion '$\supset$' holds for $n = \nu$ (≥ 2), i.e., that for $^\forall s_1, \dots, {}^\forall s_\nu > 0$,

$$\Big\{z;\ \max_{1\le k\le\nu}\Big(s_k - \sum_{i\ne k} s_i\Big)^+ \le |z| \le s_1 + \cdots + s_\nu\Big\} \subset \Big\{\sum_{k=1}^{\nu} s_k z_k;\ z_1, \dots, z_\nu \in S\Big\}.$$

Let $s_1, \dots, s_{\nu+1} > 0$, and $z \in \mathbb{C}$ satisfy

$$\max_{1\le k\le\nu+1}\Big(s_k - \sum_{i\ne k} s_i\Big)^+ \le |z| \le s_1 + \cdots + s_{\nu+1}.$$

In case $^\exists k$ s.t. $s_k > \sum_{i\ne k} s_i$,

$$s_l \le \sum_{i\ne l} s_i \quad (^\forall l \ne k).$$

For if not so, the following implications hold:

$$\begin{aligned}{}^{\exists}l \neq k \text{ s.t. } s_l > \sum_{i\neq l} s_i \Rightarrow s_k > \sum_{i\neq k} s_i = s_l + \sum_{i\notin\{k,l\}} s_i &> \sum_{i\neq l} s_i + \sum_{i\notin\{k,l\}} s_i \\ &= s_k + 2\sum_{i\notin\{k,l\}} s_i \\ &> s_k.\end{aligned}$$

Thus

$$\begin{aligned}\max_{1\le l\le \nu+1}\Big(s_l - \sum_{i\neq l} s_i\Big)^+ &= \Big(s_k - \sum_{i\neq k} s_i\Big)^+ \vee \max_{l\in\{1,\ldots,\nu+1\}\setminus\{k\}}\Big(s_l - \sum_{i\neq l} s_i\Big)^+ \\ &= s_k - \sum_{i\neq k} s_i > 0,\end{aligned}$$

so $0 < s_k - \sum_{i\neq k} s_i \le |z| \le s_k + \sum_{i\neq k} s_i$. From 1°, it follows that

$$\begin{aligned}{}^{\exists}\theta,\ {}^{\exists}\psi \text{ s.t. } & s_k e^{\sqrt{-1}\theta} + \Big(\sum_{i\neq k} s_i\Big)e^{\sqrt{-1}\psi} = z \\ & \qquad \| \\ & s_k e^{\sqrt{-1}\theta} + \sum_{i\neq k} s_i e^{\sqrt{-1}\psi}.\end{aligned}$$

In case $s_k \le \sum_{i\neq k} s_i$ (${}^{\forall}k = 1,\ldots,\nu+1$),

$$\max_{1\le k\le \nu+1}\Big(s_k - \sum_{i\neq k} s_i\Big)^+ = 0,$$

thus $|z| \le s_1 + \cdots + s_{\nu+1}$. If ${}^{\exists}l$ s.t. $0 \le \sum_{i\neq l} s_i - s_l \le |z|$, then $0 \le \sum_{i\neq l} s_i - s_l \le |z| \le \sum_{i\neq l} s_i + s_l$. By 1°,

$$\begin{aligned}{}^{\exists}\theta,\ {}^{\exists}\psi \text{ s.t. } & \Big(\sum_{i\neq l} s_i\Big)e^{\sqrt{-1}\theta} + s_l e^{\sqrt{-1}\psi} = z \\ & \qquad \| \\ & \sum_{i\neq l} s_i e^{\sqrt{-1}\theta} + s_l e^{\sqrt{-1}\psi}.\end{aligned}$$

If $\sum_{i\neq l} s_i - s_l > |z|$ (${}^{\forall}l = 1,\ldots,\nu+1$), then

$$\left|z - s_{l_0}e^{\sqrt{-1}(\psi+\pi)}\right| = \left||z|e^{\sqrt{-1}\psi} - s_{l_0}e^{\sqrt{-1}\psi}e^{\sqrt{-1}\pi}\right| = |z| + s_{l_0} < \sum_{i\neq l_0} s_i$$

where $s_{l_0} = \max_{1\le l\le\nu+1} s_l$ and $z = |z|e^{\sqrt{-1}\psi}$ ($\psi \in \mathbb{R}$). On the other hand,

$$|z| + s_{l_0} \ge \max_{l\neq l_0}\Big(s_l - \sum_{i\notin\{l,l_0\}} s_i\Big)^+$$

by $|z| + s_{l_0} \geq s_{l_0} \geq s_l > s_l - \sum_{i \notin \{l, l_0\}} s_i$ ($^\forall l \neq l_0$). By combining these,

$$\max_{l \neq l_0} \Big(s_l - \sum_{i \notin \{l, l_0\}} s_i \Big)^+ \leq \big| z - s_{l_0} e^{\sqrt{-1}(\psi+\pi)} \big| < \sum_{i \neq l_0} s_i .$$

Our assumption of induction shows that

$$^\exists\theta_1, \overset{l_0}{\cdot\overset{\vee}{\cdot}\cdot}, {}^\exists\theta_{\nu+1} \text{ s.t. } \sum_{l \neq l_0} s_l e^{\sqrt{-1}\theta_l} = z - s_{l_0} e^{\sqrt{-1}(\psi+\pi)},$$

i.e.,

$$\sum_{l \neq l_0} s_l e^{\sqrt{-1}\theta_l} + s_{l_0} e^{\sqrt{-1}(\psi+\pi)} = z.$$

Hence it is seen that

$$z \in \left\{ \sum_{k=1}^{\nu+1} s_k z_k ;\ z_1, \ldots, z_{\nu+1} \in S \right\},$$

so that the opposite inclusion '$\supset$' is valid for $n = \nu + 1$.

By induction, the proof of the lemma is complete. ∎

Proof of the second assertion of Claim 3.4. For simplicity, put $F(z) = -\log(1-z)$, $|z| < 1$. By (3.3),

$$|F(z) - z| = \Big| \sum_{m=2}^{\infty} \frac{z^m}{m} \Big| \leq \sum_{m=2}^{\infty} \frac{|z|^m}{m} \leq \frac{|z|^2}{2} \sum_{m=0}^{\infty} |z|^m = \frac{|z|^2}{2(1-|z|)}. \tag{3.16}$$

We divide the proof into two steps:

$\underline{1^\circ}$ $\sum_{p:\text{prime}} \frac{1}{p} = \infty$.

∵ From Claim 4.3 below and (3.2), it follows that for $s > 1$,

$$\begin{aligned} \zeta(s) &= \prod_{p:\text{prime}} \frac{1}{1 - \frac{1}{p^s}} = \prod_{p:\text{prime}} e^{F(\frac{1}{p^s})} \\ &= e^{\sum_{p:\text{prime}} F(\frac{1}{p^s})} = e^{\sum_{p:\text{prime}} \left(F(\frac{1}{p^s}) - \frac{1}{p^s} \right)} e^{\sum_{p:\text{prime}} \frac{1}{p^s}} . \end{aligned}$$

By (3.16),

$$\begin{aligned} e^{\sum_{p:\text{prime}} \frac{1}{p^s}} = \zeta(s) e^{-\sum_{p:\text{prime}} \left(F(\frac{1}{p^s}) - \frac{1}{p^s} \right)} &\geq \zeta(s) e^{-\sum_{p:\text{prime}} \left| F(\frac{1}{p^s}) - \frac{1}{p^s} \right|} \\ &\geq \zeta(s) e^{-\sum_{p:\text{prime}} \frac{\frac{1}{p^{2s}}}{2(1-\frac{1}{p^s})}} \\ &\geq \zeta(s) e^{-\sum_{p:\text{prime}} \frac{1}{p^2}} . \end{aligned}$$

Since $\lim_{s \searrow 1} \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n}$ [∵ By the monotone convergence theorem] $= \infty$, we have $\sum_{p:\text{prime}} \frac{1}{p} = \infty$.

2° Let $\frac{1}{2}<\sigma\le 1$. Fix $^\forall\varepsilon>0$ and $^\forall z_0\in\mathbb{C}$. Since $\sum\limits_{p:\text{prime}}\frac{1}{p^{2\sigma}}<\infty$,

$$^\exists N_0\in\mathbb{N}\ \text{ s.t. }\ \sum_{i=N_0+1}^{\infty}\frac{1}{p_i^{2\sigma}}<2\Big(1-\frac{1}{2^\sigma}\Big)\varepsilon.$$

Put $z_1:=\sum\limits_{i=1}^{N_0}F(\frac{1}{p_i^\sigma})\in\mathbb{R}$. Since $\sum\limits_{p:\text{prime}}\frac{1}{p^\sigma}=\infty$ $\big[\because$ By 1°, $\sum_{p:\text{prime}}\frac{1}{p^\sigma}\ge\sum_{p:\text{prime}}\frac{1}{p}=\infty\big]$ and $\sum\limits_{i=N+1}^{\infty}F\big(\frac{e(-\log p_i)}{p_i^\sigma}\big)\to 0$ $\mathbf{P}$-a.e. as $N\to\infty$ $\big[\because$ Theorem 3.1$\big]$,

$$^\exists N_1\in\mathbb{N}\cap(N_0+1,\infty)\ \text{ s.t. }\ \begin{cases}\displaystyle\sum_{i=N_0+1}^{N_1}\frac{1}{p_i^\sigma}>|z_1-z_0|\vee\frac{2}{2^\sigma},\\ \displaystyle\mathbf{P}\Big(\Big|\sum_{i=N_1+1}^{\infty}F\Big(\frac{e(-\log p_i)}{p_i^\sigma}\Big)\Big|<\varepsilon\Big)>0.\end{cases}$$

We observe that for $N_0+1\le{}^\forall j\le N_1$,

$$\sum_{i\in\{N_0+1,\dots,N_1\}\setminus\{j\}}\frac{1}{p_i^\sigma}=\sum_{i=N_0+1}^{N_1}\frac{1}{p_i^\sigma}-\frac{1}{p_j^\sigma}>\frac{2}{2^\sigma}-\frac{1}{p_j^\sigma}>\frac{2}{p_j^\sigma}-\frac{1}{p_j^\sigma}=\frac{1}{p_j^\sigma}.$$

Thus

$$\max_{N_0+1\le j\le N_1}\Big(\frac{1}{p_j^\sigma}-\sum_{i\in\{N_0+1,\dots,N_1\}\setminus\{j\}}\frac{1}{p_i^\sigma}\Big)^+=0\le|z_1-z_0|<\sum_{i=N_0+1}^{N_1}\frac{1}{p_i^\sigma}.$$

Apply Lemma 3.2. We can take $\theta_{N_0+1},\dots,\theta_{N_1}\in[0,2\pi)$ such that

$$\sum_{i=N_0+1}^{N_1}\frac{1}{p_i^\sigma}e^{\sqrt{-1}\theta_i}=z_0-z_1.$$

Then

$$\begin{aligned}&\Big|\sum_{i=1}^{N_1}F\Big(\frac{e^{\sqrt{-1}\theta_i}}{p_i^\sigma}\Big)-z_0\Big|\quad\big[\text{where }\theta_1=\cdots=\theta_{N_0}:=0\big]\\
&=\Big|\sum_{i=1}^{N_0}F\Big(\frac{1}{p_i^\sigma}\Big)+\sum_{i=N_0+1}^{N_1}F\Big(\frac{e^{\sqrt{-1}\theta_i}}{p_i^\sigma}\Big)-z_0\Big|\\
&=\Big|z_1+\sum_{i=N_0+1}^{N_1}\Big(F\Big(\frac{e^{\sqrt{-1}\theta_i}}{p_i^\sigma}\Big)-\frac{e^{\sqrt{-1}\theta_i}}{p_i^\sigma}\Big)+\sum_{i=N_0+1}^{N_1}\frac{e^{\sqrt{-1}\theta_i}}{p_i^\sigma}-z_0\Big|\\
&=\Big|z_1+\sum_{i=N_0+1}^{N_1}\Big(F\Big(\frac{e^{\sqrt{-1}\theta_i}}{p_i^\sigma}\Big)-\frac{e^{\sqrt{-1}\theta_i}}{p_i^\sigma}\Big)+z_0-z_1-z_0\Big|\end{aligned}$$

$$
\begin{aligned}
&= \left|\sum_{i=N_0+1}^{N_1}\left(F\Big(\frac{e^{\sqrt{-1}\theta_i}}{p_i^\sigma}\Big) - \frac{e^{\sqrt{-1}\theta_i}}{p_i^\sigma}\right)\right| \\
&\le \sum_{i=N_0+1}^{N_1}\left|F\Big(\frac{e^{\sqrt{-1}\theta_i}}{p_i^\sigma}\Big) - \frac{e^{\sqrt{-1}\theta_i}}{p_i^\sigma}\right| \\
&\le \sum_{i=N_0+1}^{N_1} \frac{\frac{1}{p_i^{2\sigma}}}{2(1-\frac{1}{p_i^\sigma})} \quad \big[\because (3.16)\big] \\
&\le \frac{1}{2(1-\frac{1}{2^\sigma})}\sum_{i=N_0+1}^{N_1}\frac{1}{p_i^{2\sigma}} \\
&< \frac{1}{2(1-\frac{1}{2^\sigma})}\sum_{i=N_0+1}^{\infty}\frac{1}{p_i^{2\sigma}} \\
&< \varepsilon \quad \big[\because \text{By the choice of } N_0\big].
\end{aligned}
\tag{3.17}
$$

Now we put $\Omega_0 \in \mathcal{B}_{\mathrm{K}}(\mathbb{R}^{\mathbb{B}})$ as

$$
\begin{aligned}
\Omega_0 := \bigcap_{i=1}^{N_1}&\left\{\left|F\Big(\frac{e(-\log p_i)}{p_i^\sigma}\Big) - F\Big(\frac{e^{\sqrt{-1}\theta_i}}{p_i^\sigma}\Big)\right| < \frac{\varepsilon}{N_1}\right\} \\
&\cap \left\{\left|\sum_{i=N_1+1}^{\infty} F\Big(\frac{e(-\log p_i)}{p_i^\sigma}\Big)\right| < \varepsilon\right\}.
\end{aligned}
$$

Note that

$$
\mathbf{P}\left(\left|F\Big(\frac{e(-\log p_i)}{p_i^\sigma}\Big) - F\Big(\frac{e^{\sqrt{-1}\theta_i}}{p_i^\sigma}\Big)\right| < \frac{\varepsilon}{N_1}\right) > 0, \quad 1 \le i \le N_1.
$$

Because, by Claim 3.1,

$$
\begin{aligned}
\text{L.H.S.} &= \frac{1}{2\pi}\int_0^{2\pi} \mathbf{1}_{\left|F\left(\frac{e^{\sqrt{-1}t}}{p_i^\sigma}\right) - F\left(\frac{e^{\sqrt{-1}\theta_i}}{p_i^\sigma}\right)\right| < \frac{\varepsilon}{N_1}}\, dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathbf{1}_{\left|F\left(\frac{e^{\sqrt{-1}(t+\theta_i)}}{p_i^\sigma}\right) - F\left(\frac{e^{\sqrt{-1}\theta_i}}{p_i^\sigma}\right)\right| < \frac{\varepsilon}{N_1}}\, dt \\
&\ge \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathbf{1}_{|t| < (p_i^\sigma - 1)\frac{\varepsilon}{N_1}}\, dt \\
&\qquad \left[\begin{array}{l} \because \text{ Since } |F(w_1) - F(w_2)| \le \frac{|w_1 - w_2|}{1 - |w_1| \vee |w_2|}, \\ \left|F\Big(\frac{e^{\sqrt{-1}(t+\theta_i)}}{p_i^\sigma}\Big) - F\Big(\frac{e^{\sqrt{-1}\theta_i}}{p_i^\sigma}\Big)\right| \le \frac{\frac{1}{p_i^\sigma}}{1-\frac{1}{p_i^\sigma}}|e^{\sqrt{-1}t} - 1| \le \frac{|t|}{p_i^\sigma - 1} \end{array}\right] \\
&> 0.
\end{aligned}
$$

Thus, by the independence of $\left\{F\big(\frac{e(-\log p_i)}{p_i^\sigma}\big)\right\}_{i=1}^{\infty}$, this, together with the choice of N_1, implies that

$$\mathbf{P}(\Omega_0) = \prod_{i=1}^{N_1} \mathbf{P}\Big(\Big|F\Big(\frac{e(-\log p_i)}{p_i^\sigma}\Big) - F\Big(\frac{e^{\sqrt{-1}\theta_i}}{p_i^\sigma}\Big)\Big| < \frac{\varepsilon}{N_1}\Big)$$
$$\times \mathbf{P}\Big(\Big|\sum_{i=N_1+1}^{\infty} F\Big(\frac{e(-\log p_i)}{p_i^\sigma}\Big)\Big| < \varepsilon\Big)$$
$$> 0.$$

On the other hand, we observe that on Ω_0,

$$\begin{aligned}
\Big|\sum_{i=1}^{\infty} F\Big(\frac{e(-\log p_i)}{p_i^\sigma}\Big) - z_0\Big| &= \Big|\sum_{i=1}^{N_1}\Big(F\Big(\frac{e(-\log p_i)}{p_i^\sigma}\Big) - F\Big(\frac{e^{\sqrt{-1}\theta_i}}{p_i^\sigma}\Big)\Big) \\
&\quad + \sum_{i=1}^{N_1} F\Big(\frac{e^{\sqrt{-1}\theta_i}}{p_i^\sigma}\Big) - z_0 + \sum_{i=N_1+1}^{\infty} F\Big(\frac{e(-\log p_i)}{p_i^\sigma}\Big)\Big| \\
&= \sum_{i=1}^{N_1}\Big|F\Big(\frac{e(-\log p_i)}{p_i^\sigma}\Big) - F\Big(\frac{e^{\sqrt{-1}\theta_i}}{p_i^\sigma}\Big)\Big| \\
&\quad + \Big|\sum_{i=1}^{N_1} F\Big(\frac{e^{\sqrt{-1}\theta_i}}{p_i^\sigma}\Big) - z_0\Big| + \Big|\sum_{i=N_1+1}^{\infty} F\Big(\frac{e(-\log p_i)}{p_i^\sigma}\Big)\Big| \\
&< N_1 \cdot \frac{\varepsilon}{N_1} + \varepsilon + \varepsilon \quad [\because \text{ By (3.17)}] \\
&= 3\varepsilon.
\end{aligned}$$

Therefore

$$\begin{array}{c}
\mathbf{P}\Big(\Big|\sum_{i=1}^{\infty} F\Big(\frac{e(-\log p_i)}{p_i^\sigma}\Big) - z_0\Big| < 3\varepsilon\Big) \geq \mathbf{P}(\Omega_0) > 0. \\
\| \\
p_\sigma\Big(\big\{(x, y); |(x, y) - (\mathrm{Re}\, z_0, \mathrm{Im}\, z_0)| < 3\varepsilon\big\}\Big)
\end{array}$$

Since $\varepsilon > 0$ and $z_0 \in \mathbb{C}$ are arbitrary, the second assertion follows at once. ∎

Before closing this chapter, we introduce some known results for the distribution of $p_\sigma(dxdy)$. The following is due to Jessen-Wintner [16]:

Fact 3.1 (i) *$p_\sigma(dxdy)$ is absolutely continuous w.r.t. the 2-dimensional Lebesgue measure, and its probability density $p_\sigma(x, y)$ is smooth.*
(ii) *When $\frac{1}{2} < \sigma \leq 1$, $p_\sigma(x, y) > 0$ ($^\forall(x, y) \in \mathbb{R}^2$). Also, when $\frac{1}{2} < \sigma < 1$, $p_\sigma(x, y)$ is real analytic.*
(iii) *For $^\forall\alpha, {}^\forall\beta \in \{0, 1, 2, \ldots\}$ and $^\forall\lambda > 0$,*

$$\Big|\Big(\frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial y^\beta} p_\sigma\Big)(x, y)\Big| = O\big(e^{-\lambda(x^2+y^2)}\big) \quad \text{as } x^2 + y^2 \to \infty.$$

Claim 3.3 is obvious from Fact 3.1(i).

For $l > 0$, put

$$R(l) = \{z \in \mathbb{C}; |\operatorname{Re} z|,\ |\operatorname{Im} z| \le l\} = \{x + \sqrt{-1}\,y; |x|,\ |y| \le l\}.$$

As $l \to \infty$,

$$\mathbf{P}\left(\sum_p -\log\big(1 - \frac{e(-\log p)}{p^\sigma}\big) \in \mathbb{C} \setminus R(l)\right) \to 0, \tag{3.18}$$

since $R(l) \nearrow \mathbb{C}$. When $\sigma > 1$, it is seen from Claim 3.4 that

$$\mathbf{P}\left(\sum_p -\log\big(1 - \frac{e(-\log p)}{p^\sigma}\big) \in \mathbb{C} \setminus R(l)\right) = 0, \quad l \gg 1.$$

So (3.18) is valid without taking the limit. On the other hand, when $\frac{1}{2} < \sigma \le 1$,

$$\mathbf{P}\left(\sum_p -\log\big(1 - \frac{e(-\log p)}{p^\sigma}\big) \in \mathbb{C} \setminus R(l)\right) > 0, \quad {}^\forall l > 0.$$

Thus we have a question of determining the speed of the convergence (3.18). This has been solved by Hattori-Matsumoto [14]:

Fact 3.2 *When* $\frac{1}{2} < \sigma < 1$,

$$\lim_{l\to\infty} \frac{1}{(1-\sigma) l^{\frac{1}{1-\sigma}} (\log l)^{\frac{\sigma}{1-\sigma}}} \log \mathbf{P}\left(\sum_p -\log\big(1 - \frac{e(-\log p)}{p^\sigma}\big) \in \mathbb{C} \setminus R(l)\right)$$
$$= -\left(\frac{1-\sigma}{\sigma}\int_0^\infty \log I_0(y^{-\sigma})dy\right)^{-\frac{\sigma}{1-\sigma}}.$$

Here

$$I_0(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{x\cos u} du.$$

$I_0(\cdot)$ *is the* 0*th modified Bessel function, i.e.,*

$$I_0(x) = \sum_{m=0}^\infty \frac{1}{(m!)^2}\left(\frac{x}{2}\right)^{2m}.$$

The case where $\sigma = 1$ remains open, i.e., when $\sigma = 1$, we have no limit theorem as above.

For the proofs of Facts 3.1 and 3.2, we need much efforts [cf. [33]].