Chapter 7

Behavior of bicharacteristics

7.1 Results

In this chapter we prove the next result which was proved in [41] for the case that the codimension of Σ is 3 and in [3], [47] in full generality.

Theorem 7.1.1 ([41], [3], [47]) *The following assertions are equivalent.*

(i) *p admits an elementary decomposition,*

(ii) *there is no null bicharacteristic of p issuing from a simple characteristic having a limit point in Σ.*

Thanks to Lemma 3.1.1 it suffices to prove

Theorem 7.1.2 ([3], [47]) *There exists a null bicharacteristic having a limit point in the doubly characteristic set if the condition* (i) *in Theorem 3.5.1 fails.*

In [41], the existence proof of such a null bicharacteristic is based on a peculiarity of 2-dimensional autonomous system. Since such a null bicharacteristic, if exists, is essentially unique it seems to be hard to show the existence of such bicharacteristic by topological arguments as in [41] when the codimension of Σ is greater than 3. In [47] we take a completely different method which we follow here assuming

$$\mathrm{Tr}^+ F_p(\rho) = 0, \quad \rho \in \Sigma. \tag{7.1.1}$$

We refer to [47] for the proof of general case without assuming (7.1.1).

The condition (7.1.1) implies

Lemma 7.1.1 *Assume* (7.1.1)*. Then we have*

$$\mathrm{rank}\,(\{\phi_i, \phi_j\}(\rho))_{0\le i,j\le r} = 2. \tag{7.1.2}$$

Proof: Let us recall

$$p=-\phi_0^2+\sum_{j=1}^r\phi_j^2,\quad \Sigma=\{\phi_j=0\}.$$

Then we have $p_\rho=-\theta_0^2+\sum_{j=1}^r\theta_j^2$ where $\theta_j(X)=d\phi_j(X)$. From the assumption (7.1.1) one can take a symplectic change of coordinates $X\mapsto TX$ such that

$$-\theta_0(TX)^2+\sum_{j=1}^r\theta_j(TX)^2=-(\xi_0^2+2\xi_0\xi_1+x_1^2)/\sqrt{2}+\sum_{j=1}^{\ell}\xi_j^2$$

where $X=(x,\xi)$. Denoting $\{\xi_j=0,0\le j\le\ell,x_1=0\}=\{\psi_j=0\}$ it is clear that $\mathrm{rank}\,(\{\psi_i,\psi_j\})=2$. Since $\psi(X)=M\theta(TX)$ with a non singular matrix M and hence $\mathrm{rank}\,(\{\psi_i,\psi_j\})=\mathrm{rank}\,(\{\theta_i,\theta_j\})$ which proves the assertion. □

Recall that p takes the form

$$p=-\xi_0^2+\sum_{j=1}^r\phi_j^2. \tag{7.1.3}$$

We set $\phi_0=\xi_0$ as before. We repeat similar arguments as in Section 4.5. Consider $\{\phi_0,\phi_j\}(\rho)$, $j=1,...,r$ and suppose $\{\phi_0,\phi_j\}(\rho)=0$, $j=1,...,r$. With $q=\sum_{j=1}^r\phi_j^2$ we see easily that $\mathrm{Ker}\,F_p^2\cap\mathrm{Im}\,F_p^2=\mathrm{Ker}\,F_q^2\cap\mathrm{Im}\,F_q^2$ but $\mathrm{Ker}\,F_q^2\cap\mathrm{Im}\,F_q^2=\{0\}$ because q is non negative, which contradicts (5.1.1). Thus this can not happen. Considering $(\tilde{\phi}_j)_{1\le j\le r}=O(\phi_j)_{1\le j\le r}$ with a suitable smooth orthogonal O we may assume that $\{\phi_0,\phi_1\}(\rho)\neq 0$ and $\{\phi_0,\phi_j\}=0$ on Σ for $j=2,...,r$. We next consider $\{\phi_1,\phi_j\}(\rho)$, $j=2,...,r$. If $\{\phi_1,\phi_j\}(\rho)=0$, $j=2,...,r$ then we have

$$p_\rho(H_{\phi_1})=\sigma(H_{\phi_1},F_p(\rho)H_{\phi_1})=-\{\phi_0,\phi_1\}(\rho)^2<0$$

and p would be effectively hyperbolic at ρ by Corollary 2.3.1. This shows that there is $2\le j\le r$ with $\{\phi_1,\phi_j\}(\rho)\neq 0$. Repeating the same arguments as above, leaving ϕ_0, ϕ_1 unchanged, we may assume that $\{\phi_1,\phi_2\}(\rho)\neq 0$ and $\{\phi_1,\phi_j\}=0$ on Σ for $j=3,...,r$. We now consider $(\{\phi_i,\phi_j\})_{0\le i,j\le r}$. It is clear that the first two rows are linearly independent. From Lemma 7.1.1 it follows that the j-th row is a linear combination of the first two rows for $j\ge 3$. Then we conclude that the third row is proportional to the first row and j-th row is zero for $j\ge 4$. Thus we have $\{\phi_2,\phi_j\}=0$, $j=3,...,r$ and $(\{\phi_i,\phi_j\})_{3\le i,j\le r}=O$ on Σ.

Lemma 7.1.2 *Assume* (7.1.1). *Then we can assume that*

$$p=-(\xi_0+\phi_1)(\xi_0-\phi_1)+\phi_2^2+\sum_{j=3}^r\phi_j^2,$$

$$\{\xi_0-\phi_1,\phi_j\}=0\ \text{ on }\ \Sigma,\ j=1,2,...,r,$$

$$\{\phi_2,\phi_j\}=0,\ j=3,...,r,\ \ (\{\phi_i,\phi_j\})_{3\le i,j\le r}=O\ \text{ on }\ \Sigma.$$

Proof: It is enough to repeat the same arguments as in Section 4.3. For completeness we give a proof. It remains to prove the second assertion. Let us write

$$p=-\xi_0^2+\phi_1^2+\phi_2^2+\sum_{j=3}^{r}\phi_j^2=r+q,\quad q=\sum_{j=3}^{r}\phi_j^2.$$

Since $\{\phi_i,\phi_j\}=0$ for $i=0,1,2$ and $j\geq 3$ and $\operatorname{Im}F_r\cap\operatorname{Im}F_q=\{0\}$ then we see that, noting that $\operatorname{Im}F_q^2\cap\operatorname{Ker}F_q^2=\{0\}$

$$\operatorname{Im}F_p^2\cap\operatorname{Ker}F_p^2=\operatorname{Im}F_r^2\cap\operatorname{Ker}F_r^2.$$

This shows that $\operatorname{Im}F_r^2\cap\operatorname{Ker}F_r^2\neq\{0\}$. Let $0\neq X=aH_{\phi_0}+bH_{\phi_1}+cH_{\phi_2}\in\operatorname{Im}F_r^2\cap\operatorname{Ker}F_r^2$. Since $X=F_r(AH_{\phi_0}+BH_{\phi_1}+CH_{\phi_2})$ it follows that

$$a=-2B\{\phi_0,\phi_1\},\ \ c=2B\{\phi_2,\phi_1\}.$$

From $F_r^2X=0$ we see that

$$b\big(\{\phi_0,\phi_1\}^2-\{\phi_1,\phi_2\}^2\big)=0,\ B\big(\{\phi_0,\phi_1\}^2-\{\phi_1,\phi_2\}^2\big)=0.$$

If $\{\phi_0,\phi_1\}^2-\{\phi_1,\phi_2\}^2\neq 0$ then we would have $X=0$ which is a contradiction. Thus we have proved

$$\{\xi_0,\phi_1\}^2=\{\phi_1,\phi_2\}^2\ \text{ on }\ \Sigma.$$

We may assume that $\{\xi_0,\phi_1\}=-\{\phi_1,\phi_2\}$ so that $\{\xi_0-\phi_2,\phi_1\}=0$ on Σ. It is clear that $\{\xi_0-\phi_2,\phi_j\}=0$, $j=2,...,r$. Writing

$$p=-(\xi_0+\phi_2)(\xi_0-\phi_2)+\phi_1^2+\sum_{j=3}^{r}\phi_j^2$$

and exchanging ϕ_1 and ϕ_2 we get the desired assertion. □

7.2 Hamilton system and formal solutions

To simplify notations little bit let us set $\Xi_0=\xi_0-\phi_1$, $X_0=x_0$ and extend to a full symplectic coordinates (X,Ξ) (see Chapter 10, Appendix). Switching the notation from (X,Ξ) to (x,ξ) one can write

$$p=-\xi_0(\xi_0+2\phi_1)+\sum_{j=2}^{r}\phi_j^2$$

where $\{\xi_0,\phi_j\}=0$ on Σ, $1\leq j\leq r$. Note that

$$F_pX=-\frac{1}{2}\sigma(H_{\xi_0},X)H_{\xi_0+2\phi_1}-\frac{1}{2}\sigma(H_{\xi_0+2\phi_1},X)H_{\xi_0}+\sum_{j=2}^{r}\sigma(H_{\phi_j},X)H_{\phi_j},$$

$$F_p^2X=-\sigma(H_{\phi_2},X)\sigma(H_{\phi_1},H_{\phi_2})H_{\xi_0}-\sigma(H_{\xi_0},X)\sigma(H_{\phi_2},H_{\phi_1})H_{\phi_2}$$

and hence $\operatorname{Ker} F_p^2 \cap \operatorname{Im} F_p^2 = \langle H_{\xi_0}, H_{\phi_2}\rangle$. Since $F_p H_{\phi_2} = -\sigma(H_{\phi_1}, H_{\phi_2}) H_{\xi_0}$ one can take $S = \phi_2$. Since $H_S^3 p = \{\phi_2, \{\phi_2, \xi_0\}\}$ the condition $H_S^3 p(\bar\rho) \neq 0$ is equivalent to

$$\{\phi_2, \{\phi_2, \xi_0\}\}(\bar\rho) \neq 0. \tag{7.2.1}$$

From the Jacobi identity it follows that $\{\phi_2, \{\phi_j, \xi_0\}\} = 0$ on Σ for $j = 3, ..., r$ and this implies that

$$\{\phi_j, \xi_0\} = C^0_{j1}\phi_1^2 + \sum_{k=2}^{r} C^0_{jk}\phi_k, \quad j = 3, ..., r. \tag{7.2.2}$$

Let us take

$$\xi_0, x_0, \phi_1, \phi_2, \phi_3, ..., \phi_r, \psi_1, ..., \psi_\ell$$

where $r + 2 + \ell = 2n + 2$ to be a system of local coordinates around $\bar\rho$. Note that we can take ψ_j so that

$$\{\xi_0, \psi_j\} = 0, \quad \{\phi_2, \psi_j\} = 0, \quad 1 \leq j \leq \ell$$

on Σ. Indeed it is clear that one can take ψ_j independent of x_0. We next replace ψ_j by $\psi_j - c_j\phi_1$ with $c_j = \{\phi_2, \psi_j\}/\{\phi_2, \phi_1\}$ we obtain desired ψ_j.

Our Hamilton system is

$$\begin{cases} \dot x = \dfrac{\partial}{\partial \xi} p(x, \xi), \\[2mm] \dot \xi = -\dfrac{\partial}{\partial x} p(x, \xi). \end{cases}$$

Let $\gamma(s) = (x(s), \xi(s))$ be a solution to the Hamilton system and we consider $\xi_0(s)$, $x_0(s)$, $\phi_j(\gamma(s))$, $\psi_k(\gamma(s))$ then we have

$$\frac{d}{ds}\phi_j(\gamma(s)) = \{p, \phi_j\}(\gamma(s)), \ \ \frac{d}{ds}\psi_j(\gamma(s)) = \{p, \psi_j\}(\gamma(s)).$$

Let us change the parameter

$$t = \frac{1}{s}$$

so that we have

$$\frac{d}{ds} = -tD, \ \ D = t\frac{d}{dt}$$

and hence $tD(t^p F) = t^{p+1}(DF + pF)$. Let us introduce new unknowns

$$\xi_0(s) = t^4\Xi_0(t), \ x_0(s) = tX_0(t), \ \phi_1(\gamma(s)) = t^2\Phi_1(t), \ \phi_2(\gamma(s)) = t^3\Phi_2(t),$$
$$\phi_j(\gamma(s)) = t^3\Phi_j(t), \ 3 \leq j \leq r, \ \psi_j(\gamma(s)) = t^2\Psi_j(t), \ 1 \leq j \leq \ell.$$

Note that

$$dx_0/ds = -(\xi_0 + 2\phi_1) - \xi_0 = -2(\xi_0 + \phi_1)$$

which gives

$$DX_0 = -X_0 + 2\Phi_1 + O(t^2). \tag{7.2.3}$$

Consider $d\xi_0/ds = \{p, \xi_0\} = -\xi_0\{\xi_0 + 2\phi_1, \xi_0\} + 2\sum_{j=2}^r \{\phi_j, \xi_0\}\phi_j$. Noting $\{\phi_2, \xi_0\} = 0$ on Σ and (7.2.2) one can write

$$\frac{d\xi_0}{ds} = -2\xi_0 \sum_{i=1}^r C_i^{10}\phi_i + 2\sum_{i=1}^r C_i^{20}\phi_i\phi_2 + 2\sum_{j=3}^r (C_{j1}^0\phi_1^2 + \sum_{k=2}^r C_{jk}^0\phi_k)\phi_j$$

which shows

$$D\Xi_0 = -4\Xi_0 - 2C_1^{20}\Phi_1\Phi_2 + O(t). \tag{7.2.4}$$

Here we note that $\{\phi_2, \{\phi_2, \xi_0\}\} = \sum_{i=1}^r C_i^{20}\{\phi_2, \phi_i\} = C_1^{20}\{\phi_2, \phi_1\}$ on Σ. This shows that

$$C_1^{20} = \frac{\{\phi_2, \{\phi_2, \xi_0\}\}}{\{\phi_2, \phi_1\}} = \frac{H_S^3 p}{\{\phi_2, \phi_1\}} \quad \text{on } \Sigma.$$

Since $\{p, \phi_1\} = -2\{\xi_0, \phi_1\}(\xi_0 + \phi_1) + 2\sum_{j=2}^r \phi_j\{\phi_j, \phi_1\}$ and $\{\phi_j, \phi_1\} = 0$ on Σ for $j \geq 3$ then

$$D\Phi_1 = -2\Phi_1 + 2\{\phi_1, \phi_2\}\Phi_2 + O(t). \tag{7.2.5}$$

Note that $\{p, \phi_2\} = 2\{\phi_2, \xi_0\}(\xi_0 + \phi_1) - 2\{\phi_1, \phi_2\}\xi_0 + 2\sum_{j=3}^r \phi_j\{\phi_j, \phi_2\}$ and $\{\phi_2, \xi_0\} = \sum_{i=1}^r C_i^{20}\phi_i$ and $\{\phi_j, \phi_2\} = 0$ on Σ we see that

$$D\Phi_2 = -3\Phi_2 - 2C_1^{20}\Phi_1^2 + 2\{\phi_1, \phi_2\}\Xi_0 + O(t). \tag{7.2.6}$$

Let us turn to $\{p, \phi_j\}$ for $j \geq 3$. Note that $\{p, \phi_j\} = 2\{\phi_j, \xi_0\}(\xi_0 + \phi_1) - 2\{\phi_1, \phi_j\}\xi_0 + 2\sum_{k=2}^r \phi_k\{\phi_k, \phi_j\}$ and $\{\phi_j, \xi_0\} = \sum_{i=1}^r C_i^{j0}\phi_i$ we get

$$D\Phi_j = -3\Phi_j - 2C_1^{j0}\Phi_1^2 + O(t), \quad 3 \leq j \leq r. \tag{7.2.7}$$

We finally study $\{p, \psi_j\}$. We remark that $\{p, \psi_j\} = -2\{\xi_0, \psi_j\}(\xi_0 + \phi_1) - 2\{\phi_1, \psi_j\}\xi_0 + 2\sum_{k=2}^r \phi_k\{\phi_k, \psi_j\}$. Since $\{\xi_0, \psi_j\} = 0$ and $\{\phi_2, \psi_j\} = 0$ on Σ we have

$$D\Psi_j = -2\Psi_j - 2\sum_{k=3}^r \{\phi_k, \psi_j\}\Phi_k + O(t). \tag{7.2.8}$$

Since we may assume that $x_0(\bar{\rho}) = 0$, $\phi_j(\bar{\rho}) = 0$, $\psi_j(\bar{\rho}) = 0$ then one sees with $\phi = (\phi_1, ..., \phi_r)$, $\psi = (\psi_1, ..., \psi_\ell)$

$$\begin{aligned} C_1^{j0}(\rho) = C_1^{j0}(\bar{\rho}) + \sum_{|\alpha|+\beta|=1} a_{\alpha\beta}(x, \xi')(x - \bar{x})^\alpha(\xi' - \bar{\xi}')^\beta \\ = C_1^{j0}(\bar{\rho}) + \tilde{a}(x_0, \phi, \psi) \end{aligned}$$

where $\tilde{a}(0,0,0)=0$. Let us write $V=(X_0,\Phi_2,\Xi_0,\Phi_1,\Phi,\Psi)$ with $\Phi=(\Phi_3,...,\Phi_r)$ and $\Psi=(\Psi_1,...,\Psi_\ell)$ then one can write

$$C_1^{j0}(\rho)=C_1^{j0}(\bar{\rho})+tG(t,V)$$

where $G(t,0)=0$. It is also clear that

$$\{\phi_j,\phi_k\}(\rho)=\{\phi_j,\phi_k\}(\bar{\rho})+tG(t,V)$$

where $G(t,V)$ verifies again $G(t,0)=0$. To simplify notations we set

$$\kappa_j=C_1^{j0}(\bar{\rho}),\ \ \delta=\{\phi_1,\phi_2\}(\bar{\rho}),\ \ \nu_{jk}=\{\psi_j,\phi_k\}(\bar{\rho}).$$

We now summarize

Proposition 7.2.1 *We have*

$$\tag{7.2.9}\begin{cases} D\Xi_0=-4\Xi_0-2\kappa_2\Phi_1\Phi_2+tG(t,V),\\ DX_0=-X_0+2\Phi_1+tG(t,V),\\ D\Phi_1=-2\Phi_1+2\delta\Phi_2+tG(t,V),\\ D\Phi_2=-3\Phi_2-2\kappa_2\Phi_1^2+2\delta\Xi_0+tG(t,V),\\ D\Phi_j=-3\Phi_j-2\kappa_j\Phi_1^2+tG(t,V),\\ D\Psi_j=-2\Psi_j+2\sum_{k=3}^r\nu_{jk}\Phi_k+tG(t,V)\end{cases}$$

where $G(t,V)$ denotes a smooth function in (t,V) such that $G(t,0)=0$.

We first look for a formal solution to (7.2.9). Let us define the class of formal series in t and $\log 1/t$ in which we look for our formal solutions to (7.2.9).

Definition 7.2.1 *We set*

$$\mathcal{E}=\{\sum_{0\le j\le i}t^i(\log 1/t)^jV_{ij}\mid V_{ij}\in\mathbb{C}^N\},$$
$$\mathcal{E}^{\#}=\{\sum_{1\le i,0\le j\le i}t^i(\log 1/t)^jV_{ij}\mid V_{ij}\in\mathbb{C}^N\}.$$

Lemma 7.2.1 *Assume that $V\in\mathcal{E}$ satisfies (7.2.9) formally and $\Phi_2(0)\neq 0$. Then $X_0(0)$, $\Xi_0(0)$, $\Phi_j(0)$ and $\Psi_j(0)$ are uniquely determined.*

Proof: Let us set

$$X_0=\sum_{0\le j\le i}t^i(\log 1/t)^j\beta_{ij}^{(0)},\ \ \Xi_0=\sum_{0\le j\le i}t^i(\log 1/t)^j\alpha_{ij}^{(0)},$$
$$\Phi_1=\sum_{0\le j\le i}t^i(\log 1/t)^j\beta_{ij}^{(1)},\ \ \Phi_2=\sum_{0\le j\le i}t^i(\log 1/t)^j\alpha_{ij}^{(1)},$$
$$\Phi_k=\sum_{0\le j\le i}t^i(\log 1/t)^j\gamma_{ij}^{(k)},\ \ \Psi_k=\sum_{0\le j\le i}t^i(\log 1/t)^j\delta_{ij}^{(k)}.$$

Equating the constant terms of both sides of (7.2.9) one has

$$-4\alpha_{00}^{(0)} - 2\kappa_2\beta_{00}^{(1)}\alpha_{00}^{(1)} = 0, \quad -\beta_{00}^{(0)} + 2\beta_{00}^{(1)} = 0,$$
$$-2\beta_{00}^{(1)} + 2\delta\alpha_{00}^{(1)} = 0, \quad -3\alpha_{00}^{(1)} - 2\kappa_2(\beta_{00}^{(1)})^2 + 2\delta\alpha_{00}^{(0)} = 0,$$
$$-3\gamma_{00}^{(j)} - 2\kappa_j(\beta_{00}^{(1)})^2 = 0, \quad -2\delta_{00}^{(j)} + 2\sum_{k=3}^{r}\nu_{jk}\gamma_{00}^{(k)} = 0.$$

Since $\alpha_{00}^{(1)} \neq 0$ then we have $\alpha_{00}^{(1)} = -(\delta^2\kappa_2)^{-1}$. Then it is clear that $\beta_{00}^{(1)}$, $\beta_{00}^{(0)}$, $\alpha_{00}^{(0)}$ are uniquely determined and hence $\gamma_{00}^{(j)}$, $3 \leq j \leq r$ and $\delta_{00}^{(j)}$, $1 \leq j \leq \ell$ are also uniquely determined. □

We now show that there exists a formal solution $V \in \mathcal{E}$ verifying (7.2.9) and $\Phi_2(0) \neq 0$. If such a solution exists then $V(0)$ is uniquely determined by Lemma 7.2.1. Taking this fact into account let us put $\bar{V} = (\bar{X}_0, \bar{\Phi}_2, \bar{\Xi}_0, \bar{\Phi}_1, \bar{\Phi}, \bar{\Psi}) = V(0)$ and we look for a formal solution to (7.2.9) in the form $\bar{V} + V$, $V \in \mathcal{E}^{\#}$. Let us denote

$$V^I = {}^t(X_0, \Phi_2, \Xi_0, \Phi_1), \quad V^{II} = \Phi, \quad V^{III} = \Psi.$$

Then (7.2.9) becomes

$$\left\{\begin{array}{l} DV^I = A_I V^I + F_I t + G_I(t, V), \\ DV^{II} = B_{II}V^I - 3V^{II} + F_{II}t + G_{II}(t, V), \\ DV^{III} = B_{III}V^{II} - 2V^{III} + F_{III}t + G_{III}(t, V) \end{array}\right. \tag{7.2.10}$$

where

$$\left\{\begin{array}{l} G_J(t, V) = \sum_{2\leq i, 0\leq j\leq i} G_{Jij}t^i(\log 1/t)^j, \\ G_{Jij} = G_{Jij}(V_{pq} \mid q \leq p \leq i-1) \end{array}\right. \tag{7.2.11}$$

and F_J are constant vectors where $J = I, II, III$. Make more precise looks on A_I. We get

$$A_I = \left[\begin{array}{cccc} -1 & 0 & 0 & 2 \\ 0 & -3 & 2\delta & -4\kappa_2\bar{\Phi}_1 \\ 0 & -2\kappa_2\bar{\Phi}_1 & -4 & -2\kappa_2\bar{\Phi}_2 \\ 0 & 2\delta & 0 & -2 \end{array}\right]. \tag{7.2.12}$$

Let us write

$$DV = AV + tF + G(t, V) \tag{7.2.13}$$

where

$$A = \left[\begin{array}{ccc} A_I & O & O \\ B_{II} & -3I & O \\ O & B_{III} & -2I \end{array}\right].$$

Lemma 7.2.2 *We have*

$$\sigma(A) = \{-6, -4, -3, -2, -1, 1\}.$$

Proof: From the proof of Lemma 7.2.1 we see that

$$\bar{\Phi}_1 = \beta_{00}^{(1)} = \delta\alpha_{00}^{(1)} = \delta\bar{\Phi}_2, \ \ \bar{\Phi}_2\kappa_2\delta^2 = \alpha_{00}^{(1)}\kappa_2\delta^2 = -1.$$

Using these relations we get

$$\det(\lambda - A_I) = (\lambda+1)\begin{vmatrix} \lambda+3 & -2\delta & 4\kappa_2\delta\bar{\Phi}_2 \\ 2\kappa_2\delta\bar{\Phi}_2 & \lambda+4 & 2\kappa_2\bar{\Phi}_2 \\ -2\delta & 0 & \lambda+2 \end{vmatrix}$$

$$= (\lambda+1)\begin{vmatrix} \lambda+3 & -2\delta & 4\kappa_2\delta\bar{\Phi}_2 \\ 0 & \lambda+4 & \kappa_2\bar{\Phi}_2(\lambda+4) \\ -2\delta & 0 & \lambda+2 \end{vmatrix}$$

$$= (\lambda-1)(\lambda+1)(\lambda+4)(\lambda+6)$$

which proves the assertion. □

Proposition 7.2.2 *There exists a formal solution* $V \in \mathcal{E}$ *to* (7.2.1) *verifying* $\Phi_2(0) \neq 0$.

Proof: Note that (7.2.13) implies that

$$(iV_{ij} - (j+1)V_{ij+1}) = AV_{ij} + \delta_{i1}\delta_{j0}F + G_{ij} \tag{7.2.14}$$

where $G_{ij} = 0$ for $i = 0, 1$. Then we have

$$\begin{cases} (I-A)V_{11} = 0, \\ (I-A)V_{10} = V_{11} + F. \end{cases} \tag{7.2.15}$$

Choose $V_{11} \in \mathrm{Ker}\,(I-A)$ so that

$$F + V_{11} \in \mathrm{Im}\,(I-A).$$

Then we can take $V_{10} \neq 0$ so that

$$(I-A)V_{10} = F + V_{11}$$

since $\mathrm{Ker}\,(I-A) \neq \{0\}$ by Lemma 7.2.2. We turn to the case $i \geq 2$

$$(iI - A)V_{ij} = (j+1)V_{ij+1} + G_{ij}. \tag{7.2.16}$$

With $j = i$, (7.2.16) turns to

$$(iI - A)V_{ii} = G_{ii}(V_{pq} \mid q \leq p \leq i-1).$$

Since $iI - A$ is non singular for $i \geq 2$ by Lemma 7.2.2 one has

$$V_{ii} = (iI - A)^{-1}G_{ii}(V_{pq} \mid q \leq p \leq i-1).$$

Recurrently one can solve V_{ij} by

$$V_{ij} = (iI - A)^{-1}\big((j+1)V_{ij+1} + G_{ij}(V_{pq} \mid q \leq p \leq i-1)\big)$$

for $j = i-1$, $i-2$,...,0. This proves the assertion. □

7.3 A singular initial value problem and bicharacteristics

In this section we first study the next system of ordinary differential equations

$$t\frac{d}{dt}u = Ku + tL(t)u + tR(t,u) + tF \tag{7.3.1}$$

where K is a $N \times N$ constant matrix and $R(t,u)$ is a C^1 function defined in a neighborhood of $(0,0) \in \mathbb{R} \times \mathbb{C}^N$ such that

$$|R(t,u)| \le B|u|$$

for $(t,u) \in \{|t| \le T\} \times \{|u| \le CT\}$ and $L(t) \in C^1((0,T])$ which verifies $\|L(t)\|_{C((0,T])} \le B$. We assume that

$$\sigma(K) \subset \{z \in \mathbb{C} \mid \mathsf{Re}\, z < -\delta\} \tag{7.3.2}$$

where $\sigma(K)$ denotes the spectrum of K. Our aim is to prove

Proposition 7.3.1 *Equation* (7.3.1) *has a solution u such that $u(0) = 0$.*

Proof: For $h \in C([0,T])$ we set

$$\mathcal{G}[h] = \int_0^t \left(\frac{t}{s}\right)^K \frac{1}{s} h(s)ds$$

so that

$$t\frac{d}{dt}\mathcal{G}[h] = K\mathcal{G}[h] + h. \tag{7.3.3}$$

We start with

Lemma 7.3.1 *Let $h \in C([0,T])$. Assume $\delta > 1$. Then we have*

$$|\mathcal{G}[h](t)| \le C\|h\|_{C([0,t])}.$$

Proof: The assertion is clear because

$$|s^{-K}v| \le Cs^{\delta}|v|, \quad s \in (0,1), \ \ v \in \mathbb{C}^N$$

and hence

$$|\mathcal{G}[h](t)| \le \int_0^1 \left(\frac{1}{s}\right)^K \frac{1}{s}|h(ts)|ds \le C\|h\|_{C([0,t])}$$

which is the assertion. □

Using (7.3.3) we rewrite (7.3.1) as an integral equation

$$u = \mathcal{G}[tL(t)u + tR(t,u) + tF]. \tag{7.3.4}$$

Let $u_0(t) = 0$ and define $u_n(t)$ successively by

$$u_{n+1}(t) = \mathcal{G}[tR(t, u_n(t)) + tF].$$

Let us assume that

$$\left|\frac{\partial R}{\partial u}(t,u)\right| \le B \tag{7.3.5}$$

for $(t, u) \in \{|t| \le T\} \times \{|u| \le CT\}$. From

$$|tR(t, u_{n-1}) - tR(t, u_{n-2})| \le Bt|u_{n-1} - u_{n-2}|$$

one gets

$$\left|\mathcal{G}[tR(t, u_{n-1}) - tR(t, u_{n-2})]\right| \le CBT\|u_{n-1} - u_{n-2}\|_{C([0,t])}.$$

It is clear that $|\mathcal{G}[tL(t)(n_{n-1} - u_{n-2})]| \le CBT\|u_{n-1} - u_{n-2}\|_{C([0,t])}$. Set

$$W_n(t) = \|u_n - u_{n-1}\|_{C([0,t])}$$

and recall that

$$u_{n+1} - u_n = \mathcal{G}[tL(t)(u_n - u_{n-1})] + \mathcal{G}[tR(t, u_n) - tR(t, u_{n-1})].$$

It is easy to see that

$$|u_{n+1} - u_n| \le 2CBTW_n(t)$$

which gives

$$W_{n+1}(t) \le 2CBTW_n(t), \quad 0 \le t \le T.$$

Taking T small so that $2CBT < 1$ we have

$$W_n(t) \le \sum_{k=1}^{n-1}(2CBT)^k W_1(t). \tag{7.3.6}$$

This proves that $\{u_n\}$ converges in $C([0, T])$ to some $u(t) \in C([0, T])$. This proves Proposition 7.3.1. □

We now apply Proposition 7.2.2 to prove Theorem 7.1.2.

Proof of Theorem 7.1.2: By Proposition 7.2.2 there exists a non trivial formal solution to (7.2.13)

$$U = \sum_{0\le j\le i} U_{ij}t^i(\log 1/t)^j.$$

This shows that for any $m \in \mathbb{N}$ there is $N = N(m)$ such that

$$U_N = \sum_{0\le j\le i\le N} U_{ij}t^i(\log 1/t)^j$$

verifies (7.2.13) modulo $O(t^{m+1})$, that is

$$DU_N - \big(AU_N + tG(t,U_N) + tF\big) = O(t^{m+1}).$$

We look for a solution V in the form

$$U_N + t^m U.$$

Note that one can write

$$\begin{aligned} G(t,U_N + t^m U) &= G(t,U_N) + t^m \sum U_j \frac{\partial G}{\partial V_j}(t,U_N) + t^{2m} R(t,U) \\ &= G(t,U_N) + t^m L(t)U + t^{2m} R(t,U). \end{aligned}$$

It is clear that $L(t) = L + O(t\log 1/t)$ so that $L(t)$ is bounded in $[0,T]$. Since

$$D(t^m v) = t^m (D+m)v$$

substituting $U_N + t^m U$ into (7.3.1) and dividing the resulting equation by t^m one has

$$DU = -mU + AU + tL(t)U + tR(t,U) + tF. \tag{7.3.7}$$

Since it is clear that (7.3.2) is verified for large m we can now apply Proposition 7.3.1 to conclude that there exists V verifying (7.2.13). Switching to the original coordinates this shows that the Hamilton system has a solution $(x(s),\xi(s))$ such that

$$\lim_{s\to\infty} (x(s),\xi(s)) \in \Sigma.$$

Since

$$\frac{d\phi_j}{dx_0}\Big|_{x_0=0} = \Big(\frac{d\phi_j}{dt}\Big/\frac{dx_0}{dt}\Big)_{x_0=0} = \frac{t^j\Phi_j(t)}{X_0(t)}\Big|_{t=0} = 0, \quad j=1,2$$

and hence we see that $(x(s),\xi(s))$ is actually tangent to Σ.

We make some comments on the general case when

$$\mathrm{Tr}^+ F_p(\rho) \neq 0. \tag{7.3.8}$$

Assuming that condition (i) in Theorem 3.5.1 fails we look for a null bicharacteristic $(x(s),\xi(s))$ such that

$$\begin{gathered} \lim_{s\to\infty} s^2(x(s),\xi(s)) = v \neq 0, \\ v \in \mathrm{Ker}\,F_p^2 \cap \mathrm{Im}\,F_p^2, \quad 0 \neq F_p v \in \mathrm{Ker}\,F_p \cap \mathrm{Im}\,F_p^3. \end{gathered}$$

To put the above conditions in evidence, we prove that one can choose symplectic coordinates so that the line spanned by $z(\rho)$ verifying

$$z(\rho) \in \mathrm{Ker}\,F_p^2(\rho) \cap \mathrm{Im}\,F_p^2(\rho), \quad 0 \neq F_p(\rho)z(\rho) \in \mathrm{Ker}\,F_p(\rho) \cap \mathrm{Im}\,F_p^3(\rho)$$

(recall that $z(\rho)$ is unique up to a multiple factor from (3.3.3) and hence proportional to v) is given by $m_j(x,\xi)=0$ on Σ and the representation of p, in these coordinates, contains the sum of m_j^2. Then our expecting solution is assumed to satisfy approximately the Hamilton system with hamiltonian $\tilde{p}$ obtained from p removing the terms m_j^2. We write down our Hamilton system supposing that m_j were unknowns. We look for a solution $(x(s),\xi(s))$ of the Hamilton system such that $\xi(s)=O(s^{-2})$, $x'(s)=O(s^{-3})$ $(x=(x_0,x'))$ and $m_j(x(s),\xi(s))=O(s^{-4})$. To do so we repeat similar arguments in this section. We first transform thus obtained system (m_j are unknowns) to another system by the change of independent variable $t=s^{-1}$ and suitable change of unknowns. The resulting system is a coupled system of a system which has $t=0$ as a singular point of the first kind and a system which has $t=0$ as a singular point of the second kind. Here the singular point of the second kind comes from positive trace (7.3.8). The main feature of the system is that all eigenvalues of the leading term of the singular point of the second kind (the coefficient matrix of t^{-2}) are simple, pure imaginary and different from zero.

The resulting system looks like

$$
(7.3.9)\quad \begin{cases} \left(t^2\dfrac{d}{dt}-i\Lambda\right)u=-mtu+L_1(t)v+Q_1(t,u,v)+tR_1(t,u,v)+tF_1, \\ t\dfrac{d}{dt}v=-mv+Lu+L_2(t)v+Q_2(t,u,v)+tR_2(t,u,v)+tF_2 \end{cases}
$$

where $Q_j(t,u,v)$ and $R_j(t,u,v)$ are C^1 functions defined near $(0,0,0)\in\mathbb{R}\times\mathbb{C}^{N_1}\times\mathbb{C}^{N_2}$ such that

$$
\begin{cases} |Q_j(t,u,v)|\le B_{1j}(|u|^2+|v|^2), \\ |R_j(t,u,v)|\le \tilde{B}_{1j}(|u|+|v|) \end{cases}
$$

for $(t,u,v)\in\{|t|\le T\}\times\{|u|\le CT\}\times\{|v|\le CT\}$ and $L_2(t)$ is a $N_2\times N_2$ square matrix and $L_1(t)$ and L (a constant matrix) are $N_1\times N_2$ and $N_2\times N_1$ matrices respectively which verifies

$$
\|L_j(t)\|_{C([0,T])},\quad \|tL_j'(t)\|_{C([0,T])}\le B.
$$

Here Λ is a constant nonsingular real diagonal matrix

$$
\Lambda=\mathrm{diag}(\lambda_1,...,\lambda_{N_1}),\quad \lambda_j\in\mathbb{R}\setminus\{0\}.
$$

Then we have

Theorem 7.3.1 *If $m\in\mathbb{R}$ is sufficiently large then* (7.3.9) *has a solution (u,v) such that $u(0)=0$, $v(0)=0$.*