CHAPTER 14

# Further problems

## 14.1. Multiplicities of spectral parameters

Suppose a Fuchsian differential equation and its middle convolution are given. Then we can analyze the corresponding transformation of a global structure of its local solution associated with an eigenvalue of the monodromy generator at a singular point if the eigenvalue is free of multiplicity.

When the multiplicity of the eigenvalue is larger than one, we have not a satisfactory result for the transformation (cf. Theorem 12.5). The value of a generalized connection coefficient defined by Definition 12.17 may be interesting. Is the procedure in Remark 12.19 always valid? In particular, is there a general result assuring Remark 12.19 (1) (cf. Remark 12.23)? Are the multiplicities of zeros of the generalized connection coefficients of a rigid Fuchsian differential equation free?

## 14.2. Schlesinger canonical form

Can we define a natural *universal* Fuchsian system of Schlesinger canonical form (1.79) with a given realizable spectral type? Here we recall Example 9.2.

Let $P_{\mathbf{m}}$ be the universal operator in Theorem 6.14. Is there a natural system of Schlesinger canonical form which is isomorphic to the equation $P_{\mathbf{m}}u = 0$ together with the explicit correspondence between them?

## 14.3. Apparent singularities

Katz [**Kz**] proved that any irreducible rigid local system is constructed from the trivial system by successive applications of middle convolutions and additions and it is proved in this paper that the system is realized by a single differential equation without an apparent singularity.

In general, an irreducible local system cannot be realized by a single differential equation without an apparent singularity but it is realized by that with apparent singularities. Hence it is expected that there exist some natural operations of single differential equations with apparent singularities which correspond to middle convolutions of local systems or systems of Schlesinger canonical form.

The Fuchsian ordinary differential equation satisfied by an important special function often hasn't an apparent singularity even if the spectral type of the equation is not rigid. Can we understand the condition that a $W(x)$-module has a generator so that it satisfies a differential equation without an apparent singularity? Moreover it may be interesting to study the existing of contiguous relations among differential equations with fundamental spectral types which have no apparent singularity.

## 14.4. Irregular singularities

Our fractional operations defined in Chapter 1 give transformations of ordinary differential operators with polynomial coefficients, which have irregular singularities in general. The reduction of ordinary differential equations under these operations is a problem to be studied. Note that versal additions and middle convolutions construct such differential operators from the trivial equation.

A similar result as in this paper is obtained for certain classes of ordinary differential equations with irregular singularities, namely, unramified irregular singularities (cf. [**Hi**], [**HiO**], [**O10**]).

A "versal" path of integral in an integral representation of the solution and a "versal" connection coefficient and Stokes multiplier should be studied. Here "versal" means a natural expression corresponding to the versal addition.

We define a complete model with a given spectral type as follows. For simplicity we consider differential operators without singularities at the origin. For a realizable irreducible tuple of partitions $\mathbf{m} = \big(m_{j,\nu}\big)_{\substack{0\le j\le p\\ 1\le\nu\le n_j}}$ of a positive integer $n$ Theorem 6.14 constructs the universal differential operator

$$P_{\mathbf{m}} = \prod_{j=1}^{p}(1-c_jx)^n \cdot \frac{d^n}{dx^n} + \sum_{k=0}^{n-1} a_k(x,c,\lambda,g)\frac{d^k}{dx^k} \tag{14.1}$$

with the Riemann scheme

$$\begin{Bmatrix} x=\infty & \frac{1}{c_1} & \cdots & \frac{1}{c_p} \\ [\lambda_{0,1}]_{(m_{0,1})} & [\lambda_{1,1}]_{(m_{1,1})} & \cdots & [\lambda_{p,1}]_{(m_{p,1})} \\ \vdots & \vdots & \vdots & \vdots \\ [\lambda_{0,n_0}]_{(m_{0,n_0})} & [\lambda_{1,n_1}]_{(m_{1,n_1})} & \cdots & [\lambda_{p,n_p}]_{(m_{p,n_p})} \end{Bmatrix}$$

and the Fuchs relation

$$\sum_{j=0}^{p}\sum_{\nu=1}^{n_j} m_{j,\nu}\lambda_{j,\nu} = n - \frac{\operatorname{idx}\mathbf{m}}{2}.$$

Here $c = (c_0,\dots,c_p)$, $\lambda = (\lambda_{j,\nu})$ and $g = (g_1,\dots,g_N)$ are parameters. We have $c_ic_j(c_i-c_j) \ne 0$ for $0 \le i < j \le p$. The parameters $g_j$ are called accessory parameters and we have $\operatorname{idx}\mathbf{m} = 2-2N$. We call the Zariski closure $\overline{P}_{\mathbf{m}}$ of $P_{\mathbf{m}}$ in $W[x]$ the *complete model* of differential operators with the spectral type $\mathbf{m}$, whose dimension equals $p + \sum_{j=0}^{p} n_j + N - 1$. It is an interesting problem to analyze the complete model $\overline{P}_{\mathbf{m}}$.

When $\mathbf{m} = 11, 11, 11$, the complete model equals

$$(1-c_1x)^2(1-c_2x)^2\tfrac{d^2}{dx^2} - (1-c_1x)(1-c_2x)(a_{1,1}x+a_{1,0})\tfrac{d}{dx} + a_{0,2}x^2 + a_{0,1}x + a_{0,0},$$

whose dimension equals 7. Any differential equation defined by the operator belonging to this complete model is transformed into a Gauss hypergeometric equation, a Kummer equation, an Hermite equation or an Airy equation by a suitable gauge transformation and a coordinate transformation. A good understanding together with a certain completion of our operators is required even in this fundamental example (cf. [**Yos**]). It is needless to say that the good understanding is important in the case when $\mathbf{m}$ is fundamental.

## 14.5. Special parameters

Let $P_{\mathbf{m}}$ be the universal operator of the form (14.1) for an irreducible tuple of partition $\mathbf{m}$. When a decomposition $\mathbf{m} = \mathbf{m}' + \mathbf{m}''$ with realizable tuples of partitions $\mathbf{m}'$ and $\mathbf{m}''$ is given, Theorem 4.19 gives the values of the parameters of $P_{\mathbf{m}}$ corresponding to the product $P_{\mathbf{m}'}P_{\mathbf{m}''}$. A $W(x,\xi)$-automorphism of $P_{\mathbf{m}}u = 0$ gives a transformation of the parameters $(\lambda, g)$, which is a contiguous relation and called Schlesinger transformation in the case of systems of Schlesinger canonical form. How can we describe the values of the parameters obtained in this way and characterize their position in all the values of the parameters when the universal operator is reducible? In general, they are not all even in a rigid differential equation. A direct decomposition $32,32,32,32 = 12,12,12,12 \oplus 2(10,10,10,10)$ of a rigid tuples $32,32,32,32$ gives this example (cf. (10.64)).

Analyse the reducible differential equation with an irreducibly realizable spectral type. This is interesting even when $\mathbf{m}$ is a rigid tuple. For example, describe the monodromy of its solutions.

Describe the characteristic exponents of the generalized Riemann scheme with an irreducibly realizable spectral type such that there exists a differential operator with the Riemann scheme which is outside the universal operator (cf. Example 5.6 and Remark 6.16). In particular, when the spectral type is not fundamental nor simply reducible, does there exist such a differential operator?

The classification of rigid and simply reducible spectral types coincides with that of indecomposable objects described in [**MWZ**, Theorem 2.4]. Is there some meaning in this coincidence?

Has the condition (6.28) a similar meaning in the case of Schlesinger canonical form? What is the condition on the local system or a (single) Fuchsian differential equation which has a realization of a system of Schlesinger canonical form?

Give the condition so that the monodromy group is finite (cf. [**BH**]).

Give the condition so that the centralizer of the monodromy is the set of scalar multiplications.

Suppose $\mathbf{m}$ is fundamental. Study the condition so that the connection coefficients is a quotient of the products of gamma functions as in Theorem 12.6 or the solution has an integral representation only by using elementary functions.

## 14.6. Shift operators

Calculate the intertwining polynomial $c_{\mathbf{m}}(\epsilon;\lambda)$ of $\lambda$ defined in Theorem 11.8. Is it square free? See Conjecture 11.12.

Is the shift operator $R_{\mathbf{m}}(\epsilon,\lambda)$ Fuchsian?

Is there a natural operator in $R_{\mathbf{m}}(\epsilon,\lambda) + W(x;\lambda)P_{\mathbf{m}}(\lambda)$?

Study the shift operators given in Theorem 11.7.

Study the condition on the characteristic exponents and accessory parameters assuring the existence of a shift operator for a Fuchsian differential operator with a fundamental spectral type.

Study the shift operator or Schlesinger transformation of a system of Schlesinger canonical form with a fundamental spectral type. When is it not defined or when is it not bijective?

## 14.7. Isomonodromic deformations

The isomonodromic deformations of Fuchsian systems of Schlesinger canonical form give Painlevé type equations and their degenerations correspond to confluence of the systems (cf. §13.1.6). Can we get a nice theory for these equations? Is it true that two Painlevé type equations corresponding to Fuchsian systems with fundamental spectral types are not isomorphic to each other if their spectral types are different?

## 14.8. Several variables

We have analyzed Appell hypergeometric equations in §13.10. What should be the geometric structure of singularities of more general system of equations when it has a good theory?

Describe or define operations of differential operators that are fundamental to analyze good systems of differential equations.

A series expansion of a local solution of a rigid ordinal differential equation indicates that it may be natural to think that the solution is a restriction of a solution of a system of differential equations with several variables (cf. Theorem 8.1 and §§13.3–13.4). Study the system.

### 14.9. Other problems

**1**. Given a rigid tuple $\mathbf{m}$ and a root $\alpha \in \Delta_+^{re}$ with $(\alpha|\alpha_{\mathbf{m}}) > 0$. Is there a good necessary and sufficient condition so that $\alpha \in \Delta(\mathbf{m})$? See Proposition 7.9 iv) and Remark 7.11 i).

For example, for a rigid decomposition $\mathbf{m} = \mathbf{m}' \oplus \mathbf{m}''$, can we determine whether $\alpha_{\mathbf{m}'} \in \Delta(\mathbf{m})$ or $\alpha_{\mathbf{m}''} \in \Delta(\mathbf{m})$?

**2**. Is there a direct expression of $\lambda(K)_{j,\ell(K)_j}$ in (12.10) for a given Riemann scheme $\{\lambda_{\mathbf{m}}\}$?

**3**. Are there analyzable series $\mathcal{L}$ of rigid tuples of partitions different from the series given in §13.9? Namely, $\mathcal{L} \subset \mathcal{P}$, the elements of $\mathcal{L}$ are rigid, the number of isomorphic classes of $\mathcal{L} \cap \mathcal{P}^{(n)}$ are bounded for $n \in \mathbb{Z}_{>0}$ and the following condition is valid.

Let $\mathbf{m} = k\mathbf{m}' + \mathbf{m}''$ with $k \in \mathbb{Z}_{>0}$ and rigid tuples of partitions $\mathbf{m}$, $\mathbf{m}'$ and $\mathbf{m}''$. If $\mathbf{m} \in \mathcal{L}$, then $\mathbf{m}' \in \mathcal{L}$ and $\mathbf{m}'' \in \mathcal{L}$. Moreover for any $\mathbf{m}'' \in \mathcal{L}$, this decomposition $\mathbf{m} = k\mathbf{m}' + \mathbf{m}''$ exists with $\mathbf{m} \in \mathcal{L}$, $\mathbf{m}' \in \mathcal{L}$ and $k \in \mathbb{Z}_{>0}$. Furthermore $\mathcal{L}$ is indecomposable. Namely, if $\mathcal{L} = \mathcal{L}' \cup \mathcal{L}''$ so that $\mathcal{L}'$ and $\mathcal{L}''$ satisfy these conditions, then $\mathcal{L}' = \mathcal{L}$ or $\mathcal{L}'' = \mathcal{L}$.

**4**. Characterize the ring of automorphisms and that of endomorphisms of the localized Weyl algebra $W(x)$. Can we find a good class of endomorphisms? These questions are more important in the case of several variables.

**5**. In general, different procedures of the reduction of the universal operator $P_{\mathbf{m}}u = 0$ give different integral representations and series expansions of its solution (cf. Example 8.2, Remark 8.3 and the last part of §13.3). Analyze the difference.

**6**. Analyse the differential equation whose solutions are spanned by the Wronskians of $k$ independent solutions of the equation $P_{\mathbf{m}}u = 0$ with a universal operator $P_{\mathbf{m}}$ such that $1 < k < \operatorname{ord}\mathbf{m}$ (cf. Remark 12.18 ii)).

**7**. Generalize our results for differential equations on some compact complex manifolds.

**8**. Generalize our results for difference equations (cf. [**Ya**]).