Chapter 2

Mathematical problem and main results

2.1 Intial boundary value problem for hydrodynamic model

By assuming the physical coefficients in (1.9) are positive constants and letting $\varepsilon' = 1$, we have a system of equations

$$\rho_s + m_x = 0, \tag{2.1a}$$

$$m_s + \left(\frac{m^2}{\rho} + \rho\theta\right)_x = \rho\phi_x - \frac{m}{\tau_m}, \tag{2.1b}$$

$$\rho\theta_s + m\theta_x + \frac{2}{3}\left(\frac{m}{\rho}\right)_x \rho\theta - \frac{2}{3}\left(\tau_m\kappa_0\theta_x\right)_x = \frac{2\tau_e - \tau_m}{3\tau_m\tau_e}\frac{m^2}{\rho} - \frac{\rho}{\tau_e}(\theta - 1), \tag{2.1c}$$

$$\phi_{xx} = \rho - D. \tag{2.1d}$$

We study the initial boundary value problem for (2.1) with a spatial variable $x \in \Omega := (0, 1)$ and a time variable $s > 0$. The unknown functions ρ, m, θ and ϕ stand for the electron density, the current density, the electron temperature and the electrostatic potential, respectively. Positive constants τ_m and τ_e are the momentum relaxation time and the energy relaxation time, respectively. From the physical point of view, it holds that $0 < \tau_m \leq \tau_e$. Positive constant $\tau_m\kappa_0$ corresponds to the thermal conductivity. A doping profile $D(x)$, which determines the electric property of semiconductors, is assumed to be a bounded continuous and positive function of the spatial variable x, that is,

$$D \in \mathcal{B}^0(\overline{\Omega}), \quad \inf_{x\in\overline{\Omega}} D(x) > 0. \tag{2.2}$$

The initial and the boundary data to the system (2.1) are prescribed as

$$(\rho, m, \theta)(0, x) = (\rho_0, m_0, \theta_0)(x). \tag{2.3}$$
$$\rho(t, 0) = \rho_l > 0, \quad \rho(t, 1) = \rho_r > 0, \tag{2.4}$$
$$\theta_x(t, 0) = \theta_x(t, 1) = 0, \tag{2.5}$$
$$\phi(t, 0) = 0, \quad \phi(t, 1) = \phi_r \geq 0 \tag{2.6}$$

with given constants ρ_l, ρ_r and ϕ_r. In typical devices (see [10]), ρ_l and ρ_r coincide with the boundary value of the doping profile, that is, $\rho_l = \rho_r = D(0) = D(1)$ however we do not assume this condition. The constant ϕ_r means voltage applied to the devices. In engineering, it is preferable to design devices to have stable steady flow with small voltage to save electricity. Thus it is admissible to assume that the difference between boundary values

$$\delta := |\rho_r - \rho_l| + |\phi_r|,$$

which is called a boundary strength, is small from physical point of view. Throughout the present paper we assume the smallness of the boundary strength δ. It is also assumed that the initial data (ρ_0, m_0, θ_0) is compatible with boundary data (2.4), (2.5) and $\rho_t(t, 0) = \rho_t(t, 1) = 0$:

$$\rho_0(0) = \rho_l, \quad \rho_0(1) = \rho_r, \tag{2.7a}$$
$$\theta_{0x}(0) = \theta_{0x}(1) = 0, \tag{2.7b}$$
$$m_{0x}(0) = m_{0x}(1) = 0 \tag{2.7c}$$

to establish the strong solution.

Solving the Poisson equation (2.1d) with using (2.6), we have an explicit formula of electrostatic potential

$$\begin{aligned} \phi(t, x) &= \Phi[\rho](t, x) \\ &:= \int_0^x \int_0^y (\rho - D)(t, z)\, dzdy + \left(\phi_r - \int_0^1 \int_0^y (\rho - D)(t, z)\, dzdy \right) x. \end{aligned} \tag{2.8}$$

The initial boundary value problem (2.1) and (2.3)–(2.6) is studied under the initial condition that

$$\inf_{x \in \Omega} \rho_0 > 0, \quad \inf_{x \in \Omega} \theta_0 > 0, \quad \inf_{x \in \Omega} \left(\theta_0 - \frac{m_0^2}{\rho_0^2} \right) > 0. \tag{2.9}$$

In fact, we construct the solution satisfying the same conditions for an arbitrary $t > 0$ under the initial assumptions as in (2.9). Namely,

$$\inf_{x\in\Omega} \rho > 0, \tag{2.10a}$$

$$\inf_{x\in\Omega} \theta > 0, \tag{2.10b}$$

$$\inf_{x\in\Omega} \left(\theta - \frac{m^2}{\rho^2}\right) > 0. \tag{2.10c}$$

The conditions (2.10a) and (2.10b) means positivity of the density and the temperature, respectively. We call the condition (2.10c) a subsonic condition in analogy of fluid dynamics. By straightforward computation, we see it is equivalent to the property that one characteristic speed of the hyperbolic equations (2.1a) and (2.1b) is negative and the other is positive:

$$\frac{m}{\rho} - \sqrt{\theta} < 0, \quad \frac{m}{\rho} + \sqrt{\theta} > 0.$$

Therefore we need one condition on each boundary $x = 0,\ 1$ for (2.1a) and (2.1b). We also need two boundary conditions for the parabolic equation (2.1c) and the elliptic equation (2.1d). In total, the three boundary conditions in (2.4)–(2.6) are necessary and sufficient for the well-posedness of the initial boundary value problem (2.1), (2.3) and (2.4)–(2.6) at least locally in time.

2.2 Formal computation of relaxation limits

In Section 1.2, we have studied formal computations of the relaxation limits for the multi-dimensional hydrodynamic model (1.1). To clarify the main mathematical problems in the present paper, we again discuss the relaxation limits for the one-dimensional model (2.1). Here we also derive the initial and the boundary data for the drift-diffusion and the energy-transport models from (2.3)–(2.6).

In the same way as in Section 1.2, we employ scaled variables

$$t := \tau_m s, \quad j := \frac{m}{\tau_m}, \quad \varepsilon := \tau_m^2, \quad \zeta := \tau_m \tau_e.$$

and substitute them in (2.1). This computation gives a system of equations

$$\rho_t + j_x = 0, \tag{2.11a}$$

$$\varepsilon j_t + \left(\varepsilon \frac{j^2}{\rho} + \rho\theta\right)_x = \rho\phi_x - j, \tag{2.11b}$$

$$\rho\theta_t + j\theta_x + \frac{2}{3}\left(\frac{j}{\rho}\right)_x \rho\theta - \frac{2}{3}\kappa_0\theta_{xx} = \left(\frac{2}{3} - \frac{\varepsilon}{3\zeta}\right)\frac{j^2}{\rho} - \frac{\rho}{\zeta}(\theta - 1), \tag{2.11c}$$

$$\phi_{xx} = \rho - D, \tag{2.11d}$$

which is called a hydrodynamic model, too. The initial data to (2.11) are derived from (2.3) as

$$\rho(0,x) = \rho_0(x), \tag{2.12a}$$
$$j(0,x) = j_0(x) := (m_0/\tau_m)(x), \tag{2.12b}$$
$$\theta(0,x) = \theta_0(x). \tag{2.12c}$$

The boundary data to (2.11) is prescribed by (2.4)–(2.6). The subsonic condition (2.10c) is rewritten as

$$\inf_{x\in\Omega} S[\rho,j,\theta] > 0, \quad S[\rho,j,\theta] := \theta - \varepsilon\frac{j^2}{\rho^2}. \tag{2.13}$$

Note the positivity of the temperature (2.10b) immediately implies that the subsonic condition (2.13) holds for the sufficiently small momentum relaxation $\tau_m = \sqrt{\varepsilon}$.

Letting the parameter ε tend to zero in (2.11) with ζ kept constant formally yields the energy-transport model

$$\rho_t + j_x = 0, \tag{2.14a}$$
$$\rho\theta_t + j\theta_x + \frac{2}{3}\left(\frac{j}{\rho}\right)_x \rho\theta - \frac{2}{3}\kappa_0\theta_{xx} = \frac{2}{3}\frac{j^2}{\rho} - \frac{\rho}{\zeta}(\theta-1), \tag{2.14b}$$
$$\phi_{xx} = \rho - D, \tag{2.14c}$$

where the electric current is explicitly given by

$$j = \rho\phi_x - (\theta\rho)_x. \tag{2.14d}$$

The initial and the boundary data to (2.14) are also prescribed by (2.12a), (2.12c) and (2.4)–(2.6). Furthermore, letting the parameter ζ tend to zero in (2.14) or letting the parameters ε and ζ tend to zero in (2.11) yields the drift-diffusion model

$$\rho_t + j_x = 0, \tag{2.15a}$$
$$\phi_{xx} = \rho - D \tag{2.15b}$$

with the electric current

$$j = \rho\phi_x - \rho_x. \tag{2.15c}$$

Here the initial and the boundary data to (2.15) are given by (2.4), (2.6) and (2.12a).

The remained case is to make the parameter ζ tend to zero with ε kept constant. This formal limit yields the isothermal hydrodynamic model

$$\begin{aligned} &\rho_t + j_x = 0, \\ &\varepsilon j_t + \left(\varepsilon\frac{2j^2}{3\rho} + \rho\right)_x = \rho\phi_x - j, \\ &\phi_{xx} = \rho - D. \end{aligned}$$

This computation, however, is not admissible from physical point of view since $\varepsilon < \zeta$. Moreover, we have the drift-diffusion model by letting the parameter ε tend to zero in the isothermal hydrodynamic model. The limit procedures above are called relaxation time limits or relaxation limits in short. They are summarized in the next figure.

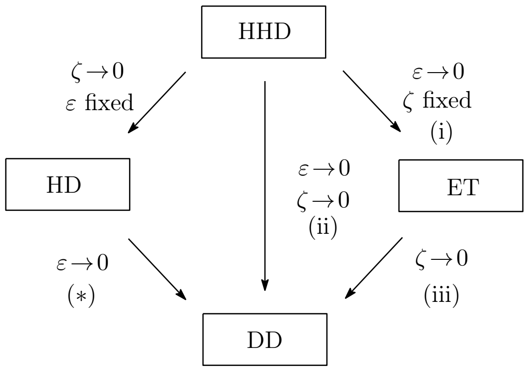

Figure 2.1: HHD, ET, DD and HD mean the hydrodynamic, the energy-transport, the drift-diffusion, the isothermal hydrodynamic models, respectively.

In the authors' paper [33], the relaxation limit $(*)$ in Figure 2.1 have been justified rigorously. Thus the main purpose of the present paper is the justification of the other limits. Precisely, we show that (i) the solution for the hydrodynamic model (2.1) converges to that for the energy-transport model (2.14) as ε tends to zero; (ii) the solution for the hydrodynamic model converges to that for the drift-diffusion model (2.15) as ε and ζ tend to zero; (iii) the solution for the energy-transport converges to that for the drift-diffusion model as ζ tends to zero. The above three assertions hold for the solutions globally in time without any restriction on the norm of the initial data as far as it belongs to the suitable Sobolev space. Hence we firstly have to establish the time global existence of the solutions to these models. In these procedures, we also show that their asymptotic behaviors are given by the corresponding stationary solutions.

2.3 Asymptotic behavior of hydrodynamic model

The asymptotic stability of a stationary solution to (2.11) for the large initial data (ρ_0, j_0, θ_0) is one of the main results. The authors have shown the stability theorem for a small initial disturbance from the stationary solution in [32], where the authors adopt the Dirichlet boundary condition for the electron temperature in place of the Neumann boundary condition (2.5), that is, $\theta(t,0) = \theta_l$ and $\theta(t,1) = \theta_r$. Precisely it is shown that the solution (ρ, j, θ, ϕ)

to the problem (2.1) converges to the corresponding stationary solution $(\tilde{\rho}, \tilde{j}, \tilde{\theta}, \tilde{\phi})$ as time tends to infinity provided that the initial disturbance $\|(\rho_0 - \tilde{\rho}, j_0 - \tilde{j}, \theta - \tilde{\theta})\|_2$ is sufficiently small. In the present research, we show the stability theorem for the boundary condition (2.5) without any smallness assumption on the initial disturbance. Instead of it, we assume that the parameters ε and ζ are sufficiently small.

The stationary solution $(\tilde{\rho}, \tilde{j}, \tilde{\phi})$ is a solution to (2.11) independent of a time variable t. Since $(1/\tilde{\rho})_x \tilde{\rho} = -(\log \tilde{\rho})_x$, it versifies

$$\tilde{j}_x = 0, \tag{2.17a}$$

$$S[\tilde{\rho}, \tilde{j}, \tilde{\theta}]\tilde{\rho}_x + \tilde{\rho}\tilde{\theta}_x = \tilde{\rho}\tilde{\phi}_x - \tilde{j}, \tag{2.17b}$$

$$\tilde{j}\tilde{\theta}_x - \frac{2}{3}\tilde{j}\tilde{\theta}\,(\log \tilde{\rho})_x - \frac{2}{3}\kappa_0 \tilde{\theta}_{xx} = \left(\frac{2}{3} - \frac{\varepsilon}{3\zeta}\right)\frac{\tilde{j}^2}{\tilde{\rho}} - \frac{\tilde{\rho}}{\zeta}(\tilde{\theta} - 1), \tag{2.17c}$$

$$\tilde{\phi}_{xx} = \tilde{\rho} - D \tag{2.17d}$$

and the boundary conditions

$$\tilde{\rho}(0) = \rho_l > 0, \quad \tilde{\rho}(1) = \rho_r > 0, \tag{2.18}$$

$$\tilde{\theta}_x(0) = \tilde{\theta}_x(1) = 0, \tag{2.19}$$

$$\tilde{\phi}(0) = 0, \quad \tilde{\phi}(1) = \phi_r > 0. \tag{2.20}$$

In Section 2.1, the unique existence of the stationary solution is proven (see Theorem 3.5). Its asymptotic stability is summarized in

Theorem 2.1. *Let* $(\tilde{\rho}, \tilde{j}, \tilde{\theta}, \tilde{\phi})$ *be the stationary solution of* (2.17)–(2.20). *Suppose that the initial data* $(\rho_0, j_0, \theta_0) \in H^2(\Omega) \times H^2(\Omega) \times H^3(\Omega)$ *and the boundary data* ρ_l, ρ_r *and* ϕ_r *satisfy the conditions* (2.4), (2.6), (2.7), (2.10a), (2.10b) *and* (2.13). *Then there exist positive constants* δ_0 *and* ζ_0 *such that if* $\delta \leq \delta_0$ *and* $\zeta \leq \zeta_0$, *there exist a positive constant* ε_0, *depending on* ζ *but independent of* δ, *such that if* $\varepsilon \leq \varepsilon_0$, *the initial boundary value problem* (2.11), (2.12) *and* (2.4)–(2.6) *has a unique solution* (ρ, j, θ, ϕ) *satisfying* $\rho,\ j \in \mathfrak{X}_2([0,\infty))$, $\theta, \theta_x \in \mathfrak{Y}([0,\infty))$, $\phi \in C^2([0,\infty); H^2(\Omega))$ *and the conditions* (2.10a), (2.10b) *and* (2.13). *Moreover, the solution* (ρ, j, θ, ϕ) *verifies the additional regularity* $\phi - \tilde{\phi} \in \mathfrak{X}_2^2([0,\infty))$ *and the decay estimate*

$$\begin{aligned}&\|(j - \tilde{j})(t)\|_1^2 + \|(\rho - \tilde{\rho}, \theta - \tilde{\theta})(t)\|_2^2 \\ &\qquad + \varepsilon\|(\partial_x^2\{j - \tilde{j}\}, \partial_x^3\{\theta - \tilde{\theta}\})(t)\|^2 + \|(\phi - \tilde{\phi})(t)\|_4^2 \leq Ce^{-\alpha t},\end{aligned} \tag{2.21}$$

where C *and* α *are positive constants depending on* ζ *but independent of* δ, ε *and* t.

Remark 2.2. *The initial data in Theorem 2.1 are able to be taken arbitrarily large in the Sobolev space* $H^2 \times H^2 \times H^3$. *Hence, we see that the original problem* (2.1) *and* (2.3)–(2.6) *has a time global solution for the large electron density* $\rho_0(x)$ *and the electron temperature* $\theta_0(x)$ *at initial time. On the other hand,* H^2*-norm of the current density* $m_0(x) = \tau_m j_0(x)$ *might be small. However, it is an admissible condition for the situation immediately after voltage is applied to the devices since there are no current of electrons before, that is,* $m_0(x) = 0$.

2.4 Relaxation time limits

The results on the relaxation limit from the hydrodynamic model (2.11) to the energy-transport model (2.14) is summarized in Theorem 2.3 below. To study them we have to establish the existence and the stability of the stationary solution for the energy-transport model, which are proven in Section 3 and 4 (see Theorems 3.5 and 4.2). Let us note that we prescribe two initial conditions for (2.14) while three initial conditions in (2.12) are necessary for (2.11). Here the initial value $j(0, x)$ to (2.14) is determined by (2.8), (2.12a), (2.12c) and (2.14d), that is,

$$j(0, x) = (-(\theta_0 \rho_0)_x + \rho_0 \{\Phi[\rho_0]\}_x)(x).$$

In general, the initial data (2.12a) to (2.11) is not necessarily in the "momeuntam equilibrium" states $j_0(x) = (-(\theta_0 \rho_0)_x + \rho_0 \{\Phi[\rho_0]\}_x)(x)$. Hence, in the justification of the relaxation limits, the difference between $j_0(x) - j(0, x)$ gives rise to the initial layer. However, it is shown that the layer decays as time t tends to infinity and/or the parameter ε tends to zero.

Theorem 2.3. *Suppose that the initial data* $(\rho_0, j_0, \theta_0) \in H^2(\Omega) \times H^2(\Omega) \times H^3(\Omega)$ *and the boundary data* ρ_l, ρ_r *and* ϕ_r *satisfy the conditions* (2.4), (2.6), (2.7), (2.10a), (2.10b) *and* (2.13). *Then there exist positive constants* δ_0 *and* ζ_0 *such that if* $\delta \leq \delta_0$ *and* $\zeta \leq \zeta_0$, *there exists a positive constant* ε_0, *depending on* ζ *but independent of* δ, *such that if* $\varepsilon \leq \varepsilon_0$, *the time global solution* $(\rho_\zeta^\varepsilon, j_\zeta^\varepsilon, \theta_\zeta^\varepsilon, \phi_\zeta^\varepsilon)$ *for the problem* (2.11), (2.12) *and* (2.4)–(2.6) *converges to the time global solution* $(\rho_\zeta^0, j_\zeta^0, \theta_\zeta^0, \phi_\zeta^0)$ *for the problem* (2.14), (2.12a), (2.12c) *and* (2.4)–(2.6) *as* ε *tends to zero. Precisely, the following estimates hold for an arbitrary* $t \in (0, \infty)$.

$$\|(\rho_\zeta^\varepsilon - \rho_\zeta^0, \theta_\zeta^\varepsilon - \theta_\zeta^0)(t)\|_1^2 + \|(\phi_\zeta^\varepsilon - \phi_\zeta^0)(t)\|_3^2 \leq C\varepsilon^\gamma, \tag{2.22}$$

$$\|(j_\zeta^\varepsilon - j_\zeta^0)(t)\|^2 \leq \|(j_0 - j_\zeta^0)(0)\|^2 e^{-t/\varepsilon} + C\varepsilon^\gamma, \tag{2.23}$$

$$\|(\partial_x^2\{\rho_\zeta^\varepsilon - \rho_\zeta^0\}, \partial_x^1\{j_\zeta^\varepsilon - j_\zeta^0\}, \partial_x^2\{\theta_\zeta^\varepsilon - \theta_\zeta^0\}, \partial_x^4\{\phi_\zeta^\varepsilon - \phi_\zeta^0\})(t)\|^2 \leq C\varepsilon^\gamma(t^{-2} + 1), \tag{2.24}$$

where γ *and* C *are positive constants depending on* ζ *but independent of* ε, δ *and* t.

Furthermore, we justify the relaxation limit from the energy-transport model (2.14) to the drift-diffusion model (2.15). Similarly as in the relaxation limit from (2.11) to (2.14), the initial layer occurs due to the difference $\theta_0 - 1$ since the initial data θ_0 is not nessesary in the "energy equilibrium" $\theta_0 = 1$. The layer is also shown to decay as time t tends to infinity and/or the parameter ε tends to zero. This result is summarized in the next lemma. Here the existence of the time global solution to (2.15) has been proven in the author's previous paper [33].

Theorem 2.4. *Suppose that the initial data* $(\rho_0, \theta_0) \in H^1(\Omega)$ *and the boundary data* ρ_l, ρ_r *and* ϕ_r *satisfy* (2.4), (2.6), (2.7a), (2.10a) *and* (2.10b). *Then there exist positive constants* δ_0 *and* ζ_0 *such that if* $\delta \le \delta_0$ *and* $\zeta \le \zeta_0$, *then the time global solution* $(\rho^0_\zeta, j^0_\zeta, \theta^0_\zeta, \phi^0_\zeta)$ *for the problem* (2.14), (2.12a), (2.12c) *and* (2.4)–(2.6) *converges to the time global solution* $(\rho^0_0, j^0_0, \phi^0_0)$ *for the problem* (2.15), (2.12a), (2.4) *and* (2.6) *as* ζ *tends to zero. Precisely, the following estimates hold for an arbitrary* $t \in [0, \infty)$.

$$\|(\rho^0_\zeta - \rho^0_0)(t)\|^2 + \|(\phi^0_\zeta - \phi^0_0)(t)\|_2^2 \le C\zeta^\gamma, \tag{2.25}$$

$$\|(\theta^0_\zeta - 1)(t)\|^2 \le C\|\theta_0 - 1\|^2 e^{-\nu t/\zeta} + C\zeta^\gamma, \tag{2.26}$$

$$\|(\{\rho^0_\zeta - \rho^0_0\}_x, \{\theta^0_\zeta\}_x, j^0_\zeta - j^0_0)(t)\|^2 \le C\zeta^\gamma(1 + t^{-1}), \tag{2.27}$$

where ν, γ *and* C *are positive constants independent of* ζ, δ *and* t.

The next corollary, concerning the relaxation limit from the hydrodynamic model (2.11) to the drift-diffusion model (2.15), immediately follows from Theorems 2.3 and 2.4.

Corollary 2.5. *Assume the same conditions as in Theorems 2.3 and 2.4. Then the time global solution* $(\rho^\varepsilon_\zeta, j^\varepsilon_\zeta, \theta^\varepsilon_\zeta, \phi^\varepsilon_\zeta)$ *for the problem* (2.11), (2.12) *and* (2.4)–(2.6) *converges to the time global solution* $(\rho^0_0, j^0_0, 1, \phi^0_0)$ *for the problem* (2.15), (2.12a), (2.4) *and* (2.6) *as* ε *and* ζ *tend to zero. Precisely, the following estimates hold for an arbitrary* $t \in (0, \infty)$.

$$\|(\rho^\varepsilon_\zeta - \rho^0_0)(t)\|^2 + \|(\phi^\varepsilon_\zeta - \phi^0_0)(t)\|_2^2 \le \overline{C}\zeta^{\overline{\gamma}} + C\varepsilon^\gamma, \tag{2.28}$$

$$\|(\theta^\varepsilon_\zeta - 1)(t)\|^2 \le \overline{C}\|\theta_0 - 1\|^2 e^{-\overline{\nu} t/\zeta} + \overline{C}\zeta^{\overline{\gamma}} + C\varepsilon^\gamma, \tag{2.29}$$

$$\|(j^\varepsilon_\zeta - j^0_0)(t)\|^2 \le \overline{C}\|(j_0 - j^0_0)(0)\|^2 e^{-t/\varepsilon} + \overline{C}\zeta^{\overline{\gamma}}(1 + t^{-1}) + C\varepsilon^\gamma, \tag{2.30}$$

$$\|(\{\rho^\varepsilon_\zeta - \rho^0_0\}_x, \{\theta^\varepsilon_\zeta\}_x)(t)\|^2 \le \overline{C}\zeta^{\overline{\gamma}}(1 + t^{-1}) + C\varepsilon^\gamma. \tag{2.31}$$

Here γ *and* C *are positive constants depending on* ζ *but independent of* ε, δ *and* t. *Also* $\overline{\nu}$, $\overline{\gamma}$ *and* $\overline{C}$ *are positive constants independent of* ε, ζ, δ *and* t.

Remark 2.6. *If we regard the time when voltage is applied on the device as initial time* $t = 0$, *then it is physically admissible that the initial temperature coincides with the ambient temperature, that is,* $\theta_0 = 1$. *In this typical setting, we can improve the results in Theorem 2.3 and/or Corollary 2.5. Precisely, if* $\theta_0 = 1$ *in Theorem 2.3 and Corollary 2.5, we can explicitly write the dependence on the constants* γ *and* C *in the inequalities* (2.22)–(2.24) *and* (2.28)–(2.31) *with respect to* ζ. *Namely, the constant* $C\varepsilon^\gamma$ *in these inequalities are replaced* *by* $\overline{C}\{\varepsilon + (\varepsilon/\zeta)^2\}^{\overline{\gamma}}$*, where* $\overline{\gamma}$ *and* $\overline{C}$ *are positive constants independent of* ε*,* ζ*,* δ *and* t*. This fact means that the time global solution to the hydrodynamic model converges to those to the energy-transport and/or the drift-diffusion models as* ε *and/or* ζ *tend to zero if the inequality* $\varepsilon^{1-a} \leq \zeta$ *hold for an arbitrarily fixed constant* $a \in (0,1)$*. See Remarks 4.18 and 5.17.*

2.5 Outline of proofs

In Section 3.1, we begin detailed discussion with the proof of the unique existence of the stationary solutions for the hydrodynamic and the energy-transport models. The existence is established by the Schauder fixed-point theorem. The uniqueness follows from the energy method. The relaxation limits for the stationary solutions are justified in Section 3.2. Precisely, we show that (i) the stationary solution for the hydrodynamic model converges to that for the energy-transport model as ε tends to zero; (ii) the stationary solution for the energy-transport converges to that for the drift-diffusion model as ζ tends to zero; (iii) the stationary solution for the hydrodynamic model converges to that for the drift-diffusion model (2.15) as ε and ζ tend to zero. These results are proven by the standard energy method and utilized to show the relaxation limits for the non-stationary solutions.

In Section 4.1–4.3, we discuss about the asymptotic stability of the stationary solution for the energy-transport model with the large initial data. The stability theorem is shown by the following three steps. It is firstly shown that there exists positive time T_* independent of ζ such that the energy-transport model has a unique solution until T_* in Section 4.1. Secondly, a "semi-global existence" of the solution is established in Section 4.2. Precisely, we prove that the solution exists until arbitrary time T by assuming ζ is sufficiently small. Then we confirm that the difference between the non-stationary and the stationary solutions becomes arbitrarily small by taking time T large enough. In Section 4.3, we thirdly prove the stability theorem for the small initial disturbance. Consequently, these three results in Sections 4.1–4.3 complete the proof of the stability theorem with the large initial data. The theorem also shows that the solution converges to the stationary solution exponentially fast.

In Section 4.4, Theorem 2.4 shows that the relaxation limit from the energy-transport model to the drift-diffusion model is justified for the time global solution. In the proof, to handle the initial layer, we employ the time weighted energy method and then show that the layer decays exponentially fast as the parameter ζ tends to zero and/or time t tends to infinity. The key of the proof is the facts that the solutions for both models converge to the corresponding stationary solutions exponentially fast and that the both stationary solutions are close to each other in the Sobolev space.

We study in Sections 5.1–5.3 the stability of the stationary solution for the hydrodynamic model with the large initial data, by the essentially same procedures in Sections 4.1–4.3. Precisely, it is shown in Section 5.1 that the existence time of the local solution is independent of the parameter ε. In Section 5.2, we prove the "semi-global existence". The stability theorem with the small initial disturbance is proven in Section 5.3. These three procedures complete the proof of Theorem 2.1.

In Section 5.4, we prove Theorem 2.3 and Corollary 2.5, which ensure the justification of the relaxation limits to the energy-transport and the drift-diffusion models, respectively. Namely, it is shown that the time global solution for the hydrodynamic model converges to that for the energy-transport and the drift-diffusion models as the relaxation times tend to zero. In these proofs, the time weighted energy method and the convergence rate toward the stationary solutions play essential roles similarly as in Sections 4.4.