CHAPTER 2

Relation between Real and Complex Secondary Classes

A natural mapping $\varphi\colon B\Gamma_q^{\mathbb{C}} \to B\Gamma_{2q}$ is obtained by forgetting transverse complex structures. There is a natural homomorphism from $H^*(\mathrm{WO}_{2q})$ to $H^*(\mathrm{WU}_q)$ which corresponds to this mapping as follows.

THEOREM 2.1 ([64], [3, Theorem 3.1]). *Let* λ *be the mapping from* WO_{2q} *to* WU_q *given by*

$$\lambda(c_k) = (\sqrt{-1})^k \sum_{j=0}^{k} (-1)^j v_{k-j}\overline{v}_j,$$

$$\lambda(h_{2k+1}) = \frac{(-1)^k}{2}\sqrt{-1}\sum_{j=0}^{2k+1}(-1)^j \widetilde{u}_{2k-j+1}(v_j+\overline{v}_j),$$

where v_0 *and* $\overline{v}_0$ *are considered as* 1*. Then* λ *induces a homomorphism from* $H^*(\mathrm{WO}_{2q})$ *to* $H^*(\mathrm{WU}_q)$*, denoted by* $[\lambda]$*. The homomorphism* $[\lambda]$ *corresponds to forgetting transverse complex structures, indeed, the following diagram commutes*:

$$\begin{array}{ccc} H^*(\mathrm{WO}_{2q}) & \xrightarrow{[\lambda]} & H^*(\mathrm{WU}_q) \\ \chi\big\downarrow & & \big\downarrow\chi^{\mathbb{C}} \\ H^*(B\Gamma_{2q}) & \xrightarrow[\varphi^*]{} & H^*(B\Gamma_q^{\mathbb{C}}). \end{array}$$

The Godbillon–Vey class and the imaginary part of the Bott class are related by the formula

$$[\lambda](\mathrm{GV}_{2q}) = \frac{(2q)!}{q!\,q!}\,\xi_q\cdot\mathrm{ch}_1^q\,,$$

where $\mathrm{ch}_1 = \dfrac{v_1 + \overline{v}_1}{2}$ *and it corresponds to the first Chern class of the complex normal bundle of the foliation. The image of* GV_{2q} *under* $[\lambda]$ *is also called the Godbillon–Vey class.*

Remark 2.2. Theorem 2.1 first appeared in [64] without proofs.

The kernel, image and cokernel of $[\lambda]$ have the following meanings:

$$\begin{aligned}\ker[\lambda] &= \left\{\begin{matrix}\text{classes in } H^*(\mathrm{WO}_{2q}) \text{ which are obstructions for foliations}\\ \text{to admit transverse holomorphic structures}\end{matrix}\right\},\\ \operatorname{im}[\lambda] &= \{\text{classes in } H^*(\mathrm{WU}_q) \text{ which are invariants as real foliations}\},\\ \operatorname{coker}[\lambda] &= \{\text{classes in } H^*(\mathrm{WU}_q) \text{ which cannot be induced from real classes}\}.\end{aligned}$$

If $H^*(\mathrm{WU}_q)$ is explicitly described, then one can write down $[\lambda]$ and determine these spaces. This is done for $q \leq 3$ in [5]. The results are given in the last part of this section (Theorems 2.6 and 2.7).

The image of GV_{2q} is non-trivial in $H^*(\mathrm{WU}_q)$. Indeed, we will construct transversely holomorphic foliations with non-trivial Godbillon–Vey classes. On the other hand, we have the following

Corollary 2.3. *The image of* GV_{2q} *is trivial in* $H^*(\mathrm{W}_q^{\mathbb{C}})$.

Proof. The equality $\mathrm{ch}_1 = d\left(\dfrac{u_1 + \overline{u}_1}{2}\right)$ holds in $\mathrm{W}_q^{\mathbb{C}}$. Therefore GV_{2q} is trivial by Theorem 2.1. □

This corollary implies that the Godbillon–Vey classes of transversely holomorphic foliations is trivial if the first Chern class of the complex normal bundle is trivial.

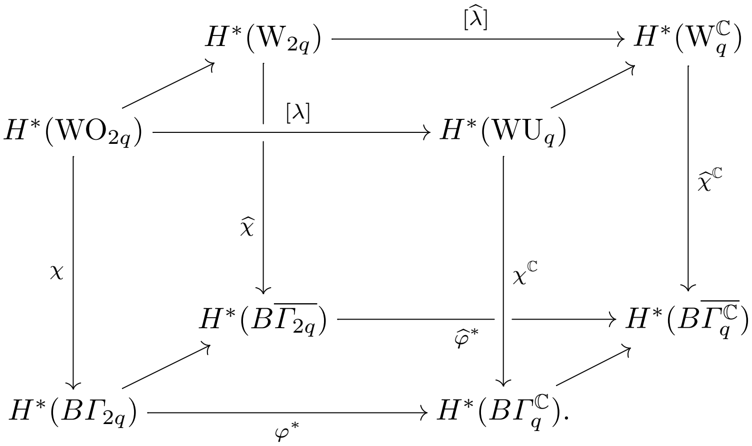
There is a version of Theorem 2.1 for foliations with trivialized normal bundles.

Theorem 2.4 ([3]). *Let* $\widehat{\lambda}\colon \mathrm{W}_{2q} \to \mathrm{W}_q^{\mathbb{C}}$ *be an extension of* λ *defined by*

$$\widehat{\lambda}(h_{2k}) = \frac{(-1)^k}{2}\sqrt{-1}\sum_{j=0}^{2k}(-1)^j(u_{2k-j}\overline{v}_j + \overline{u}_j v_{2k-j}),$$

where v_0 *and* $\overline{v}_0$ *are regarded as* 2, *and* u_0 *and* $\overline{u}_0$ *are regarded as* 0. *Then* $\widehat{\lambda}$ *induces on the cohomology a homomorphism, denoted by* $[\widehat{\lambda}]$, *which corresponds to forgetting transverse complex structures.*

Let $\widehat{\varphi}\colon B\overline{\Gamma_q^{\mathbb{C}}} \to B\overline{\Gamma_{2q}}$ be the mapping obtained by forgetting transverse complex structures. The induced homomorphisms commute as follows:



The mapping in which we are most interested in the first half of this monograph is $\chi^{\mathbb{C}} \circ [\lambda]\colon H^*(\mathrm{WO}_{2q}) \to H^*(B\Gamma_q^{\mathbb{C}})$.

Remark 2.5 (cf. Remark 1.3.13). The above diagram can be explained in terms of the Schubert varieties. Let $X_q \times X_q \to X_{2q}$ be the mapping induced by taking the direct sum. This mapping also induces a mapping from $\widetilde{X_q} = (X_q \times X_q)/\mathrm{U}(q)$ to $X_{2q}/\mathrm{O}(2q)$. Then the top square in the above diagram corresponds to the following commutative diagram:

$$\begin{array}{ccc} X_{2q} & \longleftarrow & X_q \times X_q \\ \downarrow & & \downarrow \\ X_{2q}/\mathrm{O}(2q) & \longleftarrow & \widetilde{X_q}. \end{array}$$

The following is a table of bases for $\ker[\lambda]$, $\mathrm{im}[\lambda]$ and $\mathrm{coker}[\lambda]$ for $q = 2, 3$. The image of a class $\alpha \in H^*(\mathrm{WO}_{2q})$ under $[\lambda]$ is denoted by $[\alpha]_\lambda$.

THEOREM 2.6 ([5, Theorem 1.8]).

1) *As a basis for the image of* $H^*(\mathrm{WO}_4)$ *in* $H^*(\mathrm{WU}_2)$, *we can take the following classes*:

$$[c_2]_\lambda, \quad [h_1c_2^2]_\lambda, \quad [h_1c_1^2c_2]_\lambda, \quad [h_1c_1^4]_\lambda.$$

2) *The classes in Table 2.1 form a basis for the kernel of* $[\lambda]$.
3) *The image is equal to* $\langle[\overline{v}_1]^2 - 2[\overline{v}_2]\rangle \oplus H^9(\mathrm{WU}_2)$, *where* $\langle[\overline{v}_1]^2 - 2[\overline{v}_2]\rangle$ *denotes the linear subspace spanned by* $[\overline{v}_1]^2 - 2[\overline{v}_2]$. *In particular, the subspace spanned by the classes* $[h_1c_1^4]$, $[h_1c_1^2c_2]$ *and* $[h_1c_2^2]$ *in* $H^9(\mathrm{WO}_4)$ *is mapped isomorphically to* $H^9(\mathrm{WU}_2)$.
4) *The cokernel consists of the secondary classes of* $H^*(\mathrm{WU}_2)$ *which are not of degree* 9 *and the subspace spanned by the classes* $[\overline{v}_1]$, $[\overline{v}_1]^2 + 2[\overline{v}_2]$.

THEOREM 2.7 ([5, Theorem 1.9]).

1) *A basis for the image of* $H^*(\mathrm{WO}_6)$ *in* $H^*(\mathrm{WU}_3)$ *is given by Table 2.2.*
2) *A basis for the kernel of* $[\lambda]$ *is given by Table 2.3.*
3) *The image is described as follows*:
   i) *The only Chern class in the image is* $[\overline{v}_1]^2 - 2[\overline{v}_2]$.
   ii) *The image of the secondary classes is contained in the subspace* $H^{13}(\mathrm{WU}_3) \oplus H^{17}(\mathrm{WU}_3) \oplus H^{18}(\mathrm{WU}_3)$, *more precisely,*
      - *the subspace of* $H^{13}(\mathrm{WO}_6)$ *spanned by the classes*

        $$[h_1c_1^6],\ [h_1c_1^4c_2],\ [h_1c_1^3c_3],\ [h_1c_1c_2c_3],\ [h_1c_3^2],\ [h_1c_1^2c_2^2]$$

        is mapped to the subspace of $H^{13}(\mathrm{WU}_3)$ *spanned by the classes*

        $$[\widetilde{u}_1v_1^3\overline{v}_1^3],\ [\widetilde{u}_1v_1v_2\overline{v}_1^3],\ [\widetilde{u}_1v_1v_2\overline{v}_1\overline{v}_2],\ [\widetilde{u}_1v_1v_2\overline{v}_3],\ [\widetilde{u}_1v_3\overline{v}_1^3],$$
        $$[\widetilde{u}_1v_3\overline{v}_3].$$

        The class $[h_3c_2^2]$ *is mapped to the class* $[\widetilde{u}_2v_1v_2\overline{v}_2] - [\widetilde{u}_2v_2\overline{v}_3]$ *modulo the above subspace.*
      - *The class* $[h_3c_3^2]$, *of degree* 17, *is mapped to the class* $[\widetilde{u}_3v_3\overline{v}_3]$.

4	$[c_2]^2$, $[c_4]$,
9	$[h_3c_2] - \frac{1}{2}[h_1c_1c_3]$, $[h_1c_4] - \frac{1}{2}[h_1c_2^2] + \frac{1}{12}[h_1c_1^4]$, $[h_1c_1c_3] - [h_1c_1^2c_2] + \frac{1}{3}[h_1c_1^4]$
	$[h_3c_J]$, where $\lvert J\rvert \geq 3$
	$[h_1h_3c_J]$, where $\lvert J\rvert \geq 4$

TABLE 2.1. A basis for the kernel of $[\lambda]\colon H^*(\mathrm{WO}_4) \to H^*(\mathrm{WU}_2)$.

4	$[c_2]_\lambda$
13	$[h_1c_3^2]_\lambda, [h_1c_1c_2c_3]_\lambda$, $[h_1c_1^3c_3]_\lambda$, $[h_1c_1^4c_2]_\lambda$, $[h_1c_1^2c_2^2]_\lambda$, $[h_1c_1^6]_\lambda$, $[h_3c_2^2]_\lambda$
17	$[h_3c_3^2]_\lambda$
18	$[h_1h_3c_3^2]_\lambda$, $[h_1h_3c_1c_2c_3]_\lambda$, $[h_1h_3c_1^3c_3]_\lambda$, $[h_1h_3c_1^4c_2]_\lambda$, $[h_1h_3c_1^2c_2^2]_\lambda$, $[h_1h_3c_1^6]_\lambda$

TABLE 2.2. A basis for the image of $[\lambda]\colon H^*(\mathrm{WO}_6) \to H^*(\mathrm{WU}_3)$.

- *The subspace spanned by the classes*

  $$[h_1h_3c_1^6],\ [h_1h_3c_1c_2c_3],\ [h_1h_3c_1^3c_3],\ [h_1h_3c_3^2],\ [h_1h_3c_1^4c_2],\ [h_1h_3c_1^2c_2^2],$$

  which are of degree 18, *is mapped to the subspace spanned by the classes*

  $$[\widetilde{u}_1\widetilde{u}_3v_1^3\overline{v}_1^3],\ [\widetilde{u}_1\widetilde{u}_3(v_1v_2\overline{v}_3 + v_3\overline{v}_1\overline{v}_2)],\ [\widetilde{u}_1\widetilde{u}_3(v_1^3\overline{v}_3 + v_3\overline{v}_1^3)],$$
  $$[\widetilde{u}_1\widetilde{u}_3v_3\overline{v}_3],\ [\widetilde{u}_1\widetilde{u}_3v_1v_2\overline{v}_1^3],\ [\widetilde{u}_1\widetilde{u}_3v_1v_2\overline{v}_1\overline{v}_2].$$

4) *The following classes form a basis for the cokernel, namely,*
   i) *the classes of degree not equal to* 4, 13, 17, 18,
   ii) *the class* $[\widetilde{u}_2v_1v_2\overline{v}_2] + [\widetilde{u}_2v_2\overline{v}_3]$ (*of degree* 13),
   iii) *the classes* $[\widetilde{u}_1\widetilde{u}_3(v_1^3\overline{v}_3 - v_3\overline{v}_1^3)]$, $[\widetilde{u}_1\widetilde{u}_3(v_1v_2\overline{v}_3 - v_3\overline{v}_1\overline{v}_2)]$, $[\widetilde{u}_2\widetilde{u}_3v_1v_2\overline{v}_2]$, $[\widetilde{u}_2\widetilde{u}_3v_2\overline{v}_3]$ *and* $[\widetilde{u}_2\widetilde{u}_3v_3\overline{v}_2]$ (*of degree* 18), *and*
   iv) *the class* $[\overline{v}_1]^2 + 2[\overline{v}_2]$ (*of degree* 4).

	the Pontrjagin classes other than $[c_2]$
13	$[h_1c_2^3] - \frac{1}{8}[h_1c_1^6] + \frac{3}{4}[h_1c_1^4c_2] - \frac{3}{2}[h_1c_1^2c_2^2]$, $[h_1c_2c_4] - \frac{1}{16}[h_1c_1^6] - [h_1c_1c_2c_3] + \frac{1}{4}[h_1c_1^2c_2^2] + \frac{1}{8}[h_1c_1^4c_2]$, $[h_1c_1^2c_4] - \frac{1}{4}[h_1c_1^6] + [h_1c_1^4c_2] - [h_1c_1^3c_3] - \frac{1}{2}[h_1c_1^2c_2^2]$, $[h_1c_1c_5] - [h_1c_1c_2c_3] - \frac{1}{20}[h_1c_1^6] + \frac{1}{2}[h_1c_1^2c_2^2]$, $[h_1c_6] + \frac{1}{80}[h_1c_1^6] - \frac{1}{8}[h_1c_1^4c_2] + \frac{1}{4}[h_1c_1^2c_2^2] - \frac{1}{2}[h_1c_3^2]$, $[h_3c_4] - \frac{1}{4}[h_1c_1^3c_3] + [h_1c_1c_2c_3] - [h_1c_3^2] - \frac{1}{2}[h_3c_2^2]$, $[h_5c_2] - \frac{1}{2}[h_1c_1c_5]$
15	$[h_3c_5]$, $[h_3c_2c_3]$
17	$[h_3c_6] - \frac{1}{2}[h_3c_3^2]$, $[h_5c_4]$, $[h_5c_2^2]$, $[h_3c_2^3]$, $[h_3c_2c_4]$
18	$[h_1h_3c_2^3] - \frac{1}{8}[h_1h_3c_1^6] + \frac{3}{4}[h_1h_3c_1^4c_2] - \frac{3}{2}[h_1h_3c_1^2c_2^2]$, $[h_1h_3c_2c_4] - \frac{1}{16}[h_1h_3c_1^6] - [h_1h_3c_1c_2c_3] + \frac{1}{4}[h_1h_3c_1^2c_2^2] + \frac{1}{8}[h_1h_3c_1^4c_2]$, $[h_1h_3c_1^2c_4] - \frac{1}{4}[h_1h_3c_1^6] + [h_1h_3c_1^4c_2] - [h_1h_3c_1^3c_3] - \frac{1}{2}[h_1h_3c_1^2c_2^2]$, $[h_1h_3c_1c_5] - [h_1h_3c_1c_2c_3] - \frac{1}{20}[h_1h_3c_1^6] + \frac{1}{2}[h_1h_3c_1^2c_2^2]$, $[h_1h_3c_6] + \frac{1}{80}[h_1h_3c_1^6] - \frac{1}{8}[h_1h_3c_1^4c_2] + \frac{1}{4}[h_1h_3c_1^2c_2^2] - \frac{1}{2}[h_1h_3c_3^2]$
19	$[h_5c_5]$
	the secondary classes of degree greater than 20

TABLE 2.3. A basis for the kernel of $[\lambda]\colon H^*(\mathrm{WO}_6) \to H^*(\mathrm{WU}_3)$.