Chapter 7

Frequencies around multiplicities

Finally, let us turn to finding estimates for the first term of (6.2.3), which we may write in the form

$$\int_{\Omega} e^{ix\cdot\xi}\Big(\sum_{k=1}^{L} e^{i\tau_k(\xi)t} A_j^k(t,\xi)\Big)\chi(\xi)\widehat{f}(\xi)\,d\xi\,,$$

where the characteristic roots $\tau_1(\xi),\dots,\tau_L(\xi)$ coincide in a set $\mathcal{M}\subset\Omega$ of codimension ℓ (in the sense of Section 2.1), $\Omega\subset\mathbb{R}^n$ is a bounded open set and $\chi\in C_0^\infty(\Omega)$.

As before, we must consider the cases where the image of the phase function(s) either lie on the real axis, are separated from the real axis or meet the real axis. One additional thing to note in this case is that in principle the order of contact at points of multiplicity may be infinite as the roots are not necessarily analytic at such points; we have no examples of such a situation occurring, so it is not worth studying too deeply unless such an example can be found—for now, we can use the same technique as if the point(s) were points where the roots lie entirely on the real axis, and the results in these two situations are given together in Theorem 2.4.1. We study this very briefly nevertheless to ensure the completeness of the obtained results.

Unlike in the case away from multiplicities of characteristic roots, we have no explicit representation for the coefficients $A_j^k(t,\xi)$ (as we have in Lemma 6.1.1 away from the multiplicities), which in turn means we cannot split this into L separate integrals. To overcome this, we first show, in Section 7.1, that a useful representation for the above integral does exist that allows us to use techniques from earlier. Using this alternative representation, it is a simple matter to find estimates in the case where the image of the set $\mathcal{M}$

mapped by the characteristic roots is separated from the real axis (this is Theorem 2.1.2) and when it arises on the real axis as a result of all the roots meeting the axis with finite order, and these are done in Sections 7.2 and 7.3, respectively.

The situations where the roots meet on the real axis and at least one has a zero of infinite order there (either because it fully lies on the axis, or because it meets the axis with infinite order) is slightly more complicated; this is discussed in Section 7.4.

7.1 Resolution of multiple roots

In this section, we find estimates for

$$\sum_{k=1}^{L} e^{i\tau_k(\xi)t} A_j^k(t,\xi)\,,$$

corresponding to (2.1.4), where $\tau_1(\xi),\ldots,\tau_L(\xi)$ coincide in a set $\mathcal{M}$ of codimension ℓ. For simplicity, first consider the simplest case of two roots intersecting at a single point, so that we have $L=2$ and $\mathcal{M}=\{\xi^0\}$; the general case works in a similar way, and we shall show how it differs below. So, assume

$$\tau_1(\xi^0)=\tau_2(\xi^0) \text{ and } \tau_k(\xi^0)\neq\tau_1(\xi^0) \text{ for } k=3,\ldots,m\,;$$

by continuity, there exists a ball of radius $\varepsilon>0$ about ξ^0, $B_\varepsilon(\xi^0)$, in which the only root which coincides with $\tau_1(\xi)$ is $\tau_2(\xi)$. Then:

Lemma 7.1.1. *For all $t\geq 0$ and $\xi\in B_\varepsilon(\xi^0)$, we have*

$$\Big|\sum_{k=1}^{2} e^{i\tau_k(\xi)t} A_j^k(t,\xi)\Big| \leq C(1+t)e^{-\min(\operatorname{Im}\tau_1(\xi),\operatorname{Im}\tau_2(\xi))t}\,, \tag{7.1.1}$$

where the minimum is taken over $\xi\in B_\varepsilon(\xi^0)$.

Proof. First, note that in the set

$$S:=\{\xi\in\mathbb{R}^n : \tau_1(\xi)\neq\tau_k(\xi)\ \forall k=2,\ldots,m \text{ and } \tau_2(\xi)\neq\tau_l(\xi) \quad \forall l=3,\ldots,m\}$$

the formula (6.1.3) is valid for $A_j^1(\xi)$ and $A_j^2(\xi)$. Now, recall that the sum $E_j(t,\xi)=\sum_{k=1}^m e^{i\tau_k(\xi)t}A_j^k(t,\xi)$ is the solution to the Cauchy problem (6.1.1a), (6.1.1c), and thus is continuous; therefore, for all $\eta\in\mathbb{R}^n$ such that $\tau_1(\eta)\neq\tau_k(\eta)$ and $\tau_2(\eta)\neq\tau_k(\eta)$ for $k=3,\ldots,m$ (but allow $\tau_1(\eta)=\tau_2(\eta)$), we have

$$\sum_{k=1}^{2} e^{i\tau_k(\eta)t} A_j^k(t,\eta) = \lim_{\xi\to\eta}\left(e^{i\tau_1(\xi)t}A_j^1(\xi)+e^{i\tau_2(\xi)t}A_j^2(\xi)\right),$$

provided ξ varies in the set S (thus, ensuring $e^{i\tau_1(\xi)t}A_j^1(\xi) + e^{i\tau_2(\xi)t}A_j^2(\xi)$ is well-defined). Hence, to obtain (7.1.1) for all $\xi \in B_\varepsilon(\xi^0)$, it suffices to show

$$\left|e^{i\tau_1(\xi)t}A_j^1(\xi) + e^{i\tau_2(\xi)t}A_j^2(\xi)\right| \le C(1+t)e^{-\min(\operatorname{Im}\tau_1(\xi),\operatorname{Im}\tau_2(\xi))t}$$

for all $t \ge 0$, $\xi \in B'_\varepsilon(\xi^0) = B_\varepsilon(\xi^0) \setminus \{\xi^0\}$.

Now, note the following trivial equality:

$$\begin{aligned}K_1e^{iy_1} + K_2e^{iy_2} &= K_1e^{iy_2}e^{i(y_1-y_2)} + K_2e^{iy_1}e^{-i(y_1-y_2)}\\ &= \frac{e^{i(y_1-y_2)} - e^{-i(y_1-y_2)}}{2}K_1e^{iy_2} + \frac{e^{i(y_1-y_2)} + e^{-i(y_1-y_2)}}{2}K_1e^{iy_2}\\ &\quad + \frac{e^{-i(y_1-y_2)} - e^{i(y_1-y_2)}}{2}K_2e^{iy_1} + \frac{e^{-i(y_1-y_2)} + e^{i(y_1-y_2)}}{2}K_2e^{iy_1}\\ &= \sinh(y_1-y_2)[K_1e^{iy_2} - K_2e^{iy_1}] + \cosh(y_1-y_2)[K_1e^{iy_2} + K_2e^{iy_1}]\,.\end{aligned}$$

Using this, we have, for all $\xi \in B'_\varepsilon(\xi^0)$, $t \ge 0$,

$$\begin{aligned}&e^{i\tau_1(\xi)t}A_j^1(\xi) + e^{i\tau_2(\xi)t}A_j^2(\xi)\\ &\qquad = \sinh[(\tau_1(\xi)-\tau_2(\xi))t](e^{i\tau_2(\xi)t}A_j^1(\xi) - e^{i\tau_1(\xi)t}A_j^2(\xi))\\ &\qquad\quad + \cosh[(\tau_1(\xi)-\tau_2(\xi))t](e^{i\tau_2(\xi)t}A_j^1(\xi) + e^{i\tau_1(\xi)t}A_j^2(\xi))\,.\end{aligned}\tag{7.1.2}$$

We estimate each of these terms:

(a) <u>"sinh" term</u>: The first term is simple to estimate: since

$$\frac{\sinh[(\tau_1(\xi)-\tau_2(\xi))t]}{(\tau_1(\xi)-\tau_2(\xi))} \to t \quad \text{as} \quad (\tau_1(\xi)-\tau_2(\xi)) \to 0\,,$$

or, equivalently, as $\xi \to \xi^0$ through S, and $A_j^k(\xi)(\tau_1(\xi)-\tau_2(\xi))$ is continuous in $B_\varepsilon(\xi^0)$ for $k = 1, 2$, it follows that, for all $\xi \in B'_\varepsilon(\xi^0)$, $t \ge 0$, we have

$$\begin{aligned}&\left|\sinh[(\tau_1(\xi)-\tau_2(\xi))t](A_j^1(\xi)e^{i\tau_2(\xi)t} - A_j^2(\xi)e^{i\tau_1(\xi)t})\right|\\ &\qquad \le Ct[|e^{i\tau_2(\xi)t}| + |e^{i\tau_1(\xi)t}|] \le Cte^{-\min(\operatorname{Im}\tau_1(\xi),\operatorname{Im}\tau_2(\xi))t}\,.\end{aligned}\tag{7.1.3}$$

(b) <u>"cosh" term</u>: Estimating the second term is slightly more complicated. First, recall the explicit representation (6.1.3) for the $A_j^k(\xi)$ at points away from multiplicities of $\tau_k(\xi)$

$$A_j^k(\xi) = \frac{(-1)^j \displaystyle\sum_{1\le s_1<\cdots<s_{m-j-1}\le m}^{k} \prod_{q=1}^{m-j-1} \tau_{s_q}(\xi)}{\displaystyle\prod_{l=1,l\ne k}^{m} (\tau_l(\xi)-\tau_k(\xi))}\,.$$

So, we can write

$$\cosh[(\tau_1(\xi)-\tau_2(\xi))t](A_j^1(\xi)e^{i\tau_2(\xi)t}+A_j^2(\xi)e^{i\tau_1(\xi)t})$$
$$=\frac{\cosh[(\tau_1(\xi)-\tau_2(\xi))t]}{\prod_{k=3}^m(\tau_k(\xi)-\tau_1(\xi))(\tau_k(\xi)-\tau_2(\xi))}\frac{e^{i\tau_2(\xi)t}F_{j+1}^{1,2}(\xi)-e^{i\tau_1(\xi)t}F_{j+1}^{2,1}(\xi)}{\tau_1(\xi)-\tau_2(\xi)},$$

where

$$F_i^{\rho,\sigma}(\xi):=\left(\sum_{1\le s_1<\cdots<s_{m-i}\le m}^{\rho}\prod_{q=1}^{m-i}\tau_{s_q}(\xi)\right)\prod_{k=1,k\neq\rho,\sigma}^{m}(\tau_k(\xi)-\tau_\sigma(\xi)).$$

Now, $\big(\cosh[(\tau_1(\xi)-\tau_2(\xi))t]\big)/\big(\prod_{k=3}^m(\tau_k(\xi)-\tau_1(\xi))(\tau_k(\xi)-\tau_2(\xi))\big)$ is continuous in S, hence it is bounded there, and, thus, absolutely converges to a constant, $C\ge 0$ say, as $\xi\to\xi^0$ through S. This leaves the $[e^{i\tau_2(\xi)t}F_{j+1}^{1,2}(\xi)-e^{i\tau_1(\xi)t}F_{j+1}^{2,1}(\xi)]/(\tau_1(\xi)-\tau_2(\xi))$ term.

For this, write

$$F_i^{\rho,\sigma}(\xi)=\sum_{\kappa=0}^{m-1}Q_{\kappa,i}^{\rho,\sigma}(\xi)\tau_\sigma(\xi)^\kappa,$$

where the $Q_{\kappa,i}^{\rho,\sigma}(\xi)$ are polynomials in the $\tau_k(\xi)$ for $k\neq\rho,\sigma$ (which depend on i); also, note $Q_{\kappa,i}^{\rho,\sigma}(\xi)=Q_{\kappa,i}^{\sigma,\rho}(\xi)$. Then, we have

$$\frac{e^{i\tau_2(\xi)t}F_{j+1}^{1,2}(\xi)-e^{i\tau_1(\xi)t}F_{j+1}^{2,1}(\xi)}{\tau_1(\xi)-\tau_2(\xi)}=\frac{\sum_{\kappa=0}^{m-1}\left[Q_{\kappa,j+1}^{1,2}(\xi)(\tau_2(\xi)^\kappa e^{i\tau_2(\xi)t}-\tau_1(\xi)^\kappa e^{i\tau_1(\xi)t})\right]}{\tau_1(\xi)-\tau_2(\xi)}. \quad (7.1.4)$$

Let us show that this is continuous in $B_\varepsilon(\xi^0)$ and is bounded absolutely by $Cte^{-\min\{\lambda_1,\lambda_2\}}$: for $y_1\neq y_2$, and for all $r,s\in\mathbb{N}$, $t\ge 0$, we have

$$\frac{y_2^sy_1^re^{iy_2t}-y_1^sy_2^re^{iy_1t}}{y_1-y_2}=\frac{y_2^sy_1^r(e^{iy_2t}-e^{iy_1t})}{y_1-y_2}+\frac{y_2^se^{iy_1t}(y_1^r-y_2^r)}{y_1-y_2}+\frac{e^{iy_1t}y_2^r(y_2^s-y_1^s)}{y_1-y_2}.$$

Furthermore, for all $y_1,y_2\in\mathbb{C}$, $t\in[0,\infty)$, $s\in\mathbb{N}$,

$$\left|\frac{e^{iy_2t}-e^{iy_1t}}{y_1-y_2}\right|\le C_0te^{-\min(\operatorname{Im}y_1,\operatorname{Im}y_2)t}\quad\text{and}\quad\left|\frac{y_1^s-y_2^s}{y_1-y_2}\right|\le C_s,$$

for some constants C_0, C_s. Using these with $y_1 = \tau_1(\xi)$, $y_2 = \tau_2(\xi)$, $r = \kappa$, and s chosen appropriately for $Q^{1,2}_{\kappa,j+1}(\xi)$, the continuity and upper bound follow immediately. Thus, for all $\xi \in B'_\varepsilon(\xi^0)$, $t \geq 0$,

$$|\cosh[(\tau_1(\xi) - \tau_2(\xi))t](A^1_j(\xi)e^{i\tau_2(\xi)t} + A^2_j(\xi)e^{i\tau_1(\xi)t})| \\ \leq Cte^{-\min(\operatorname{Im}\tau_1(\xi),\operatorname{Im}\tau_2(\xi))t}. \quad (7.1.5)$$

Combining (7.1.2), (7.1.3) and (7.1.5) we have (7.1.1), which completes the proof of the lemma. □

Now we show that a similar result holds in the general case: suppose the characteristic roots $\tau_1(\xi), \ldots, \tau_L(\xi)$, $2 \leq L \leq m$, coincide in a set $\mathcal{M}$, and that $\tau_1(\xi) \neq \tau_k(\xi)$ for all $\xi \in \mathcal{M}$ when $k = L+1, \ldots, m$. By continuity, we may take $\varepsilon > 0$ so that the set $\mathcal{M}^\varepsilon = \{\xi \in \mathbb{R}^n : \operatorname{dist}(\xi, \mathcal{M}) < \varepsilon\}$ contains no points η at which $\tau_1(\eta), \ldots, \tau_L(\eta) = \tau_k(\eta)$ for $k = L+1, \ldots, m$. With this notation, we have:

Lemma 7.1.2. *For all $t \geq 0$ and $\xi \in \mathcal{M}^\varepsilon$, we have the estimate*

$$\Big|\sum_{k=1}^{L} e^{i\tau_k(\xi)t}A^k_j(t,\xi)\Big| \leq C(1+t)^{L-1}e^{-t\min_{k=1,\ldots,L}\operatorname{Im}\tau_k(\xi)}, \quad (7.1.6)$$

where the minimum is taken over $\xi \in \mathcal{M}^\varepsilon$.

Note that this estimate does not depend on the codimension of $\mathcal{M}$.

Proof. First note that, just as in the previous proof, for all $\eta \in \mathbb{R}^n$ such that $\tau_1(\eta) \ldots, \tau_L(\eta) \neq \tau_k(\eta)$ when $k = L+1, \ldots, m$ (but allowing any or all of $\tau_1(\eta), \ldots, \tau_L(\eta)$ to be equal),

$$\sum_{k=1}^{L} e^{i\tau_k(\eta)t}A^k_j(t,\eta) = \lim_{\xi\to\eta}\left(e^{i\tau_1(\xi)t}A^1_j(\xi) + \cdots + e^{i\tau_L(\xi)t}A^L_j(\xi)\right),$$

provided ξ to varies the set $S := \bigcup_{l=1}^{L} S_l$, where

$$S_l := \{\xi \in \mathbb{R}^n : \tau_l(\xi) \neq \tau_k(\xi)\ \forall k \neq l\},$$

to ensure that each term of the sum on the right-hand side is well-defined. Note that Lemma 6.2.1 ensures every point in $\mathcal{M}$ is the limit of a sequence of points in S in the case of differential operators. Thus, we must simply show, for all $t \geq 0$, $\xi \in (\mathcal{M}^\varepsilon)' = \mathcal{M}^\varepsilon \setminus \mathcal{M}$, that we have the estimate

$$\left|e^{i\tau_1(\xi)t}A^1_j(\xi) + \cdots + e^{i\tau_L(\xi)t}A^L_j(\xi)\right| \leq C(1+t)^{L-1}e^{-t\min_{k=1,\ldots,L}\operatorname{Im}\tau_k(\xi)}.$$

Now, we claim that we can write $\sum_{k=1}^{L} e^{i\tau_k(\xi)t} A_j^k(t,\xi)$, for $\xi \in (\mathcal{M}^\varepsilon)'$ and $t \geq 0$, as a sum of terms involving products of $\frac{(L-1)L}{2}$ sinh and cosh terms of differences of coinciding roots; to clarify, (7.1.2) is this kind of representation for $L = 2$, while for $L = 3$, we want sums of terms such as

$$\sinh[\alpha_1(\tau_1(\xi) - \tau_2(\xi))t] \cosh[\alpha_2(\tau_1(\xi) - \tau_3(\xi))t] \sinh[\alpha_3(\tau_2(\xi) - \tau_3(\xi))t]\,,$$

where the α_i are appropriately chosen constants; incidentally, a comparison to the $L = 2$ case suggests that the term above is multiplied by

$$\left(A_j^1(\xi) e^{i\tau_2(\xi)t} - A_j^2(\xi) e^{i\tau_1(\xi)t}\right)$$

in the full representation.

To show this, we do induction on L; Lemma 7.1.1 gives us the case $L = 2$ (note that the proof holds with ξ^0 and $B_\varepsilon(\xi^0)$ replaced throughout by $\mathcal{M}$ and $\mathcal{M}^\varepsilon$, respectively). Assume there is such a representation for $L = K \leq m-1$. Observe,

$$\sum_{k=1}^{K+1} e^{i\tau_k(\xi)t} A_j^k(\xi) = \frac{1}{K} \sum_{k=1}^{K} e^{i\tau_k(\xi)t} A_j^k(\xi) + \frac{1}{K} \sum_{k=1, k\neq K}^{K+1} e^{i\tau_k(\xi)t} A_j^k(\xi) + \cdots + \frac{1}{K} \sum_{k=2}^{K+1} e^{i\tau_k(\xi)t} A_j^k(\xi)\,;$$

by the induction hypothesis, there is a representation for each of these terms by means of products of $\frac{(K-1)K}{2}$

$$\sinh[\alpha_{k,l}(\tau_k(\xi) - \tau_l(\xi))t] \text{ and } \cosh[\beta_{k,l}(\tau_k(\xi) - \tau_l(\xi))t] \text{ terms,}$$

where $1 \leq k, l \leq K+1$ and the $\alpha_{k,l}, \beta_{k,l}$ are some non-zero constants. Next, note that we can write $(\tau_1(\xi) - \tau_2(\xi))$ (or, indeed, the difference of any pair of roots from $\tau_1(\xi), \ldots, \tau_{K+1}(\xi)$) as a linear combination of the $\frac{K(K+1)}{2}$ differences $\tau_k(\xi) - \tau_l(\xi)$ such that $1 \leq k < l \leq K+1$; that is

$$\sinh[\alpha_{1,2}(\tau_1(\xi) - \tau_2(\xi))t] = \sinh\Big[\sum_{1\leq k<l\leq K+1} \alpha'_{k,l}(\tau_k(\xi) - \tau_l(\xi))t\Big]\,,$$

for some non-zero constants $\alpha'_{k,l}$; similarly, there is such a representation for $\cosh[\beta_{1,2}(\tau_1(\xi) - \tau_2(\xi))t]$. Lastly, repeated application of the double angle formulae

$$\sinh(a \pm b) = \sinh a \cosh b \pm \cosh a \sinh b\,,$$
$$\cosh(a \pm b) = \cosh a \cosh b \pm \sinh a \sinh b\,,$$

yields products of $\frac{K(K+1)}{2}$ terms, which completes the induction step.

Now, as in the previous proof, each of these terms must be estimated. The key fact to observe is that

$$A_j^k(\xi) \prod_{l=1, l\neq k}^{L} (\tau_l(\xi) - \tau_k(\xi))$$

is continuous in $\mathcal{M}^\varepsilon$ for all $k = 1, \dots, L$. Then, using the same arguments as for each of the terms in the earlier proof, and observing that the exponent of t is determined by the products involving either

(a) $(\sinh[\alpha_{k,l}(\tau_k(\xi) - \tau_l(\xi)t)])/(\tau_k(\xi) - \tau_l(\xi))$ terms, or

(b) $(e^{i\tau_k(\xi)t} - e^{i\tau_l(\xi)t})/(\tau_k(\xi) - \tau_l(\xi))$ terms (see (7.1.4)),

the estimate (7.1.6) is immediately obtained. □

7.2 Phase separated from the real axis: Theorem 2.1.2

We now turn back to finding $L^p - L^q$ estimates for

$$\int_\Omega e^{ix\cdot\xi}\Big(\sum_{k=1}^{L} e^{i\tau_k(\xi)t} A_j^k(t,\xi)\Big)\chi(\xi)\widehat{f}(\xi)\,d\xi\,,$$

when $\tau_1(\xi), \dots, \tau_L(\xi)$ coincide in a set $\mathcal{M}$ of codimension ℓ; choose $\varepsilon > 0$ so that these roots do not intersect with any of the roots $\tau_{L+1}(\xi), \dots, \tau_m(\xi)$ in $\mathcal{M}^\varepsilon$. The set Ω is bounded, and we may take $\chi \in C_0^\infty(\mathcal{M}^\epsilon)$.

In this section (under assumptions of Theorem 2.1.2), we assume that there exists $\delta > 0$ such that $\operatorname{Im}\tau_k(\xi) \geq \delta$ for all $\xi \in \mathcal{M}^\varepsilon$—so, $\min_k \operatorname{Im}\tau_k(\xi) \geq \delta > 0$. For this, we use the same approach as in Section 6.10, but using Lemma 7.1.2 to estimate the sum. Firstly, the $L^1 - L^\infty$ estimate:

$$\begin{aligned}
&\Big\|D_t^r D_x^\alpha\Big(\int_\Omega e^{ix\cdot\xi}\Big(\sum_{k=1}^{L} e^{i\tau_k(\xi)t}A_j^k(t,\xi)\Big)\chi(\xi)\widehat{f}(\xi)\,dx\Big)\Big\|_{L^\infty(\mathbb{R}^n_x)} \\
&\qquad= \Big\|\int_\Omega e^{ix\cdot\xi}\Big(\sum_{k=1}^{L} e^{i\tau_k(\xi)t}A_j^k(t,\xi)\tau_k(\xi)^r\Big)\xi^\alpha\chi(\xi)\widehat{f}(\xi)\,dx\Big\|_{L^\infty(\mathbb{R}^n_x)} \\
&\qquad\leq \max_k \sup_\Omega|\tau_k(\xi)|^r \int_{\mathcal{M}^\varepsilon}\Big|\sum_{k=1}^{L} e^{i\tau_k(\xi)t}A_j^k(t,\xi)\Big||\xi|^{|\alpha|}|\widehat{f}(\xi)|\,dx \\
&\qquad\leq C(1+t)^{L-1}e^{-\delta t}\|\widehat{f}\|_{L^\infty(\mathcal{M}^\varepsilon)} \leq C(1+t)^{L-1}e^{-\delta t}\|f\|_{L^1}\,.
\end{aligned}$$

Similarly, the $L^2 - L^2$ estimate:

$$\begin{aligned}\Big\|D_t^r D_x^\alpha\Big(\int_\Omega e^{ix\cdot\xi}\Big(\sum_{k=1}^{L} e^{i\tau_k(\xi)t}A_j^k(t,\xi)\Big)\chi(\xi)\widehat{f}(\xi)\,dx\Big)\Big\|_{L^2(\mathbb{R}^n_x)}\\ = \Big\|\Big(\sum_{k=1}^{L} e^{i\tau_k(\xi)t}A_j^k(t,\xi)\tau_k(\xi)^r\Big)\xi^\alpha\chi(\xi)\widehat{f}(\xi)\Big\|_{L^2(\Omega)}\\ \le C(1+t)^{L-1}e^{-\delta t}\|\widehat{f}\|_{L^2(\Omega)} \le C(1+t)^{L-1}e^{-\delta t}\|f\|_{L^2}\,.\end{aligned}$$

Then, Theorem 6.2.3 yields

$$\begin{aligned}\Big\|D_t^r D_x^\alpha\Big(\int_\Omega e^{ix\cdot\xi}\Big(\sum_{k=1}^{L} e^{i\tau_k(\xi)t}A_j^k(t,\xi)\Big)\chi(\xi)\widehat{f}(\xi)\,dx\Big)\Big\|_{L^q(\mathbb{R}^n_x)}\\ \le C(1+t)^{L-1}e^{-\delta t}\|f\|_{L^p}\,,\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$, $1\le p\le 2$. Once again, we have exponential decay. This, together with (6.10.1) gives the statement when there are multiplicities away from the axis and completes the proof of Theorem 2.1.2.

7.3 Phase meeting the real axis: Theorem 2.3.1

We next look at the case where the characteristic roots $\tau_1(\xi),\ldots,\tau_L(\xi)$ that coincide in the C^1 set $\mathcal{M}$ of codimension ℓ meet the real axis in $\mathcal{M}$ with finite orders. If there are more points in $\mathcal{M}$ at which the above roots meet the axis with finite order (or even with infinite order/lying on the axis), they may be considered separately in the same way (or using the method below where necessary), while away from such points, the roots are separated from the axis, and the previous arguments and results of Section 2.1 may be used.

Since the characteristic roots are not necessarily analytic (or even differentiable) in $\mathcal{M}$, we must look at each branch of the roots as they approach the real axis; set s_k to be the maximal order of the contact with the real axis for $\tau_k(\xi)$, that is, the maximal value for which there exist constant $c_0>0$ such that

$$c_0\,\mathrm{dist}(\xi,Z_k)^{s_k}\le \operatorname{Im}\tau_k(\xi)\,,$$

for all ξ sufficiently near Z_k, where $Z_k=\{\xi\in\mathbb{R}^n:\operatorname{Im}\tau_k(\xi)=0\}$. By assumptions of Theorem 2.3.1, we have the estimate

$$c_0\,\mathrm{dist}(\xi,\mathcal{M})^{s}\le \operatorname{Im}\tau_k(\xi)\,,$$

for some $c_0 > 0$ and $s \geq \max(s_1, \ldots, s_L)$, for ξ close to $\mathcal{M}$. We will need the following extension of Proposition 6.11.1. Its proof is similar to the proof of Proposition 6.11.1 if we consider the C^1 coordinate system associated to $\mathcal{M}$. As usual $\mathcal{M}^\epsilon = \{\xi \in \mathbb{R}^n : \operatorname{dist}(\xi, \mathcal{M}) < \epsilon\}$.

Proposition 7.3.1. *Let $U \subset \mathbb{R}^n$ be open and let $\phi : U \to \mathbb{R}$ be a continuous function. Suppose $\mathcal{M} \subset U$ is a C^1 set of codimension ℓ such that*

$$c_0 \operatorname{dist}(\xi, \mathcal{M})^s \leq \phi(\xi)\,,$$

for some $c_0 > 0$, and all $\xi \in \mathcal{M}^\epsilon$ for sufficiently small $\epsilon > 0$. Then, for any function $a(\xi)$ that is bounded and compactly supported in U, and for all $t \geq 0$, $f \in C_0^\infty(\mathbb{R}^n)$, and $r \in \mathbb{R}$, we have

$$\int_{\mathcal{M}^\epsilon} e^{-\phi(\xi)t} \operatorname{dist}(\xi, \mathcal{M})^r |a(\xi)||\widehat{f}(\xi)|\, d\xi \leq C(1+t)^{-(\ell+r)/s} \|f\|_{L^1}\,,$$

and

$$\left\| e^{-\phi(\xi)t} \operatorname{dist}(\xi, \mathcal{M})^r a(\xi) \widehat{f}(\xi) \right\|_{L^2(\mathcal{M}^\epsilon)} \leq C(1+t)^{-r/s} \|f\|_{L^2}\,.$$

The proof of this proposition is similar to the proof of Proposition 6.11.1 and is omitted. Theorem 2.3.1 states that we must have the estimate (2.3.1), which is

$$\begin{aligned}\Big\| D_t^r D_x^\alpha \Big(\int_{\mathcal{M}^\varepsilon} e^{ix\cdot\xi} \Big(\sum_{k=1}^{L} e^{i\tau_k(\xi)t} A_j^k(t,\xi) \Big) \chi(\xi) \widehat{f}(\xi)\, d\xi \Big) \Big\|_{L^q(\mathbb{R}^n_x)} \\ \leq C(1+t)^{-\frac{\ell}{s}\left(\frac{1}{p}-\frac{1}{q}\right)+L-1} \|f\|_{L^p}\,.\end{aligned}$$

By Lemma 7.1.2 and Proposition 7.3.1, to estimate the sum in the amplitude, for all $t \geq 0$, we have

$$\begin{aligned}&\Big\| D_t^r D_x^\alpha \Big(\int_{\mathcal{M}^\varepsilon} e^{ix\cdot\xi} \Big(\sum_{k=1}^{L} e^{i\tau_k(\xi)t} A_j^k(t,\xi) \Big) \chi(\xi) \widehat{f}(\xi)\, d\xi \Big) \Big\|_{L^\infty(\mathbb{R}^n_x)} \\ &\quad \leq C \Big\| \int_{\mathcal{M}^\varepsilon} e^{ix\cdot\xi} \Big(\sum_{k=1}^{L} e^{i\tau_k(\xi)t} A_j^k(t,\xi) \tau_k(\xi)^r \Big) \xi^\alpha \chi(\xi) \widehat{f}(\xi)\, d\xi \Big\|_{L^\infty(\mathbb{R}^n_x)} \\ &\quad \leq C \int_{\mathcal{M}^\varepsilon} (1+t)^{L-1} e^{-t \min_{k=1,\ldots,L} \operatorname{Im} \tau_k(\xi)} |\chi(\xi)||\widehat{f}(\xi)|\, d\xi \\ &\quad \leq C(1+t)^{L-1-(\ell/s)} \|f\|_{L^1}\,.\end{aligned}$$

Also, using the Plancherel's theorem, we have

$$\begin{aligned}
&\Big\| D_t^r D_x^\alpha \Big(\int_{\mathcal{M}^\varepsilon} e^{ix\cdot\xi} \Big(\sum_{k=1}^{L} e^{i\tau_k(\xi)t} A_j^k(t,\xi) \Big) \chi(\xi) \widehat{f}(\xi)\, d\xi \Big) \Big\|_{L^2(\mathbb{R}^n_x)} \\
&\quad = \Big\| \int_{\mathcal{M}^\varepsilon} e^{ix\cdot\xi} \Big(\sum_{k=1}^{L} e^{i\tau_k(\xi)t} A_j^k(t,\xi) \tau_k(\xi)^r \Big) \xi^\alpha \chi(\xi) \widehat{f}(\xi)\, d\xi \Big\|_{L^2(\mathbb{R}^n_x)} \\
&\quad = \Big\| \Big(\sum_{k=1}^{L} e^{i\tau_k(\xi)t} A_j^k(t,\xi) \tau_k(\xi)^r \Big) \xi^\alpha \chi(\xi) \widehat{f}(\xi) \Big\|_{L^2(\mathcal{M}^\varepsilon)} \\
&\quad \le C(1+t)^{L-1} \big\| e^{-t \min_{k=1,\dots,L} \operatorname{Im} \tau_k(\xi)} |\chi(\xi)| |\widehat{f}(\xi)| \big\|_{L^2(\mathcal{M}^\varepsilon)} \\
&\quad \le C(1+t)^{L-1} \|f\|_{L^2}\,.
\end{aligned}$$

Therefore, interpolation Theorem 6.2.3 yields, for all $t \ge 0$,

$$\begin{aligned}
&\Big\| D_t^r D_x^\alpha \Big(\int_{\mathcal{M}^\varepsilon} e^{ix\cdot\xi} \Big(\sum_{k=1}^{L} e^{i\tau_k(\xi)t} A_j^k(t,\xi) \Big) \chi(\xi) \widehat{f}(\xi)\, d\xi \Big) \Big\|_{L^q(\mathbb{R}^n_x)} \\
&\qquad\qquad \le C(1+t)^{-\frac{\ell}{s}\left(\frac{1}{p}-\frac{1}{q}\right)+L-1} \|f\|_{L^p}\,,
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$, $1 \le p \le 2$; this, together with (6.11.3) proves Theorem 2.3.1 for roots meeting the axis with finite order.

7.4 Phase on the real axis for bounded frequencies

Recall that in the division of the integral in Section 6.2, we have

$$\int_{B_{2M}(0)} e^{ix\cdot\xi} \Big(\sum_{k=1}^{m} e^{i\tau_k(\xi)t} A_j^k(t,\xi) \Big) \widehat{f}(\xi)\, d\xi\,,$$

which we then subdivide around and away from multiplicities. The cases where the root or roots are either separated from the real axis or meet it with finite order have already been discussed; here we shall complete the analysis by proving estimates for the situation where a root or roots lie on the real axis. These results can be also applied to the case of multiple roots.

We note that in the case of nonhomogeneous symbols this analysis is essential since time genuinely interacts with frequencies. Unlike in the case of homogeneous symbols in Section 1.2, where one could eliminate time completely from estimates by rescaling, here it is present in phases and amplitude and causes them to blow up even for low frequencies. Thus, we must carry out a detailed investigation of the structure of solutions for low frequencies, and it will be done in this section.

A number of estimates can be already obtained using our results on multiple roots from Section 7.1. To have any possibility of obtaining better estimates, we must impose additional conditions on the characteristic roots at low frequencies—for large $|\xi|$, these properties were obtained by using perturbation results, but naturally such results are no longer valid for $|\xi| \leq M$. Also, we can impose the convexity condition on the roots to obtain a better result than the general case. We will give different formulation of possible results in this section.

Again, throughout we assume that either $\tau(\xi) \geq 0$ for all ξ or $\tau(\xi) \leq 0$ for all ξ. The key point is to use a carefully chosen cut-off function to isolate the multiplicities and then use Theorem 4.3.1 or Theorem 5.1.2 to estimate the integrals where there are no multiplicities (and hence the coefficients $A_j^k(t,\xi)$ are independent of t) and use suitable adjustments around the singularities. For these purposes, let us first assume that the only multiplicity is at a point $\xi^0 \in B_{2M}(0)$ and $\tau_1(\xi^0) = \tau_2(\xi^0)$ are the only coinciding roots, and let χ be a cut-off function around ξ^0. Then, we must consider the sum of the first two roots, where we have a multiplicity at ξ^0,

$$I = \int_{B_{2M}(0)} e^{ix\cdot\xi} \Big(\sum_{k=1}^{2} e^{i\tau_k(\xi)t} A_j^k(t,\xi) \Big) \chi(\xi) \widehat{f}(\xi)\, d\xi \,, \tag{7.4.1}$$

and terms involving the remaining roots, which are all distinct,

$$II = \sum_{k=3}^{m} \int_{B_{2M}(0)} e^{i(x\cdot\xi + \tau_k(\xi))t} A_j^k(t,\xi) \chi(\xi) \widehat{f}(\xi)\, d\xi \,.$$

Case of no multiplicities: Theorem 2.2.6

For the second of these integrals II, we wish to apply Theorem 4.3.1 if $\tau_k(\xi)$ satisfies the convexity condition, and Theorem 5.1.2 otherwise.

In order to ensure the hypotheses of these theorems are satisfied, however, we need to impose an additional regularity condition on the behaviour of the characteristic roots for the relevant frequencies (i.e. $\xi \in B_{2M}(0)$) to avoid pathological situations:

$$\text{Assume } |\partial_\omega \tau_k(\lambda\omega)| \geq C_0 \text{ for all } \omega \in \mathbb{S}^{n-1},\ 2M \geq \lambda > 0. \tag{7.4.2}$$

Since this is satisfied for large $|\xi|$ (see Proposition 3.2.4) and always satisfied for roots of operators with homogeneous symbols, it is quite a natural extra assumption.

The other hypotheses of these theorems hold: hypothesis (i) is satisfied because $|\partial_\xi^\alpha \tau_k(\xi)| \le C_\alpha$ for all ξ since the characteristic roots are smooth in $\mathbb{R}^n$; hypothesis (ii) only requires information about high frequencies; and hypotheses (iv) holds by the same argument as for large $|\xi|$, where only Part II of Proposition 3.2.1 is needed, and that holds for all $\xi \in \mathbb{R}^n$. Also, the coefficients $A_k^j(\xi)$ are smooth away from multiplicities, so the symbolic behaviour (i.e. decay, or bounded for small frequencies) holds.

Now $L^1 - L^\infty$ and $L^2 - L^2$ estimates can be found as in the case for large $|\xi|$, and the interpolation theorem used to give the desired results. Thus, with condition (7.4.2), we have proved the on axis, no multiplicities case of Theorem 2.2.6.

Multiplicities: shrinking neighborhoods

Now we can turn to the other integral given by (7.4.1). Here we will analyse what happens in certain shrinking neighborhoods of multiplicities. First we will assume that only two roots intersect at an isolated point, and then we will indicate what happens in the general situation.

To continue the analysis of an isolated point of multiplicity as in (7.4.1), we introduce a cut-off function $\psi \in C_0^\infty([0,\infty))$, $0 \le \psi(s) \le 1$, which is identically 0 for $s > 1$ and 1 for $s < \frac{3}{4}$; then (7.4.1) can be rewritten as the sum of two integrals $I = I_1 + I_2$, where

$$I_1 = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \psi(t|\xi - \xi^0|)\chi(\xi) \sum_{k=1}^{2} A_j^k(t,\xi) e^{i\tau_k(\xi)t} \widehat{f}(\xi)\, d\xi\,,$$

$$I_2 = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} (1-\psi)(t|\xi - \xi^0|)\chi(\xi) \sum_{k=1}^{2} A_j^k(t,\xi) e^{i\tau_k(\xi)t} \widehat{f}(\xi)\, d\xi\,.$$

We study $L^1 - L^\infty$ estimates for I_1 and $L^2 - L^2$ estimates for both I_1 and I_2 in this section.

$L^1 - L^\infty$ estimates: For this, we use the resolution of multiplicities technique of Section 7.1. By Lemma 7.1.1, we have, in particular,

$$\left| \sum_{k=1}^{2} A_j^k(t,\xi) e^{i\tau_k(\xi)t} \right| \le C(1+t),$$

for $|\xi - \xi^0| < t^{-1}$. Now, we may estimate the integral using the compactness of the support of $\psi(s)$: for $0 \le t \le 1$, I_1 is clearly bounded; for $t > 1$, we have

$$\begin{aligned}|I_1| &\leq Ct\int_{\mathbb{R}^n}|\psi(t|\xi-\xi^0|)||\widehat{f}(\xi)|\,d\xi\\ &= Ct^{1-n}\|\widehat{f}\|_{L^\infty}\int_{\mathbb{R}^n}\psi(|\eta|)\,d\eta \leq C(1+t)^{1-n}\|f\|_{L^1}.\end{aligned}$$

This argument can be extended to the case when L roots meet on a set of codimension ℓ. In the following proposition we will change the notation for the cut-off function to avoid any confusion with point multiplicities in the case above.

Proposition 7.4.1. *Suppose that L roots intersect in a set $\mathcal{M}$ of codimension ℓ. Let $\mathcal{M}^\epsilon = \{\xi \in \mathbb{R}^n : \operatorname{dist}(\xi,\mathcal{M}) < \epsilon\}$, and let $\theta \in C_0^\infty(\mathcal{M}^\epsilon)$ for sufficiently small $\epsilon > 0$. Then we have the estimate*

$$\left|\int_{\mathbb{R}^n} e^{ix\cdot\xi}\theta(t\operatorname{dist}(\xi,\mathcal{M}))\sum_{k=1}^{L} A_j^k(t,\xi)e^{i\tau_k(\xi)t}\widehat{f}(\xi)\,d\xi\right| \leq C(1+t)^{L-1-\ell}. \quad (7.4.3)$$

Proof. By using Lemma 7.1.2 in the (bounded) neighborhood $\mathcal{M}^\epsilon$ of $\mathcal{M}$, we obtain

$$\left|\sum_{k=1}^{L} e^{i\tau_k(\xi)t}A_j^k(t,\xi)\right| \leq C(1+t)^{L-1}\,.$$

The size of the support of $\theta(t\operatorname{dist}(\xi,\mathcal{M}))$ can be bounded by $(1+t)^{-\ell}$, which implies estimate (7.4.3). □

$L^2 - L^2$ estimates: Let us now analyse the L^2-estimate. This analysis will apply not only in a shrinking, but in a fixed neighborhood of the set of multiplicities. We will discuss first the case of two roots intersecting at a point in more detail, thus analysing mainly integral I in (7.4.1). We can have several versions of L^2-estimates dependent on conditions on multiplicities and on the Cauchy data that we can impose. For example, by Lemma 7.1.1 and Plancherel's theorem we get

$$\|I\|_{L^2} \leq C(1+t)\|f\|_{L^2}. \quad (7.4.4)$$

On the other hand we can improve the time behaviour of the L^2-estimate (7.4.4) if we make additional regularity assumptions for the data. For example, we can eliminate time from estimate (7.4.4) if we work in suitable

Sobolev type spaces taking the singularity into account. Let us rewrite

$$\begin{aligned} I &= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\chi(\xi) \sum_{k=1}^{2} A_j^k(t,\xi) e^{i\tau_k(\xi)t} \widehat{f}(\xi)\, d\xi \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\chi(\xi) \left[(\tau_1(\xi)-\tau_2(\xi)) \sum_{k=1}^{2} A_j^k(t,\xi) e^{i\tau_k(\xi)t} \right] \times \\ &\qquad\qquad \times (\tau_1(\xi)-\tau_2(\xi))^{-1} \widehat{f}(\xi)\, d\xi. \end{aligned}$$

Using the representation from Lemma 6.1.1 we see that the expression in the square brackets is bounded. Hence by the Plancherel's theorem we get that

$$\begin{aligned} &\|I\|_{L^2} \leq \\ &\quad \|(\tau_1(\xi)-\tau_2(\xi))^{-1}\chi(\xi)\widehat{f}(\xi)\|_{L^2} = \|(\tau_1(D)-\tau_2(D))^{-1}\chi(D)f\|_{L^2}. \end{aligned} \tag{7.4.5}$$

An example of this is the appearance of homogeneous Sobolev spaces for small frequencies in the analysis of the wave equations, or more general equations with homogeneous symbols. For example, in the case of the wave equation we have $\tau_1(\xi) = |\xi|$ and $\tau_2(\xi) = -|\xi|$, so that (7.4.5) means that we have the low frequency estimate for the solution of the form

$$\|I\|_{L^2} \leq \|f\|_{\dot{H}^{-1}},$$

with the homogeneous Sobolev space $\dot{H}^{-1}$.

In the case of several roots intersecting in a set $\mathcal{M}$, we have similarly:

Proposition 7.4.2. *Suppose that L roots intersect in a set $\mathcal{M}$. Let $\mathcal{M}^\epsilon = \{\xi \in \mathbb{R}^n : \operatorname{dist}(\xi, \mathcal{M}) < \epsilon\}$, and let $\theta \in C_0^\infty(\mathcal{M}^\epsilon)$ for sufficiently small $\epsilon > 0$. Let J denote the part of solution corresponding to these roots microlocalised near the set $\mathcal{M}$ of multiplicities:*

$$J(t,x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi}\theta(\xi) \sum_{k=1}^{L} A_j^k(t,\xi) e^{i\tau_k(\xi)t} \widehat{f}(\xi)\, d\xi.$$

Then we have the estimate

$$||J||_{L^2(\mathbb{R}^n_x)} \leq C(1+t)^{L-1}||f||_{L^2(\mathbb{R}^n_x)}. \tag{7.4.6}$$

Moreover, let us assume without loss of generality that intersecting L roots are labeled by $\tau_1, \cdots, \tau_L$. Then we also have

$$\left|\left| \prod_{1\leq l<k\leq L} (\tau_l(D)-\tau_k(D))^{-1} J \right|\right|_{L^2(\mathbb{R}^n_x)} \leq C||f||_{L^2(\mathbb{R}^n_x)}. \tag{7.4.7}$$

Estimate (7.4.6) follows from Lemma 7.1.2 and Plancherel's theorem. Estimate (7.4.7) follows from Plancherel's theorem and formula (6.1.3).

Interpolating between Propositions 7.4.3 and 7.4.6, we can obtain different versions of the dispersive estimate in a region shrinking around $\mathcal{M}$, depending on whether we use (7.4.6) or (7.4.7).

Multiplicities: fixed neighborhoods

Here, for simplicity, we will concentrate on the case of two roots τ_1 and τ_2 intersecting at an isolated point ξ^0. We will discuss both $L^1 - L^\infty$ and $L^2 - L^2$ estimates under additional assumptions on the roots τ_1 and τ_2.

$L^1 - L^\infty$ estimates: For I_2 we are away from the singularity, so we can use that

$$\sum_{k=1}^{2} A_j^k(t,\xi)e^{i\tau_k(\xi)t} = A_j^1(\xi)e^{i\tau_1(\xi)t} + A_j^2(\xi)e^{i\tau_2(\xi)t}.$$

Now, we would like to apply Theorem 4.3.1 (for the case where the root satisfies the convexity condition) and Theorem 5.1.2 (for the general case), as in the case of simple roots; however, the proximity of the multiplicity brings the additional cut-off function, $(1-\psi)(t|\xi-\xi^0|)$, into play, and this depends on t. Therefore, the aforementioned results cannot be used directly. However, a similar result does hold, provided we impose some additional conditions, producing analogues of Theorems 4.3.1 and 5.1.2 in this case.

Proposition 7.4.3. *Let $\chi \in C_0^\infty(\mathbb{R}^n)$. Suppose $\tau_k(\xi)$, $k=1,2$, satisfy the following assumptions on* $\operatorname{supp}\chi$*:*

(i) *for each multi-index α there exists a constant $C_\alpha > 0$ such that, for some $\delta > 0$,*

$$|\partial_\eta^\alpha[(\nabla_\xi\tau_k)(\xi^0+s\eta)]| \le C_\alpha(1+|\eta|)^{-|\alpha|}, \text{ for small } s>0 \text{ and } |\eta|>\delta;$$

(ii) *there exists a constant $C_0 > 0$ such that $|\partial_\omega\tau_k(\xi^0+\lambda\omega)| \ge C > 0$ for all $\omega \in \mathbb{S}^{n-1}$ and $\lambda > 0$; in particular, each of the level sets*

$$\lambda\Sigma'_\lambda \equiv \Sigma_\lambda = \left\{\eta \in \mathbb{R}^n : \tau_k(\xi^0+\eta) = \lambda\right\}$$

is non-degenerate;

(iii) *there exists a constant $R_1 > 0$ such that, for all $\lambda > 0$,*

$$\Sigma'_\lambda := \frac{1}{\lambda}\Sigma_\lambda(\tau_k) \subset B_{R_1}(0).$$

Furthermore, assume that $A_j^k(\xi)$ satisfies the following condition: for each multi-index α there exists a constant $C_\alpha > 0$ such that

(iv) *we have the estimate*

$$|\partial_\eta^\alpha[A_j^k(\xi^0+s\eta)]| \le C_\alpha s^{-j}(1+|\eta|)^{-j-|\alpha|}\,, \text{ for small } s>0 \text{ and } |\eta|>\delta\,.$$

Finally, assume that $\psi \in C_0^\infty((-\delta,\delta))$ is such that $\psi(\sigma) = 1$ for $|\sigma| \le \delta/2$. Then, the following estimate holds for all $x \in \mathbb{R}^n$, $t \ge 0$:

$$\Big|\sum_{k=1}^{2}\int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau_k(\xi)t)}A_j^k(\xi)(1-\psi)(t|\xi-\xi^0|)\chi(\xi)\,d\xi\Big| \le C(1+t)^{j-n}\,, \quad (7.4.8)$$

for $j \ge n - \frac{n-1}{\gamma}$, where $\gamma := \sup_{\lambda>0}\gamma(\Sigma_\lambda(\tau_k))$, if $\tau_k(\xi)$ satisfies the convexity condition; and for $j \ge n - \frac{1}{\gamma_0}$, where $\gamma_0 := \sup_{\lambda>0}\gamma_0(\Sigma_\lambda(\tau_k))$, if it does not.

Remark 7.4.4. *Conditions* (i), (ii) *and* (iv) *appear and are satisfied naturally when roots $\tau_k(\xi)$ are homogeneous functions of order one—for example, the wave equation, or for homogeneous equations.*

Remark 7.4.5. *Assumption* (iv) *is needed because $A_j^k(\xi)$ has a singularity at ξ^0, so we must ensure we are away from that—this is the role of the cut-off function $(1-\psi)(|\eta|)$ in this proposition;*

Remark 7.4.6. *As usual, for example in the convex case, taking $j = n-\frac{n-1}{\gamma}$, we get the time decay estimate*

$$|\text{ Left hand side of (7.4.8) }| \le C(1+t)^{-\frac{n-1}{\gamma}}\,.$$

Proof. As before, cut-off near the wave front: let $\kappa \in C_0^\infty(\mathbb{R}^n)$ be a cut-off function supported in $B(0,r)$. Then, consider

$$I_1(t,x) := \sum_{k=1}^{2}\int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau_k(\xi)t)}A_j^k(\xi)(1-\psi)(t|\xi-\xi^0|)\chi(\xi)\kappa\big(t^{-1}x+\nabla\tau_k(\xi)\big)\,d\xi,$$

and

$$I_2(t,x) := \sum_{k=1}^{2}\int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau_k(\xi)t)}A_j^k(\xi)(1-\psi)(t|\xi-\xi^0|)\chi(\xi) \\ (1-\kappa)\big(t^{-1}x+\nabla\tau_k(\xi)\big)\,d\xi.$$

Away from the wave front set: First, we estimate $I_2(t,x)$; we claim that

$$|I_2(t,x)| \le C_r(1+t)^{j-n} \text{ for all } t>0\,, x\in\mathbb{R}^n\,. \tag{7.4.9}$$

In order to show this, we consider each term of the sum separately,

$$I_2^k(t,x) = \int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau_k(\xi)t)} A_j^k(\xi)(1-\psi)(t|\xi-\xi^0|)\chi(\xi)(1-\kappa)\big(\tfrac{x}{t}+\nabla\tau_k(\xi)\big)\,d\xi\,,$$

and imitate the proof of Lemma 4.3.3 (in which the corresponding term was estimated in Theorem 4.3.1), but noting that in place of $g_R(\xi)\in C_0^\infty(\mathbb{R}^n)$ we have $(1-\psi)(t(\xi-\xi^0))$, which depends also on t; in particular, this means that care must be taken when carrying out the integration by parts when derivatives fall on $(1-\psi)(t|\xi-\xi^0|)$. To take this into account, use the change of variables $\xi=\xi^0+t^{-1}\eta$:

$$\begin{aligned} I_2^k(t,x) = e^{ix\cdot\xi^0} \int_{\mathbb{R}^n} & e^{i(t^{-1}x\cdot\eta+\tau_k(\xi^0+t^{-1}\eta)t)} A_j^k(\xi^0+t^{-1}\eta)(1-\psi)(|\eta|) \\ & \chi(\xi^0+t^{-1}\eta)(1-\kappa)\big(t^{-1}x+(\nabla_\xi\tau_k)(\xi^0+t^{-1}\eta)\big)t^{-n}\,d\eta. \end{aligned}$$

Integrating by parts, with respect to η gives

$$\begin{aligned} I_2^k(t,x) = e^{ix\cdot\xi^0}t^{-n} \int_{\mathbb{R}^n} & e^{i(t^{-1}x\cdot\eta+\tau_k(\xi^0+t^{-1}\eta)t)} P^*\big[A_j^k(\xi^0+t^{-1}\eta)(1-\psi)(|\eta|) \\ & \chi(\xi^0+t^{-1}\eta)(1-\kappa)\big(t^{-1}x+(\nabla_\xi\tau_k)(\xi^0+t^{-1}\eta)\big)\big]\,d\eta\,, \end{aligned}$$

where P^* is the adjoint operator to $P=\frac{t^{-1}x+(\nabla_\xi\tau_k)(\xi^0+t^{-1}\eta)}{i|t^{-1}x+(\nabla_\xi\tau_k)(\xi^0+t^{-1}\eta)|^2}\cdot\nabla_\eta$; this integration by parts is valid as $|t^{-1}x+(\nabla_\xi\tau_k)(\xi^0+t^{-1}\eta)|\ge r>0$, in the support of $(1-\kappa)\big(t^{-1}x+\nabla\tau_k(\xi^0+t^{-1}\eta)\big)$. For suitable functions $f\equiv f(\eta;x,t)$, and $\xi=\xi^0+t^{-1}\eta$, we have

$$\begin{aligned} P^*f =& \nabla_\eta\cdot\Big[\frac{t^{-1}x+(\nabla_\xi\tau_k)(\xi)}{i|t^{-1}x+(\nabla_\xi\tau_k)(\xi)|^2}f\Big] \\ =& \frac{\nabla_\eta\cdot(\nabla_\xi\tau_k)(\xi)}{i|t^{-1}x+(\nabla_\xi\tau_k)(\xi)|^2}f + \frac{t^{-1}x+(\nabla_\xi\tau_k)(\xi)}{i|t^{-1}x+(\nabla_\xi\tau_k)(\xi)|^2}\cdot\nabla_\eta f \\ & - \frac{2(t^{-1}x+(\nabla_\xi\tau_k)(\xi))\cdot[\nabla_\eta[(\nabla_\xi\tau_k)(\xi)]\cdot(t^{-1}x+(\nabla_\xi\tau_k)(\xi))]}{i|t^{-1}x+(\nabla_\xi\tau_k)(\xi)|^4}f. \end{aligned}$$

Comparing this to (4.3.5), observe that the first and third terms have one power of t fewer in the denominator due to the transformation; this is critical in this case where we are approaching a singularity in $A_j^k(\xi^0+t^{-1}\eta)$ when

$t \to \infty$. By hypothesis (i), for η in the support of the integrand of $I_2^k(t,x)$, we get

$$\frac{\nabla_\eta \cdot [(\nabla_\xi \tau_k)(\xi^0 + t^{-1}\eta)]}{|t^{-1}x + (\nabla_\xi \tau_k)(\xi^0 + t^{-1}\eta)|^2} \le C_r(1+|\eta|)^{-1};$$

thus, we have

$$|P^* f| \le C_r[(1+|\eta|)^{-1}|f| + |\nabla_\eta f|].$$

In Lemma 4.3.3, we carried out this integration by parts repeatedly in order to estimate the integral. Here, however, note that differentiating $(1-\psi)(|\eta|)$ once is sufficient: by definition of $\psi(s)$,

$$\partial_{\eta_j}[(1-\psi)(|\eta|)] = -\frac{\eta_j}{|\eta|}(\partial_s \psi)(|\eta|)$$

is supported in $\frac{3}{4} \le |\eta| \le 1$, so

$$|\partial_{\eta_j}[(1-\psi)(|\eta|)]| \le C\mathbf{1}_{\mathbf{1} \ge |\eta| \ge \mathbf{3/4}}(\eta),$$

where $\mathbf{1}_{\mathbf{1} \ge |\eta| \ge \mathbf{3/4}}(\eta)$ denotes the characteristic function of the set

$$\{\eta \in \mathbb{R}^n : 1 \ge |\eta| \ge 3/4\};$$

hence, by hypothesis (iv), for large t we have

$$\begin{aligned} \int_{\mathbb{R}^n} \left| \frac{t^{-1}x + (\nabla_\xi \tau_k)(\xi^0 + t^{-1}\eta)}{i|t^{-1}x + (\nabla_\xi \tau_k)(\xi^0 + t^{-1}\eta)|^2} \right| |A_j^k(\xi^0 + t^{-1}\eta)| |\partial_{\eta_j}[(1-\psi)(|\eta|)]| \\ |\chi(\xi^0 + t^{-1}\eta)| |(1-\kappa)\big(t^{-1}x + \nabla \tau_k(\xi^0 + t^{-1}\eta)\big)| t^{-n}\, d\eta \\ \le C_r \int_{\frac{3}{4} \le |\eta| \le 1} |A_j^k(\xi^0 + t^{-1}\eta)| t^{-n}\, d\eta \\ \le C_r t^j \int_{\frac{3}{4} \le |\eta| \le 1} \frac{1}{(1+|\eta|)^j} t^{-n}\, d\eta \le C_r t^{j-n}, \end{aligned} \tag{7.4.10}$$

which is the desired estimate (7.4.9).

On the other hand, if, when integrating by parts, the derivative does not fall on $\psi(|\eta|)$, we use a similar argument to that in the earlier proof; let us look at the effect of differentiating each of the other terms: in the support of $\psi(|\eta|)$, for each multi-index α and $t > 0$,

- $|\partial_\eta^\alpha [A_j^k(\xi^0 + t^{-1}\eta)]| \le C_\alpha t^j (1+|\eta|)^{-j-|\alpha|}$ by hypothesis (iv);
- $|\partial_\eta^\alpha [\chi(\xi^0 + t^{-1}\eta)]| \le C_\alpha (1+|\eta|)^{-|\alpha|}$: for $\alpha = 0$, take $C_\alpha = 1$; for $|\alpha| \ge 1$, note that

  $$\partial_\eta^\alpha [\chi(\xi^0 + t^{-1}\eta)] = t^{-|\alpha|}(\partial_\xi^\alpha \chi)(\xi^0 + t^{-1}\eta),$$

  and that $(\partial_\xi^\alpha \chi)(\xi^0 + t^{-1}\eta)$ is supported in $N \le |\xi^0 + t^{-1}\eta| \le 2N$, so $t^{-1} \le C_{N,\xi^0}|\eta|^{-1}$;

- $|\partial_\eta^\alpha[(1-\kappa)\big(t^{-1}x+(\nabla_\xi\tau_k)(\xi^0+t^{-1}\eta)\big)]| \leq C_\alpha(1+|\eta|)^{-|\alpha|}$: obvious for $\alpha=0$; for $|\alpha|\geq 1$, note

$$\partial_\eta^\alpha[(1-\kappa)(t^{-1}x+(\nabla_\xi\tau_k)(\xi^0+t^{-1}\eta))] \\ = -(\partial_\xi^\alpha\kappa)(t^{-1}x+\nabla_\xi\tau_k(\xi))\partial_\eta^\alpha[(\nabla_\xi\tau_k)(\xi^0+t^{-1}\eta)]\,,$$

which yields the desired estimate by hypothesis (i).

Summarising, this means

$$\big|(1-\psi)(|\eta|)\partial_\eta^\alpha\big[A_j^k(\xi^0+t^{-1}\eta)\chi(\xi^0+t^{-1}\eta)(1-\kappa)\big(t^{-1}x+(\nabla_\xi\tau_k)(\xi^0+t^{-1}\eta)\big)\big]\big| \\ \leq C_r(1+|\eta|)^{-j-|\alpha|}t^j\mathbf{1}_{|\eta|\geq\frac{3}{4}}(\eta)\,.$$

So, repeatedly integrating by parts we find that either a derivative falls on $(1-\psi)(|\eta|)$ (in which case a similar argument to that in (7.4.10) above works) or we eventually get the integrable function $Ct^j(1+|\eta|)^{-n-1}\mathbf{1}_{|\eta|\geq\mathbf{3/4}}(\eta)$ as an upper bound; in either case, we have (7.4.9).

On the wave front set: Next, we look at the term supported around the wave front set, $I_1(t,x)$. As in the case away from the wave front, set $\xi=\xi^0+t^{-1}\eta$: consider, for $k=1,2$,

$$I_1^k(t,x):=e^{ix\cdot\xi^0}\int_{\mathbb{R}^n}e^{i(t^{-1}x\cdot\eta+\tau_k(\xi^0+t^{-1}\eta)t)}A_j^k(\xi^0+t^{-1}\eta)(1-\psi)(|\eta|) \\ \chi(\xi^0+t^{-1}\eta)\kappa\big(t^{-1}x+(\nabla_\xi\tau_k)(\xi^0+t^{-1}\eta)\big)t^{-n}\,d\eta\,.$$

As in the proof of Theorems 4.3.1 and 5.1.2, let $\{\Psi_\ell(\eta)\}_{\ell=1}^L$ be a conic partition of unity, where the support of $\Psi_\ell(\eta)$ is a cone K_ℓ, and each cone can be mapped by rotation onto K_1, which contains $e_n=(0,\dots,0,1)$. Then, it suffices to estimate

$$t^{-n}\int_{\mathbb{R}^n}e^{i(t^{-1}x\cdot\eta+\tau_k(\xi^0+t^{-1}\eta)t)}A_j^k(\xi^0+t^{-1}\eta)(1-\psi)(|\eta|) \\ \Psi_1(\eta)\chi(\xi^0+t^{-1}\eta)\kappa\big(t^{-1}x+(\nabla_\xi\tau_k)(\xi^0+t^{-1}\eta)\big)\,d\eta\,,$$

for $k=1,2$.

Let us parameterise the cone K_1: it follows from hypothesis (ii) that each of the level sets

$$\Sigma_{\lambda,t}\equiv\big\{\eta\in\mathbb{R}^n:\tau_k(\xi^0+t^{-1}\eta)=t^{-1}\lambda\big\}$$

is non-degenerate; so, for some $U \subset \mathbb{R}^{n-1}$, and smooth function $h_k(t, \lambda, \cdot) : U \to \mathbb{R}$,

$$K_1 = \{(\lambda y, \lambda h_k(t, \lambda, y)) : \lambda > 0, y \in U\} .$$

If $\tau_k(\xi)$ satisfies the convexity condition, then h_k is also a concave function in y. Now, we change variables $\eta \mapsto (\lambda y, \lambda h_k(t, \lambda, y))$ and will often omit t from the notation of h_k since the dependence on t will be uniform. We obtain:

$$\begin{aligned} t^{-n} \int_0^\infty \int_U & e^{i\lambda(t^{-1}x'\cdot y + t^{-1}x_n h_k(\lambda,y)+1)} A_j^k(\xi^0 + t^{-1}\lambda(y, h_k(\lambda, y))) \\ & (1-\psi)(\lambda|(y, h_k(\lambda, y))|)\Psi_1(\lambda(y, h_k(\lambda, y)))\chi(\xi^0 + t^{-1}\lambda(y, h_k(\lambda, y))) \\ & \qquad \kappa\big(t^{-1}x + (\nabla_\xi \tau_k)(\xi^0 + t^{-1}\lambda(y, h_k(\lambda, y)))\big) \frac{d\eta}{d(\lambda, y)}\, d\lambda dy, \end{aligned} \tag{7.4.11}$$

where we have used $\tau_k(\xi^0 + t^{-1}(\lambda y, \lambda h_k(\lambda, y))) = t^{-1}\lambda$. As in the earlier proofs, we ensure x_n is away from zero in the cone—this requires hypotheses (i) and (iii)). So, in the general case, we can write this as, with $\widetilde{x} = t^{-1}x$, $\widetilde{\lambda} = \lambda \widetilde{x}_n = \lambda t^{-1} x_n$,

$$\begin{aligned} t^{-n} \int_0^\infty \int_U & e^{i\lambda x_n(t^{-1}\widetilde{x}_n^{-1}x'\cdot y + t^{-1}h_k(\lambda,y)+\widetilde{x}_n^{-1})} A_j^k(\xi^0 + t^{-1}\lambda(y, h_k(\lambda, y))) \\ & (1-\psi)(\lambda|(y, h_k(\lambda, y))|)\Psi_1(\lambda(y, h_k(\lambda, y)))\chi(\xi^0 + t^{-1}\lambda(y, h_k(\lambda, y))) \\ & \qquad \kappa\big(t^{-1}x + (\nabla_\xi \tau_k)(\xi^0 + t^{-1}\lambda(y, h_k(\lambda, y)))\big) \frac{d\eta}{d(\lambda, y)}\, d\lambda dy . \end{aligned}$$

If the convexity condition holds, then, as in the proof of Theorem 4.3.1, we have the Gauss map

$$\underline{\mathbf{n}}_k : K_1 \cap \Sigma'_\lambda \to S^{n-1},\ \underline{\mathbf{n}}_k(\zeta) = \frac{\nabla_\zeta[\tau_k(\xi^0 + t^{-1}\zeta)]}{|\nabla_\zeta[\tau_k(\xi^0 + t^{-1}\zeta)]|} = \frac{(\nabla_\xi \tau_k)(\xi^0 + t^{-1}\zeta)}{|(\nabla_\xi \tau_k)(\xi^0 + t^{-1}\zeta)|},$$

and, as before, can define $z_k(\lambda) \in U$ so that

$$\underline{\mathbf{n}}_k(z_k(\lambda), h_k(\lambda, z(\lambda))) = -x/|x| .$$

Then,

$$\frac{x'}{x_n} = -\nabla_y h_k(\lambda, z(\lambda)) .$$

So, in this case, (7.4.11) becomes:

$$(I_1^k)' := t^{-n} \int_0^\infty \int_U e^{i\lambda x_n[-t^{-1}\nabla_y h_k(\lambda,z(\lambda))\cdot y + t^{-1}h_k(\lambda,y)+\widetilde{x}_n^{-1}]}$$
$$A_j^k(\xi^0 + t^{-1}\lambda(y,h_k(\lambda,y)))(1-\psi)(\lambda|(y,h_k(\lambda,y))|)\Psi_1(\lambda(y,h_k(\lambda,y)))$$
$$\chi(\xi^0 + t^{-1}\lambda(y,h_k(\lambda,y)))\kappa\big(\widetilde{x} + (\nabla_\xi \tau_k)(\xi^0 + t^{-1}\lambda(y,h_k(\lambda,y)))\big)\frac{d\eta}{d(\lambda,y)}\, d\lambda dy,$$

Let us estimate this integral in the case where the convexity condition holds. We have:

- The same argument as in the earlier proof (which uses hypothesis (ii)), shows
$$\left|\frac{d\eta}{d(\lambda,y)}\right| \le C\lambda^{n-1}.$$
The constant C here is independent of t;

- Now, with $\widetilde{A}_k^j(\nu) = A_k^j(\nu)\chi(\nu)\kappa\big(\widetilde{x} + (\nabla_\xi\tau_k)(\nu)\big)\Psi_1(\lambda(y,h_k(\lambda,y)))$, where $\nu = \xi^0 + t^{-1}\lambda(y,h_k(\lambda,y))$, we have
$$|(I_1^k)'| \le t^{j-n}\int_0^\infty \Big|\int_U e^{i\lambda\widetilde{x}_n[-(y-z(\lambda))\cdot\nabla_y h_k(\lambda,z(\lambda))+h_k(\lambda,y)+h_k(\lambda,z(\lambda))]}$$
$$t^{-j}\lambda^j\widetilde{A}_j^k(\xi^0 + t^{-1}\lambda(y,h_k(\lambda,y)))(1-\psi)(\lambda|(y,h_k(\lambda,y))|)\,dy\Big|\lambda^{n-1-j}\,d\lambda\,.$$

- Now, applying Theorem 4.1.1— this may be used due to the properties of $A_j^k(\xi)$ and $\tau_k(\xi)$ stated in hypotheses (iv) and (i)—we find that
$$\Big|\int_U e^{i\lambda\widetilde{x}_n[-(y-z(\lambda))\cdot\nabla_y h_k(\lambda,z(\lambda))+h_k(\lambda,y)+h_k(\lambda,z(\lambda))]}$$
$$t^{-j}\lambda^j\widetilde{A}_j^k(\xi^0 + t^{-1}\lambda(y,h_k(\lambda,y)))(1-\psi)(\lambda|(y,h_k(\lambda,y))|)\,dy\Big| \le C\lambda^{j-n}\widetilde{\chi}(\lambda)\,,$$
where $\widetilde{\chi}(\lambda)$ is a compactly supported smooth function that is zero in a neighbourhood of the origin.

- Hence,
$$|(I_1^k)'| \le t^{j-n}\int_0^\infty \widetilde{\chi}(\lambda)\lambda^{-1}\,d\lambda \le Ct^{j-n}\,.$$

Finally, the general case without convexity can be estimated in a similar way, with the necessary changes used in the proof of Theorem 5.1.2 to account for the change in the phase function—in particular, the use of the Van der Corput Lemma, Lemma 5.0.5, in place of Theorem 4.1.1. This completes the proof of (7.4.8). □

Using Proposition 7.4.3, it is clear that

$$\Big\| \int_{\mathbb{R}^n} e^{ix\cdot\xi}(1-\psi)(t|\xi-\xi^0|)\chi(\xi)\sum_{k=1}^{2} A_j^k(t,\xi)e^{i\tau_k(\xi)t}\widehat{f}(\xi)\,d\xi\Big\|_{L^\infty(\mathbb{R}^n_x)} \le C(1+t)^{-\frac{n-1}{\gamma}}\|f\|_{L^1}$$

if the roots satisfy the convexity condition, and

$$\Big\| \int_{\mathbb{R}^n} e^{ix\cdot\xi}(1-\psi)(t|\xi-\xi^0|)\chi(\xi)\sum_{k=1}^{2} A_j^k(t,\xi)e^{i\tau_k(\xi)t}\widehat{f}(\xi)\,d\xi\Big\|_{L^\infty(\mathbb{R}^n_x)} \le C(1+t)^{-\frac{1}{\gamma_0}}\|f\|_{L^1}$$

otherwise. In comparison to (6.6.6), here we have L^1-norms on the right hand sides, since χ is a cut-off function to bounded frequencies.

Finally, we must consider the case where L roots intersect; the above proof can easily be adapted for such a case, giving corresponding results.

$L^2 - L^2$ **estimates:** For the L^2-estimates on the support of $(1-\psi)(t|\xi-\xi^0|)\chi(\xi)$ we only need assumption (iv) of Proposition 7.4.3 with $\alpha = 0$ for the amplitude, namely that

$$|A_j^k(\xi^0+s\eta)| \le C_\alpha s^{-j}(1+|\eta|)^{-j}\,,\ \text{for small } s>0 \ \text{ and } \ |\eta|>\delta\,. \tag{7.4.12}$$

Then, for the left hand side of (7.4.8), we have

$$\begin{aligned} &\left\|\sum_{k=1}^{2}\int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau_k(\xi)t)}A_j^k(\xi)(1-\psi)(t|\xi-\xi^0|)\chi(\xi)\widehat{f}(\xi)\,d\xi\right\|_{L^2(\mathbb{R}^n_x)} \\ &= \left\|\sum_{k=1}^{2} e^{i\tau_k(\xi)t}A_j^k(\xi)(1-\psi)(t|\xi-\xi^0|)\chi(\xi)\widehat{f}(\xi)\right\|_{L^2(\mathbb{R}^n_\xi)} \\ &\le ||t^j(1+|\eta|)^{-j}\widehat{f}(\xi^0+t^{-1}\eta)||_{L^2(\mathbb{R}^n_\eta)}, \end{aligned}$$

where we used Plancherel's theorem, (7.4.12), and the notation $s = t^{-1}$, $\xi = \xi^0 + t^{-1}\eta$, so that $\eta = t(\xi-\xi^0)$. Then we can easily estimate

$$\begin{aligned} ||t^j(1+|\eta|)^{-j}\widehat{f}(\xi^0+t^{-1}\eta)||_{L^2(\mathbb{R}^n_\eta)} &= ||t^j(1+t|\xi-\xi^0|)^{-j}\widehat{f}(\xi)||_{L^2(\mathbb{R}^n_\xi)} \\ &= ||(t^{-1}+|\xi-\xi^0|)^{-j}\widehat{f}(\xi)||_{L^2(\mathbb{R}^n_\xi)} \\ &\le ||\,|\xi-\xi^0|^{-j}\widehat{f}(\xi)||_{L^2(\mathbb{R}^n_\xi)} \\ &= ||\,|D-D_0|^{-j}f||_{L^2(\mathbb{R}^n_x)}, \end{aligned}$$

where $D-D_0$ is a Fourier multiplier with symbol $\xi-\xi^0$. So, we finally obtain the estimate

$$\left|\left|\sum_{k=1}^{2}\int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau_k(\xi)t)}A_j^k(\xi)(1-\psi)(t|\xi-\xi^0|)\chi(\xi)\widehat{f}(\xi)\,d\xi\right|\right|_{L^2(\mathbb{R}^n_x)} \leq C||\,|D-D_0|^{-j}f||_{L^2(\mathbb{R}^n_x)}.$$

In the case of equations with homogeneous symbols (like for the wave equation), when roots are homogeneous, we have $\xi^0=0$, so that the right hand side becomes just the norm in the corresponding homogeneous Sobolev space.

Due to the earlier bound near the multiplicity, we can combine the results with the interpolation Theorem 6.2.3.