Chapter 5

Oscillatory integrals without convexity

Theorem 4.3.1 requires the phase function to satisfy the convexity condition of Definition 2.2.3; however, we will also investigate solutions to hyperbolic equations for which the characteristic roots do not necessarily satisfy such a condition. In this section we state and prove a theorem for this case. First, we give the key results that replaces Theorem 4.1.1 in the proof, the well-known van der Corput Lemma. We recall the standard van der Corput Lemma as given in, for example, [Sog93, Lemma 1.1.2], or in [Ste93, Proposition 2, Ch VIII]:

Lemma 5.0.5. *Let $\Phi \in C^\infty(\mathbb{R})$ be real-valued, $a \in C_0^\infty(\mathbb{R})$ and $m \geq 2$ be an integer such that $\Phi^{(j)}(0) = 0$ for $0 \leq j \leq m-1$ and $\Phi^{(m)}(0) \neq 0$; then*

$$\left| \int_0^\infty e^{i\lambda\Phi(x)} a(x)\, dx \right| \leq C(1+\lambda)^{-1/m} \quad \text{for all} \quad \lambda \geq 0,$$

provided the support of a is sufficiently small. The constant on the right-hand side is independent of λ and Φ.

If $m = 1$, then the same result holds provided $\Phi'(x)$ is monotonic on the support of a.

5.1 Real-valued phase function

In the case when the convexity condition holds the estimate of Theorem 4.3.1 is given in terms of the constant γ; as in the case of the homogeneous operators (see Introduction, Section 1.2) we introduce an analog to this in the case where the convexity condition does not hold. Let Σ be a hypersurface in $\mathbb{R}^n$; we set

$$\gamma_0(\Sigma) := \sup_{\sigma\in\Sigma} \inf_P \gamma(\Sigma;\sigma,P) \leq \gamma(\Sigma)$$

where $\gamma(\Sigma;\sigma,P)$ is as in Definition 2.2.4.

An important result for calculating this value is the following:

Lemma 5.1.1 ([Sug96]). *Suppose $\Sigma = \{(y,h(y)) : y\in U\}$, $h\in C^\infty(U)$, $U\subset\mathbb{R}^{n-1}$ is an open set, and let*

$$F(\rho) = h(\eta+\rho\omega) - h(\eta) - \rho\nabla h(\eta)\cdot\omega$$

where $\eta\in U$, $\omega\in\mathbb{S}^{n-2}$. Taking $\sigma = (\eta,h(\eta))\in\Sigma$, $\omega\in\mathbb{S}^{n-2}$ and

$$P = \{\sigma + s(\omega,\nabla h(\eta)\cdot\omega) + t(-\nabla h(\eta),1)\in\mathbb{R}^n : s,t\in\mathbb{R}\}\ ,$$

then

$$\gamma(\Sigma;\sigma,P) = \min\left\{k\in\mathbb{N} : F^{(k)}(0)\neq 0\right\} =: \gamma(h;\eta,\omega)\,.$$

Therefore,

$$\gamma(\Sigma) = \sup_\eta \sup_\omega \gamma(h;\eta,\omega),$$
$$\gamma_0(\Sigma) = \sup_\eta \inf_\omega \gamma(h;\eta,\omega)\,.$$

Now we are in a position to state and prove the result for oscillatory integrals with a real-valued phase function that does not satisfy the earlier introduced convexity condition. This is a parameter dependent version of Corollary 2.2.11.

Theorem 5.1.2. *Let $a(\xi)$ be a symbol of order $\frac{1}{\gamma_0}-n$ of type $(1,0)$ on $\mathbb{R}^n$. Let $\tau:\mathbb{R}^n\to\mathbb{R}$ be smooth on $\operatorname{supp} a$, set $\gamma_0 := \sup_{\lambda>0}\gamma_0(\Sigma_\lambda(\tau))$ and assume it is finite; furthermore, on $\operatorname{supp} a$, we also assume the following conditions:*

(i) *for all multi-indices α there exists a constant $C_\alpha>0$ such that*

$$|\partial_\xi^\alpha\tau(\xi)| \leq C_\alpha(1+|\xi|)^{1-|\alpha|};$$

(ii) *there exist constants $M,C>0$ such that for all $|\xi|\geq M$ we have $|\tau(\xi)|\geq C|\xi|$;*

(iii) *there exists a constant $C_0>0$ such that $|\partial_\omega\tau(\lambda\omega)|\geq C_0$ for all $\omega\in\mathbb{S}^{n-1}$, $\lambda>0$;*

(iv) *there exists a constant* $R_1 > 0$ *such that, for all* $\lambda > 0$,

$$\frac{1}{\lambda}\Sigma_\lambda(\tau) \subset B_{R_1}(0)\,.$$

Then, the following estimate holds for all $R \geq 0$, $x \in \mathbb{R}^n$, $t > 1$:

$$\Big|\int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)} a(\xi) g_R(\xi)\, d\xi\Big| \leq Ct^{-\frac{1}{\gamma_0}}\,,$$

where $g_R(\xi)$ *is as given in (4.3.1) and* $C > 0$ *is independent of* R.

Proof. We follow the proof of Theorem 4.3.1 as far as possible, and shall show how the absence of the convexity condition affects the estimate. Thus, as in the proof of Theorem 4.3.1, we may first assume, without loss of generality, that either $\tau(\xi) \geq 0$ for all $\xi \in \mathbb{R}^n$ or $\tau(\xi) \leq 0$ for all $\xi \in \mathbb{R}^n$. We will always work on the support of a, so by writing $\xi \in \mathbb{R}^n$ we will mean $\xi \in \operatorname{supp} a$.

Divide the integral into two parts:

$$\begin{aligned} I_1(t,x) &:= \int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)} a(\xi) g_R(\xi) \kappa\big(t^{-1}x + \nabla\tau(\xi)\big)\, d\xi\,, \\ I_2(t,x) &:= \int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)} a(\xi) g_R(\xi) (1-\kappa)\big(t^{-1}x + \nabla\tau(\xi)\big)\, d\xi\,, \end{aligned}$$

where $\kappa \in C_0^\infty(\mathbb{R}^n)$, $0 \leq \kappa(y) \leq 1$, which is identically 1 in the ball of radius $r > 0$ centred at the origin, $B_r(0)$, and identically 0 outside the ball of radius $2r$, $B_{2r}(0)$. By Lemma 4.3.3 (which does not require the phase function to satisfy the convexity condition), we have

$$|I_2(t,x)| \leq C_r t^{-1/\gamma_0} \quad \text{for all } t > 1.$$

To estimate $|I_1(t,x)|$ we introduce, as before, a partition of unity $\{\Psi_\ell(\xi)\}_{\ell=1}^L$ and restrict attention to

$$I_1'(t,x) = \int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)} a(\xi) g_R(\xi) \Psi_1(\xi) \kappa\big(t^{-1}x + \nabla\tau(\xi)\big)\, d\xi\,,$$

where $\Psi_1(\xi)$ is supported in a sufficiently narrow cone, K_1, that contains $e_n = (0, \ldots, 0, 1)$. Parameterise this cone in the same way as above: with $U \subset \mathbb{R}^{n-1}$,

$$K_1 = \begin{cases} \{(\lambda y, \lambda h_\lambda(y)) : \lambda > 0,\, y \in U\} & \text{if } \tau(\xi) \geq 0 \text{ for all } \xi \in \mathbb{R}^n \\ \{(\lambda y, \lambda h_\lambda(y)) : \lambda < 0,\, y \in U\} & \text{if } \tau(\xi) \leq 0 \text{ for all } \xi \in \mathbb{R}^n\,. \end{cases}$$

Here the Implicit Function Theorem ensures the existence of a smooth function $h_\lambda : U \to \mathbb{R}$ for each $\lambda > 0$, but there is one major difference: the functions h_λ are not necessarily concave, in contrast to the earlier proof. Using the change of variables $\xi \mapsto (\lambda y, \lambda h_\lambda(y))$—note that

$$0 < C \le \left| \frac{d\xi}{d(\lambda, y)} \right| \le C\lambda^{n-1}$$

by the same argument as in the proof of Theorem 4.3.1, providing the width of K_1 is taken to be sufficiently small—gives

$$\begin{aligned} I_1'(t,x) = \int_0^\infty \int_U & e^{i[\lambda x'\cdot y + \lambda x_n h_\lambda(y) + \tau(\lambda y, \lambda h_\lambda(y))t]} a(\lambda y, \lambda h_\lambda(y)) \\ & g_R(\lambda y, \lambda h_\lambda(y))\Psi_1(\lambda y, \lambda h_\lambda(y))\kappa\big(t^{-1}x + \nabla\tau(\lambda y, \lambda h_\lambda(y))\big)\frac{d\xi}{d(\lambda, y)}\, dy\, d\lambda\,. \end{aligned}$$

Once again, let $G \in C_0^\infty(\mathbb{R})$ so that $g_R(\xi) = g_R(\xi)G(\tau(\xi)/\mathcal{R})$ (where $\mathcal{R} = \max(R, 1)$) and $\widetilde{a}(\xi) = a(\xi)g_R(\xi)\Psi_1(\xi)$, which is a symbol of order $\frac{1}{\gamma_0} - n$ supported in K_1 and with all the constants in the symbolic estimates independent of R. So, recalling that $\tau(\lambda y, \lambda h_\lambda(y)) = \lambda$ and writing $h(\lambda, y) \equiv h_\lambda(y)$, we get

$$\begin{aligned} I_1'(t,x) &= \int_0^\infty \int_U e^{i\lambda[x'\cdot y + x_n h_\lambda(y) + t]}\widetilde{a}(\lambda y, \lambda h_\lambda(y)) \\ &\qquad\qquad G(\lambda/\mathcal{R})\kappa\big(t^{-1}x + \nabla\tau(\lambda y, \lambda h_\lambda(y))\big)\frac{d\xi}{d(\lambda, y)}\, dy\, d\lambda \\ &= \int_0^\infty \int_U e^{i\widetilde{\lambda}[\frac{\widetilde{x}'}{\widetilde{x}_n}\cdot y + h\left(\frac{\widetilde{\lambda}}{\widetilde{x}_n t}, y\right) + \widetilde{x}_n^{-1}]}\widetilde{a}\Big(\tfrac{\widetilde{\lambda}}{\widetilde{x}_n t}y, \tfrac{\widetilde{\lambda}}{\widetilde{x}_n t}h\Big(\tfrac{\widetilde{\lambda}}{\widetilde{x}_n t}, y\Big)\Big) \\ &\qquad G\Big(\tfrac{\widetilde{\lambda}}{\mathcal{R}\widetilde{x}_n t}\Big)\kappa\Big(\widetilde{x} + \nabla\tau\Big(\tfrac{\widetilde{\lambda}}{\widetilde{x}_n t}y, \tfrac{\widetilde{\lambda}}{\widetilde{x}_n t}h\Big(\tfrac{\widetilde{\lambda}}{\widetilde{x}_n t}, y\Big)\Big)\Big)\frac{d\xi}{d(\lambda, y)}\widetilde{x}_n^{-1}t^{-1}\, dy\, d\widetilde{\lambda}\,, \end{aligned}$$

where $x = t\widetilde{x}$ and $\widetilde{\lambda} = \lambda x_n = \lambda\widetilde{x}_n t$. Thus, using $|\kappa(\eta)| \le 1$, we have

$$|I_1'(t,x)| \le C|\widetilde{x}_n|^{-1/\gamma_0}t^{-1/\gamma_0}\int_0^\infty \Big| I\Big(\widetilde{\lambda}, \tfrac{\widetilde{\lambda}}{\widetilde{x}_n t}; \widetilde{x}_n^{-1}\widetilde{x}\Big)G\Big(\tfrac{\widetilde{\lambda}}{\mathcal{R}\widetilde{x}_n t}\Big)\widetilde{\lambda}^{-1+(1/\gamma_0)}\Big|\, d\widetilde{\lambda} \tag{5.1.1}$$

where

$$I\Big(\widetilde{\lambda}, \tfrac{\widetilde{\lambda}}{\widetilde{x}_n t}; \widetilde{x}_n^{-1}\widetilde{x}'\Big) = \int_U e^{i\widetilde{\lambda}\left[\widetilde{x}_n^{-1}\widetilde{x}'\cdot y + h\left(\frac{\widetilde{\lambda}}{\widetilde{x}_n t}, y\right)\right]}\widetilde{a}\Big(\tfrac{\widetilde{\lambda}}{\widetilde{x}_n t}y, \tfrac{\widetilde{\lambda}}{\widetilde{x}_n t}h\Big(\tfrac{\widetilde{\lambda}}{\widetilde{x}_n t}, y\Big)\Big)\Big(\tfrac{\widetilde{\lambda}}{|\widetilde{x}_n| t}\Big)^{n-\frac{1}{\gamma_0}}\, dy\,.$$

At this point, we diverge from the proof of the earlier theorem since we cannot apply Theorem 4.1.1; instead, note that, for some $b \in C_0^\infty(\mathbb{R}^{n-1})$

with support contained in U, we have

$$\left|I\left(\widetilde{\lambda},\tfrac{\widetilde{\lambda}}{\widetilde{x}_n t};\widetilde{x}_n^{-1}\widetilde{x}'\right)\right| \le \int_{\mathbb{R}^{n-2}}\left|\int_{\mathbb{R}} e^{i\widetilde{\lambda}\left[\widetilde{x}_n^{-1}\widetilde{x}'\cdot y+h\left(\frac{\widetilde{\lambda}}{\widetilde{x}_n t},y\right)\right]}\right.$$
$$\left.\widetilde{a}\left(\tfrac{\widetilde{\lambda}}{\widetilde{x}_n t}y,\tfrac{\widetilde{\lambda}}{\widetilde{x}_n t}h\left(\tfrac{\widetilde{\lambda}}{\widetilde{x}_n t},y\right)\right)\left(\tfrac{\widetilde{\lambda}}{|\widetilde{x}_n|t}\right)^{n-\frac{1}{\gamma_0}} b(y)\,dy_1\right|\,dy'\,.$$

We wish to apply the van der Corput Lemma, Lemma 5.0.5, to the inner integral. Set $\Phi(y,\mu;z) := z\cdot y + h_\mu(y)$, which is real-valued, and consider the integral

$$\int_{\mathbb{R}} e^{i\lambda\Phi(y,\mu;z)}a_0(y,\mu)b(y)\,dy_1$$

where $a_0(y,\mu) := \mu^{n-(1/\gamma_0)}\widetilde{a}(\mu y,\mu h_\mu(y))$. Recall that

$$\Sigma_\mu = \{(y,h_\mu(y)) : y\in U\}\ ,$$

so by Lemma 5.1.1,

$$\min\left\{k\in\mathbb{N} : \partial_{y_1}^k\Phi(y,\mu;z)\Big|_{y_1=0}\neq 0\right\} = \gamma(h_\mu;0,(1,0,\ldots,0)) =: m\,.$$

Fixing the size of U so that $|\partial_{y_1}^{(m)}\Phi(y,\mu;z)|\ge\varepsilon>0$ for all $y\in U$ ensures that the hypotheses of Lemma 5.0.5 are satisfied. Thus, since the support of b is compact in $\mathbb{R}^{n-1}$, is contained in U, and a_0 is smooth, we obtain

$$\left|\int_{\mathbb{R}} e^{i\lambda\Phi(y,\mu;z)}a_0(y,\mu)b(y)\,dy_1\right| \le C\lambda^{-1/m}\,.$$

Carry out a suitable change of coordinates so that $m=\inf_\omega\gamma(h_\mu;0,\omega)$ (this is possible due to the rotational invariance of all properties used); then, since $m\le\gamma_0$ by definition, we have

$$\left|I\left(\widetilde{\lambda},\tfrac{\widetilde{\lambda}}{\widetilde{x}_n t};\widetilde{x}_n^{-1}\widetilde{x}'\right)\right| \le C\widetilde{\lambda}^{-1/\gamma_0}\,,$$

for all $\widetilde{\lambda}$ such that $\frac{\widetilde{\lambda}}{\mathcal{R}\widetilde{x}_n t}\in\operatorname{supp}G$ (this is to ensure $\widetilde{\lambda}$ is away from the origin). Combining this with (5.1.1) then gives the required estimate:

$$\begin{aligned}|I_1'(t,x)| &\le C|\widetilde{x}_n|^{-1/\gamma_0}t^{-1/\gamma_0}\int_0^\infty\left|\widetilde{\lambda}^{-1}G\left(\tfrac{\widetilde{\lambda}}{\mathcal{R}\widetilde{x}_n t}\right)\right|d\widetilde{\lambda}\\ &= C|\widetilde{x}_n|^{-1/\gamma_0}t^{-1/\gamma_0}\int_0^\infty(\nu\mathcal{R}\widetilde{x}_n t)^{-1}G(\nu)\mathcal{R}\widetilde{x}_n t\,d\nu \le Ct^{-\frac{1}{\gamma_0}}\,.\end{aligned}$$ $\square$