Chapter 3

Properties of hyperbolic polynomials

In order to study the solution $u(t,x)$ to (1.0.1), we must first know some properties of the characteristic roots $\tau_1(\xi),\dots,\tau_m(\xi)$. Naturally, we do not have explicit formulae for the roots, unlike in the cases of the dissipative wave equation and the Klein–Gordon equation (i.e. for second order equations), but we do know some properties for the roots of the principal symbol. For general hyperbolic operators, the roots $\varphi_1(\xi),\dots,\varphi_m(\xi)$ of the characteristic polynomial of the *principal part* are homogeneous functions of order 1 since the principal part is homogeneous. Furthermore, for strictly hyperbolic polynomials these roots are distinct when $\xi \neq 0$. Since these two properties are very useful when studying homogeneous (strictly) hyperbolic equations, it is useful to know whether the characteristic roots of the full equation, $\tau_1(\xi),\dots,\tau_m(\xi)$, have similar properties. Indeed, if we regard the full equation as a perturbation of the principal part by lower order terms, we can show that similar properties hold for large $|\xi|$; these results are the focus of this section. In the outline of the method in Section 2.5, we subdivided the phase space into large $|\xi|$ and bounded $|\xi|$, and it is these properties that motivate this step.

3.1 General properties

First, we give some properties of general polynomials which are useful to us. For constant coefficient polynomials, the following result holds:

Lemma 3.1.1. *Consider the polynomial over* $\mathbb{C}$ *with complex coefficients*

$$z^m + c_1 z^{m-1} + \cdots + c_{m-1} z + c_m = \prod_{k=1}^{m} (z - z_k).$$

If there exists $M > 0$ *such that* $|c_j| \le M^j$ *for each* $j = 1, \ldots, m$, *then* $|z_k| \le 2M$ *for all* $k = 1, \ldots, m$.

Proof. Assume that $|z| > 2M$. Then

$$\begin{aligned} |z^m + c_1 z^{m-1} + \cdots + c_{m-1} z + c_m| &\ge |z|^m \left(1 - \frac{|c_1|}{|z|} - \cdots - \frac{|c_{m-1}|}{|z|^{m-1}} - \frac{|c_m|}{|z|^m}\right) \\ &\ge (2M)^m (1 - 2^{-1} - \cdots - 2^{-(m-1)} - 2^{-m}) > 0. \end{aligned}$$

That is, no zero of the polynomial lies outside of the ball about the origin of radius $2M$; hence $|z_k| \le 2M$ for each $k = 1, \ldots, m$. □

Remark 3.1.2. *If we replace the hypothesis* $|c_j| \le M^j$ *by* $|c_j| \le M$ *for each* $j = 1, \ldots, m$, *then by a similar argument we obtain that* $|z_k| \le \max\{2, 2M\}$. *The quantity* $\max\{2, 2M\}$ *appears because we need* $M \ge 1$ *for the sum on the right hand side to be positive.*

For general polynomials with variable coefficients, we have continuous dependence of roots on coefficients (we give an independent proof of this result here for the sake of completeness and for referencing, but analogue of this result can be found in many monographs dealing with hyperbolic polynomials).

Lemma 3.1.3. *Consider the* m^{th} *order polynomial with coefficients depending on* $\xi \in \mathbb{R}^n$

$$p(\tau, \xi) = \tau^m + a_1(\xi)\tau^{m-1} + \cdots + a_m(\xi).$$

If each of the coefficient functions $a_j(\xi)$, $j = 1, \ldots, m$, *is continuous in* $\mathbb{R}^n$ *then each of the roots* $\tau_1(\xi), \ldots, \tau_m(\xi)$ *with respect to* τ *of* $p(\tau, \xi) = 0$ *is also continuous in* $\mathbb{R}^n$.

Proof. Define $\rho : \mathbb{C}^m \to \mathbb{C}^m$ by $\rho(z_1, \ldots, z_m) = (c_1, \ldots, c_m)$ where the c_j satisfy

$$z^m + c_1 z^{m-1} + \cdots + c_m = \prod_{j=1}^{m} (z - z_j).$$

By the fundamental theorem of algebra ρ is invertible (but the inverse is not unique modulo permutation of roots), and, moreover, ρ is:

(a) surjective by the Fundamental Theorem of Algebra;

(b) continuous since each of the c_j may be written as polynomials of the z_j (by the Vièta formulae);

(c) proper (that is, the preimage of each compact set is compact) by Remark 3.1.2;

properties (b) and (c) imply that ρ is a closed mapping.

Now, fix $\xi^0 \in \mathbb{R}^n$. For any given $\varepsilon > 0$, consider the set

$$U = \bigcup_{\alpha \in S_m} \bigcap_{k=1}^{m} \left\{ \zeta = (\zeta_1, \ldots, \zeta_m) \in \mathbb{C}^m : |\zeta_{\alpha_k} - \tau_k(\xi^0)| < \varepsilon \right\} ,$$

where $\alpha = (\alpha_1, \ldots, \alpha_m) \in S_m$ denotes the set of permutations of $\{1, \ldots, m\}$ (see Fig. 3.1 for a diagram of this). Note that U is, by construction, symmet-

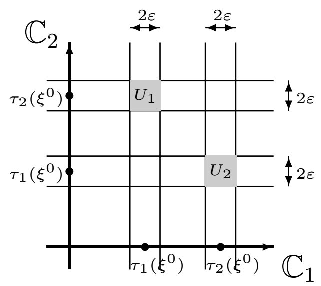

Figure 3.1: $U = U_1 \cup U_2$

ric, i.e. if $(z_1, \ldots, z_m) \in U$ then $(z_{\alpha_1}, \ldots, z_{\alpha_m}) \in U$ for all $(\alpha_1, \ldots, \alpha_m) \in S_m$. Let F denote the complement to U:

$$F = \bigcap_{\alpha \in S_m} \left\{ \zeta = (\zeta_1, \ldots, \zeta_m) \in \mathbb{C}^m : |\zeta_{\alpha_k} - \tau_k(\xi^0)| \geq \varepsilon \; \exists \, k = 1, \ldots, m \right\} .$$

We need to show that there exists $\delta > 0$ such that $(\tau_1(\xi), \ldots, \tau_m(\xi)) \in U$ whenever $|\xi - \xi^0| < \delta$; note:

- $\rho^{-1}(\rho(F)) = F$ by construction—if $\rho(w) = \rho(w')$ then both w and w' give rise to the same polynomial, and hence their entries are permutations of each other, and so either both or neither lie in F;

- by the surjectivity of ρ,

$$\rho(U) = \rho(F^c) = \rho([\rho^{-1}(\rho(F))]^c) = \rho(\rho^{-1}(\rho(F)^c)) = \rho(F)^c ;$$

- $\rho(F)$ is closed since F a closed set and ρ is a closed mapping;

therefore, $\rho(U)$ is open. Thus, there exists an open ball in $\rho(U)$ of radius δ' (for some $\delta' > 0$) about $a(\xi^0) \equiv (a_1(\xi^0), \ldots, a_m(\xi^0)) = \rho(\tau_1(\xi^0), \ldots, \tau_m(\xi^0))$:

$$B_{\delta'}(a(\xi^0)) = \left\{(c_1, \ldots, c_m) \in \mathbb{C}^m : |c_j - a_j(\xi^0)| < \delta' \, \forall\, j = 1, \ldots, m\right\} \subset \rho(U).$$

By the continuity of the $a_j(\xi)$, there exists $\delta > 0$ such that

$$|\xi - \xi^0| < \delta \implies |a_j(\xi) - a_j(\xi^0)| < \delta' \text{ for all } j = 1, \ldots, m\,;$$

hence,

$$|\xi - \xi^0| < \delta \implies (a_1(\xi), \ldots, a_m(\xi)) \in B_{\delta'}(a(\xi^0)) \subset \rho(U)\,.$$

Finally, since $\rho(\tau_1(\xi), \ldots, \tau_m(\xi)) = (a_1(\xi), \ldots, a_m(\xi))$ and U is symmetric (this is needed as different root orderings give the same coefficients), we find that we have $(\tau_1(\xi), \ldots, \tau_m(\xi)) \in U$ when $|\xi - \xi^0| < \delta$ as required; this completes the proof of the lemma. □

Now, let us turn to proving properties of the characteristic roots.

Proposition 3.1.4. *Let $L = L(D_t, D_x)$ be a linear m^{th} order constant coefficient differential operator in D_t with coefficients that are pseudo-differential operators in x, with symbol*

$$L(\tau, \xi) = \tau^m + \sum_{j=1}^{m} P_j(\xi)\tau^{m-j} + \sum_{j=1}^{m} a_j(\xi)\tau^{m-j},$$

where $P_j(\lambda\xi) = \lambda^j P_j(\xi)$ for all $\lambda >> 1$, $|\xi| >> 1$, and $a_j \in S^{j-\epsilon}$, for some $\epsilon > 0$.

Then each of the characteristic roots of L, denoted $\tau_1(\xi), \ldots, \tau_m(\xi)$, is continuous in $\mathbb{R}^n$; furthermore, for each $k = 1, \ldots, m$, the characteristic root $\tau_k(\xi)$ is smooth away from multiplicities, and analytic if the operator $L(D_t, D_x)$ is differential.

If operator $L(D_t, D_x)$ is strictly hyperbolic, then there exists a constant M such that, if $|\xi| \geq M$ then the characteristic roots $\tau_1(\xi), \ldots, \tau_m(\xi)$ of L are pairwise distinct.

Proof. The first part of Proposition is simple. Let us now investigate the structure of the characteristic determinant. We use the notation and results from Chapter 12 of [GKZ94] concerning the discriminant Δ_p of the polynomial $p(x) = p_m x^m + \cdots + p_1 x + p_0$,

$$\Delta_p \equiv \Delta(p_0, \ldots, p_m) := (-1)^{\frac{m(m-1)}{2}} p_m^{2m-2} \prod_{i<j} (x_i - x_j)^2\,,$$

where the x_j $(j = 1, \ldots, m)$ are the roots of $p(x)$; that is, the irreducible polynomial in the coefficients of the polynomial which vanishes when the polynomial has multiple roots. We note that Δ_p is a continuous function of the coefficients $p_0, \ldots, p_m$ of $p(x)$ and it is a homogeneous function of degree $2m-2$ in them; in addition, it satisfies the quasi-homogeneity property:

$$\Delta(p_0, \lambda p_1, \lambda^2 p_2, \ldots, \lambda^m p_m) = \lambda^{m(m-1)} \Delta(p_0, \ldots, p_m).$$

Furthermore, $\Delta_p = 0$ if and only if $p(x)$ has a double root.

We write $L(\tau, \xi)$ in the form

$$L(\tau, \xi) = L_m(\tau, \xi) + a_1(\xi)\tau^{m-1} + a_2(\xi)\tau^{m-2} + \cdots + a_{m-1}(\xi)\tau + a_m(\xi),$$

where

$$L_m(\tau, \xi) = \tau^m + \sum_{j=1}^{m} P_j(\xi)\tau^{m-j}$$

is the principal part of $L(\tau, \xi)$; note that the $P_j(\xi)$ are homogeneous of degree j and the $a_j(\xi)$ are symbols of degree $< j$. By the homogeneity and quasi-homogeneity properties of Δ_L, we have, for $\lambda \neq 0$,

$$\begin{aligned}
\Delta_L(\lambda\xi) &= \Delta(P_m(\lambda\xi) + a_m(\lambda\xi), \ldots, P_1(\lambda\xi) + a_1(\lambda\xi), 1) \\
&= \Delta(\lambda^m[P_m(\xi) + \tfrac{a_m(\lambda\xi)}{\lambda^m}], \ldots, \lambda[P_1(\xi) + \tfrac{a_1(\lambda\xi)}{\lambda}], 1) \\
&= \lambda^{m(2m-2)} \Delta(P_m(\xi) + \tfrac{a_m(\lambda\xi)}{\lambda^m}, \ldots, \lambda^{-(m-1)}[P_1(\xi) + \tfrac{a_1(\lambda\xi)}{\lambda}], \lambda^{-m}) \\
&\qquad\qquad\text{(using that } \Delta \text{ is homogenous of degree } 2m-2) \\
&= \lambda^{m(m-1)} \Delta(P_m(\xi) + \tfrac{a_m(\lambda\xi)}{\lambda^m}, \ldots, P_1(\xi) + \tfrac{a_1(\lambda\xi)}{\lambda}, 1) \\
&\qquad\qquad\text{(by quasi-homogeneity).}
\end{aligned}$$

Now, since L is strictly hyperbolic, the characteristic roots $\varphi_1(\xi), \ldots, \varphi_m(\xi)$ of L_m are pairwise distinct for $\xi \neq 0$, so

$$\Delta_{L_m}(\xi) = \Delta(P_m(\xi), \ldots, P_1(\xi), 1) \neq 0 \text{ for } \xi \neq 0.$$

Since the discriminant is continuous in each argument, there exists $\delta > 0$ such that if $\left|\frac{a_j(\lambda\xi)}{\lambda^j}\right| < \delta$ for all $j = 1, \ldots, m$ then

$$\left|\Delta(P_m(\xi) + \tfrac{a_m(\lambda\xi)}{\lambda^m}, \ldots, P_1(\xi) + \tfrac{a_1(\lambda\xi)}{\lambda}, 1)\right| \neq 0,$$

and hence the roots of the associated polynomial are pairwise distinct. So, fix $\xi \in \{\xi \in \mathbb{R}^n : |\xi| = 1\}$ and let $\lambda \to \infty$. Since the $a_j(\xi)$ are polynomials of degree $< j$ it follows that when $|\xi| \geq M$, the characteristic roots of L are pairwise distinct. $\square$

3.2 Symbolic properties

In this section we will establish a number of useful properties of characteristic roots which will be important for the subsequent analysis. In particular, we will show that asymptotically roots behave like symbols, and we will show the relation between roots of the full symbol of a strictly hyperbolic operator with homogeneous roots of the principal part.

Proposition 3.2.1 (Symbolic properties of roots). *Let $L = L(D_t, D_x)$ be a hyperbolic operator of the following form*

$$L(D_t, D_x) = D_t^m + \sum_{j=1}^{m} P_j(D_x) D_t^{m-j} + \sum_{j=1}^{m} \sum_{|\alpha|+m-j=K} c_{\alpha,j}(D_x) D_t^{m-j},$$

where $P_j(\lambda\xi) = \lambda^j P_j(\xi)$ for $\lambda >> 1$, $|\xi| >> 1$, and $c_{\alpha,j} \in S^{|\alpha|}$. Here $0 \le K \le m-1$ is the maximum order of the lower order terms of L. Let $\tau_1(\xi), \dots, \tau_m(\xi)$ denote its characteristic roots; *then*

I. *for each $k = 1, \dots, m$, there exists a constant $C > 0$ such that*

$$|\tau_k(\xi)| \le C(1 + |\xi|) \quad \textit{for all } \xi \in \mathbb{R}^n .$$

Furthermore, if we insist that L is strictly *hyperbolic, and denote the roots of the principal part $L_m(\tau, \xi)$ by $\varphi_1(\xi), \dots, \varphi_m(\xi)$, then we have the following properties as well*:

II. *For each $\tau_k(\xi)$, $k = 1, \dots, m$, there exists a corresponding root of the principal symbol $\varphi_k(\xi)$ (possibly after reordering) such that*

$$|\tau_k(\xi) - \varphi_k(\xi)| \le C(1 + |\xi|)^{K+1-m} \quad \textit{for all } \xi \in \mathbb{R}^n . \tag{3.2.1}$$

In particular, for arbitrary lower terms, we have

$$|\tau_k(\xi) - \varphi_k(\xi)| \le C \quad \textit{for all } \xi \in \mathbb{R}^n . \tag{3.2.2}$$

III. *There exists $M > 0$ such that, for each characteristic root of L and for each multi-index α, we can find constants $C = C_{k,\alpha} > 0$ such that*

$$\left|\partial_\xi^\alpha \tau_k(\xi)\right| \le C|\xi|^{1-|\alpha|} \quad \textit{for all } |\xi| \ge M , \tag{3.2.3}$$

In particular, there exists a constant $C > 0$ such that

$$|\nabla \tau_k(\xi)| \le C \quad \textit{for all } |\xi| \ge M . \tag{3.2.4}$$

IV. *There exists $M > 0$ such that, for each $\tau_k(\xi)$ a corresponding root of the principal symbol $\varphi_k(\xi)$ can be found (possibly after reordering) which satisfies, for each multi-index α and $k = 1, \dots, m$,*

$$\left|\partial_\xi^\alpha \tau_k(\xi) - \partial_\xi^\alpha \varphi_k(\xi)\right| \leq C|\xi|^{K+1-m-|\alpha|} \quad \text{for all } |\xi| \geq M \tag{3.2.5}$$

In particular, since $K \leq m-1$, we have

$$\left|\partial_\xi^\alpha \tau_k(\xi) - \partial_\xi^\alpha \varphi_k(\xi)\right| \leq C|\xi|^{-|\alpha|} \quad \text{for all } |\xi| \geq M\,, \tag{3.2.6}$$

for each multi-index α and $k = 1, \dots, m$.

First, we need the following lemma about perturbation properties of general smooth functions. Clearly, we do not need to require that functions are smooth, but this will be the case in our application.

Lemma 3.2.2. *Let $p, q : \mathbb{C} \to \mathbb{C}$ be smooth functions and suppose z_0 is a simple zero of $p(z)$ (i.e. $p(z_0) = 0$, $p'(z_0) \neq 0$). Consider, for each $\varepsilon > 0$, the following "perturbation" of $p(z)$:*

$$p_\varepsilon(z) := p(z) + \varepsilon q(z)\,,$$

and suppose z_ε is a root of $p_\varepsilon(z)$; then, for all sufficiently small $\varepsilon > 0$, we have

$$|z_\varepsilon - z_0| \leq C\varepsilon\left|\frac{q(z_0)}{p'(z_0)}\right|. \tag{3.2.7}$$

Proof. By Taylor's theorem, we have, near z_0,

$$\begin{aligned} p_\varepsilon(z) &= p_\varepsilon(z_0) + p_\varepsilon'(z_0)(z - z_0) + O(|z - z_0|^2) \\ &= \varepsilon q(z_0) + (p'(z_0) + \varepsilon q'(x_0))(z - z_0) + O(|z - z_0|^2)\,. \end{aligned}$$

Thus, setting $z = z_\varepsilon$, we get

$$0 = \varepsilon q(z_0) + (p'(z_0) + \varepsilon q'(z_0))(z_\varepsilon - z_0) + O(|z_\varepsilon - z_0|^2)\,. \tag{3.2.8}$$

Now, consider the function of ε, $z(\varepsilon) := z_\varepsilon$; this is clearly smooth since p and q are smooth and z_0 is a simple zero of $p(z)$. Indeed, $p_\varepsilon'(z_\varepsilon) \approx p'(z_0) \neq 0$ for small ε, hence z_ε is a simple root of p_ε. Thus, near the origin,

$$z(\varepsilon) = z(0) + \varepsilon z'(0) + O(\varepsilon^2)\,. \tag{3.2.9}$$

Combining (3.2.8) and (3.2.9), we get

$$0 = \varepsilon q(z_0) + (p'(z_0) + \varepsilon q'(z_0))(\varepsilon z'(0) + O(\varepsilon^2)) + O(\varepsilon^2)\,,$$

or,

$$0 = q(z_0) + p'(z_0)z'(0) + O(\varepsilon)\,,$$

for small ε. Therefore, by the triangle inequality, for each $\varepsilon > 0$ small enough,

$$|z'(0)| \le \frac{C\varepsilon}{|p'(z_0)|} + \left|\frac{q(z_0)}{p'(z_0)}\right|,$$

and, thus,

$$|z'(0)| \le C\left|\frac{q(z_0)}{p'(z_0)}\right|. \tag{3.2.10}$$

Finally, combining (3.2.10) with (3.2.9), we obtain (3.2.7) as required. □

Proof of Proposition 3.2.1.

Part I: We may write $L(\tau,\xi)$ in the form

$$L(\tau,\xi) = \tau^m + a_1(\xi)\tau^{m-1} + \cdots + a_{m-1}(\xi)\tau + a_m(\xi),$$

where $|a_j(\xi)| \le C\langle\xi\rangle^j$. Hence for all k we have $|\tau_k(\xi)| \le C\langle\xi\rangle$ by Lemma 3.1.1.

Part II: In the proof of this part, let us write $L(\tau,\xi)$ in the form

$$L(\tau,\xi) = \sum_{i=0}^{R} L_{m-r_i}(\tau,\xi)\,,$$

where $r_0 = 0$, $m - r_1 = K$ (the maximum order of the lower order terms), $1 \le r_1 < \cdots < r_R \le m$,

$$L_m(\tau,\xi) = \tau^m + \sum_{j=1}^{m} P_j(\xi)\tau^{m-j}$$

$$\text{and } L_{m-r_i}(\tau,\xi) = \sum_{|\alpha|+j=m-r_i} c_{\alpha,j}(\xi)\tau^j \quad \text{for } 1 \le i \le R;$$

here, as usual, the $P_j(\xi)$ are homogeneous in ξ of order j.

Denote the roots of

$$\mathcal{L}_l(\tau,\xi) := \sum_{i=0}^{l} L_{m-r_i}(\tau,\xi)\,, \quad 0 \le l \le R\,,$$

with respect to τ by $\tau_1^l(\xi),\ldots,\tau_m^l(\xi)$. Note that $\mathcal{L}_0(\tau,\xi) = L_m(\tau,\xi)$, i.e. $\mathcal{L}_0(\tau,\xi)$ is the principal symbol with no lower order terms. Since $\mathcal{L}_l(\tau,\xi)$ are strictly hyperbolic, we will look at $|\xi| \geq M_0$, where all $\tau_1^l(\xi), \dots, \tau_m^l(\xi)$ are distinct, for all l.

We shall show that there exists $M \geq M_0$ so that, possibly after reordering the roots, for all $k = 1, \dots, m$,

$$|\tau_k^{l+1}(\xi) - \tau_k^l(\xi)| \leq C|\xi|^{-r_{l+1}+1} \text{ for all } l = 0, \dots, R-1 \text{ and } |\xi| \geq M\,. \quad (3.2.11)$$

Assuming this, and noting that $\tau_k^0(\xi) = \varphi_k(\xi)$ and $\tau_k^R(\xi) = \tau_k(\xi)$ for each $k = 1, \dots, m$ (possibly after reordering), we obtain

$$|\tau_k(\xi) - \varphi_k(\xi)| \leq \sum_{l=0}^{R-1} |\tau_k^{l+1}(\xi) - \tau_k^l(\xi)| \leq C|\xi|^{-r_1+1} \text{ when } |\xi| \geq M;$$

this, together with the continuity of the $\tau_k(\xi)$ and $\varphi_k(\xi)$—and thus the boundedness of $|\tau_k(\xi) - \varphi_k(\xi)|$ in $B_M(0)$, gives (3.2.1). Then, (3.2.2) follows by setting $K = m - 1$. Here we also used $r_1 = m - K$.

So, with the aim of proving (3.2.11), we first introduce some notation: set

$$\begin{aligned} \widetilde{L}_{m-r_i} : \mathbb{C} \times \mathbb{S}^{n-1} \to \mathbb{C} :\ & \widetilde{L}_{m-r_i}(\tau, \omega) = L_{m-r_i}(\tau, \omega)\,, \quad i = 0, \dots, R, \\ \widetilde{\mathfrak{L}}_l : (M_0, \infty) \times \mathbb{C} \times \mathbb{S}^{n-1} \to \mathbb{C} :\ & \widetilde{\mathfrak{L}}_l(\rho, \tau, \omega) = \rho^{-m} \mathfrak{L}_l(\rho\tau, \rho\omega), \quad l = 0, \dots, R; \end{aligned}$$

observe that $\widetilde{L}_{m-r_i}$ is just the restriction of $L_{m-r_i}(\tau, \xi)$ to $\mathbb{C} \times \mathbb{S}^{n-1}$. Denote by $\widetilde{\varphi}_1(\omega), \widetilde{\varphi}_2(\omega), \dots, \widetilde{\varphi}_m(\omega)$ the roots of $\widetilde{L}_m(\tau, \omega) = \widetilde{\mathfrak{L}}_0(\rho, \tau, \omega)$ with respect to τ, and by $\widetilde{\tau}_1^k(\rho, \omega), \widetilde{\tau}_2^k(\rho, \omega), \dots, \widetilde{\tau}_m^k(\rho, \omega)$ those of $\widetilde{\mathfrak{L}}_k(\rho, \tau, \omega)$.

We denote $\widetilde{\tau} = \frac{\tau}{|\xi|}$. Since,

$$\widetilde{L}_m\big(\widetilde{\tau}, \tfrac{\xi}{|\xi|}\big) = L_m\big(\widetilde{\tau}, \tfrac{\xi}{|\xi|}\big) = |\xi|^{-m} L_m(\tau, \xi) = |\xi|^{-m} \mathfrak{L}_0(\tau, \xi) = \widetilde{\mathfrak{L}}_0\big(|\xi|, \widetilde{\tau}, \tfrac{\xi}{|\xi|}\big)$$

for $\xi \in \mathbb{R}^n$, $\tau \in \mathbb{C}$, and

$$\begin{aligned} \widetilde{\mathfrak{L}}_{l+1}(\rho, \tau, \omega) = \rho^{-m} \mathfrak{L}_{l+1}(\rho\tau, \rho\omega) &= \rho^{-m} \sum_{i=0}^{l+1} L_{m-r_i}(\rho\tau, \rho\omega) \\ &= \rho^{-m} \sum_{i=0}^{l} L_{m-r_i}(\rho\tau, \rho\omega) + \rho^{-m} \sum_{|\alpha|+j=m-r_{l+1}} c_{\alpha,j}(\rho\omega)(\rho\tau)^j \\ &= \widetilde{\mathfrak{L}}_l(\rho, \tau, \omega) + \rho^{-r_{l+1}} \sum_{|\alpha|+j=m-r_{l+1}} \frac{c_{\alpha,j}(\rho\omega)}{\rho^{|\alpha|}} \tau^j \\ &= \widetilde{\mathfrak{L}}_l(\rho, \tau, \omega) + \rho^{-r_{l+1}} L^0_{m-r_{l+1}}(\rho, \tau, \omega) \end{aligned}$$

for $\omega \in S^{n-1}$, $\rho > M_0$, $\tau \in \mathbb{C}$, $l = 0, \dots, R-1$. Here

$$L^0_{m-r_{l+1}}(\rho, \tau, \omega) = \sum_{|\alpha|+j=m-r_{l+1}} \frac{c_{\alpha,j}(\rho\omega)}{\rho^{|\alpha|}} \tau^j.$$

We also have

$$|\xi|^{-m} \mathfrak{L}_L(\tau, \xi) = \widetilde{\mathfrak{L}}_l\big(|\xi|, \tfrac{\xi}{|\xi|}, \widetilde{\tau}\big).$$

As the left-hand side of this is zero when $\tau = \tau^l_k(\xi)$, $k = 1, \dots, m$, and the right-hand side is zero when $\widetilde{\tau} = \widetilde{\tau}^l_k(|\xi|, \frac{\xi}{|\xi|})$, $k = 1, \dots, m$, we see that $|\xi|\widetilde{\tau}^l_k(|\xi|, \frac{\xi}{|\xi|}) = \tau^l_k(\xi)$ for each $k = 1, \dots, m$ (possibly after reordering). Hence, for all $|\xi| \geq M_0$, $k = 1, \dots, m$ and $l = 0, \dots, R-1$, we have

$$|\tau^{l+1}_k(\xi) - \tau^l_k(\xi)| = |\widetilde{\tau}^{l+1}_k\big(|\xi|, \tfrac{\xi}{|\xi|}\big) - \widetilde{\tau}^l_k\big(|\xi|, \tfrac{\xi}{|\xi|}\big)||\xi|.$$

Next, observe that applying Lemma 3.2.2 with $\varepsilon = \rho^{-r_{l+1}}$ to

$$\widetilde{\mathfrak{L}}_l(\rho, \tau, \omega) + \rho^{-r_{l+1}} L^0_{m-r_{l+1}}(\rho, \tau, \omega)$$

yields, for all $\omega \in \mathbb{S}^{n-1}$ and $k = 1, \dots, m$,

$$|\widetilde{\tau}^{l+1}_k(\rho, \omega) - \widetilde{\tau}^l_k(\rho, \omega)| \leq C\rho^{-r_{l+1}} \left| \frac{L^0_{m-r_{l+1}}(\rho, \widetilde{\tau}^l_k(\rho, \omega), \omega)}{\partial_\tau \widetilde{\mathfrak{L}}_l(\rho, \widetilde{\tau}^l_k(\rho, \omega), \omega)} \right|.$$

provided we take $\rho \geq M'$ for a sufficiently large constant $M' \geq M_0$. Therefore, for all $|\xi| \geq M'$, $k = 1, \dots, m$ and $l = 0, \dots, R-1$, we have

$$|\tau^{l+1}_k(\xi) - \tau^l_k(\xi)| \leq C|\xi|^{-r_{l+1}+1} \left| \frac{L^0_{m-r_{l+1}}\big(|\xi|, \frac{\tau^l_k(\xi)}{|\xi|}, \frac{\xi}{|\xi|}\big)}{\partial_\tau \widetilde{\mathfrak{L}}_l\big(|\xi|, \frac{\tau^l_k(\xi)}{|\xi|}, \frac{\xi}{|\xi|}\big)} \right|. \tag{3.2.12}$$

Thus, it suffices to show the following two inequalities when $|\xi| \geq M$ for some $M \geq M'$:

- there exists a constant C_1 so that, for all $1 \leq i \leq R$,

$$\left| L^0_{m-r_i}\big(|\xi|, \tfrac{\tau^l_k(\xi)}{|\xi|}, \tfrac{\xi}{|\xi|}\big) \right| = \left| \sum_{|\alpha|+j=m-r_i} \frac{c_{\alpha,j}(\xi)}{|\xi|^{|\alpha|}} \left(\frac{\tau^l_k(\xi)}{|\xi|} \right)^j \right| \leq C_1; \tag{3.2.13}$$

and

- there exists a constant $C_2 > 0$ so that, for all $0 \leq l \leq R-1$,

$$\left| \partial_\tau \widetilde{\mathfrak{L}}_l\big(|\xi|, \tfrac{\tau^l_k(\xi)}{|\xi|}, \tfrac{\xi}{|\xi|}\big) \right| = |\xi|^{-m+1} |\partial_\tau \mathfrak{L}_l(\tau^l_k(\xi), \xi)| \geq C_2. \tag{3.2.14}$$

Then, combining (3.2.12), (3.2.13) and (3.2.14) gives (3.2.11).

The first estimate (3.2.13) follows immediately from Part I since the $\tau_k^l(\xi)$ are roots of strictly hyperbolic equations, and from the fact that $c_{\alpha,j} \in S^{|\alpha|}$.

The second, (3.2.14), in the case $l = 0$ is clear: the homogeneity of $L_m(\tau, \xi)$ and its roots give

$$|\xi|^{-m+1}|\partial_\tau \mathfrak{L}_0(\tau_k^0(\xi), \xi)| = \left|\partial_\tau L_m\big(\varphi_k\big(\tfrac{\xi}{|\xi|}\big), \tfrac{\xi}{|\xi|}\big)\right|,$$

which is never zero due to the strict hyperbolicity of L_m and hence (using that the sphere S^{n-1} is compact and $L_m(\tau, \xi)$ is continuous and thus achieves its minimum) is bounded below by some positive constant as required.

For $1 \le l \le R-1$, we know that $\tau_k^l(\xi)$, $k = 1, \dots, m$, are simple zeros of $\mathfrak{L}_L(\tau, \xi)$ for $|\xi| \ge M_0$ by the earlier choice of M_0. Observe,

$$\frac{(\partial_\tau \mathfrak{L}_l)(\tau_k^l(\xi), \xi)}{|\xi|^{m-1}} = \frac{(\partial_\tau L_m)(\tau_k^l(\xi), \xi)}{|\xi|^{m-1}} + \sum_{i=1}^{l} \frac{(\partial_\tau L_{m-r_i})(\tau_k^l(\xi), \xi)}{|\xi|^{m-1}}.$$

Now,

$$\frac{(\partial_\tau L_{m-r_i})(\tau_k^l(\xi), \xi)}{|\xi|^{m-1}} \to 0 \text{ as } |\xi| \to \infty$$

for $i = 1, \dots, l$, because $\partial_\tau L_{m-r_i}(\tau, \xi)$ is a symbol of order $m - r_i - 1$. Also, using the Mean Value Theorem,

$$\begin{aligned}(\partial_\tau L_m)(\tau_k^l(\xi), \xi) &= (\partial_\tau L_m)(\varphi_k(\xi), \xi) + [(\partial_\tau L_m)(\tau_k^l(\xi), \xi) - (\partial_\tau L_m)(\varphi_k(\xi), \xi)] \\ &= (\partial_\tau L_m)(\varphi_k(\xi), \xi) + (\partial_\tau^2 L_m)(\bar{\tau}_k^l(\xi), \xi),\end{aligned}$$

where $\bar{\tau}_k^l(\xi)$ lies on the line connecting $\varphi_k(\xi)$ and $\tau_k^l(\xi)$ for each $\xi \in \mathbb{R}^n$, $k = 1, \dots, m$ and $l = 1, \dots, R-1$, and

$$\frac{\left|(\partial_\tau^2 L_m)(\bar{\tau}_k^l(\xi), \xi)\right|}{|\xi|^{m-1}} \le C|\xi|^{-1} \to 0 \text{ as } |\xi| \to \infty.$$

Therefore, for a sufficiently large constant $M \ge M'$, there exists a constant $C_2 > 0$ such that

$$\frac{\left|\partial_\tau L_m(\tau_k^l(\xi), \xi)\right|}{|\xi|^{m-1}} \ge C\frac{|\partial_\tau L_m(\varphi_k(\xi), \xi)|}{|\xi|^{m-1}} \ge C_2, \text{ when } |\xi| \ge M.$$

This completes the proof of (3.2.13) and thus of Part II.

Part III: We take $M > 0$ so that for $|\xi| \geq M$, the roots $\tau_1(\xi), \dots, \tau_m(\xi)$ are distinct.

To prove the statement, we do induction on $|\alpha|$.

First, assume $|\alpha| = 1$. Since $L(\tau_k(\xi), \xi) = 0$ for each $k = 1, \dots, m$, we have, for each $i = 1, \dots, n$,

$$\frac{\partial L}{\partial \xi_i}(\tau_k(\xi), \xi) + \frac{\partial L}{\partial \tau}(\tau_k(\xi), \xi)\frac{\partial \tau_k}{\partial \xi_i}(\xi) = 0\,.$$

The first term is a symbol of order $m-1$ in $(\tau_k(\xi), \xi)$, hence, by Part I, there exists a constant C such that, when $|\xi| \geq M_1$ for some suitably large constant $M_1 \geq M$,

$$\Big|\frac{\partial L}{\partial \xi_i}(\tau_k(\xi), \xi)\Big| \leq C|\xi|^{m-1}\,.$$

The inequality (3.2.3) for $|\alpha| = 1$ (i.e. (3.2.4)) then follows immediately from:

Lemma 3.2.3. *There exists constants $C > 0$, $M_2 \geq M$ such that, for each $k = 1, \dots, m$,*

$$\Big|\frac{\partial L}{\partial \tau}(\tau_k(\xi), \xi)\Big| \geq C|\xi|^{m-1} \quad \textit{when } |\xi| \geq M_2\,.$$

Proof. Note that

$$\Big|\frac{\partial L}{\partial \tau}(\tau_k(\xi), \xi)\Big| \geq \Big|\frac{\partial L_m}{\partial \tau}(\varphi_k(\xi), \xi)\Big| - \Big|\frac{\partial L}{\partial \tau}(\tau_k(\xi), \xi) - \frac{\partial L_m}{\partial \tau}(\varphi_k(\xi), \xi)\Big|\,, \tag{3.2.15}$$

where $L_m(\tau, \xi)$ is the principal symbol of L and $\varphi_1(\xi), \dots, \varphi_m(\xi)$ are the corresponding characteristic roots, ordered in the same way as in Part II. We look at each of the terms on the right-hand side in turn:

- By strict hyperbolicity, $\frac{\partial L_m}{\partial \tau}(\varphi_k(\xi), \xi)$ is non-zero for $\xi \neq 0$. Thus, for all $\xi \neq 0$,

$$\Big|\frac{\partial L_m}{\partial \tau}(\varphi_k(\xi), \xi)\Big| = |\xi|^{m-1}\Big|\frac{\partial L_m}{\partial \tau}\Big(\tfrac{\xi}{|\xi|}, \varphi\big(\tfrac{\xi}{|\xi|}\big)\Big)\Big| \geq C|\xi|^{m-1}\,. \tag{3.2.16}$$

- Observe,

$$\begin{aligned}&\frac{\partial L}{\partial \tau}(\tau_k(\xi), \xi) - \frac{\partial L_m}{\partial \tau}(\varphi_k(\xi), \xi)\\ &\qquad = \frac{\partial L_m}{\partial \tau}(\tau_k(\xi), \xi) - \frac{\partial L_m}{\partial \tau}(\varphi_k(\xi), \xi) + \sum_{r=0}^{m-1}\sum_{|\alpha|+l=r} l\; c_{\alpha,l}(\xi)\tau_k(\xi)^{l-1}\,.\end{aligned}$$

Now,

$$\frac{\partial L_m}{\partial \tau}(\tau_k(\xi),\xi) - \frac{\partial L_m}{\partial \tau}(\varphi_k(\xi),\xi)$$
$$= m(\tau_k(\xi)^{m-1} - \varphi_k(\xi)^{m-1}) + \sum_{j=1}^{m}(m-j)P_j(\xi)(\tau_k(\xi)^{m-j-1} - \varphi_k(\xi)^{m-j-1}),$$

and

$$|\tau_k(\xi)^r - \varphi_k(\xi)^r| = |\tau_k(\xi) - \varphi_k(\xi)||\tau_k(\xi)^{r-1} + \tau_k(\xi)^{r-2}\varphi_k(\xi) + \cdots + \varphi_k(\xi)^{r-1}|\,.$$

So, by Part I and Part II (specifically inequality (3.2.2)) and the fact that the $P_j(\xi)$ are homogeneous in ξ of order j, we have, for some suitably large $M_2 \geq M$,

$$\left|\frac{\partial L_m}{\partial \tau}(\tau_k(\xi),\xi) - \frac{\partial L_m}{\partial \tau}(\varphi_k(\xi),\xi)\right| \leq C|\xi|^{m-2} \quad \text{when } |\xi| \geq M_2\,.$$

This, together with

$$\Big|\sum_{|\alpha|+l=r} l\, c_{\alpha,r}(\xi)\tau_k(\xi)^{l-1}\Big| \leq C|\xi|^{r-1} \leq C|\xi|^{m-2}$$
$$\text{when } |\xi| \geq M_2,\ r = 0,\ldots,m-1\,,$$

which again follows straight from Part I, yields

$$\left|\frac{\partial L}{\partial \tau}(\tau_k(\xi),\xi) - \frac{\partial L_m}{\partial \tau}(\varphi_k(\xi),\xi)\right| \leq C|\xi|^{m-2} \quad \text{for } |\xi| \geq M_2\,. \tag{3.2.17}$$

The result now follows by combining (3.2.15), (3.2.17) and (3.2.16). The proof of Lemma 3.2.3 is complete. □

For $|\alpha| = J > 1$, assume inductively that,

$$\left|\partial_\xi^\alpha \tau_k(\xi)\right| \leq C|\xi|^{1-|\alpha|} \quad \text{when } |\xi| \geq M,\ |\alpha| \leq J-1\,,$$

for some fixed $M \geq \max(M_1, M_2)$.

Then, for $|\alpha| = J$, we use $\partial_\xi^\alpha[L(\tau_k(\xi),\xi)] = 0$, i.e.

$$\partial_\xi^\alpha \tau_k(\xi)\partial_\tau L(\tau_k(\xi),\xi)$$
$$+ \sum_{\substack{\beta^1+\cdots+\beta^r\leq\alpha,\\ \beta^j\neq 0,\beta^j\neq\alpha}} c_{\alpha,\beta^1,\ldots,\beta^r}\Big(\prod_{j=1}^{r}\partial_\xi^{\beta^j}\tau_k(\xi)\Big)\partial_\xi^{\alpha-\beta^1-\cdots-\beta^r}\partial_\tau^r L(\tau_k(\xi),\xi) = 0\,.$$

By the inductive hypothesis and the fact that $\partial_\xi^\beta \partial_\tau^j L(\tau_k(\xi), \xi)$ is a symbol of order $m - j - |\beta|$, we have, for all multi-indices $\beta^1, \dots, \beta^r \neq 0$ or α satisfying $\beta^1 + \dots + \beta^r \leq \alpha$,

$$\left| \Big(\prod_{j=1}^{r} \partial_\xi^{\beta^j} \tau_k(\xi) \Big) \partial_\xi^{\alpha - \beta^1 - \dots - \beta^r} \partial_\tau^r L(\tau_k(\xi), \xi) \right| \leq C_{k,\alpha} |\xi|^{m-|\alpha|} \text{ when } |\xi| \geq M.$$

Thus, using Lemma 3.2.3 again, we have

$$|\partial_\xi^\alpha \tau_k(\xi)| \leq \frac{C_\alpha |\xi|^{m-|\alpha|}}{|\partial_\tau L(\tau_k(\xi), \xi)|} \leq C_{k,\alpha} |\xi|^{1-|\alpha|} \text{ when } |\xi| \geq M,$$

which completes the proof of the induction step.

Part IV: Once again, assume that the roots $\tau_k(\xi)$, $k = 1, \dots, m$, correspond to $\varphi_k(\xi)$, $k = 1, \dots, m$, in the manner of Part II.

The proof of this part for general multi-index α is quite technical, so we first give the proof in the case $|\alpha| = 1$ to demonstrate the main ideas required, and then show how it can be extended when $|\alpha| > 1$.

From $L(\tau_k(\xi), \xi) = 0 = L_m(\varphi_k(\xi), \xi)$, we have for each $i = 1, \dots, n$,

$$\begin{aligned} \frac{\partial L}{\partial \xi_i}(\tau_k(\xi), \xi) + \frac{\partial L}{\partial \tau}(\tau_k(\xi), \xi) \frac{\partial \tau_k}{\partial \xi_i}(\xi) &= 0\,, \\ \frac{\partial L_m}{\partial \xi_i}(\varphi_k(\xi), \xi) + \frac{\partial L_m}{\partial \tau}(\varphi_k(\xi), \xi) \frac{\partial \varphi_k}{\partial \xi_i}(\xi) &= 0\,. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\partial L}{\partial \tau}(\tau_k(\xi), \xi) \Big(\frac{\partial \tau_k}{\partial \xi_i}(\xi) - \frac{\partial \varphi_k}{\partial \xi_i}(\xi) \Big) &= \frac{\partial L_m}{\partial \xi_i}(\varphi_k(\xi), \xi) - \frac{\partial L_m}{\partial \xi_i}(\tau_k(\xi), \xi) \\ &+ \frac{\partial \varphi_k}{\partial \xi_i} \Big[\frac{\partial L_m}{\partial \tau}(\varphi_k(\xi), \xi) - \frac{\partial L}{\partial \tau}(\tau_k(\xi), \xi) \Big] - \frac{\partial (L - L_m)}{\partial \xi_i}(\tau_k(\xi), \xi)\,. \end{aligned} \tag{3.2.18}$$

It suffices to show that the right-hand side is bounded absolutely by $C|\xi|^{m-2}$ when $|\xi| \geq M_1$ for some suitably large $M_1 \geq M_0$; this is because an application of Lemma 3.2.3 then yields

$$\Big| \frac{\partial \tau_k}{\partial \xi_i}(\xi) - \frac{\partial \varphi_k}{\partial \xi_i}(\xi) \Big| \leq \frac{C|\xi|^{m-2}}{\big| \frac{\partial L}{\partial \tau}(\tau_k(\xi), \xi) \big|} \leq C|\xi|^{-1} \quad \text{for } |\xi| \geq M\,,$$

where $M = \max(M_1, M_2)$.

Since $\partial_{\xi_i}(L - L_m)(\tau, \xi)$ is a symbol of order $\leq m - 2$ in (τ, ξ), it is immediately clear that the final term of (3.2.18) is bounded by $C|\xi|^{m-2}$;

here we have also used Part I. Also, noting that $|\partial_{\xi_i}\varphi_k(\xi)| \leq C$ by the homogeneity of $\varphi_k(\xi)$, we have, by (3.2.17),

$$\Big|\frac{\partial\varphi_k}{\partial\xi_i}(\xi)\Big|\Big|\frac{\partial L_m}{\partial\tau}(\varphi_k(\xi),\xi) - \frac{\partial L_m}{\partial\tau}(\tau_k(\xi),\xi)\Big| \leq C|\xi|^{m-2}\,.$$

Finally, by the Mean Value Theorem,

$$\Big|\frac{\partial L_m}{\partial\xi_i}(\varphi_k(\xi),\xi) - \frac{\partial L_m}{\partial\xi_i}(\tau_k(\xi),\xi)\Big| \leq C\Big|\frac{\partial^2 L_m}{\partial\tau\partial\xi_i}(\xi,\bar\tau)\Big||\varphi_k(\xi) - \tau_k(\xi)|\,,$$

where $\bar\tau$ lies on the linear path between $\varphi_k(\xi)$ and $\tau_k(\xi)$—which means that (using Part I once more) $|\bar\tau| \leq C|\xi|$ for $|\xi| \geq M$. Since $\partial_\tau\partial_{\xi_i}L_m(\tau,\xi)$ is a symbol of order $m-2$ in (τ,ξ), and $|\varphi_k(\xi) - \tau_k(\xi)| \leq C$ by Part II, this term is bounded by $C|\xi|^{m-2}$, completing the proof in the case $|\alpha| = 1$.

For $|\alpha| = J > 1$, we assume inductively that

$$\big|\partial_\xi^\alpha\tau_k(\xi) - \partial_\xi^\alpha\varphi_k(\xi)\big| \leq C|\xi|^{-|\alpha|} \quad \text{for } |\xi| \geq M\,, |\alpha| \leq J-1\,.$$

As in the proof of Part III, we have

$$\partial_\xi^\alpha\tau_k(\xi)\partial_\tau L(\tau_k(\xi),\xi) \\ + \sum_{\substack{\beta^1+\cdots+\beta^r\leq\alpha,\\ \beta^j\neq 0,\beta^j\neq\alpha}} c_{\alpha,\beta^1,\ldots,\beta^r}\Big(\prod_{j=1}^r \partial_\xi^{\beta^j}\tau_k(\xi)\Big)\partial_\xi^{\alpha-\beta^1-\cdots-\beta^r}\partial_\tau^r L(\tau_k(\xi),\xi) = 0\,;$$

similarly,

$$\partial_\xi^\alpha\varphi_k(\xi)\partial_\tau L_m(\varphi_k(\xi),\xi) \\ + \sum_{\substack{\beta^1+\cdots+\beta^r\leq\alpha,\\ \beta^j\neq 0,\beta^j\neq\alpha}} c_{\alpha,\beta^1,\ldots,\beta^r}\Big(\prod_{j=1}^r \partial_\xi^{\beta^j}\varphi_k(\xi)\Big)\partial_\xi^{\alpha-\beta^1-\cdots-\beta^r}\partial_\tau^r L_m(\varphi_k(\xi),\xi) = 0\,.$$

Thus,

$$
\begin{aligned}
(\partial_\xi^\alpha \tau_k(\xi) - \partial_\xi^\alpha \varphi_k(\xi))\partial_\tau L(\tau_k(\xi),\xi) = \\
&\partial_\xi^\alpha \varphi_k(\xi)\big(\partial_\tau L_m(\varphi_k(\xi),\xi) - \partial_\tau L(\tau_k(\xi),\xi)\big) \\
&+ \sum_{\substack{\beta^1+\cdots+\beta^r\le\alpha,\\ \beta^j\neq 0,\beta^j\neq\alpha}} c_{\alpha,\beta^1,\ldots,\beta^r}\Big(\prod_{j=1}^r \partial_\xi^{\beta^j}\varphi_k(\xi)\Big)[\partial_\xi^{\alpha-\beta^1-\cdots-\beta^r}\partial_\tau^r L_m(\varphi_k(\xi),\xi) - \\
&\qquad \partial_\xi^{\alpha-\beta^1-\cdots-\beta^r}\partial_\tau^r L_m(\tau_k(\xi),\xi)] \\
&+ \sum_{\substack{\beta^1+\cdots+\beta^r\le\alpha,\\ \beta^j\neq 0,\beta^j\neq\alpha}} c_{\alpha,\beta^1,\ldots,\beta^r}\Big(\prod_{j=1}^r [\partial_\xi^{\beta^j}\varphi_k(\xi) - \partial_\xi^{\beta^j}\tau_k(\xi)]\Big)\partial_\xi^{\alpha-\beta^1-\cdots-\beta^r}\partial_\tau^r L_m(\tau_k(\xi),\xi) \\
&- \sum_{\substack{\beta^1+\cdots+\beta^r\le\alpha,\\ \beta^j\neq 0,\beta^j\neq\alpha}} c_{\alpha,\beta^1,\ldots,\beta^r}\Big(\prod_{j=1}^r \partial_\xi^{\beta^j}\tau_k(\xi)\Big)\partial_\xi^{\alpha-\beta^1-\cdots-\beta^r}\partial_\tau^r (L-L_m)(\tau_k(\xi),\xi)\,.
\end{aligned}
$$

We claim the right-hand side is then bounded absolutely by $C_\alpha|\xi|^{m-1-|\alpha|}$, which, together with Lemma 3.2.3, yields the desired estimate.

To see this, let us look at each of the terms in turn:

- $|\partial_\xi^\alpha \varphi_k(\xi)| \le C_\alpha|\xi|^{1-|\alpha|}$ by the homogeneity of $\varphi_k(\xi)$; using this with (3.2.17) gives the desired bound.

- Using the Mean Value Theorem as in the case $|\alpha| = 1$, we get

$$
\begin{aligned}
\big|[\partial_\xi^{\alpha-\beta^1-\cdots-\beta^r}\partial_\tau^r L_m(\varphi_k(\xi),\xi) - \partial_\xi^{\alpha-\beta^1-\cdots-\beta^r}\partial_\tau^r L_m(\tau_k(\xi),\xi)]\big| \\
\le C_\alpha|\xi|^{m-|\alpha|+|\beta^1|+\cdots+|\beta^r|-r-1}\,;
\end{aligned}
$$

  coupled with $|\partial_\xi^\beta \varphi_k(\xi)| \le C_\alpha|\xi|^{1-|\beta|}$, this gives the correct bound.

- By the inductive hypothesis,

$$
|\partial_\xi^{\beta^j}\varphi_k(\xi) - \partial_\xi^{\beta^j}\tau_k(\xi)| \le C_\beta|\xi|^{1-|\beta^j|}\,;
$$

  together with

$$
|\partial_\xi^{\alpha-\beta^1-\cdots-\beta^r}\partial_\tau^r L_m(\tau_k(\xi),\xi)| \le C_\alpha|\xi|^{m-|\alpha|+|\beta^1|+\cdots+|\beta^r|-r}\,,
$$

  which follows from Part I and the homogeneity of $L_m(\tau,\xi)$, this gives the correct estimate.

- To show the final term is bounded absolutely by $|\xi|^{m-1-|\alpha|}$, first note that

$$\partial_\xi^{\alpha-\beta^1-\cdots-\beta^r}\partial_\tau^r(L-L_m)(\tau_k(\xi),\xi)$$

is a symbol of order $\leq m-|\alpha|+|\beta^1|+\cdots+|\beta^r|-r-1$; applying Part III to estimate the $\partial_\xi^{\beta^j}\tau_k(\xi)$ terms, we have the required result.

This completes the proof of (3.2.6); (3.2.5) is proved in a similar way in the proof using the set-up of the proof of Part II. The proof of Proposition 3.2.1 is now complete. □

We will now establish further symbolic properties of characteristic roots. A refinement of this proposition concerning real and imaginary parts of complex roots τ is given in Proposition 6.8.2.

Proposition 3.2.4. *Suppose that the characteristic roots ϕ_k, $k=1,\ldots,m$, of the principal part $L_m(\tau,\xi)$ of a strictly hyperbolic operator $L(\tau,\xi)$ in (2.0.1) are non-zero for all $\xi\neq 0$. Then the roots $\tau(\xi)$ of the full symbols satisfy the following properties:*

(i) *for all multi-indices α there exists a constants $M, C_\alpha>0$ such that*

$$|\partial_\xi^\alpha\tau(\xi)|\leq C_\alpha|\xi|^{1-|\alpha|};$$

for all $|\xi|\geq M$.

(ii) *there exist constants $M,C>0$ such that for all $|\xi|\geq M$ we have $|\tau(\xi)|\geq C|\xi|$;*

(iii) *there exists a constant $C_0>0$ such that $|\partial_\omega\tau(\lambda\omega)|\geq C_0$ for all $\omega\in\mathbb{S}^{n-1}$, $\lambda>0$; in particular, $|\nabla\tau(\xi)|\geq C_0$ for all $\xi\in\mathbb{R}^n\setminus\{0\}$;*

(iv) *there exists a constant $R_1>0$ such that, for all $\lambda>0$,*

$$\frac{1}{\lambda}\Sigma_\lambda(\tau)\equiv\frac{1}{\lambda}\{\xi\in\mathbb{R}^n:\ \tau(\xi)=\lambda\}\subset B_{R_1}(0)\,.$$

Proof. • Property (i): by Proposition 3.2.1, Part III,

$$|\partial_\xi^\alpha\tau(\xi)|\leq C_\alpha|\xi|^{1-|\alpha|}\quad\text{for all }|\xi|\geq M\,,$$

for all multi-indices α.

- Properties (ii) and (iii): these follow by using perturbation methods. By Proposition 3.2.1, Part IV, there exists a homogeneous function $\varphi(\xi)$ of order 1 such that, for all $|\xi| \geq M$ and $k = 1, \ldots, n$,

$$|\tau(\xi) - \varphi(\xi)| \leq C_0 \text{ and } |\partial_{\xi_k}\tau(\xi) - \partial_{\xi_k}\varphi(\xi)| \leq C_k|\xi|^{-1}\,,$$

for some constants $C_0, C_k > 0$. Now, the homogeneity of $\varphi(\xi)$ implies that $\varphi(\xi) = |\xi|\varphi\big(\frac{\xi}{|\xi|}\big)$ and $e_k \cdot \nabla\varphi(e_k) = \varphi(e_k)$, where $e_k = (\underbrace{0, \ldots, 0, 1}_{k}, 0, \ldots, 0)$, so

$$|\varphi(\xi)| \geq C'|\xi| \text{ for all } \xi \in \mathbb{R}^n \text{ and } |\partial_\omega\varphi(\lambda\omega)| \geq C' \text{ for all } \omega \in \mathbb{S}^{n-1},\ \lambda > 0\,,$$

for some constant $C' > 0$. Thus,

$$|\tau(\xi)| \geq |\varphi(\xi)| - |\tau(\xi) - \varphi(\xi)| \geq C'|\xi| - C_0 \geq C|\xi| \text{ for } |\xi| \geq M\,, \quad (3.2.19)$$

for some constants $M, C > 0$, and

$$|\partial_\omega\tau(\lambda\omega)| \geq |\partial_\omega\varphi(\lambda\omega)| - |\partial_\omega\varphi(\lambda\omega) - \partial_\omega\tau(\lambda\omega)| \geq C' - C_k\lambda^{-1} \geq C > 0$$

for all $\omega \in \mathbb{S}^{n-1}$ and suitably large λ; for small $\lambda > 0$, $\partial_\omega\tau(\lambda\omega)$ is separated from 0 by the convexity condition, so $|\partial_\omega\tau(\lambda\omega)| \geq C > 0$ for all $\omega \in \mathbb{S}^{n-1}$, $\lambda > 0$, as required.

- Property (iv)—there exists a constant $R_1 > 0$ such that, for all $\lambda > 0$, $\frac{1}{\lambda}\Sigma_\lambda(\tau) \subset B_{R_1}(0)$—holds by Proposition 3.2.1, Part II, and the fact that $\frac{1}{\lambda}\Sigma_\lambda(\varphi) = \Sigma_1(\varphi)$ for the characteristic root of the principal symbol φ corresponding to τ.

$\square$