Supplements.

§1. Borel's density theorem for fuchsian groups.[1] Let $G_{\mathbf{R}} = PSL_2(\mathbf{R})$. Then all finite dimensional irreducible ordinary representations of $G_{\mathbf{R}}$ are given by ρ_n ($n = 0, 1, 2, \cdots$), defined in Chapter 3, §3. Since they are algebraic representations, it is clear that if Δ is a subgroup of $G_{\mathbf{R}}$ not contained in any proper algebraic subgroup of $G_{\mathbf{R}}$, then $\rho_n|\Delta$ is also irreducible. In particular, if Δ is a discrete subgroup of $G_{\mathbf{R}}$ whose quotient has finite invariant volume, then Δ is Zariski dense in $G_{\mathbf{R}}$ (a special case of Borel's density theorem [**1**]; but since $\dim G_{\mathbf{R}} = 3$ is small, it can also be checked directly); hence $\rho_n|\Delta$ is irreducible.

In particular, $\rho_1|\Delta$ is irreducible. Since ρ_1 is equivalent to the adjoint representation Ad. of $G_{\mathbf{R}}$ in its Lie algebra $\mathfrak{g}_{\mathbf{R}}$, this shows that no proper Lie subalgebra $\neq \{0\}$ of $\mathfrak{g}_{\mathbf{R}}$ is invariant by $\mathrm{Ad}\,\Delta$. Now if $H_{\mathbf{R}}$ is a closed subgroup of $G_{\mathbf{R}}$ containing Δ with $(H_{\mathbf{R}} : \Delta) = \infty$, then $H_{\mathbf{R}}$ is non-discrete, and hence the corresponding Lie subalgebra $\mathfrak{h}_{\mathbf{R}}$ is non-trivial. But $\mathfrak{h}_{\mathbf{R}}$ is invariant by $\mathrm{Ad}\,H_{\mathbf{R}}$, and hence also by $\mathrm{Ad}\,\Delta$. Therefore $\mathfrak{h}_{\mathbf{R}} = \mathfrak{g}_{\mathbf{R}}$; hence $H_{\mathbf{R}} = G_{\mathbf{R}}$ (since $G_{\mathbf{R}}$ is connected).

Therefore, if $\tilde{\Delta}$ is a group with $G_{\mathbf{R}} \supset \tilde{\Delta} \supset \Delta$ and with $(\tilde{\Delta} : \Delta) = \infty$, then $\tilde{\Delta}$ is dense in $G_{\mathbf{R}}$.

Supplements to Chapter 1.

§2. A generalization of Lemma 10 of Chapter 1. Here, we shall verify the following assertion.[2]

The Lemma 10 of Chapter 1 remains valid if we weaken the compactness assumption of the quotient $G_{\mathbf{R}}/\Delta$ and replace it by the finiteness of volume, and if we assume that $f(z)$ is a cusp form. (*also by Kuga.*)

PROOF. As in §21 (Chapter 1), put

$$F(g) = f(g(\sqrt{-1})) \cdot j(g, \sqrt{-1}) \quad (g \in G_{\mathbf{R}}), \tag{1}$$

so that $F(g)$ is a Δ-invariant continuous function on $G_{\mathbf{R}}$. In this case, the quotient $G_{\mathbf{R}}/\Delta$ may not be compact, but we shall check that $|F(g)|$ still achieves its maximum value on

[1] This is referred to in the following places: Chapter 2, §7, §24, Chapter 3, §1, §8.

[2] This is used in the proofs of Theorem 7 (Chapter 1, Part 2) (the inequality (171)), and the Theorem in Supplement §6.

$G_{\mathbf{R}}$. Since the compactness assumption of the quotient $G_{\mathbf{R}}/\Delta$ is not used in the rest of the proof of Lemma 10 (Chapter 1), this would settle our assertion.

Let $\mathfrak{F}$ be a closed fundamental domain of Δ on $\mathfrak{H}$, so that $\widetilde{\mathfrak{F}} = \{g \in G_{\mathbf{R}} \mid g(\sqrt{-1}) \in \mathfrak{F}\}$ is a closed fundamental domain for $\Delta\backslash G_{\mathbf{R}}$. Let $T_1, \cdots, T_t$ be all the non-equivalent cusps lying on the borders of $\mathfrak{F}$. It is enough to check

$$\lim_{g\in\widetilde{\mathfrak{F}}, g(\sqrt{-1})\to T_i} |F(g)| = 0 \quad (i = 1, \cdots, t). \tag{2}$$

(In fact, assuming (2), take sufficiently small neighborhoods $U_1, \cdots, U_t$ of $T_1, \cdots, T_t$ on $\mathfrak{F}$. Then $\sup_{g\in\widetilde{\mathfrak{F}}} |F(g)| = \sup_{g\in\widetilde{\mathfrak{F}}, g(\sqrt{-1})\notin U_i\text{'s}} |F(g)|$, and $\mathfrak{F} - \bigcup_{i=1}^{t} U_i$ is compact; hence $|F(g)|$ achieves its maximum value on $\widetilde{\mathfrak{F}}$.) Now to check (2), we may assume, without loss of generality, that the cusp in question is $\sqrt{-1}\infty$. Then by the definition of cusp forms, (2) is easily reduced to the following trivial equality:

$$\lim_{\delta\to 0} \delta^{-k/2} e^{-\mu/\delta} = 0, \tag{3}$$

where μ is a positive real constant. □

Supplements to Chapter 2.

§3. Lemma 8 (Chapter 2) for "p-side", and application. Let Γ be a discrete subgroup of $G = G_{\mathbf{R}} \times G_{\mathfrak{p}}$ whose quotient G/Γ has finite invariant volume and whose projections $\Gamma_{\mathbf{R}}, \Gamma_{\mathfrak{p}}$ are dense in $G_{\mathbf{R}}, G_{\mathfrak{p}}$ respectively.

First, let Γ' be another such subgroup of G, satisfying $\Gamma'_{\mathfrak{p}} = \Gamma_{\mathfrak{p}}$. For each $\gamma_{\mathfrak{p}} \in \Gamma_{\mathfrak{p}}$, let $\gamma = \gamma_{\mathbf{R}} \times \gamma_{\mathfrak{p}} \in \Gamma, \gamma' = \gamma'_{\mathbf{R}} \times \gamma_{\mathfrak{p}} \in \Gamma'$, and put $\gamma'_{\mathbf{R}} = \varphi(\gamma_{\mathbf{R}})$. Then, it is clear that φ satisfies the conditions stated in Lemma 8 (Chapter 2). Therefore, there exists $x \in G'_{\mathbf{R}} = PL_2(\mathbf{R})$ such that $\varphi(\gamma_{\mathbf{R}}) = x^{-1}\gamma_{\mathbf{R}}x$ for all $\gamma_{\mathbf{R}} \in \Gamma_{\mathbf{R}}$. Therefore, if Γ and Γ' are moreover commensurable with each other, then x must commute with all elements of $(\Gamma \cap \Gamma')_{\mathbf{R}}$; hence $x = 1$; hence $\varphi = 1$; hence we get $\Gamma' = \Gamma$.

Here, we shall prove the following:

Proposition. *Let Γ be as above, and let Γ' be another such subgroup of G, satisfying $\Gamma'_{\mathbf{R}} = \Gamma_{\mathbf{R}}$. For each $\gamma_{\mathbf{R}} \in \Gamma_{\mathbf{R}}$, let $\gamma = \gamma_{\mathbf{R}} \times \gamma_{\mathfrak{p}} \in \Gamma$, $\gamma' = \gamma_{\mathbf{R}} \times \gamma'_{\mathfrak{p}} \in \Gamma'$, and put $\gamma'_{\mathfrak{p}} = \varphi(\gamma_{\mathfrak{p}})$. Then, there is a topological automorphism σ of $G_{\mathfrak{p}}$ such that $\varphi(\gamma_{\mathfrak{p}}) = \sigma(\gamma_{\mathfrak{p}})$ for all $\gamma_{\mathfrak{p}} \in \Gamma_{\mathfrak{p}}$.*

Corollary 1. *Let Γ, Γ' be as in the above proposition, and assume that Γ and Γ' are moreover commensurable with each other. Then $\Gamma' = \Gamma$.*

Corollary 2. *Let Γ be as above. Then there exist only finitely many subgroups Δ of G satisfying $\Delta \supset \Gamma$ and $(\Delta : \Gamma) < \infty$.*

First, let us prove the corollaries, assuming the proposition.

Proof of Corollary 1. The automorphism σ must be trivial on $(\Gamma \cap \Gamma')_{\mathfrak{p}}$, but $(\Gamma \cap \Gamma')_{\mathfrak{p}}$ is dense in $G_{\mathfrak{p}}$ (since it is of finite index in $\Gamma_{\mathfrak{p}}$). Therefore $\sigma = 1$. Therefore $\Gamma' = \Gamma$. □

Proof of Corollary 2. Let V be an open compact subgroup of $G_\mathfrak{p}$, and put $\Gamma^V = \Gamma \cap (G_\mathbf{R} \times V)$, $\Delta^V = \Delta \cap (G_\mathbf{R} \times V)$. Then $\Delta_\mathfrak{p} = \Delta_\mathfrak{p}^V \cdot \Gamma_\mathfrak{p}$ (since $\Gamma_\mathfrak{p}$ is dense in $G_\mathfrak{p}$); hence $\Delta = \Delta^V \cdot \Gamma$. But $\Gamma_\mathbf{R}^V$ is a discrete subgroup of $G_\mathbf{R}$ whose quotient has finite invariant volume, and $\Delta_\mathbf{R}^V \supset \Gamma_\mathbf{R}^V$, $(\Delta_\mathbf{R}^V : \Gamma_\mathbf{R}^V) = (\Delta : \Gamma) < \infty$. Therefore, there are only finitely many possibilities for $\Delta_\mathbf{R}^V$; hence there are only finitely many possibilities for $\Delta_\mathbf{R} = \Delta_\mathbf{R}^V \cdot \Gamma_\mathbf{R}$. But if $\Delta, \Delta' \supset \Gamma, (\Delta : \Gamma) < \infty, (\Delta' : \Gamma) < \infty$ and $\Delta_\mathbf{R} = \Delta'_\mathbf{R}$ then by the above Corollary 1, we get $\Delta = \Delta'$. Therefore for each $\Delta_\mathbf{R}$, there exists at most *one* Δ. Therefore, there are at most finitely many groups Δ. □

Remark 1. The above proof shows that the number of such Δ is at most equal to the number of fuchsian groups containing $\Gamma_\mathbf{R}^V$ for $V = PSL_2(O_\mathfrak{p})$.

Proof of the Proposition. This is completely parallel to the proof of Lemma 8 (Chapter 2). Call $g_\mathfrak{p} \in G_\mathfrak{p}$ $\mathfrak{p}$-*elliptic* if its eigenvalues are *not* contained in $k_\mathfrak{p}$. Then, $g_\mathfrak{p} \in G_\mathfrak{p}$ is $\mathfrak{p}$-elliptic if and only if its centralizer in $G_\mathfrak{p}$ is compact. Therefore, by an argument completely parallel to the proof of Lemma 9 (Chapter 2), we can prove that if $\gamma = \gamma_\mathbf{R} \times \gamma_\mathfrak{p} \in \Gamma$, then $\gamma_\mathfrak{p}$ is $\mathfrak{p}$-elliptic if and only if the centralizer of $\gamma_\mathbf{R}$ in $\Gamma_\mathbf{R}$ is discrete in $G_\mathbf{R}$; hence also if and only if $\gamma'_\mathfrak{p}$ is $\mathfrak{p}$-elliptic, where $\gamma' = \gamma_\mathbf{R} \times \gamma'_\mathfrak{p} \in \Gamma'$. Therefore, φ and φ^{-1} preserve the $\mathfrak{p}$-ellipticity of elements of $\Gamma_\mathfrak{p}, \Gamma'_\mathfrak{p}$. We can also prove the assertion corresponding to Lemma 10 (Chapter 2) by an argument completely parallel to that used in the proof of Lemma 10 (Chapter 2). Namely, we assert that if $\gamma_1, \gamma_2, \cdots$ is any sequence in $\Gamma_\mathfrak{p}$, then it tends to 1 if and only if for any $\mathfrak{p}$-elliptic element $\delta \in \Gamma_\mathfrak{p}$, $\gamma_n\delta$ are $\mathfrak{p}$-elliptic for all sufficiently large n. The proof runs as follows. The "only if" part is trivial, since $\mathfrak{p}$-elliptic elements of $G_\mathfrak{p}$ form an open subset of $G_\mathfrak{p}$. To prove the "if" part, we first remark that there exist four elements $\delta_1, \delta_2, \delta_3, \delta_4 \in \Gamma_\mathfrak{p}$ such that δ_i $(1 \le i \le 4)$ are $\mathfrak{p}$-elliptic and are (additively) linearly independent over $k_\mathfrak{p}$. In fact, put

$$g_1 = \begin{pmatrix} 0 & -1 \\ 1 & \alpha \end{pmatrix}, g_2 = \begin{pmatrix} 0 & -1 \\ 1 & \beta \end{pmatrix}, g_3 = \begin{pmatrix} \alpha & -1 \\ 1 & 0 \end{pmatrix}, g_4 = \begin{pmatrix} \alpha & -\frac{1}{2} \\ 2 & 0 \end{pmatrix},$$

where

$$\alpha, \beta \in k_\mathfrak{p},\ \alpha \neq \beta,\ \alpha,\ \beta \neq 0; \quad \alpha^2 - 4, \beta^2 - 4 \notin k_\mathfrak{p}^2.$$

Then they are $\mathfrak{p}$-elliptic elements of $G_\mathfrak{p}$ and are linearly independent over $k_\mathfrak{p}$. Since $\Gamma_\mathfrak{p}$ is dense in $G_\mathfrak{p}$, we can take $\delta_i \in \Gamma_\mathfrak{p}$ $(1 \le i \le 4)$ to be sufficiently near g_i $(1 \le i \le 4)$ respectively. Then δ_i $(1 \le i \le 4)$ satisfy the desired conditions. Put $\Pi = \{x \in G_\mathfrak{p} \mid x\delta_i\ (1 \le i \le 4) \text{ are } \mathfrak{p}\text{-elliptic}\}$. Then, since the map

$$M_2(k_\mathfrak{p}) \ni x \mapsto (\mathrm{tr}(x\delta_1), \cdots, \mathrm{tr}(x\delta_4)) \in k_\mathfrak{p}^4$$

gives an isomorphism of the two vector spaces over $k_\mathfrak{p}$, and since the image of Π is contained in $O_\mathfrak{p}^4$ (eigenvalues of $\mathfrak{p}$-elliptic elements of $G_\mathfrak{p}$ are integers since their norms over $k_\mathfrak{p}$ are 1), we see that Π is relatively compact. Now, let $\gamma_1, \gamma_2, \cdots$ be a sequence in $\Gamma_\mathfrak{p}$ such that for any $\mathfrak{p}$-elliptic element $\delta \in \Gamma_\mathfrak{p}$, $\gamma_n\delta$ are $\mathfrak{p}$-elliptic for all sufficiently large n. Then, $\gamma_n \in \Pi$ holds for all large n. Therefore, it suffices to show that if $\xi \in G_\mathfrak{p}$ is an accumulating point of $\gamma_1, \gamma_2, \cdots$ (hence $\xi \in \overline{\Pi}$), then $\xi = 1$. Let ξ be an accumulating point of $\gamma_1, \gamma_2, \cdots$. Then, for each $\mathfrak{p}$-elliptic element $\delta \in \Gamma_\mathfrak{p}$, $\xi\delta$ is an accumulating point of $\mathfrak{p}$-elliptic elements $\gamma_n\delta$ ($n \gg 0$). Hence $\operatorname{tr}(\xi\delta) \in O_\mathfrak{p}$ holds for all such δ. Therefore, $\operatorname{tr}(\xi g_\mathfrak{p}) \in O_\mathfrak{p}$ holds for all $\mathfrak{p}$-elliptic elements $g_\mathfrak{p} \in G_\mathfrak{p}$. Put $\xi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $g_\mathfrak{p} = \begin{pmatrix} 0 & -z \\ \frac{1}{z} & \alpha \end{pmatrix}$, with

$$\alpha \in k_\mathfrak{p},\ \alpha^2 - 4 \notin k_\mathfrak{p}^2 \text{ and } z \in k_\mathfrak{p}^\times.$$

Then $g_\mathfrak{p}$ is $\mathfrak{p}$-elliptic, and we have $\operatorname{tr}(\xi g_\mathfrak{p}) = \frac{b}{z} - cz + d\alpha$. Now if $b \neq 0$, let $|z|$ be sufficiently small; and if $c \neq 0$, let $|z|$ be sufficiently large. Then in either case we get a contradiction to $\operatorname{tr}(\xi g_\mathfrak{p}) \in O_\mathfrak{p}$. Therefore $b = c = 0$; hence $\xi = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$. Now we have shown that if $\delta \in \Gamma_\mathfrak{p}$ is $\mathfrak{p}$-elliptic, then $\xi\delta$ is an accumulating point of $\mathfrak{p}$-elliptic elements. But there are $\mathfrak{p}$-elliptic elements $\delta \in \Gamma_\mathfrak{p}$ which are arbitrarily near 1. Therefore, ξ itself is an accumulating point of $\mathfrak{p}$-elliptic elements. But ξ being of the form $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$, this is possible only when $a = \pm 1$. Therefore $\xi = 1$.

Therefore, the convergence (to 1) of sequences $\gamma_1, \gamma_2, \cdots$ in $\Gamma_\mathfrak{p}$ is characterized in terms of $\mathfrak{p}$-ellipticity, which is invariant by φ. Therefore, *φ is bicontinuous*. Therefore, φ can be extended to a topological automorphism σ of $G_\mathfrak{p}$. □

Remark 2. By a slight modification of the above argument, we can also prove that if $G'_\mathfrak{p}$ is a subgroup of $PL_2(k_\mathfrak{p})$ with $PSL_2(k_\mathfrak{p}) = G_\mathfrak{p} \subset G'_\mathfrak{p} \subset PL_2(k_\mathfrak{p})$, and if Γ' is a discrete subgroup of $G' = G_\mathbf{R} \times G'_\mathfrak{p}$ whose quotient has finite volume and whose projections $\Gamma'_\mathbf{R}, \Gamma'_\mathfrak{p}$ are dense in $G_\mathbf{R}, G'_\mathfrak{p}$ respectively, and if moreover $\Gamma'_\mathbf{R} = \Gamma_\mathbf{R}$ holds, then $G'_\mathfrak{p} = G_\mathfrak{p}$, and Γ' is mapped onto Γ by some topological automorphism σ of $G_\mathfrak{p}$. This remark is needed in Chapter 4, §4.

§4. Proof of Lemma 1 (Chapter 2). In E. B. Dynkin [8], the following Theorem is proved (cf. [8] Part 1, Theorems I, II).

Theorem (Dynkin). *Let K be a non-discrete simple normed field, and let R_1 and R_2 be normed Lie algebras over K. Then a homomorphism of R_1 into R_2 is at the same time a local homomorphism of the local group $G(R_1)$ into the local group $G(R_2)$. Conversely, an arbitrary local homomorphism of $G(R_1)$ into $G(R_2)$ is equivalent to a certain homomorphism of R_1 into R_2. In this manner, homomorphisms $R_1 \to R_2$ and local homomorphism classes $G(R_1) \to G(R_2)$ correspond in a one-to-one manner.*

Here, by a non-discrete simple normed field, he means either the real number field $\mathbf{R}$ or the p-adic number field $\mathbf{Q}_p$. The definition of normed Lie algebra over a normed field (cf. [8]) is given in [8]. In particular, every finite dimensional Lie algebra over a normed field K is a normed Lie algebra over K (in a natural manner). For each normed Lie algebra R_1 over a normed field K, the local group $G(R_1)$ is defined (cf. [8]). As a set, $G(R_1)$ is a certain neighborhood of 0 in R_1.

Consider the following special cases. Let $K = \mathbf{R}$ or $= \mathbf{Q}_p$, and let K' be a finite extension of K. Let $\mathfrak{G}$ be a Lie subalgebra of $M_n(K')$ over K', and *consider $\mathfrak{G}$ as a Lie algebra over K*. Then $G(\mathfrak{G})$ is a neighborhood of 0 in $\mathfrak{G}$, with the product law given by $A \cdot B = \log(\exp A \exp B)$ $(A, B \in G(\mathfrak{G}))$ whenever it exists and is contained in $G(\mathfrak{G})$. Therefore, if $\mathfrak{G} = \{X \in M_n(K') | \operatorname{tr} X = 0\}$, then $G(\mathfrak{G})$ can be identified (by exp, log) with the local group of $PL_n(K')$. But we can verify the following lemma without any difficulty.

Lemma . *Let $\mathfrak{G} = \{X \in M_2(K') | \operatorname{tr} X = 0\}$ be considered as a Lie algebra over K. Then the non-zero endomorphisms of $\mathfrak{G}$ over K are automorphisms of $\mathfrak{G}$, and are composites of inner automorphisms (by $PL_2(K')$) and automorphisms of $\mathfrak{G}$ induced by field automorphisms of K' over K.*

Now $G(\mathfrak{G})$ being identified with the local group of $PL_2(K')$, it is clear that the corresponding local automorphisms of $G(\mathfrak{G})$ are the restrictions to $G(\mathfrak{G})$ of the global automorphisms $\varphi = \operatorname{Int}(t) \circ \sigma$ of $PL_2(K')$, where $t \in PL_2(K')$ and σ are the automorphisms of $PL_2(K')$ induced by the field automorphisms of K' over K. Therefore, by the above Dynkin's theorem, *every local automorphism of* $PL_2(K')$ is equivalent to a global automorphism φ of $PL_2(K')$ of the form $\varphi = \operatorname{Int}(t) \circ \sigma$ $(t \in PL_2(K'), \sigma \in \operatorname{Aut}_K(K'))$. By putting $K' = k_\mathfrak{p}$, Lemma 1 (i) of Chapter 2 is settled. To prove the remaining parts of this lemma, we shall show that

(I) A global automorphism of $PSL_2(k_\mathfrak{p})$ which is an identity on some neighborhood of 1 is itself the identity map.

(II) A global automorphism of $PL_2(k_\mathfrak{p})$ which is an identity on $PSL_2(k_\mathfrak{p})$ is itself the identity map.

It is clear that (I) and (II) settle (ii) and (iii) of Lemma 1 (Chapter 2). In fact, if ψ is any global automorphism of $PSL_2(k_\mathfrak{p})$, then it is equivalent to a global automorphism of $PL_2(k_\mathfrak{p})$ of the form $\varphi = \operatorname{Int}(t) \circ \sigma$. Then $\psi \circ \varphi^{-1}$ is a global automorphism of $PSL_2(k_\mathfrak{p})$ which is an identity on some neighborhood of 1. Therefore by (I), we get $\psi = \varphi$ on $PSL_2(k_\mathfrak{p})$, which settles (iii). In particular, if ψ is a global automorphism of $PL_2(k_\mathfrak{p})$, then its restriction [3] to $PSL_2(k_\mathfrak{p})$ coincides with some $\varphi = Int(t) \circ \sigma$. Hence by (II) $\psi = \varphi$ on $PL_2(k_\mathfrak{p})$, which settles (ii).

Proof of (I). Let ψ be a global automorphism of $PSL_2(k_\mathfrak{p})$ which is an identity on some neighborhood of 1. We claim that ψ is then an identity on $N = \left\{\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} | a \in k_\mathfrak{p}\right\}$. To show this, put $x = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ with $a \in k_\mathfrak{p}, a \neq 0$. Let p be the characteristic of the residue class field $O_\mathfrak{p}/\mathfrak{p}$, and let n be a sufficiently large integer such that $x^{p^n} = \begin{pmatrix} 1 & p^n a \\ 0 & 1 \end{pmatrix}$ is invariant by ψ. Then $\psi(x)^{p^n} = x^{p^n}$. But x commutes with x^{p^n}; hence $\psi(x)$ commutes with $\psi(x^{p^n}) = x^{p^n}$; hence $\psi(x)$ is also of the form $\pm\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ with $b \in k_\mathfrak{p}$. Therefore by $\psi(x)^{p^n} = x^{p^n}$ we get $b = a$; hence $\psi(x) = x$. Therefore, ψ is an identity on N. In the same manner, ψ is an

[3] Since $PSL_2(k_\mathfrak{p})$ is a characteristic subgroup of $PL_2(k_\mathfrak{p})$ (since $PSL_2(k_\mathfrak{p})$ is infinite and simple; see proof of Corollary 2 of Theorem 3, §15, Chap. 2), we have $\psi(PSL_2(k_\mathfrak{p})) = PSL_2(k_\mathfrak{p})$.

identity on $'N = \left\{\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} | a \in k_\mathfrak{p}\right\}$. But $PSL_2(k_\mathfrak{p})$ is generated by N and $'N$. Therefore, ψ is the identity map on $PSL_2(k_\mathfrak{p})$. □

Proof of (II). Let ψ be a global automorphism of $PL_2(k_\mathfrak{p})$ which is an identity on $PSL_2(k_\mathfrak{p})$. Let $x \in PL_2(k_\mathfrak{p})$. Then for any $y \in PSL_2(k_\mathfrak{p})$, we have $x^{-1}yx = y' \in PSL_2(k_\mathfrak{p})$. Therefore, $\psi(x)^{-1}y\psi(x) = \psi(y') = y'$. Therefore, $\psi(x)x^{-1}$ commutes with y. Since y is an arbitrary element of $PSL_2(k_\mathfrak{p})$, we get $\psi(x)x^{-1} = 1$; hence $\psi(x) = x$ for all $x \in PL_2(k_\mathfrak{p})$. □

This completes the proof of Lemma 1 (Chapter 2).

§5. On admissible extensions. Let k be any field, and let K be an algebraic function field of one variable over k. Let $e = e(P)$ be a $\{1, 2, \cdots; \infty\}$-valued function defined on the set of all prime divisors P of K such that

(i) $e(P) = 1$ for almost all P,

(ii) if $e(P) < \infty$, then $e(P)$ is not divisible by the characteristic p of k,

and

(iii) $2g - 2 + \sum_P \left(1 - \frac{1}{e(P)}\right) > 0$, where g is the genus of K.

A separable extension K' of K will be called an admissible extension of $\{K, e\}$ if for each prime divisor P of K and its prime factor P' of K', P'/P is (at most) tamely ramified[4] and moreover if the ramification index of P'/P divides $e(P)$ (if $e(P) < \infty$).

We shall give here the proofs of the following two (probably well-known) facts (1), (2), to supplement the arguments of the main text.[5]

(1) The composite of two admissible extensions of $\{K, e\}$ is also admissible.

Thus, to each $\{K, e\}$, there exists a (unique) maximum admissible extension. Call it M. It is clear that a conjugate field (over K) of an admissible extension of $\{K, e\}$ is also admissible; hence M/K is a Galois extension.

(2) If $k = \mathbf{C}$, then for each prime divisor P of K and its prime factor P' of M, the ramification index of P'/P *coincides* with $e(P)$.

Proof of (1). It is enough to prove in the case of finite extensions. Let K_1, K_2 be two finite admissible extensions of $\{K, e\}$, let $K_1, \cdots, K_m$ be all the conjugate fields of K_1, K_2 over K, and let K' be the composite of $K_1, \cdots, K_m$ (so that all K_i are also admissible, and K' is the smallest Galois extension of K containing K_1, K_2). It is enough to prove that K' is an admissible extension of $\{K, e\}$. Let P' be any prime divisor of K', and put $P'|_K = P$. Since P'/P is tamely ramified in all K_i/K, it is tamely ramified in K'/K. So, the inertia group T' of P'/P is cyclic. Put $e = e(P)$, $e' = (T' : 1)$, and let H_i be the subgroup of $G(K'/K)$ corresponding to K_i. Since the inertia group of P' in K'/K_i is $T' \cap H_i$, the ramification index of P' in K_i/K is given by $(T' : T' \cap H_i)$; hence by assumption we have $(T' : T' \cap H_i)|e$. Let e_0 be the greatest common divisor of e and e'. Then since $(T' : 1) = e'$, we get $(T' : T' \cap H_i)|e_0$. Let σ be a generator of T'. Then this implies that

[4] i.e., the residue field extension is also separable and the ramification index is coprime to p.

[5] (1) for Chap.5 §30; (2) for Chap.2 §42.

$T' \cap H_i$ contains σ^{e_0}. But we have $\bigcap_{i=1}^{m} H_i = 1$; hence $\sigma^{e_0} = 1$; hence $e_0 = e'$; hence $e'|e$. Since this holds for all P', K' is an admissible extension of $\{K, e\}$. □

Proof of (2). Since $\{K, e\}$ over $\mathbf{C}$ corresponds to a fuchsian group Δ, and finite admissible extensions of $\{K, e\}$ correspond to subgroups of Δ with finite indices in a natural manner (see Chapter 2, §40), it is enough to prove here the following: Let $z_1, \cdots, z_s$ be all the non-equivalent cusps of Δ, and let H_i be the stabilizer of z_i in Δ (so that H_i are infinite cyclic). Then for any positive number N, there exists a subgroup Δ' of Δ of finite index such that (i) Δ' is torsion-free, and (ii) $(H_i : \Delta' \cap H_i) \geq N$ for all i. But this is easily checked as follows: Let $w_1, \cdots, w_t$ be all the non-equivalent elliptic fixed points of Δ and let E_j be the stabilizer of w_j in Δ (so that E_j are finite cyclic). Let $\Delta = \Delta_1 \supset \Delta_2 \supset \cdots \supset \Delta_n \supset \cdots$ be any descending series of normal subgroups of Δ of finite indices such that $\bigcap_{k=1}^{n} \Delta_k = \{1\}$. Then $\Delta_k \cap H_i$ $(k = 1, 2, \cdots)$ and $\Delta_k \cap E_j$ $(k = 1, 2, \cdots)$ are also descending series with trivial intersection. Hence for some k, $\Delta_k \cap E_j = 1$ for all j (hence Δ_k is torsion-free), and $(H_i : \Delta_k \cap H_i) \geq N$ for all i. □

Supplements to Chapter 3.

§6. The vanishing of $H^1(\Gamma_{\mathbf{R}}, \rho_n)$ without compactness assumption for G/Γ. Here we shall give a generalization of the Corollary of Theorem 1 of Chapter 3 (§5) to the case where G/Γ is non-compact; namely,

Theorem . *Let Γ be a discrete subgroup of $G = G_{\mathbf{R}} \times G_{\mathfrak{p}}$ such that $\Gamma_{\mathbf{R}}, \Gamma_{\mathfrak{p}}$ are dense in $G_{\mathbf{R}}, G_{\mathfrak{p}}$ respectively and that the quotient G/Γ has finite invariant volume. Let ρ_n $(n = 0, 1, 2, \cdots)$ be the symmetric tensor representations of degree $2n$ defined in Chapter 3 (§3). Then*

$$H^1(\Gamma_{\mathbf{R}}, \rho_n) = 0 \quad (n = 0, 1, 2, \cdots). \tag{4}$$

Corollary . *The group Γ being as above, the commutator quotient $\Gamma/[\Gamma, \Gamma]$ is finite.*

This is a generalization of (a part of) Theorem 2 of Chapter 3 (§6), and is an immediate consequence of the above Theorem for $n = 0$ (see §6).

For the proof of this Theorem, our study of parabolic elements of Γ (Part 2 of Chapter 1) is basic.

Proof of the Theorem. Let $\Gamma^0_{\mathbf{R}}$ be any fuchsian group and let ρ_{n0} $(n = 0, 1, 2, \cdots)$ be the restriction of ρ_n to $\Gamma^0_{\mathbf{R}}$. Let V_n be the representation space of ρ_n. A 1-cocycle $a(\gamma)$ with respect to $\Gamma^0_{\mathbf{R}}$ and ρ_{n0} is called a *parabolic cocycle* if for each parabolic element $\varepsilon \in \Gamma^0_{\mathbf{R}}$, there exists some $b = b_\varepsilon \in V_n$, which may depend on ε, such that $a(\varepsilon) = b - \rho_{n0}(\varepsilon)b$. It is clear that the set of all parabolic cocycles forms a group containing the group of all coboundaries. Let H'_n be the factor group, so that H'_n can be considered as a subgroup of $H^1(\Gamma^0_{\mathbf{R}}, \rho_{n0})$. Then by Shimura [**31**], all results recalled in §4 hold (without the compactness assumption of the quotient $G_{\mathbf{R}}/\Gamma^0_{\mathbf{R}}$) if we replace $H^1(\Gamma^0_{\mathbf{R}}, \rho_{n0})$ by H'_n and if $\mathfrak{M}_{2n+2}$ is the space of all *cusp* forms (of weight $2n + 2$ with respect to $\Gamma^0_{\mathbf{R}}$).

Now let Γ be as in the above theorem, and put $\Gamma^0 = \Gamma \cap (G_{\mathbf{R}} \times PSL_2(O_\mathfrak{p}))$, so that $\Gamma^0_{\mathbf{R}}$ is a fuchsian group. Let φ be the restriction homomorphism of $H^1(\Gamma_{\mathbf{R}}, \rho_n)$ into $H^1(\Gamma^0_{\mathbf{R}}, \rho_{n0})$. Then φ is injective. In fact, this follows exactly in the same manner as in §5, Chap. 3. Now we claim that *the image of* φ *is contained in* H'_n. To prove this, let $a(\gamma)$ be any cocycle with respect to $\Gamma_{\mathbf{R}}, \rho_n$, and let ε be a parabolic element of $\Gamma^0_{\mathbf{R}}$. Our purpose is to prove that $a(\varepsilon)$ is contained in $(1-\rho_n(\varepsilon))V_n$. By the Corollary 2 of Theorem 3 of Chapter 1 (Part 2, §25), there exists an element $\delta \in \Gamma_{\mathbf{R}}$ and an integer $d > 1$ such that $\delta^{-1}\varepsilon\delta = \varepsilon^d$. Take $t \in G_{\mathbf{R}}$ such that $\varepsilon' = t^{-1}\varepsilon t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and put $\delta' = t^{-1}\delta t$. Then by $\delta'^{-1}\varepsilon'\delta' = \varepsilon'^d$, δ' stabilizes $i\infty$ but is not parabolic; hence δ' is of the form $\delta' = \begin{pmatrix} a^{-1} & b \\ 0 & a \end{pmatrix}$, with $a = \sqrt{d} > 1$. If $b \neq 0$, replace t by the matrix $\begin{pmatrix} 1 & -b(a-a^{-1})^{-1} \\ 0 & 1 \end{pmatrix} t$, and assume from the beginning that $\delta' = \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix}$. Now by the definition of ρ_n (§3, Chap. 3), we have

$$(5) \qquad \rho_n(\varepsilon') = \begin{pmatrix} 1 & 2n & & & * \\ & 1 & 2n-1 & & \\ & & \ddots & \ddots & \\ & O & & \ddots & 1 \\ & & & & 1 \end{pmatrix}, \quad \rho_n(\delta') = \begin{pmatrix} d^{-n} & & & & \\ & d^{1-n} & & O & \\ & & d^{2-n} & & \\ & O & & \ddots & \\ & & & & d^n \end{pmatrix}.$$

Therefore, the rank of $\rho_n(\varepsilon')$ is $2n (= \dim V_n - 1)$, and ${}^t(x_1, \cdots, x_{2n+1}) \in V_n$ is contained in $(1-\rho_n(\varepsilon'))V_n$ if and only if $x_{2n+1} = 0$. Put $A' = 1 - \rho_n(\delta')\left\{1 + \rho_n(\varepsilon') + \cdots + \rho_n(\varepsilon')^{d-1}\right\}$. Then A' is upper triangular, and the i-th diagonal component is $1 - d^{i-n}$ $(1 \le i \le 2n+1)$. We claim now that if $x \in V_n$ with $A'x \in (1-\rho_n(\varepsilon'))V_n$, then $x \in (1-\rho_n(\varepsilon'))V_n$. Put $x = {}^t(x_1, \cdots, x_{2n+1})$. Then $A'x \in (1-\rho_n(\varepsilon'))V_n$ implies $(1-d^{n+1})x_{2n+1} = 0$; hence $x_{2n+1} = 0$; hence $x \in (1-\rho_n(\varepsilon'))V_n$. Now put $A = 1 - \rho_n(\delta)\left\{1 + \rho_n(\varepsilon) + \cdots + \rho_n(\varepsilon)^{d-1}\right\}$, so that $A' = \rho_n(t)^{-1}A\rho_n(t)$; hence

$$(6) \qquad Ax \in (1-\rho_n(\varepsilon))V_n \quad \text{implies} \quad x \in (1-\rho_n(\varepsilon))V_n.$$

Therefore it is enough to prove that $Aa(\varepsilon) \in (1-\rho_n(\varepsilon))V_n$. But by $\varepsilon\delta = \delta\varepsilon^d$, we have

$$\begin{aligned} a(\varepsilon) + \rho_n(\varepsilon)a(\delta) &= a(\delta) + \rho_n(\delta)a(\varepsilon^d) \\ &= a(\delta) + \rho_n(\delta)\left\{1 + \rho_n(\varepsilon) + \cdots + \rho_n(\varepsilon)^{d-1}\right\}a(\varepsilon); \end{aligned}$$

hence $Aa(\varepsilon) = (1-\rho_n(\varepsilon))a(\delta)$. Therefore, by (8), we obtain $a(\varepsilon) \in (1-\rho_n(\varepsilon))V_n$ for each parabolic element ε of $\Gamma^0_{\mathbf{R}}$; hence the image of φ is contained in H'_n.

Now the proof is completed exactly in the same manner as in the proof of Theorem 1 in §5, Chapter 3, if we use Supplement §2 instead of Lemma 10 of Chapter 1. □