Part 1. The $G_{\mathfrak{p}}$-fields over $\mathbf{C}$.

The $G_{\mathfrak{p}}$-fields.

§1. Let L be a discrete field, on which the group $G_{\mathfrak{p}} = PSL_2(k_{\mathfrak{p}})$ acts effectively and continuously as a group of field-automorphisms; namely, each $g_{\mathfrak{p}} \in G_{\mathfrak{p}}$ gives a field automorphism $x \mapsto g_{\mathfrak{p}}(x)$ of L, and the induced map $G_{\mathfrak{p}} \to \operatorname{Aut} L$ is an injective homomorphism;

$$(1)\qquad \begin{aligned} &(g_{\mathfrak{p}}h_{\mathfrak{p}})(x) = g_{\mathfrak{p}}(h_{\mathfrak{p}}(x)) \quad \forall g_{\mathfrak{p}}, h_{\mathfrak{p}} \in G_{\mathfrak{p}},\ x \in L; \\ &g_{\mathfrak{p}}(x) = x \ \ (\forall x \in L) \leftrightarrow g_{\mathfrak{p}} = 1. \end{aligned}$$

Since L is a discrete field, the continuity of the actions of $G_{\mathfrak{p}}$ amounts to saying that, for each $x \in L$, its stabilizer in $G_{\mathfrak{p}}$ is open. For each open compact subgroup V of $G_{\mathfrak{p}}$, put

$$(2)\qquad L_V = \{x \in L \mid v(x) = x,\ \forall v \in V\}.$$

Since open compact subgroups form a basis of neighborhoods of the identity of $G_{\mathfrak{p}}$, we get $L = \bigcup_V L_V$. Moreover, it follows that for each V, L/L_V is separably algebraic, V is the group of *all* automorphisms of L/L_V, and the topology of V induced by that of $G_{\mathfrak{p}}$ coincides with the Krull topology of $V = \operatorname{Aut}(L/L_V)$. In fact, let $x \in L$, and let V' be its stabilizer in $G_{\mathfrak{p}}$. Then since V' is open, we have $(V : V' \cap V) < \infty$. Put $V = \sum_{i=1}^{d} \sigma_i(V \cap V')$. Then $\sigma_1(x), \cdots, \sigma_d(x)$ are mutually distinct, and their elementary symmetric functions are all contained in L_V; hence L/L_V is separably algebraic. Now consider $\operatorname{Aut}(L/L_V)$ as equipped with the Krull topology. Then the injection $\varphi : V \to \operatorname{Aut}(L/L_V)$ is continuous, since the action of $G_{\mathfrak{p}}$ on L is so; hence $\varphi(V)$ is also compact. On the other hand, $\varphi(V)$ is dense in $\operatorname{Aut}(L/L_V)$, since for any $\sigma \in \operatorname{Aut}(L/L_V)$, we have $\sigma(x) = \sigma_i(x)$ for some i (σ_i being as above, for *this* x). Therefore, $\varphi(V) = \operatorname{Aut}(L/L_V)$, and φ is bicontinuous (since V is compact).

Let k be the fixed field of $G_{\mathfrak{p}}$;

$$(3)\qquad k = \{x \in L \mid g_{\mathfrak{p}}(x) = x\ \forall g_{\mathfrak{p}} \in G_{\mathfrak{p}}\}.$$

We shall call L *a one-dimensional $G_{\mathfrak{p}}$-field over k*, or simply, a *$G_{\mathfrak{p}}$-field over* k, if

(L1) $\dim_k L = 1$,

and if for every open compact subgroup V of $G_{\mathfrak{p}}$, the condition:

(L2) *L_V is finitely generated over k, and almost all prime divisors of L_V over k are unramified in L;*

is satisfied. We note that since L/L_V is algebraic, (L1) implies $\dim_k L_V = 1$; hence L_V is an algebraic function field of one variable over k, in the sense that L_V/k is finitely generated and is of dimension one. By a prime divisor of L_V over k, we mean an equivalence class of non-trivial discrete valuations of L_V over k or equivalently, an equivalence class of non-trivial places of L_V over k. Since open compact subgroups of $G_\mathfrak{p}$ are commensurable with each other, the condition (L2) is satisfied for *all* V if it is satisfied for *one* V.

The subfield k of L given by (3) will be called *the constant field of* L. Two $G_\mathfrak{p}$-fields L, L' with the common constant field k are called *isomorphic* if there exists an isomorphism of the field L onto L' which is trivial on k and which commutes with the actions of all elements of $G_\mathfrak{p}$.

§2.

Example. Let p be a prime number, and put

$$\Delta^{(n)} = \{x \in SL_2(\mathbf{Z}) \mid x \equiv \pm 1 \ (\mathrm{mod} p^n)\}/\pm 1 \quad (n \geq 0). \tag{4}$$

Consider $\Delta^{(n)}$ as fuchsian groups acting on the complex upper half plane $\mathfrak{H}$, and let L_n ($n \geq 0$) be the field of automorphic functions with respect to $\Delta^{(n)}$. Put $L = \bigcup_{n=0}^{\infty} L_n$. Define the action of the group $PSL_2(\mathbf{Q}_p)$ on L in the following manner. As in Chapter 1, §2 (Example), put $\mathbf{Z}^{(p)} = \bigcup_{n=0}^{\infty} p^{-n}\mathbf{Z}$, $\Gamma = PSL_2(\mathbf{Z}^{(p)})$, and consider Γ as a discrete subgroup of $G = G_\mathbf{R} \times G_p$, with $G_\mathbf{R} = PSL_2(\mathbf{R})$ and $G_p = PSL_2(\mathbf{Q}_p)$. Then, $\Gamma_p \cong \Gamma_\mathbf{R} \subset G_\mathbf{R}$ acts on L as

$$\Gamma_p \ni \gamma_p : L \ni f(z) \mapsto \gamma_p(f(z)) = f(\gamma_\mathbf{R}^{-1} \cdot z) \in L, \tag{5}$$

where $\gamma_\mathbf{R}$ is the element of $\Gamma_\mathbf{R}$ which corresponds to γ_p. Now, we lift the action of Γ_p to that of G_p; namely, for each $f(z) \in L_n$ and $g_p \in G_p$, put $g_p(f(z)) = \gamma_p(f(z))$ with any $\gamma_p \in \Gamma_p \cap g_p U_p^{(n)}$, where

$$U_p^{(n)} = \{x \in SL_2(\mathbf{Z}_p) \mid x \equiv \pm 1 \ (\mathrm{mod} p^n)\}/\pm 1 \quad (n \geq 0). \tag{6}$$

Then, this defines an action of G_p on L, which is effective and continuous. By noting that L_n is the fixed field of $U_p^{(n)}$ ($n \geq 0$), we see immediately that L is a $PSL_2(\mathbf{Q}_p)$-field over the complex number field $\mathbf{C}$.

We shall show later (§5-§9) that all $G_\mathfrak{p}$-fields over $\mathbf{C}$ are obtained in this manner from discrete subgroups Γ of $G = G_\mathbf{R} \times G_\mathfrak{p}$ such that $\Gamma_\mathbf{R}$, $\Gamma_\mathfrak{p}$ are dense in $G_\mathbf{R}$, $G_\mathfrak{p}$ respectively and G/Γ have finite invariant volumes.

§3.

Proposition 1. *Let L be a $G_\mathfrak{p}$-field over k. Then k is algebraically closed in L.*

Proof. In fact, let $x \in L$ be algebraic over k. Then for any $g_\mathfrak{p} \in G_\mathfrak{p}$, $g_\mathfrak{p}(x)$ is conjugate to x over k. This shows that the stabilizer of x in $G_\mathfrak{p}$ is of finite index in $G_\mathfrak{p}$. But if H is a subgroup of $G_\mathfrak{p}$ of finite index, then $N = \bigcap_{g_\mathfrak{p} \in G_\mathfrak{p}} g_\mathfrak{p}^{-1} H g_\mathfrak{p}$ is a normal subgroup of $G_\mathfrak{p}$ with finite index; hence by the simplicity of the group $G_\mathfrak{p} = PSL_2(k_\mathfrak{p})$ we get $N = G_\mathfrak{p}$; and hence $H = G_\mathfrak{p}$. Therefore, the stabilizer of x in $G_\mathfrak{p}$ must be $G_\mathfrak{p}$ itself; hence $x \in k$. □

$$\begin{tikzcd}
 & L \arrow[d, dash] \\
L' \arrow[r, dash] \arrow[d, dash] & L'k \arrow[d, dash] \\
k' \arrow[r, dash] & k
\end{tikzcd}$$

Proposition 2. *Let L be a $G_\mathfrak{p}$-field over k. Let L' be a $G_\mathfrak{p}$-invariant subfield of L, not contained in k. Put $k' = L' \cap k$. Then,*

(i) *L' and k are linearly disjoint over k'.*
(ii) $[L : L' \cdot k] < \infty$.
(iii) *With the restricted action of $G_\mathfrak{p}$ on L', L' is a $G_\mathfrak{p}$-field over k'.*

Proof. (i). Suppose, on the contrary, that L' and k were not linearly disjoint over k'. Then, there exists a set of elements $c_1, \cdots, c_n \in k$ that are linearly independent over k', but not over L'. We can assume that $c_1, \cdots, c_{n-1}$ are linearly independent over L', since otherwise, we can replace $c_1, \cdots, c_n$ by $c_1, \cdots, c_{n-1}$. Now, $c_1, \cdots, c_{n-1}$ being linearly independent over L' and $c_1, \cdots, c_n$ not being so, we get $c_n = x_1c_1 + \cdots + x_{n-1}c_{n-1}$ with some $x_1, \cdots, x_{n-1} \in L'$, and with, say, $x_1 \notin k'$. Since $x_1 \in L'$ and $k' = L' \cap k$, this implies $x_1 \notin k$, and hence there exists some $g_\mathfrak{p} \in G_\mathfrak{p}$ for which $g_\mathfrak{p}(x_1) \neq x_1$. Now by $c_n = x_1c_1 + \cdots + x_{n-1}c_{n-1}$, we get $c_n = g_\mathfrak{p}(x_1)c_1 + \cdots + g_\mathfrak{p}(x_{n-1})c_{n-1}$; hence

$$(x_1 - g_\mathfrak{p}(x_1))c_1 + \cdots + (x_{n-1} - g_\mathfrak{p}(x_{n-1}))c_{n-1} = 0.$$

Since L' is $G_\mathfrak{p}$-invariant, all coefficients of c_i are contained in L', and $x_1 - g_\mathfrak{p}(x_1) \neq 0$. This contradicts the assumption on linear independence of $c_1, \cdots, c_{n-1}$ over L'. Thus (i) is settled.

(ii), (iii). It is clear that $G_\mathfrak{p}$ acts continuously on L', and that the fixed field is k'. Let Δ be the kernel of the action of $G_\mathfrak{p}$ on L'. Then, Δ is a normal subgroup of $G_\mathfrak{p}$, and since $L' \supsetneq k'$, Δ is not $G_\mathfrak{p}$ itself. Hence, by the simplicity of the group $G_\mathfrak{p} = PSL_2(k_\mathfrak{p})$, we get $\Delta = \{1\}$; hence the action of $G_\mathfrak{p}$ on L' is effective. By $k \subsetneq L'k \subset L$ and by Proposition 1, we get $\dim_k L'k = 1$, and hence by the linear disjointness of L' and k over k', we get $\dim_{k'} L' = 1$. Now let V be any open compact subgroup of $G_\mathfrak{p}$, and let L'_V be the fixed field of $V|_{L'}$. It is clear that $L'_V = L' \cap L_V$. Moreover, the argument of §1 shows that L'/L'_V is separably algebraic (hence L'_V/k' is of dimension one), and that $\mathrm{Aut}(L'/L'_V) = V|_{L'}$. Also, since $k \subsetneq L'_V \cdot k \subset L_V$, $L'_V \cdot k/k$ is finitely generated and is of dimension one (by Proposition 1). Hence, by the linear disjointness of L' and k over k', we see that L'_V/k' is also finitely generated. Since $L_V/L'_V \cdot k$ is finitely generated and algebraic, we get $[L_V : L'_V \cdot k] < \infty$.

Now, put $M = L' \cdot k$ and $M_V = M \cap L_V$. We claim that $M \cdot L_V = L$ and $M_V = L'_V \cdot k$. In fact, since $G_\mathfrak{p}$ acts effectively on $M = L' \cdot k$, the subgroup of $G_\mathfrak{p}$ which acts trivially on $M \cdot L_V$ is $\{1\}$. On the other hand, $V = \mathrm{Aut}(L/L_V)$ with the Krull topology (see §1), and the Galois theory is valid between compact subgroups of V and intermediate fields of L/L_V (Krull's Galois theory). This shows $M \cdot L_V = L$. Let us now check $M_V = L'_V \cdot k$. First, every element x of $M = L' \cdot k$ can be written in the form:

$$x = a\sum_{i=1}^{n} c_ix_i, \quad \text{with } a \in L'_V \cdot k,\ c_i \in k,\ x_i \in L'\ (1 \leq i \leq n).$$

This follows easily [5] from the fact that L'/L'_V is normal and separable, and that L' and k are linearly disjoint over k'.

Now let x be contained in M_V. We can assume that $c_1, \cdots, c_n$ are linearly independent over k', and hence also over L'. By $v(x) = x$ $(v \in V)$, we get $v(x_i) = x_i$ $(1 \le i \le n)$ for all $v \in V$. This shows $x_i \in L'_V$; hence we get $x \in L'_V \cdot k$. Hence $M_V \subset L'_V \cdot k$. On the other hand, the inclusion $M_V \supset L'_V \cdot k$ is obvious. Hence we get $M_V = L'_V \cdot k$.

So, we get the following diagram, in which every "branch" is linearly disjoint: (M, L_V are linearly disjoint over M_V since M/M_V is Galois, L_V/M_V is algebraic, and since $M \cap L_V = M_V$.)

$$\begin{tikzcd}[column sep=small]
 & & L = M \cdot L_V \arrow[dd, dash] \\
L' \arrow[r, dash] \arrow[dd, dash] & L'k = M \arrow[ur, dash] \arrow[dd, dash] & \\
 & & L_V \\
L'_V \arrow[r, dash] \arrow[dd, dash] & L'_V \cdot k = M_V \arrow[ur, dash] \arrow[dd, dash] & \\
 & & \\
k' \arrow[r, dash] & k &
\end{tikzcd}$$

So, by $[L_V : M_V] = [L_V : L'_V \cdot k] < \infty$, we get $[L : M] = [L_V : M_V] < \infty$. This settles (ii). Finally, since almost all prime divisors of L_V over k are unramified in L, and since $[L_V : M_V] < \infty$, it follows immediately that almost all prime divisors of M_V over k are unramified in L, and hence a priori in M. Thus, by the linear disjointness of L' and k over k', it follows immediately that almost all prime divisors of L'_V over k' are unramified in L'. Therefore, together with what we have proved already, we have completed the proof that L' is a $G_\mathfrak{p}$-field over k'. □

We have also proved the following:

Corollary . *The situation being as in Proposition 2, let V be an open compact subgroup of $G_\mathfrak{p}$ and let $L'_V = L' \cap L_V$. Then $L'_V \cdot k$ consists of all V-invariant elements of $L' \cdot k$, and we have $[L : L'k] = [L_V : L'_V \cdot k] < \infty$.*

[5] First, express x in the form $\sum_{i=1}^n c_i x_i / \sum_{i=1}^n d_i x_i$, with c_i, $d_i \in k$, $x_i \in L'$ $(1 \le i \le n)$, and then consider the norm over $L'_V \cdot k$ of the denominator.

§4. Let L be a $G_\mathfrak{p}$-field over k, and let L' be a $G_\mathfrak{p}$-invariant subfield of L, with $L' \not\subset k$. Put $k' = L' \cap k$. Such L' will be called a *$G_\mathfrak{p}$-subfield (of L) over k'*. Thus, if L' is such, and if $k' \subset k_1 \subset k$, then $L' \cdot k_1$ is a $G_\mathfrak{p}$-subfield over k_1. In particular, $L' \cdot k$ is a $G_\mathfrak{p}$-subfield over k, and by Proposition 2, we have $[L : L'k] < \infty$.

We shall call L' *a full $G_\mathfrak{p}$-subfield over k'*, if moreover the condition $L' \cdot k = L$ is satisfied. Since L' and k are linearly disjoint over k', it implies that L is identified with the *constant field extension* $L' \otimes_{k'} k$ of L'. We shall call a $G_\mathfrak{p}$-field L over k *irreducible* if L has no $G_\mathfrak{p}$-subfields *over* k other than L itself, i.e., there is no proper intermediate $G_\mathfrak{p}$-invariant subfield between k and L. Thus, if L is irreducible, then all $G_\mathfrak{p}$-subfields of L are full $G_\mathfrak{p}$-subfields.

$$\begin{tikzcd}[arrows=dash] & L \arrow[d] \\ L' \arrow[r] \arrow[d] & L'k \arrow[d] \\ k' \arrow[r] & k \end{tikzcd}$$

We shall prove in Part 2 of this Chapter that if L is a $G_\mathfrak{p}$-field over the complex number field $\mathbf{C}$, then it contains a full $G_\mathfrak{p}$-subfield L_k *over an algebraic number field* k. We shall prove, moreover, that under a certain condition on L which is always satisfied if L is irreducible, such L_k is *essentially unique*, in the sense that among them there is a smallest field L_{k_0} over k_0 and that all other L_k are obtained as $L_k = L_{k_0} \cdot k$, $k \supset k_0$. In other words, all $G_\mathfrak{p}$-fields over $\mathbf{C}$ are the constant field extensions of some $G_\mathfrak{p}$-fields over an algebraic number field of finite degree, and if the former is irreducible, then the latter is essentially unique.

This will be proved by using the one-to-one correspondence between $G_\mathfrak{p}$-fields over $\mathbf{C}$ and certain discrete subgroups Γ of $G = G_\mathbf{R} \times G_\mathfrak{p}$ (Theorem 1), and then by using some group theory of $G_\mathfrak{p}$ and analysis of Γ (Part 2).

Analytic construction of $G_\mathfrak{p}$-fields over C.

§5. Let Γ be a discrete subgroup of $G = G_\mathbf{R} \times G_\mathfrak{p} = PSL_2(\mathbf{R}) \times PSL_2(k_\mathfrak{p})$ such that the projections $\Gamma_\mathbf{R}, \Gamma_\mathfrak{p}$ are dense in $G_\mathbf{R}, G_\mathfrak{p}$ respectively and that the quotient G/Γ has finite invariant volume. For each open compact subgroup V of $G_\mathfrak{p}$, put

$$\Gamma^V = \Gamma \cap (G_\mathbf{R} \times V), \tag{7}$$

and let $\Gamma_\mathbf{R}^V$ be its projection to $G_\mathbf{R}$. By Proposition 2 (Chapter 1, §3) $\Gamma_\mathbf{R}^V$ is a discrete subgroup of $G_\mathbf{R}$ and the quotient $G_\mathbf{R}/\Gamma_\mathbf{R}^V$ has finite invariant volume. Let L_V be the field of automorphic functions with respect to the fuchsian group $\Gamma_\mathbf{R}^V$ acting on the complex upper half plane $\mathfrak{H}$. Put $L = \bigcup_V L_V$. Then it is obvious that $\dim_\mathbf{C} L = 1$, that L_V is finitely generated over $\mathbf{C}$, and that almost all prime divisors of L_V over $\mathbf{C}$ are unramified in L. In fact, if $z \in \mathfrak{H}$ is not an elliptic fixed point of $\Gamma_\mathbf{R}^V$, then the prime divisor of L_V given by z (mod $\Gamma_\mathbf{R}^V$) is unramified in L.

Now we define the action of the group $G_\mathfrak{p}$ on L in the following manner. Let $g_\mathfrak{p} \in G_\mathfrak{p}$ and $f(z) \in L$. Take V such that $f(z) \in L_V$, and take $\gamma \in \Gamma \cap (G_\mathbf{R} \times g_\mathfrak{p} V)$ ($\neq \phi$, since $\Gamma_\mathfrak{p}$ is dense in $G_\mathfrak{p}$). Put $g_\mathfrak{p}\{f(z)\} = f(\gamma_\mathbf{R}^{-1} \cdot z)$. Then, it is easy to check that $g_\mathfrak{p}\{f(z)\}$ is well-defined (does not depend on the choice of V or γ), that $g_\mathfrak{p}\{f(z)\} \in L$, and that

$$L \ni f(z) \mapsto g_\mathfrak{p}\{f(z)\} \in L$$

gives a field-automorphism of L. Moreover, $(h_\mathfrak{p}g_\mathfrak{p})\{f(z)\} = h_\mathfrak{p}\{g_\mathfrak{p}f(z)\}$ holds for all $g_\mathfrak{p}, h_\mathfrak{p} \in G_\mathfrak{p}$; hence $G_\mathfrak{p}$ acts as an automorphism group on L. We see easily that L_V is the fixed field of V. In fact, first, it is clear that all elements of L_V are fixed by V. Conversely, if $f(z) \in L$ is fixed by V, then $f(z)$ is invariant by $\Gamma_\mathbf{R}^V$; hence $f(z) \in L_V$. Hence L_V is the fixed field of V. This shows in particular, that the action of $G_\mathfrak{p}$ on L is continuous. If $f(z) \in L$ is fixed by the whole group $G_\mathfrak{p}$, then we get $f(\gamma_\mathbf{R}^{-1} \cdot z) = f(z)$ for all $\gamma \in \Gamma$. But since $\Gamma_\mathbf{R}$ is dense in $G_\mathbf{R}$, this implies $f(z) \in \mathbf{C}$. Hence the fixed field of $G_\mathfrak{p}$ is $\mathbf{C}$. Finally, the action of $G_\mathfrak{p}$ on L is effective. In fact, the kernel of the action is a normal subgroup of $G_\mathfrak{p}$; hence by the simplicity of $G_\mathfrak{p}$, it must be $G_\mathfrak{p}$ itself if not $\{1\}$. But that is impossible, since the fixed field of $G_\mathfrak{p}$ is $\mathbf{C}$. Therefore, $G_\mathfrak{p}$ acts effectively on L. Thus, starting from Γ, we have constructed a $G_\mathfrak{p}$-field L over $\mathbf{C}$.

§6. Now, we shall show that conversely, given any $G_\mathfrak{p}$-field L over $\mathbf{C}$, we can define Γ, and that the $G_\mathfrak{p}$-fields over $\mathbf{C}$ (up to isomorphisms) are in one-to-one correspondence with Γ (up to conjugacy in G).

Let L be a $G_\mathfrak{p}$-field over $\mathbf{C}$, and let Σ be the set of all non-equivalent, non-trivial *discrete* valuations of L over $\mathbf{C}$. To give Σ more explicitly, let V_0 be an open compact subgroup of $G_\mathfrak{p}$ which has no elements ($\neq 1$) of finite order,[6] and let

$$V_0 \supset V_1 \supset \cdots \supset V_n \supset \cdots$$

be a decreasing sequence of open compact subgroups of $G_\mathfrak{p}$, such that $\bigcap_{n=0}^\infty V_n = \{1\}$. Put $L_n = L_{V_n}$ $(n \geq 0)$, and let $\mathfrak{R}_n$ $(n \geq 0)$ be the Riemann surface of L_n. Then, we get a sequence of coverings

$$\mathfrak{R}_0 \xleftarrow{\varphi_1} \mathfrak{R}_1 \xleftarrow{\varphi_2} \cdots \leftarrow \mathfrak{R}_n \xleftarrow{\varphi_{n+1}} \cdots . \tag{8}$$

Let T_i $(1 \leq i \leq N)$ be the points on $\mathfrak{R}_0$ that are ramified in the covering sequence (8). For each i, consider T_i as a discrete valuation of L_0, and let $\mathfrak{T}_i$ be a valuation of L with $\mathfrak{T}_i|_{L_0} = T_i$. Then, since $\mathrm{Aut}(L/L_0)$ has no non-trivial finite subgroup, the inertia group of $\mathfrak{T}_i$ over L_0 is infinite; hence its ramification index in L/L_0 is infinite. Therefore, $\mathfrak{T}_i$ is *not* a discrete valuation of L. Put

$$\mathfrak{R}_0' = \mathfrak{R}_0 - \{T_1, \cdots, T_N\}, \quad \mathfrak{R}_1' = \varphi_1^{-1}(\mathfrak{R}_0'), \quad \mathfrak{R}_2' = \varphi_2^{-1}(\mathfrak{R}_1'), \cdots \text{etc.}$$

Denoting $\varphi_n|_{\mathfrak{R}_n'}$ again by φ_n, we get a sequence of unramified coverings

$$\mathfrak{R}_0' \xleftarrow{\varphi_1} \mathfrak{R}_1' \xleftarrow{\varphi_2} \cdots \leftarrow \mathfrak{R}_n' \xleftarrow{\varphi_{n+1}} \cdots . \tag{9}$$

It is now clear that the set Σ can be identified with the set of all sequences of points

$$P_0 \xleftarrow{\varphi_1} P_1 \xleftarrow{\varphi_2} \cdots \leftarrow P_n \xleftarrow{\varphi_{n+1}} \cdots . \tag{10}$$

[6] It is well-known, and easy to prove, that such V_0 exists.

with $P_n \in \mathfrak{R}'_n$ and $P_n = \varphi_{n+1}(P_{n+1})$ for all $n \geq 0$. So, we shall denote the elements of Σ simply as

$$\Sigma \ni P = \{P_0 \overset{\varphi_1}{\leftarrow} P_1 \overset{\varphi_2}{\leftarrow} \cdots\}. \tag{11}$$

Now we shall define a complex structure on Σ. Let $\Sigma \ni P = \{P_0 \leftarrow P_1 \leftarrow \cdots\}$, and let U_0 be any simply connected neighborhood of P_0 on $\mathfrak{R}'_0$. For each $n \geq 0$, let U_n be the connected component of $(\varphi_1 \circ \cdots \circ \varphi_n)^{-1} U_0$ containing P_n;

$$\begin{array}{ccccccccc} U_0 & \leftarrow & U_1 & \leftarrow & \cdots & \leftarrow & U_n & \leftarrow & \cdots \\ \Downarrow & & \Downarrow & & & & \Downarrow & & \\ P_0 & \leftarrow & P_1 & \leftarrow & \cdots & \leftarrow & P_n & \leftarrow & \cdots . \end{array} \tag{12}$$

Since U_0 is simply connected, $U_0 \leftarrow U_n$ is a simple covering; hence φ_n induces an isomorphism of U_n onto U_{n-1}; and each point $P'_0 \in U_0$ defines a unique element $P' = \{P'_0 \leftarrow P'_1 \leftarrow \cdots\}$ of Σ, with $P'_n \in U_n$ for all $n \geq 0$. Therefore, by taking such $U_0 \cong \{U_0 \leftarrow U_1 \leftarrow \cdots\}$ as a coordinate neighbourhood of P, we can define a complex structure on Σ, by which Σ is a one-dimensional complex manifold.

An important point is that *this complex structure of Σ is independent of the choice of the sequence $V_0 \supset V_1 \supset \cdots$ of open compact subgroups of $G_{\mathfrak{p}}$*. To check this, let $V'_0 \supset V'_1 \supset \cdots$ be another sequence such that V'_0 is torsion-free and $\bigcap_{n=0}^{\infty} V'_n = \{1\}$. Then, they are cofinal; i.e., every V_n contains some V'_m, and vice versa. So, we get a new sequence $V_{i_1} \supset V'_{j_1} \supset V_{i_2} \supset V'_{j_2} \supset \cdots$, with $i_1 < i_2 < \cdots$ and $j_1 < j_2 < \cdots$. Now it is clear that the complex structure of Σ defined by the sequence $V_0 \supset V_1 \supset \cdots$ is equivalent to that defined by $V_{i_1} \supset V'_{j_1} \supset V_{i_2} \supset V'_{j_2} \supset \cdots$, and hence is also equivalent to that defined by $V'_0 \supset V'_1 \supset \cdots$.

Let σ be an automorphism of the field L over $\mathbf{C}$. Then, the action $\Sigma \ni P \mapsto \sigma \cdot P \in \Sigma$ is defined by

$$v_{\sigma P}(x) = v_P(\sigma^{-1}(x)) \quad (x \in L),$$

where v_P, $v_{\sigma P}$ are the normalized additive discrete valuations of L contained in the classes $P, \sigma P$ respectively. Now, by this action, *σ leaves the complex strucure of Σ invariant.* In fact, consider the sequence $\sigma(L_0) \subset \sigma(L_1) \subset \cdots$, and let n be sufficiently large. Then $\sigma(L_n)$ contains L_0; hence there is an open compact subgroup V'_n of V_0 such that $\sigma(L_n) = L_{V'_n}$. Therefore, by the above remark, the complex structure of Σ defined with respect to $\sigma(L_n) \subset \sigma(L_{n+1}) \subset \cdots$ is equivalent to the original one. But this implies that σ leaves the complex structure of Σ invariant. In particular, $G_{\mathfrak{p}}$ acts on Σ as a group of automorphisms *of the complex manifold* Σ, and hence $G_{\mathfrak{p}}$ also acts on the set of all connected components of Σ as a permutation group.

§7. We shall now prove that:

(i) *Each open compact subgroup of $G_{\mathfrak{p}}$ acts transitively on the set of all connected components of Σ.*

(ii) *Each connected component of Σ is isomorphic to the complex half plane $\mathfrak{H}$.*

PROOF OF (i). Let $P = \{P_0 \leftarrow P_1 \leftarrow \cdots\}$ and $Q = \{Q_0 \leftarrow Q_1 \leftarrow \cdots\}$ be any two elements of Σ, and let Σ_P, Σ_Q be the connected components of Σ containing P, Q respectively. It is enough to prove that for each $n \geq 0$, there is an element $v \in V_n$ such that $v(\Sigma_P) = \Sigma_Q$. Now fix n, and let $P_n(t)$ $(0 \leq t \leq 1)$ be a curve on $\mathfrak{R}'_n$, with $P_n(0) = P_n$ and $P_n(1) = Q_n$. Then there is a unique (continuous) curve $P(t)$ $(0 \leq t \leq 1)$ on Σ satisfying $P(0) = P$ and $P(t) = \{\cdots \leftarrow P_n(t) \leftarrow \cdots\}$ for all t $(0 \leq t \leq 1)$. Then $P(1) \in \Sigma_P$, and $P(1) = \{Q_0 \leftarrow \cdots \leftarrow Q_n \leftarrow Q'_{n+1} \leftarrow \cdots\}$. Since the restrictions to L_n of $P(1)$ and Q coincide with each other, there is an element $v \in \mathrm{Aut}(L/L_n) = V_n$ such that $Q = v(P(1))$; hence $\Sigma_Q = v(\Sigma_P)$. This settles (i).

PROOF OF (ii). First of all, we note that the universal covering surface $\widetilde{\mathfrak{R}}'_n$ of $\mathfrak{R}'_n$ $(n \geq 0)$ is isomorphic to $\mathfrak{H}$. In fact, since the covering sequence (9) is unramified and non-trivial, $\widetilde{\mathfrak{R}}'_n$ cannot be the Riemann sphere. Moreover, since V_n is nonabelian, (9) is a nonabelian covering sequence. Hence $\widetilde{\mathfrak{R}}'_n$ cannot be the whole complex plane. Therefore, $\widetilde{\mathfrak{R}}'_n \cong \mathfrak{H}$.

Now let Σ_0 be an arbitrary connected component of Σ, and let $\widetilde{\Sigma}_0$ be the universal covering surface of Σ_0. Then, by the unramified covering $\mathfrak{R}'_n \leftarrow \Sigma_0 \leftarrow \widetilde{\Sigma}_0$, we can identify $\widetilde{\mathfrak{R}}'_n$ with $\widetilde{\Sigma}_0$. Therefore, we get a sequence of unramified coverings :

$$\mathfrak{R}'_0 \xleftarrow{\varphi_1} \cdots \leftarrow \mathfrak{R}'_n \xleftarrow{\varphi_{n+1}} \cdots \leftarrow \Sigma_0 \leftarrow \widetilde{\Sigma}_0 \cong \mathfrak{H}. \tag{13}$$

Now fix an isomorphism $\widetilde{\Sigma}_0 \cong \mathfrak{H}$. Then we get an isomorphism $\mathrm{Aut}\,\widetilde{\Sigma}_0 = \mathrm{Aut}\,\mathfrak{H} = G_{\mathbf{R}}$. Let Δ_n $(n \geq 0)$ be the covering group of $\mathfrak{R}'_n \leftarrow \widetilde{\Sigma}_0$, considered as a subgroup of $G_{\mathbf{R}}$. Then it is a torsion-free discrete subgroup of $G_{\mathbf{R}}$, and the quotient $G_{\mathbf{R}}/\Delta_n$ has finite invariant volume (the quotient is compact if and only if $\{T_1, \cdots, T_N\} = \phi$). It is also clear that the covering group Δ of $\Sigma_0 \leftarrow \widetilde{\Sigma}_0$ is the intersection of all Δ_n $(n \geq 0)$;

$$G_{\mathbf{R}} \supset \Delta_0 \supset \Delta_1 \supset \Delta_2 \supset \cdots; \quad \Delta = \bigcap_{n=0}^{\infty} \Delta_n. \tag{14}$$

We shall identify $\mathfrak{R}'_n$ with $\mathfrak{H}/\Delta_n$, and Σ_0 with $\mathfrak{H}/\Delta$.

Now put

$$\Gamma_\mathfrak{p} = \{g_\mathfrak{p} \in G_\mathfrak{p} \mid g_\mathfrak{p}(\Sigma_0) = \Sigma_0\}. \tag{15}$$

Then by (i), we get $G_\mathfrak{p} = V \cdot \Gamma_\mathfrak{p}$ for any open compact subgroup V of $G_\mathfrak{p}$; hence $\Gamma_\mathfrak{p}$ is a *dense* subgroup of $G_\mathfrak{p}$. On the other hand, $\Gamma_\mathfrak{p}$ acts on $\Sigma_0 = \mathfrak{H}/\Delta$ as a group of automorphisms, and hence we can also consider $\Gamma_\mathfrak{p}$ as a subgroup of $\mathrm{Aut}(\mathfrak{H}/\Delta)$. We shall denote this subgroup of $\mathrm{Aut}(\mathfrak{H}/\Delta)$, identified with $\Gamma_\mathfrak{p}$, by $\Gamma_{\mathbf{R}}$;

$$\mathrm{Aut}(\mathfrak{H}/\Delta) \supset \Gamma_{\mathbf{R}} \underset{\text{identified}}{\cong} \Gamma_\mathfrak{p} \subset G_\mathfrak{p}. \tag{16}$$

Let $N(\Delta)$ be the normalizer of Δ in $G_{\mathbf{R}}$. Then we have $\mathrm{Aut}(\mathfrak{H}/\Delta) = N(\Delta)/\Delta$; hence we can put $\Gamma_{\mathbf{R}} = \widehat{\Gamma}_{\mathbf{R}}/\Delta$, with $\Delta \subset \widehat{\Gamma}_{\mathbf{R}} \subset N(\Delta)$. Further, put $\Gamma_\mathfrak{p}^n = \Gamma_\mathfrak{p} \cap V_n$ $(n \geq 0)$, and denote by $\Gamma_{\mathbf{R}}^n$ the corresponding subgroup of $\Gamma_{\mathbf{R}}$. Then we have

$$\Gamma_{\mathbf{R}}^n = \Delta_n/\Delta \quad (n \geq 0). \tag{17}$$

Now, we have

$$(\widehat{\Gamma}_{\mathbf{R}} : \Delta_n) = (\Gamma_{\mathbf{R}} : \Gamma_{\mathbf{R}}^n) = (\Gamma_\mathfrak{p} : \Gamma_\mathfrak{p}^n) = (G_\mathfrak{p} : V_n) = \infty,$$

and Δ_n is a discrete subgroup of $G_{\mathbf{R}}$ whose quotient is of finite invariant volume. Therefore,[7] $\widehat{\Gamma}_{\mathbf{R}}$ is dense in $G_{\mathbf{R}}$; hence $N(\Delta)$ is dense in $G_{\mathbf{R}}$. But since Δ is discrete in $G_{\mathbf{R}}$, $N(\Delta)$ is closed in $G_{\mathbf{R}}$; hence $N(\Delta) = G_{\mathbf{R}}$. Therefore, Δ is a discrete normal subgroup of $G_{\mathbf{R}}$. But $G_{\mathbf{R}}$ is a simple group; hence we get $\Delta = \{1\}$. Therefore $\Sigma_0 \cong \mathfrak{H}$; which settles (ii). We have also proved that $\Gamma_{\mathbf{R}} = \widehat{\Gamma}_{\mathbf{R}}$ and that it is dense in $G_{\mathbf{R}}$;

$$G_{\mathbf{R}} \underset{\text{dense}}{\supset} \Gamma_{\mathbf{R}} \underset{\text{identified}}{\cong} \Gamma_{\mathfrak{p}} \underset{\text{dense}}{\subset} G_{\mathfrak{p}}. \tag{18}$$

§8. Now let Γ be the subgroup of $G = G_{\mathbf{R}} \times G_{\mathfrak{p}}$ formed of all elements $\gamma_{\mathbf{R}} \times \gamma_{\mathfrak{p}}$ such that $\gamma_{\mathbf{R}}, \gamma_{\mathfrak{p}}$ are corresponding elements of $\Gamma_{\mathbf{R}}, \Gamma_{\mathfrak{p}}$ respectively. Then the projection maps $\Gamma \to \Gamma_{\mathbf{R}}, \Gamma \to \Gamma_{\mathfrak{p}}$ are obviously injective, and it was shown that $\Gamma_{\mathbf{R}}, \Gamma_{\mathfrak{p}}$ are dense in $G_{\mathbf{R}}, G_{\mathfrak{p}}$ respectively. Moreover,

$$\{\Gamma \cap (G_{\mathbf{R}} \times V_n)\}_{\mathbf{R}} = \Gamma_{\mathbf{R}}^n = \Delta_n \quad (n \geq 0),$$

and Δ_n is a discrete subgroup of $G_{\mathbf{R}}$ whose quotient has finite invariant volume. Therefore by Proposition 2 of Chapter 1 (§3), Γ is a discrete subgroup of G, and the quotient G/Γ has finite invariant volume. The quotient G/Γ is compact if and only if $G_{\mathbf{R}}/\Delta_n$ is so; hence if and only if L/L_n is unramified.

Finally, it can be checked immediately that these two processes of defining L from Γ, and of defining Γ from L are the inverse of each other. We have thus proved the following Theorem.

§9.

THEOREM 1. *The $G_{\mathfrak{p}}$fields L over $\mathbf{C}$ are in one-to-one correspondence with the discrete subgroups Γ of $G = G_{\mathbf{R}} \times G_{\mathfrak{p}}$ whose quotients G/Γ are of finite invariant volume and whose projections $\Gamma_{\mathbf{R}}, \Gamma_{\mathfrak{p}}$ are dense in $G_{\mathbf{R}}, G_{\mathfrak{p}}$ respectively. Here, L are counted up to isomorphisms of $G_{\mathfrak{p}}$-fields (§1), and Γ are counted up to conjugacy in G.*

More precisely, if Γ is given, then L is obtained as the union of fields of automorphic functions with respect to $\Gamma_{\mathbf{R}}^V$ (§5). Conversely, if L is given, the set Σ of all non-equivalent non-trivial discrete valuations of L over $\mathbf{C}$ can be considered as a one-dimensional complex manifold, on which $G_{\mathfrak{p}}$ acts as an automorphism group. If V is an open compact subgroup of $G_{\mathfrak{p}}$, then V acts transitively on the set of all connected components of Σ. Take any connected component Σ_0 of Σ, and let $\Gamma_{\mathfrak{p}}$ be the stabilizer of Σ_0 in $G_{\mathfrak{p}}$. Then Σ_0 is isomorphic to the complex upper half plane $\mathfrak{H}$, and hence $\Gamma_{\mathfrak{p}}$ can also be identified with a subgroup $\Gamma_{\mathbf{R}}$ of $G_{\mathbf{R}} =$ Aut$(\mathfrak{H})$*. In this manner, by the identification $\Gamma \cong \Gamma_{\mathbf{R}} \cong \Gamma_{\mathfrak{p}}$ and by the diagonal embedding, we get the discrete subgroup Γ of G (§6, §7, §8).*

REMARK 1. Let L be a $G_{\mathfrak{p}}$-field over $\mathbf{C}$, and let $\{\Gamma\}_G$ be the corresponding G-conjugacy class of discrete subgroups of $G = G_{\mathbf{R}} \times G_{\mathfrak{p}}$. Then, *choosing one Γ from among $\{\Gamma\}_G$ is equivalent to choosing one connected component Σ_0 of Σ together with an isomorphism* $\Sigma_0 \cong \mathfrak{H}$. In fact, if Γ is given, we can identify L as the union of the fields L_V of automorphic

[7] See Supplement §1.

functions $f(z)$ with respect to $\Gamma_{\mathbf{R}}^{V}$ (see §5), and moreover, each point $z_0 \in \mathfrak{H}$ defines a discrete valuation v_{z_0} of L by $v_{z_0}(f(z)) = \mathrm{ord}_{z_0} f(z)$. Therefore, we can regard $\mathfrak{H}$ as a connected component of Σ. Conversely, if L, Σ_0, and an isomorphism $\Sigma_0 \cong \mathfrak{H}$ are given, then we get a discrete subgroup Γ in the above described manner (Σ_0 defines $\Gamma_{\mathfrak{p}}$ by (15), and the isomorphism $\Sigma_0 \cong \mathfrak{H}$ defines the isomorphism $\Gamma_{\mathfrak{p}} = \Gamma_{\mathbf{R}} \subset G_{\mathbf{R}}$).

Remark 2. The cardinality of the set of all connected components of Σ is $\aleph$-infinity, since it is in one-to-one correspondence with $G_{\mathfrak{p}}/\Gamma_{\mathfrak{p}}$, and $\Gamma_{\mathfrak{p}}$ is countable (since Γ is finitely generated; see §30).

Remark 3. The quotient G/Γ is compact if and only if L/L_V is unramified for *some* open compact subgroup V of $G_{\mathfrak{p}}$. When this is satisfied, Γ is torsion-free if and only if L/L_V is unramified for *all* open compact subgroups V of $G_{\mathfrak{p}}$.

§10. Now we shall show that given Γ and the corresponding $G_{\mathfrak{p}}$-field L over $\mathbf{C}$, the subgroups Δ of $G = G_{\mathbf{R}} \times G_{\mathfrak{p}}$ containing Γ with $(\Delta : \Gamma) < \infty$ and the $G_{\mathfrak{p}}$-subfields M of L over $\mathbf{C}$ correspond naturally in a one-to-one manner. We begin by proving the following:

Proposition 3. *Let L be a $G_{\mathfrak{p}}$-field over $\mathbf{C}$, and let M be a $G_{\mathfrak{p}}$-subfield of L over $\mathbf{C}$. Let P be a non-trivial discrete valuation of L over $\mathbf{C}$. Then P is unramified in L/M.*

Proof. Suppose, on the contrary, that there exists a discrete valuation P_0 of L over $\mathbf{C}$ which is ramified in L/M. Since L and M are $G_{\mathfrak{p}}$-invariant, this implies that $g_{\mathfrak{p}}(P_0)$, for any $g_{\mathfrak{p}} \in G_{\mathfrak{p}}$, is also ramified in L/M. Let V be a *torsion-free* open compact subgroup of $G_{\mathfrak{p}}$, and let L_V be the fixed field of V. Let P be any discrete valuation of L over $\mathbf{C}$. Then, by the discreteness of P, the inertia group of P in L/L_V is a finite subgroup of $V = \mathrm{Aut}(L/L_V)$; hence it must be $\{1\}$. Hence P is unramified in L/L_V. This implies that if P is ramified in L/M, then $P|_{L_V}$ must be ramified in L_V/M_V, where $M_V = M \cap L_V$. Therefore, $g_{\mathfrak{p}}(P_0)|_{L_V}$, for every $g_{\mathfrak{p}} \in G_{\mathfrak{p}}$, must be ramified in L_V/M_V. Since $[L_V : M_V] < \infty$ (see Corollary of Proposition 2, §3), there are only finitely many discrete valuations (up to equivalence) of L_V over $\mathbf{C}$, which are ramified in L_V/M_V. Therefore, the set

$$\{g_{\mathfrak{p}}(P_0)|_{L_V};\ g_{\mathfrak{p}} \in G_{\mathfrak{p}}\} \tag{19}$$

must be finite. We shall show that this is a contradiction. Let Σ_0 be the connected component of Σ containing P_0, and let $\Gamma_{\mathfrak{p}}$ be the stabilizer of Σ_0 in $G_{\mathfrak{p}}$. We know that $\Sigma_0 \cong \mathfrak{H}$, and that by this $\Gamma_{\mathfrak{p}}$ can be identified with a subgroup $\Gamma_{\mathbf{R}}$ of $\mathrm{Aut}\,\Sigma_0 \cong \mathrm{Aut}\,\mathfrak{H} = G_{\mathbf{R}}$. By putting

$$\Gamma = \{\gamma_{\mathbf{R}} \times \gamma_{\mathfrak{p}} \in G = G_{\mathbf{R}} \times G_{\mathfrak{p}} \mid \gamma_{\mathbf{R}}, \gamma_p \text{ are corresponding elements of } \Gamma_{\mathbf{R}}, \Gamma_p\},$$

L can be considered as the union of fields of automorphic functions with respect to $\Gamma_{\mathbf{R}}^{V'}$, for all open compact subgroups V' of $G_{\mathfrak{p}}$. A discrete valuation $P^* \in \Sigma_0$ of L over $\mathbf{C}$ is given by the corresponding point $z^* \in \mathfrak{H}$ as $L \ni f(z) \underset{P^*}{\rightarrow} \mathrm{ord}_{z^*} f$. If $P^*, P^{**} \in \Sigma_0$, then $P^*|_{L_V} = P^{**}|_{L_V}$ is valid if and only if the corresponding points $z^*, z^{**} \in \mathfrak{H}$ are $\Gamma_{\mathbf{R}}^{V}$-equivalent. Now, since $\Gamma_{\mathfrak{p}}$ is dense in $G_{\mathfrak{p}}$, the set (19) is the same as the set

$$\{\gamma_{\mathfrak{p}}(P_0)|_{L_V};\ \gamma_{\mathfrak{p}} \in \Gamma_{\mathfrak{p}}\}. \tag{19'}$$

Let z_0 be the point on $\mathfrak{H}$ corresponding to $P_0 \in \Sigma_0$. Then the points corresponding to $\gamma_{\mathfrak{p}}(P_0)$ are $\gamma_{\mathbf{R}}(z_0)$; and since $\Gamma_{\mathbf{R}}$ is dense in $G_{\mathbf{R}}$, $\gamma_{\mathbf{R}}(z_0)$ ($\gamma_{\mathbf{R}} \in \Gamma_{\mathbf{R}}$) give infinitely many non-equivalent points modulo $\Gamma^V_{\mathbf{R}}$. This shows that the set (19′), and hence the set (19), is infinite. Therefore, the finiteness of the set (19) is a contradiction. Hence P_0 must be unramified in L/M. □

Now we are in the situation to prove the one-to-one correspondence between Δ and M stated at the beginning of this section. First, we shall show that Δ gives M. Let Δ be a subgroup of G containing Γ with $(\Delta : \Gamma) < \infty$. Then, it is clear that Δ is also a discrete subgroup of G, G/Δ has finite invariant volume, and that $\Delta_{\mathbf{R}}, \Delta_{\mathfrak{p}}$ are dense in $G_{\mathbf{R}}, G_{\mathfrak{p}}$ respectively. Therefore, for each open compact subgroup V of $G_{\mathfrak{p}}$, the projection $\Delta^V_{\mathbf{R}}$ of $\Delta^V = \Delta \cap (G_{\mathbf{R}} \times V)$ is a discrete subgroup of $G_{\mathbf{R}}$, and the quotient $G_{\mathbf{R}}/\Delta^V_{\mathbf{R}}$ has finite invariant volume. Let M_V be the field of automorphic functions with respect to $\Delta^V_{\mathbf{R}}$, and put $M = \bigcup_V M_V$. Since $\Delta^V_{\mathbf{R}} \supset \Gamma^V_{\mathbf{R}}$, we have $M_V \subset L_V$; hence $\mathbf{C} \subsetneq M \subset L$. We shall check that M is $G_{\mathfrak{p}}$-invariant in L, and that the restriction to M of the action of $G_{\mathfrak{p}}$ on L gives *the $G_{\mathfrak{p}}$-field M corresponding to* Δ. To check this, let $f(z) \in M$ and $g_{\mathfrak{p}} \in G_{\mathfrak{p}}$. Take V such that $f(z) \in M_V$. By definition, $g_{\mathfrak{p}}\{f(z)\} = f(\gamma_{\mathbf{R}}^{-1} \cdot z)$ with $\gamma \in \Gamma \cap (G_{\mathbf{R}} \times g_{\mathfrak{p}} V)$; hence $g_{\mathfrak{p}}\{f(z)\}$ is an automorphic function with respect to $\gamma_{\mathbf{R}} \Delta^V_{\mathbf{R}} \gamma_{\mathbf{R}}^{-1} = \Delta^{V'}_{\mathbf{R}}$, with $V' = \gamma_{\mathfrak{p}} V \gamma_{\mathfrak{p}}^{-1}$. Hence $g_{\mathfrak{p}}\{f(z)\} \in M_{V'} \subset M$, which shows the $G_{\mathfrak{p}}$-invariance of M. Since γ is also in $\Delta \cap (G_{\mathbf{R}} \times g_{\mathfrak{p}} V)$, our second assertion is obvious (see also §5). Thus we have shown that Δ gives M. We note that

$$(\Delta : \Gamma) = (\Delta^V : \Gamma^V) = [L_V : M_V] = [L : M]$$

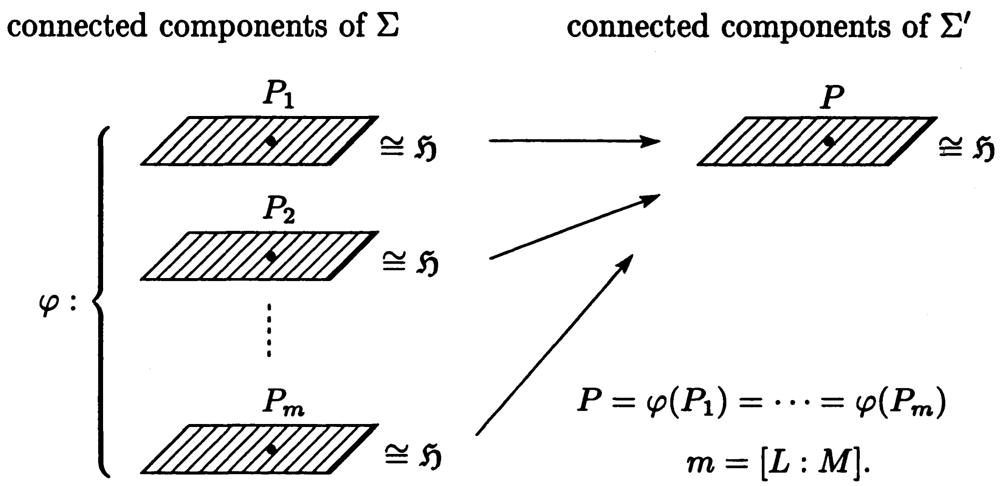
holds for each V. In fact, the first equality is an immediate consequence of $\Delta_{\mathfrak{p}} = \Delta^V_{\mathfrak{p}} \cdot \Gamma_{\mathfrak{p}}$ (since $\Gamma_{\mathfrak{p}}$ is dense in $G_{\mathfrak{p}}$), the second is obvious, and the last equality follows from the corollary of Proposition 2 (§3).

Conversely, let M be a $G_{\mathfrak{p}}$-subfield of L over $\mathbf{C}$; i.e., M is $G_{\mathfrak{p}}$-invariant and $\mathbf{C} \subsetneq M \subset L$. Then by §3, $[L : M] < \infty$, and M is also a $G_{\mathfrak{p}}$-field over $\mathbf{C}$. Let Σ (resp.Σ') be the space of all non-equivalent non-trivial discrete valuations of L (resp.M) over $\mathbf{C}$. They are one dimensional complex manifolds, of which each connected component is isomorphic to $\mathfrak{H}$. Consider the restriction map

$$\varphi : \Sigma \to \Sigma'. \tag{20}$$

Then it is clear that φ is holomorphic, and by Proposition 3, φ is unramified and gives an $[L : M]$-fold map of Σ onto Σ'. In fact, φ induces an $[L : M]$-fold map $\widetilde{\varphi}$ of the set $C(\Sigma)$ of all connected components of Σ onto the set $C(\Sigma')$ of all connected components of Σ'; and if $\Sigma_0 \in C(\Sigma)$, $\Sigma_0' \in C(\Sigma')$ and $\Sigma_0' = \widetilde{\varphi}(\Sigma_0)$, then φ gives an isomorphism of Σ_0 onto Σ_0'.

(21)



These can be checked immediately by recalling the definition of the complex structure of Σ. We also note that the actions of $G_\mathfrak{p}$ on Σ and on Σ' are consistent with the map φ. Now let Σ_1 be any connected component of Σ, and let Σ_i ($1 \leq i \leq m$, $m = [L : M]$) be the connected components of Σ such that $\varphi(\Sigma_i) = \varphi(\Sigma_1)$. Put

$$(22) \qquad \begin{cases} \Gamma_\mathfrak{p} = & \{g_\mathfrak{p} \in G_\mathfrak{p} \mid g_\mathfrak{p}(\Sigma_1) = \Sigma_1\} \\ \Delta_\mathfrak{p} = & \{g_\mathfrak{p} \in G_\mathfrak{p} \mid g_\mathfrak{p}(\Sigma_1) = \Sigma_i \text{ for some } i \ (1 \leq i \leq m)\}. \end{cases}$$

Then $\Delta_\mathfrak{p}$ can be identified with

$$(23) \qquad \Delta_\mathfrak{p} = \{g_\mathfrak{p} \in G_\mathfrak{p}|_{\Sigma'} \mid g_\mathfrak{p}(\varphi(\Sigma_1)) = \varphi(\Sigma_1)\}.$$

Therefore, $\Gamma_\mathfrak{p} \cong \Gamma_\mathbf{R} \subset \mathrm{Aut}(\Sigma_1) \cong G_\mathbf{R}$, and $\Delta_\mathfrak{p} \cong \Delta_\mathbf{R} \subset \mathrm{Aut}(\varphi(\Sigma_1)) \cong G_\mathbf{R}$. Define Γ, Δ by these identifications and by the diagonal embeddings into $G = G_\mathbf{R} \times G_\mathfrak{p}$. Then, it can be checked immediately that $\Delta \supset \Gamma$, $(\Delta : \Gamma) < \infty$, that M is the $G_\mathfrak{p}$-field corresponding to Δ, and that Δ and M correspond in a one-to-one manner in such a way that the Galois theory holds between them.

So, we have proved the following Theorem.

Theorem 2. *Let L be a $G_\mathfrak{p}$-field over $\mathbf{C}$, and let Γ be the corresponding discrete subgroup of G. Then, the $G_\mathfrak{p}$-subfields M such that $\mathbf{C} \subsetneq M \subset L$ and groups Δ such that $\Gamma \subset \Delta \subset G$ with $(\Delta : \Gamma) < \infty$ correspond naturally in a one-to-one manner satisfying the Galois theory (in particular, we have $[L : M] = (\Delta : \Gamma)$). Moreover, the group Δ is the discrete subgroup of G which corresponds to M in the sense of Theorem 1.*

We shall call Γ *maximal* if there is no such Δ other than Γ itself. Thus we obtain the following:

Corollary 1. *The $G_\mathfrak{p}$-field L over $\mathbf{C}$ is irreducible if and only if the corresponding group Γ is maximal.*

Corollary 2.[8] *Let L be a $G_\mathfrak{p}$-field over $\mathbf{C}$. Then, L contains an irreducible $G_\mathfrak{p}$-subfield over $\mathbf{C}$.*

[8]We can prove further that L contains only finitely many $G_\mathfrak{p}$-subfields over $\mathbf{C}$ (see Supplement §3 (Corollary 2)).

PROOF. Let Γ be the discrete subgroup of G corresponding to L. Let Δ be any subgroup of G containing Γ with $(\Delta : \Gamma) < \infty$. Let V be any fixed open compact subgroup of $G_{\mathfrak{p}}$, and put $\Gamma^V = \Gamma \cap (G_{\mathbf{R}} \times V)$, $\Delta^V = \Delta \cap (G_{\mathbf{R}} \times V)$. Then since $\Gamma_{\mathfrak{p}} \cdot \Delta_{\mathfrak{p}}^V = \Delta_{\mathfrak{p}}$, we have $(\Delta : \Gamma) = (\Delta^V : \Gamma^V) = (\Delta_{\mathbf{R}}^V : \Gamma_{\mathbf{R}}^V)$. But $\Gamma_{\mathbf{R}}^V$ is a discrete subgroup of $G_{\mathbf{R}}$ and the quotient $G_{\mathbf{R}}/\Gamma_{\mathbf{R}}^V$ has finite invariant volume. Hence the index $(\Delta_{\mathbf{R}}^V : \Gamma_{\mathbf{R}}^V)$ is bounded. It follows then that the index $(\Delta : \Gamma)$ is also bounded. Therefore, among all Δ, there is a maximal one. Now, our Corollary is a direct consequence of Theorem 2. □

The above argument also shows the following:

COROLLARY 3. *If there is an open compact subgroup V of $G_{\mathfrak{p}}$ such that $\Gamma_{\mathbf{R}}^V$ is a maximal fuchsian group, then the $G_{\mathfrak{p}}$-field over* **C** *which corresponds to Γ is irreducible.*

The full automorphism group of L over C.

§11. Let L be a $G_{\mathfrak{p}}$-field over **C**, and let $\mathrm{Aut}_{\mathbf{C}}\, L$ be the group of *all* automorphisms of the (abstract) field L which are trivial on **C**. Then $G_{\mathfrak{p}}$ can be considered as a subgroup of $\mathrm{Aut}_{\mathbf{C}}\, L$, and in general, may not coincide with the whole group $\mathrm{Aut}_{\mathbf{C}}\, L$. However, we can prove that $G_{\mathfrak{p}}$ is of finite index in $\mathrm{Aut}_{\mathbf{C}}\, L$. This fact will be basic for our later studies.

THEOREM 3. *Let L be a $G_{\mathfrak{p}}$-field over* **C***, and let* $\mathrm{Aut}_{\mathbf{C}}\, L$ *be the group of all automorphisms of the abstract field L which are trivial on* **C***. Then $G_{\mathfrak{p}}$ is a subgroup of* $\mathrm{Aut}_{\mathbf{C}}\, L$ *with finite index.*

For the proof, we need some preliminaries (§12, §13).

§12. Let Γ be the discrete subgroup of G which corresponds to L. Let $V_1 \supset V_2 \supset \cdots \supset V_n \supset \cdots$ be any descending sequence of open compact subgroups of $G_{\mathfrak{p}}$ satisfying $\bigcap_{n=1}^{\infty} V_n = \{1\}$. Put $\Gamma^n = \Gamma \cap (G_{\mathbf{R}} \times V_n)$ $(n \geq 1)$. Then we get a descending sequence $\Gamma_{\mathbf{R}}^1 \supset \Gamma_{\mathbf{R}}^2 \supset \cdots \supset \Gamma_{\mathbf{R}}^n \supset \cdots$ of discrete subgroups of $G_{\mathbf{R}}$ whose quotient spaces have finite invariant volumes. Now let Γ' be the subgroup of $G_{\mathbf{R}}$ formed of all elements $x \in G_{\mathbf{R}}$ such that for any $n \geq 1$, there exists some $m \geq 1$ for which the inclusions $x^{-1}\Gamma_{\mathbf{R}}^n x \supset \Gamma_{\mathbf{R}}^m$ and $x\Gamma_{\mathbf{R}}^n x^{-1} \supset \Gamma_{\mathbf{R}}^m$ hold;

$$\Gamma' = \{x \in G_{\mathbf{R}} \mid \forall n, \exists m;\ x^{-1}\Gamma_{\mathbf{R}}^n x \supset \Gamma_{\mathbf{R}}^m,\ x\Gamma_{\mathbf{R}}^n x^{-1} \supset \Gamma_{\mathbf{R}}^m \}. \tag{24}$$

It is obvious that Γ' contains $\Gamma_{\mathbf{R}}$;

$$G_{\mathbf{R}} \supset \Gamma' \supset \Gamma_{\mathbf{R}} \supset \Gamma_{\mathbf{R}}^1 \supset \Gamma_{\mathbf{R}}^2 \supset \cdots;\quad \bigcap_{n=1}^{\infty} \Gamma_{\mathbf{R}}^n = \{1\}. \tag{25}$$

Now $\Gamma_{\mathfrak{p}}$ carries a topology induced by that of $G_{\mathfrak{p}}$, and by the identification of $\Gamma_{\mathbf{R}}$ with $\Gamma_{\mathfrak{p}}$, we shall consider $\Gamma_{\mathbf{R}}$ as a topological group ($\mathfrak{p}$-adic topology; *not* the real topology). So, the subgroups $\Gamma_{\mathbf{R}}^n$ $(n \geq 1)$ form a basis of neighborhoods of 1. By the definition of Γ', we see that $\Gamma_{\mathbf{R}}^n$ $(n \geq 1)$ satisfies the axioms for a basis of neighborhoods of 1 *for* Γ', and hence by taking $\Gamma_{\mathbf{R}}^{n}$ ($n \geq 1$) as a basis of neighborhoods 1, Γ' becomes a topological group which contains $\Gamma_{\mathbf{R}}$ as an *open* subgroup. Take completions with respect to this topology, and let $\widetilde{\Gamma}'$ be the completion of Γ'.

$$\text{(26)}\qquad \text{completions}\quad \begin{array}{ccccccccc} \Gamma' & \supset & \Gamma_{\mathbf{R}} & \supset & \Gamma_{\mathbf{R}}^{1} & \supset & \cdots & \supset \Gamma_{\mathbf{R}}^{n} & \supset \cdots \\ \downarrow & & \downarrow & & \downarrow & & & \downarrow & \\ \widetilde{\Gamma}' & \supset & G_{\mathfrak{p}} & \supset & V_1 & \supset & \cdots & \supset V_n & \supset \cdots \end{array}$$

Thus $\widetilde{\Gamma}'$ contains $G_{\mathfrak{p}}$ as an open subgroup. Now we claim the following:

PROPOSITION 4. *The group* $\mathrm{Aut}_{\mathbf{C}}\, L$ *is canonically isomorphic to* $\widetilde{\Gamma}'$.

PROOF. We can identify L with the union of the fields of automorphic functions L_n with respect to $\Gamma_{\mathbf{R}}^{n}$; $L = \bigcup_{n=1}^{\infty} L_n$. Now by the following action, the group Γ' acts on L as an automorphism group over $\mathbf{C}$:

$$\text{(27)}\qquad \Gamma' \ni x : L \ni f(z) \mapsto f(x^{-1} \cdot z) \in L.$$

(It follows immediately from the definition of Γ' that $f(x^{-1} \cdot z) \in L$). As in §5 where we defined the action of $G_{\mathfrak{p}}$ on L, we can also lift the action of Γ' (on L) to that of $\widetilde{\Gamma}'$ on L. Namely, for each $\tilde{x} \in \widetilde{\Gamma}'$ and $f(z) \in L$, take n such that $f(z) \in L_n$, take $x \in \Gamma' \cap \tilde{x}V_n$ (where Γ', V_n are considered as subgroups of $\widetilde{\Gamma}'$), and put $\tilde{x}\{f(z)\} = f(x^{-1} \cdot z)$. Then,

$$\widetilde{\Gamma}' \ni \tilde{x} :\ L \ni f(z) \mapsto \tilde{x}\{f(z)\} \in L$$

defines an action of $\widetilde{\Gamma}'$ on L. It is clear that its restriction to $G_{\mathfrak{p}}$ coincides with the original action. We shall show that this action is effective. Suppose that $\tilde{x} \in \widetilde{\Gamma}'$ acts trivially on L. Then for any $n \geq 1$ and for any $x \in \Gamma' \cap \tilde{x}V_n$, we get $f(x^{-1} \cdot z) = f(z)$ for all $f(z) \in L_n$. Fix any $z = z_0 \in \mathfrak{H}$. Then since $f(x^{-1} \cdot z_0) = f(z_0)$ holds for all $f(z) \in L_n$, there exists $\delta \in \Gamma_{\mathbf{R}}^{n}$ such that $x^{-1} \cdot z_0 = \delta \cdot z_0$. If z_0 is not an elliptic fixed point of $\Gamma_{\mathbf{R}}^{n}$, then δ is uniquely determined by z_0. Consider δ as a $\Gamma_{\mathbf{R}}^{n}$-valued function of z_0 defined on $\mathfrak{H} - E$, where E is the discrete subset of $\mathfrak{H}$ formed of all elliptic fixed points of $\Gamma_{\mathbf{R}}^{n}$. Since $x^{-1} \cdot z_0$ is a continuous function of z_0, δ must also be continuous. But $\Gamma_{\mathbf{R}}^{n}$ is discrete. Therefore, δ must be a constant on $\mathfrak{H} - E$. So, put $\delta = \gamma_{\mathbf{R}} \in \Gamma_{\mathbf{R}}^{n}$. Then, $x \cdot \gamma_{\mathbf{R}}$ stabilizes all points of $\mathfrak{H} - E$; hence $x = (\gamma_{\mathbf{R}})^{-1} \in \Gamma_{\mathbf{R}}^{n}$; hence $\tilde{x} \in V_n$. Since n is arbitrary, we get $\tilde{x} = 1$. Therefore, the action of $\widetilde{\Gamma}'$ on L is effective. Thus we get

$$\text{(28)}\qquad \widetilde{\Gamma}' \subset \mathrm{Aut}_{\mathbf{C}}\, L.$$

Now we shall show that they are in fact equal. First, each point z^* on $\mathfrak{H}$ gives a discrete valuation of L by $f(z) \mapsto \mathrm{ord}_{z^*} f(z)$; and in this manner, $\mathfrak{H}$ can be considered as a connected component of Σ. Let this be denoted by Σ_0. We have seen (§6) that $\mathrm{Aut}_{\mathbf{C}}\, L$ acts on Σ, leaving the complex structure of Σ invariant. Hence, if σ is any element of $\mathrm{Aut}_{\mathbf{C}}\, L$, then $\sigma(\Sigma_0)$ is another connected component of Σ. Since $G_{\mathfrak{p}}$ acts transitively on the set of all connected component of Σ, we get $\sigma(\Sigma_0) = g_{\mathfrak{p}}(\Sigma_0)$ with some $g_{\mathfrak{p}} \in G_{\mathfrak{p}}$. Put $g_{\mathfrak{p}}^{-1} \cdot \sigma = \sigma_0$. Then, $\sigma_0(\Sigma_0) = \Sigma_0$; hence σ_0 induces an automorphism of $\Sigma_0 \cong \mathfrak{H}$. So, we can consider σ_0 also as an element of $G_{\mathbf{R}} = \mathrm{Aut}(\mathfrak{H}) \cong \mathrm{Aut}\,\Sigma_0$. By the definition of the action of $\mathrm{Aut}_{\mathbf{C}}\, L$ on Σ, we have $\sigma_0\{f(z)\} = f(\sigma_0^{-1} \cdot z)$ for any $f(z) \in L$ (z is a variable on $\mathfrak{H}$). Hence we have $f(\sigma_0^{-1} \cdot z) \in L$ for any $f(z) \in L$. Now let $n \geq 1$, and let $f_1(z), \cdots, f_N(z)$ be a generator of L_n over $\mathbf{C}$. Then $f_i(\sigma_0^{-1}\cdot z)\in L$ for all i. Take m for which L_m contains $f_i(\sigma_0^{-1}\cdot z)$ for all i $(1\le i\le N)$. Then $f(z)\in L_n$ implies $f(\sigma_0^{-1}\cdot z)\in L_m$. Since $\{f(\sigma_0^{-1}\cdot z)|f(z)\in L_n\}$ is the field of automorphic functions with respect to $\sigma_0\Gamma_{\mathbf{R}}^n\sigma_0^{-1}$, we get $\sigma_0\Gamma_{\mathbf{R}}^n\sigma_0^{-1}\supset\Gamma_{\mathbf{R}}^m$. By applying the same argument for σ_0^{-1} instead of σ_0, we get $\sigma_0^{-1}\Gamma_{\mathbf{R}}^n\sigma_0\supset\Gamma_{\mathbf{R}}^{m'}$ for some $m'\ge 0$. Therefore,

$$\sigma_0\Gamma_{\mathbf{R}}^n\sigma_0^{-1}\cap\sigma_0^{-1}\Gamma_{\mathbf{R}}^n\sigma_0\supset\Gamma_{\mathbf{R}}^l\ \text{ holds for } l=\mathrm{Max}(m,m').$$

This implies $\sigma_0\in\Gamma'$; hence $\sigma\in G_{\mathfrak{p}}\cdot\Gamma'=\widetilde{\Gamma}'$; hence we get $\mathrm{Aut}_{\mathbf{C}}L\subset\widetilde{\Gamma}'$. □

§13. Now, by (26) and by Proposition 4, we have

$$(\mathrm{Aut}_{\mathbf{C}}L:G_{\mathfrak{p}})=(\Gamma':\Gamma_{\mathbf{R}});\tag{29}$$

hence our problem is to prove the finiteness of $(\Gamma':\Gamma_{\mathbf{R}})$. For this purpose, we need a lemma on local automorphisms of the group $PL_2(k_{\mathfrak{p}})$. Here, by *a local automorphism* of a topological group X, we mean any isomorphism σ of an open subgroup U_1 of X onto another open subgroup U_2 of X; and if σ' is another local automorphism of X; σ' : $U_1'\cong U_2'$, then σ and σ' are called *equivalent* if they coincide on some open subgroup $U_1''\subset U_1\cap U_1'$. To distinguish from local automorphisms, we use the terminology "global automorphisms" for usual (topological) automorphisms of X. Of course, every global automorphism of X defines an equivalence class of local automorphisms.

LEMMA 1.

(i) *Every local automorphism of $PL_2(k_{\mathfrak{p}})$ is equivalent to a global automorphism of $PL_2(k_{\mathfrak{p}})$.*
(ii) *Every global automorphism of $PL_2(k_{\mathfrak{p}})$ is a product of an inner automorphism and an automorphism of $PL_2(k_{\mathfrak{p}})$ obtained by a field automorphism of $k_{\mathfrak{p}}$ over $\mathbf{Q}_p$.*
(iii) *Every global automorphism of $G_{\mathfrak{p}}=PSL_2(k_{\mathfrak{p}})$ is induced from that of $PL_2(k_{\mathfrak{p}})$.*

The proof is given in Supplement §4. It is a direct consequence of Dynkin's theory of normed Lie algebras [**8**].

REMARK 1. (ii) implies that the group of all global automorphisms of $PL_2(k_{\mathfrak{p}})$ is isomorphic to the semi-direct product of $PL_2(k_{\mathfrak{p}})$ and $\mathrm{Aut}_{\mathbf{Q}_p}k_{\mathfrak{p}}$, where $\mathrm{Aut}_{\mathbf{Q}_p}k_{\mathfrak{p}}$ is the group of all automorphisms of the field $k_{\mathfrak{p}}$ over $\mathbf{Q}_p$, which acts on $PL_2(k_{\mathfrak{p}})$ in a natural manner;

$$\mathrm{Aut}\,PL_2(k_{\mathfrak{p}})\cong PL_2(k_{\mathfrak{p}})\cdot\mathrm{Aut}_{\mathbf{Q}_p}k_{\mathfrak{p}}\quad\text{(semi-direct)}.\tag{30}$$

This also shows that distinct global automorphisms of $PL_2(k_{\mathfrak{p}})$ give distinct (equivalence) classes of local automorphisms of $PL_2(k_{\mathfrak{p}})$. Hence, if we denote by $\mathrm{Aut}'X$ the group of all equivalence classes of local automorphisms of a topological group X, then we get

$$\mathrm{Aut}'G_{\mathfrak{p}}=\mathrm{Aut}'PL_2(k_{\mathfrak{p}})\cong PL_2(k_{\mathfrak{p}})\cdot\mathrm{Aut}_{\mathbf{Q}_p}k_{\mathfrak{p}},\tag{31}$$

where $G_{\mathfrak{p}}=PSL_2(k_{\mathfrak{p}})$.

REMARK 2. The automorphism $x \mapsto {}^t x^{-1}$ of $PL_2(k_\mathfrak{p})$ is an inner automorphism. In fact, we have ${}^t x^{-1} = b^{-1}xb$ for any $x \in PL_2(k_\mathfrak{p})$ with $b = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. But for $PL_n(k_\mathfrak{p})$ with $n > 2$, $x \mapsto {}^t x^{-1}$ is not inner.

§14.

PROOF OF THEOREM 3. As a descending sequence $V_1 \supset V_2 \supset \cdots$ of open compact subgroups of $G_\mathfrak{p}$ with $\bigcap_{n=1}^{\infty} V_n = \{1\}$, we shall take

$$(32)\qquad \begin{cases} V_1 = & PSL_2(O_\mathfrak{p}), \\ V_n = & \{x \in V_1 \mid x \equiv 1 \ (\mathrm{mod}\,\mathfrak{p}^{n-1})\} \quad (n > 1). \end{cases}$$

Now, since $G_\mathfrak{p}$ *is* an open subgroup of $\widetilde{\Gamma}'$, *an inner automorphism of* $\widetilde{\Gamma}'$ *induces a local automorphism of* $G_\mathfrak{p}$. Thus we get a homomorphism $\widetilde{\varphi}$ of $\widetilde{\Gamma}'$ into the group $\mathrm{Aut}'\,G_\mathfrak{p}$ of all equivalence classes of local automorphisms of $G_\mathfrak{p}$;

$$(33)\qquad \widetilde{\varphi} : \widetilde{\Gamma}' \to \mathrm{Aut}'\,G_\mathfrak{p}.$$

By (31), we can identify $\mathrm{Aut}'\,G_\mathfrak{p}$ with $PL_2(k_\mathfrak{p}) \cdot \mathrm{Aut}_{\mathbf{Q}_p} k_\mathfrak{p}$, and hence by the restriction of $\widetilde{\varphi}$ to Γ', we get a homomorphism:

$$(34)\qquad \varphi : \Gamma' \to PL_2(k_\mathfrak{p}) \cdot \mathrm{Aut}_{\mathbf{Q}_p} k_\mathfrak{p}.$$

Consider the subgroup $Y = PSL_2(k_\mathfrak{p})PL_2(O_\mathfrak{p})\cdot\mathrm{Aut}_{\mathbf{Q}_p} k_\mathfrak{p}$ of $PL_2(k_\mathfrak{p})\cdot\mathrm{Aut}_{\mathbf{Q}_p} k_\mathfrak{p}$. It is of index two. Let Γ'' be the inverse image of Y by φ. Since $\varphi(\Gamma_\mathbf{R})$ is contained in $G_\mathfrak{p} = PSL_2(k_\mathfrak{p})$, Γ'' contains $\Gamma_\mathbf{R}$, and we get

$$(35)\qquad \Gamma' \supset \Gamma'' \supset \Gamma_\mathbf{R}, \quad (\Gamma' : \Gamma'') \leq 2.$$

So, to prove $(\Gamma' : \Gamma_\mathbf{R}) < \infty$, it suffices to prove $(\Gamma'' : \Gamma_\mathbf{R}) < \infty$.

For this purpose, let $x \in \Gamma''$. Then there exist $n_0 \geq 1$, $g_\mathfrak{p} \in G_\mathfrak{p} = PSL_2(k_\mathfrak{p})$, $w_\mathfrak{p} \in PL_2(O_\mathfrak{p})$, and $\sigma \in \mathrm{Aut}_{\mathbf{Q}_p} k_\mathfrak{p}$ acting on $PL_2(k_\mathfrak{p})$, such that $x^{-1}vx = g_\mathfrak{p}^{-1}w_\mathfrak{p}^{-1}v^\sigma w_\mathfrak{p} g_\mathfrak{p}$ for all $v \in V_{n_0}$. Take $\gamma \in \Gamma_\mathbf{R} \cap V_1 g_\mathfrak{p}$, and put $\gamma = v_1 g_\mathfrak{p}$ with $v_1 \in V_1$. Since $PL_2(O_\mathfrak{p})\cdot\mathrm{Aut}_{\mathbf{Q}_p} k_\mathfrak{p}$ normalizes all V_n $(n \geq 1)$, we have $w_\mathfrak{p}^{-1}v^\sigma w_\mathfrak{p} \in V_{n_0}$ for all $v \in V_{n_0}$; hence $g_\mathfrak{p}x^{-1} = v_1^{-1}\gamma x^{-1}$ normalizes all V_n for $n \geq n_0$. Since v_1 also normalizes all V_n, it follows that γx^{-1} normalizes V_n for all $n \geq n_0$. But since $\gamma x^{-1} \in \Gamma'' \subset \Gamma'$, it normalizes $\Gamma' \cap V_n = \Gamma_\mathbf{R} \cap V_n = \Gamma_\mathbf{R}^n$ for all $n \geq n_0$. So, if we put

$$(36)\qquad H^m = \{x \in G_\mathbf{R} \mid x^{-1}\Gamma_\mathbf{R}^n x = \Gamma_\mathbf{R}^n \text{ for all } n \geq m\} \quad (m \geq 1),$$

this implies that $\gamma x^{-1} \in H^{n_0}$; hence every element of Γ'' is contained in $H^{n_0} \cdot \Gamma_\mathbf{R}$ for some $n_0 \geq 1$.

Now, since $\Gamma_\mathbf{R}^1$ normalizes all $\Gamma_\mathbf{R}^n$ $(n \geq 1)$, we get

$$(37)\qquad \cdots \supset H^2 \supset H^1 \supset \Gamma_\mathbf{R}^1 \supset \Gamma_\mathbf{R}^2 \supset \cdots$$

But, in general, if Δ is a discrete subgroup of $G_\mathbf{R}$ whose quotient space has finite invariant volume, then its normalizer $N(\Delta)$ in $G_\mathbf{R}$ satisfies $(N(\Delta) : \Delta) < \infty$; and there exist only finitely many subgroups Δ' of $G_\mathbf{R}$ such that $\Delta' \supset \Delta$ and $(\Delta' : \Delta) < \infty$.

Apply this to $\Delta = \Gamma^m_{\mathbf{R}}$. Since $H^m \subset N(\Gamma^m_{\mathbf{R}})$, we get $(H^m : \Gamma^1_{\mathbf{R}}) \leq (H^m : \Gamma^m_{\mathbf{R}}) < \infty$, and hence we also get $H^m = H^{m+1} = \cdots$ for sufficiently large m. Now put $H = \bigcup_{n=1}^{\infty} H^n$. Then $(H : \Gamma^1_{\mathbf{R}}) < \infty$, and by what we have shown we have $\Gamma'' \subset H^{n_0}\Gamma_{\mathbf{R}}$ for some n_0, and hence $\Gamma'' \subset H \cdot \Gamma_{\mathbf{R}}$. Put $H = \sum_{i=1}^{t} M_i\Gamma^1_{\mathbf{R}}$ with $M_i \in H$ $(1 \leq i \leq t)$. Then we get $\Gamma'' \subset H \cdot \Gamma_{\mathbf{R}} = \sum_{i=1}^{t} M_i\Gamma_{\mathbf{R}}$; hence $(\Gamma'' : \Gamma_{\mathbf{R}}) \leqq (H : \Gamma^1_{\mathbf{R}}) < \infty$. □

So, we have also proved:

Corollary 1. *We have*

$$(\mathrm{Aut}_{\mathbf{C}} L : G_{\mathrm{p}}) = (\Gamma' : \Gamma_{\mathbf{R}}) \leq 2(H : \Gamma^1_{\mathbf{R}}), \tag{38}$$

where Γ' *is given by* (24), $\Gamma^n_{\mathbf{R}} = [\Gamma \cap (G_{\mathbf{R}} \times V_n)]_{\mathbf{R}}$ $(n \geq 1)$, *with* V_n *defined by* (32), *and* H *is the subgroup of all elements of* $G_{\mathbf{R}}$ *which normalize* $\Gamma^n_{\mathbf{R}}$ *for all sufficiently large* n.

§15. Some direct consequences of Theorem 3.

Corollary 2. *The group* G_{p} *is a characteristic subgroup of* $\mathrm{Aut}_{\mathbf{C}} L$.

Proof. Since G_{p} is a simple group (as an abstract group), it is enough to show that if a group A contains a subgroup B with $(A : B) < \infty$ and if B is an infinite simple group, then B is invariant by every automorphism of A. Let σ be any automorphism of A. Then $B \cap B^\sigma$ is of finite index in B. Therefore, $\bigcap_{x\in B} x^{-1}(B \cap B^\sigma)x$ is a normal subgroup of B with finite index. But B is infinite and simple. Therefore, $\bigcap_{x\in B} x^{-1}(B \cap B^\sigma)x = B$; hence $B \cap B^\sigma = B$; hence $B^\sigma \supset B$. Since $(A : B) = (A^\sigma : B^\sigma) = (A : B^\sigma)$, we get $B^\sigma = B$. □

Corollary 3. *Let* $\mathfrak{Z}$ *be the centralizer of* G_{p} *in* $\mathrm{Aut}_{\mathbf{C}} L$. *Then*

(i) $\mathfrak{Z}$ *is finite.*
(ii) $\mathfrak{Z}$ *reduces to* $\{1\}$ *if and only if* L *contains no* G_{p}*-subfield* M *over* $\mathbf{C}$ *such that* L/M *is normal. In particular, if* L *is irreducible, then* $\mathfrak{Z} = \{1\}$.

Proof. (i) Since G_{p} has no center, we get $\mathfrak{Z} \cap G_{\mathrm{p}} = \{1\}$; hence by $(\mathrm{Aut}_{\mathbf{C}} L : G_{\mathrm{p}}) < \infty$, we get the finiteness of $\mathfrak{Z}$.

(ii) If $\mathfrak{Z} \neq \{1\}$, let M be the fixed field of $\mathfrak{Z}$ in L. Since $\mathfrak{Z}$ centralizes G_{p}, M is G_{p}-invariant. Also it is clear that L/M is normal and $[L : M] = (\mathfrak{Z} : 1) \neq 1$. Conversely, let $M(\neq L)$ be a G_{p}-subfield of L over $\mathbf{C}$ such that L/M is normal, and put $\mathfrak{Z}' = \mathrm{Aut}(L/M) \neq \{1\}$. Then for each $g_{\mathrm{p}} \in G_{\mathrm{p}}$, the fixed field of $g_{\mathrm{p}}^{-1}\mathfrak{Z}'g_{\mathrm{p}}$ is $g_{\mathrm{p}}^{-1}(M) = M$. Hence $g_{\mathrm{p}}^{-1}\mathfrak{Z}'g_{\mathrm{p}} = \mathfrak{Z}'$; hence G_{p} normalizes $\mathfrak{Z}'$. Let X be the centralizer of $\mathfrak{Z}'$ in G_{p}. Then X is a normal subgroup of G_{p} with finite index (since $\mathfrak{Z}'$ is finite). But G_{p} is a simple group. Therefore, $X = G_{\mathrm{p}}$; hence G_{p} centralizes $\mathfrak{Z}'$; hence $\mathfrak{Z}$ contains $\mathfrak{Z}' \neq \{1\}$. Therefore, $\mathfrak{Z} \neq \{1\}$. □

Corollary 4. *Let* $N(\Gamma)$ *be the normalizer of* Γ *in* G. *Then*

(i) $(N(\Gamma) : \Gamma) < \infty$;
(ii) $N(\Gamma) = \Gamma$ *holds if and only if* L *contains no* G_{p}*-subfield* $M \neq L$ *over* $\mathbf{C}$ *such that* L/M *is normal.*

Proof. First, we claim that the projections $N(\Gamma) \to N(\Gamma)_{\mathbf{R}}$ and $N(\Gamma) \to N(\Gamma)_{\mathfrak{p}}$ are injective. In fact, let $\delta = \delta_{\mathbf{R}} \times \delta_{\mathfrak{p}} \in N(\Gamma)$ with, say, $\delta_{\mathbf{R}} = 1$. Let $\gamma = \gamma_R \times \gamma_{\mathfrak{p}} \in \Gamma$. Then $\delta^{-1}\gamma\delta = \gamma_{\mathbf{R}} \times \delta_{\mathfrak{p}}^{-1}\gamma_{\mathfrak{p}}\delta_{\mathfrak{p}} \in \Gamma$. But by the injectivity of $\Gamma \to \Gamma_{\mathbf{R}}$, we get $\delta_{\mathfrak{p}}^{-1}\gamma_{\mathfrak{p}}\delta_{\mathfrak{p}} = \gamma_{\mathfrak{p}}$. Therefore, $\delta_{\mathfrak{p}}$ commutes with all elements of $\Gamma_{\mathfrak{p}}$; hence $\delta_{\mathfrak{p}} = 1$.

Now, it is clear that $N(\Gamma)_{\mathbf{R}} \subset \Gamma'$, where Γ' is as in §12. Hence $(N(\Gamma) : \Gamma) = (N(\Gamma)_{\mathbf{R}} : \Gamma_{\mathbf{R}}) < \infty$; which settles (i). Now (ii) is a direct consequence of (i) and Theorem 2. □

§16. A $G_{\mathfrak{p}}$-field L over $\mathbf{C}$ will be called *quasi-irreducible* if L contains no $G_{\mathfrak{p}}$-subfields $M \neq L$ over $\mathbf{C}$ such that L/M is normal. By Corollary 3, L is quasi-irreducible if and only if the centralizer of $G_{\mathfrak{p}}$ in $\mathrm{Aut}_{\mathbf{C}} L$ is trivial, and by Corollary 4, if and only if $N(\Gamma) = \Gamma$. In particular, if L is irreducible or if $G_{\mathfrak{p}} = \mathrm{Aut}_{\mathbf{C}} L$, then L is quasi-irreducible. Quasi-irreducible $G_{\mathfrak{p}}$-fields over $\mathbf{C}$ play central roles in Part 2 of this Chapter.

§17.

Example . Let L be the $G_{\mathfrak{p}}$-field over $\mathbf{C}$ which corresponds to $\Gamma = PSL_2(\mathbf{Z}^{(p)})$ (see §2). Here, $G_{\mathfrak{p}} = G_p = PSL_2(\mathbf{Q}_p)$. Since $\Gamma^1_{\mathbf{R}} = PSL_2(\mathbf{Z})$ is a maximal fuchsian group, we get $H = \Gamma^1_{\mathbf{R}}$; hence by (38) we get $(\mathrm{Aut}_{\mathbf{C}} L : G_p) = (\Gamma' : \Gamma_{\mathbf{R}}) \leq 2$. But if we put

(39) $$\Gamma^* = \{x \in GL_2(\mathbf{Z}^{(p)}) | \det x = \text{ powers of } p\} / \pm \{\text{ powers of } p\},$$

then Γ^* can be considered as a subgroup of $G_{\mathbf{R}}$ with $\Gamma^* \supset \Gamma_{\mathbf{R}}$, $(\Gamma^* : \Gamma_{\mathbf{R}}) = 2$, and it is clear that $\Gamma^* \subset \Gamma'$. Therefore, we get $\Gamma' = \Gamma^*$. Therefore, $\mathrm{Aut}_{\mathbf{C}} L = \widetilde{\Gamma^*}$, where $\widetilde{\Gamma^*}$ is the p-adic completion of Γ^* given by

(40) $$\widetilde{\Gamma^*} = \{x \in GL_2(\mathbf{Q}_p) | \det x = \text{ powers of } p\} / \pm \{\text{ powers of } p\}.$$

Since $\Gamma^1_{\mathbf{R}} = PSL_2(\mathbf{Z})$ is maximal, L is irreducible by Corollary 3 of Theorem 2.