7. Notes.

General.

At present there seems to be no systematic formal introduction to geometric group theory as such, though the key ideas with many interesting examples are described in de la Harpe's book [Harp]. The fundamental notions are also described in [BriH]. Earlier, more traditional treatments of combinatorial group theory are [MagKS] and [LS]. A list of open problems in geometric group theory can be found in [Bes3].

The theory of non-positively curved spaces grew out of work of Aleksandrov, Toponogov, Busemann and more recently, Gromov. The standard introductory text is [BriH].

7.0. Section 0 :

An overview of basic "small cancellation" theory can be found in Strebel's appendix to [GhH]. The paper [ScW] was influential in the introduction of topological ideas into group theory. The works of Stallings and Dunwoody were influential in introducing methods from 3-manifold theory — see the notes on Section 3. Thurston gave an outline of his geometrisation programme in [Th].

7.1. Section 1 :

The basic material of most of this section is standard. More detailed accounts can be found in [MagKS] and [LS]. One important result that illustrates the power of combinatorial methods is Higman's embedding theorem [Hi] (see also [LS]).

A geometrical picture of the Heisenberg group is described in [GhH].

A brief survey of the Andews-Curtis conjecture can be found in [BuM].

7.2. Section 2 :

Again, much of this material can be found in [MagKS], [LS] and other introductory texts.

7.3. Section 3 :

The significance of quasi-isometries, in particular, a version of the key result, Theorem 3.6, was known to Efremovich, Schwarz amd Milnor in the 1960s. The subject was developed through work of Gromov, see for example [Gro2] and [Gro3]. A more general version of Proposition 3.1 is proven in [Gro4]. A proof of Theorem 3.6 (sometimes called the "Schwarz-Milnor Lemma") along the lines given here can be found in [BriH].

The statement that a group quasi-isometric to **Z** is virtually **Z** is an immediate consequence of the result of Hopf [Ho] that a two-ended finitely generated group is virtually **Z** — at least given the relatively simple fact that the property of being "two-ended" is quasi-isometry invariant. Here "two-ended" can be taken to mean that the complement of any sufficiently large finite subgraph of the Cayley graph has exactly two unbounded components.

A proof of the Borsuk-Ulam theorem can be found in many texts on topology, for example [Ar].

Gromov's theorem on groups of polynomial growth is given in [Gro1]. It uses the solution to Hilbert's fifth problem by Montgomery and Zippin [MonZ]. The "Gromov-Hausdorff" limit argument used in [Gro1] can be conveniently expressed in terms of asymptotic cones [Gro3], which have, in themselves become an important tool in geometric group theory, see for example [Dr]. While there are variations on the theme, this seems to be essentially the only proof known. A quite different approach to deal specifically with virtually abelian groups has been given by Shalom [Sha].

The results of Stallings [St] and Dunwoody [Du] relating to group splittings are good examples of the adaptation of ideas from 3-manifolds.

Alonso gives an account of the quasi-isometry invariance of isoperimetric inqualities, and Shapiro's observation conserning the invarians of the word problem, in [Alo].

7.4. Section 4 :

Just about any introductory text on topology will have an account of fundamental groups, covering spaces etc. Our "nice" spaces can all be given the stucture of a simplicial complex. A deeper systematic treatment of such complexes is given in [Sp].

A combinatorial proof of Theorem 4.1 can be found in [LS].

7.5. Section 5 :

Introductory texts on hyperbolic geometry include [Iv] and [An], and a general introduction is also included in [Bea]. The book [W] gives an overview of this subject in connection with Thurston's programme. See also [CanFKP]. The foundational principles of non-euclidean geometry with many historical references can be found in [Gre].

An account of Poincaré's theorem for tessellations in dimension 2 is given in [Bea].

Perelman's account of geometrisation is given in [Pe1,Pe2]. A commentary can be found in [KlL], and a survey in [Mor].

With regards to the characterisation of virtual surface groups, a seminal piece of work was Mess's paper on the Seifert conjecture [Me]. This used earlier work of Tukia [Tu1], but left open the particular and difficult case of a virtual triangle group. This was resolved in subsequent and independent work of Gabai [Gaba] and Casson and Jungreis [CasJ]. To take care of the "euclidean" case, Mess relies on the theorem of Varopoulos [V] that a group with a recurrent random walk is virtually $\mathbf{Z}^n$ for $n = 0, 1, 2$. This in turn relies on Gromov's result of polynomial growth (see notes on Section 3). An argument that bypasses this, and gives some other characterisations of virtual surface groups, can be found in [Bow5]. Some of the results therein have since been generalised by Kleiner.

For introductions to Teichmüller theory see [Ab] or [ImT].

The Mostow rigity theorem [Mo] tells us that any finite-volume hyperbolic structure on a 3-manifold is unique. The "stable trace field" of such a manifold $M = \mathbf{H}^3/\Gamma$ is the field generated by the squares of traces of elements of $\Gamma \subseteq PSL(2, \mathbf{C})$. A consequence of Mostow rigidity and a little algebraic geometry is that such a field is a finite extention of the rationals. It turns out to be a commensurability invariant [Re]. It is not hard to find explicit examples of closed hyperbolic 3-manifolds with different stable trace fields. There is a major project of enumerating small volume hyperbolic 3-manifolds and computing their stable trace fields and other invariants, see [CouGHN].

7.6. Section 6 :

Gromov introduced the notion of a hyperbolic group in [Gro2]. Several expositions of various aspects of this work appeared in the few years that followed: see [GhH], [CooDP], [Sho] or [Bowl]. Since then, the subject has developed in many different directions, though there seems to have been no new systematic general introduction to the subject. Some aspects of hyperbolic groups are discussed in [BriH] and in [Harp].

An introduction to complex hyperbolic geometry can be found in [Go]. Complex hyperbolic n-space has $2n$ real dimensions, and its boundary is homeomorphic to the $(2n-1)$-sphere. However, more refined invariants show that it is not quasi-isometric to $\mathbf{H}^{2n}$.

The notion of an $\mathbf{R}$-tree was introduced by Morgan and Shalen [MorS], in order to prove certain compactness results that formed part of Thurston's work on hyperbolic 3-manifolds. A more geometric approach to their construction was described by Bestvina [Be1]. The subject was then developed by Rips, and elaborations and generalisations of that work can be found in [GaboLP] and [BeF]. $\mathbf{R}$-trees have now become a central tool in geometric group theory. Surveys can be found in [Pau,Bes2] and a general introduction in [Chi]. A key point is that the asymptotic cone (see notes on Section 3) of a hyperbolic space is an $\mathbf{R}$-tree (see for example [Dr]). A typical $\mathbf{R}$-tree can be a quite complicated object. For example, we note that for any cardinal $c \geq 2$ there is a unique complete $\mathbf{R}$-tree with every point of valence c [DyP].

A discussion of spanning trees that approximate distances in a hyperbolic space is given in [Gro2], and some elaborations are described in [Bowl], including a construction of logarithmic spanning trees.

Proposition 6.17 is a generalisation of the corresponding lemma for hyperbolic space which is a standard ingredient for the argument proving Mostow rigidity (see notes on Section 5). The proof we present here is based on that given in [Sho].

7.6.1. Subgroups of hyperbolic groups.

The fact that the stable length of an infinite-order element is positive can be found, for example, in [GhH]. In fact, it turns out that they are uniformly rational [Gro2,Del]. For a general group, the stable length of an infinite order element is positive if and only if the cyclic group it generates is quasi-isometrically embedded. There are many examples for which this fails, for example, the centre of the Heisenberg group.

An example of a hyperbolic group with a finitely generated subgroup that is not finitely presented is given in [BowM], and is based on a related example in [Kap-mP].

7.6.2. Finiteness conditions.

An account of the Rips complex can be found in [GhH].

Gromov outlined a proof that a subquadratic isoperimetric inequality implies hyperbolicity in [Gro2], and this argument was elaborated upon in [CooDP]. Other arguments are given in [O,Pap,Bow2]. Examples of non-hyperbolic groups with more exotic isoperimetric inequalities (or "Dehn functions") are given in [Bri,BraB] and [SaBR]. This subject has expanded in many directions since.

Automatic structures provide a link with the theory of formal languages. The standard introductory text is [ECHLPT]. Cannon's argument, which can now be interpreted as a proof that a hyperbolic group is automatic, appeared in [Can].

7.6.3. Boundaries.

A general survey of boundaries of hyperbolic groups is [Kap-iB]. A seminal article on the subject was [BesM].

Convergence groups were introduced in the context of Kleinian group by Gehring and Martin [GeM]. A general discussion, applicable to the boundaries of hyperbolic groups is given in [Tu2]. The topological characterisation of a hyperbolic group as a uniform convergence group is given in [Bow4].

The fact that any compact metrisable (topological) space can be realised as the boundary of a proper hyperbolic space can be seen as follows. First embed the space in the unit sphere of a separable Hilbert space. We can view this sphere as the boundary of a Klein model for an infinite dimensional hyperbolic geometry. We take the euclidean convex hull of our set, which is also the hyperbolic convex hull. We use the fact that the convex hull of a compact subset of a Banach space is compact. Thus the convex hull gives us a proper hyperbolic space compactified by our original set. This set is then also the Gromov boundary. Delails are left to the reader familiar with the Klein model of hyperbolic space.

A discussion of "generic" properties of hyperbolic groups is given in [Cha]. This uses the notion of a "geodesic flow" (see below).

7.6.4. Other directions.

The JSJ splitting was introduced by Sela [Se1]. It is another example of a construction inspired by 3-manifold theory, in particular, work of Waldhausen, Johannson [Jo] Jaco and Shalen [JaS]. An account for hyperbolic groups, via boundaries, is given in [Bow3]. There are a number of generalisations, for example, [RiS,DuS,FuP].

An account of the geodesic flow on a hyperbolic group is given in [Mine], and connections with the Baum-Connes conjecture can be found in [MineY]. For earlier work on the Novikov conjecture for hyperbolic groups, see [ConM] and [KasS]. (The former requires a version of the geodesic flow.)

Sela's work on the Tarski problem appears in a series of articles, starting with [Se2]. It makes much use of the JSJ splittling in constructing "Makanin-Razborov" trees. He characterises groups with the same first order theory as free groups as "limit groups". Such groups were shown to be relatively hyperbolic in [Da] (see also [Ali]).

Accounts of relatively hyperbolic groups can be found in [Fa], [Sz] and [Bow8]. A topological characterisation in terms of convergence groups is given in [Y]. Many results about hyperbolic groups have now been generalised to relatively hyperbolic groups.

Harvey introduced the curve complex in [Harv]. It was shown to be hyperbolic in [MasM] (see also [Bow6,Ham]). This fact was central to the proof of Thurston's ending lamination conjecture given in [Mins,BroCM], and has wider implications for Teichmüller theory and mapping class groups. This is a particularly active area at the moment. See [Bow7] for a survey of some of this material.