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General Theory of DPN surfaces and K3 surfaces with non-symplectic involution

2.1. General remarks

As it was shown in Chapter 1, a description of log del Pezzo surfaces of index ≤ 2 is reduced to a description of rational surfaces Y containing a nonsingular curve $C \in |-2K_Y|$ and a certain configuration of exceptional curves. Such surfaces Y and exceptional curves on them were studied in the papers [Nik79, Nik83, Nik84a, Nik87] of the second author. They are one of possible generalizations of del Pezzo surfaces.

Many other generalizations of del Pezzo surfaces were proposed, see e.g. [Dem80, Har85a, Har85b, Loo81], and most authors call their surfaces "generalized del Pezzo surfaces". Therefore, we decided following [Nik87] to call our generalization DPN surfaces. One can consider DPN surfaces to be some appropriate non-singular models of log del Pezzo surfaces of index ≤ 2 and some their natural generalizations.

Definition 2.1. A nonsingular projective algebraic surface Y is called a **DPN surface** if its irregularity $q(Y) = 0$, $K_Y \neq 0$ and there exists an effective divisor $C \in |-2K_Y|$ with only simple rational, i.e. A, D, E-singularities. Such a pair (Y, C) is called a **DPN pair.** A DPN surface Y is called **right** if there exists a nonsingular divisor $C \in |-2K_Y|$; in this case the pair (Y, C) is called **right DPN pair** or nonsingular DPN pair.

The classification of algebraic surfaces implies that if $C = \emptyset$ then a DPN surface Y is an Enriques surface ($\varkappa = p = q = 0$). If $C \neq \emptyset$ then Y is a rational surface ($\varkappa = -1, p = q = 0$), e.g. see [Shaf65].

Using the well-known properties of blowups, the following results are easy to prove. Let (Y, C) be a DPN pair, $E \subset Y$ be an exceptional curve of the 1st kind on Y and $\sigma : Y \to Y'$ be the contraction of E. Then $(Y', \sigma(C))$ is also a DPN pair. In this way, by contracting exceptional curves of the 1st kind, one can always arrive at a DPN pair (Y', C') where Y' is a relatively minimal (i.e. without exceptional curves of the 1st kind) rational surface. In this case, the only possibilities for Y' are $\mathbb{P}^2$, $\mathbb{F}_0$, $\mathbb{F}_2$, $\mathbb{F}_3$ and $\mathbb{F}_4$, since only for them $|-2K_{Y'}|$ contains a reduced divisor. If $Y' = \mathbb{P}^2$ then C' is a curve of degree 6; if $Y' = \mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ then C' is a curve of bidegree $(4,4)$; if $Y' = \mathbb{F}_2$ then $C' \in |8f + 4s_2|$; if $Y' = \mathbb{F}_3$ then $D' = C_1 + s_3$, where $C_1 \in |10f + 3s_3|$; if $Y' = \mathbb{F}_4$ then $C' = D_1 + s_4$, where $D_1 \in |12f + 3s_4|$. Here, the linear system $|f|$ is a pencil of rational curves on surface $\mathbb{F}_n$ with a section s_n, $s_n^2 = -n$. Vice versa, if (Y', C') is a DPN pair, P is a singular point of C' and $\sigma : Y \to Y'$ is a blowup of P with an exceptional (-1)-curve E then (Y, C) is a DPN pair, where

$$C = \begin{cases} \sigma_*^{-1}(C') & \text{if } P \text{ has multiplicity 2 on } C' \\ \sigma_*^{-1}(C') + E & \text{if } P \text{ has multiplicity 3 on } C' \end{cases}$$

Here $\sigma_*^{-1}(C')$ denotes the proper preimage (or the strict transform) of C', i. e. $\sigma_*^{-1}(C')$ is the closure of the set-theoretic preimage $\sigma^{-1}(C' - \{P\})$ in Y.

In this way, by blowups, from an arbitrary DPN pair (Y', C') one can always pass to a right DPN pair (Y, C), i.e. with a nonsingular C. A description of arbitrary DPN pairs and surfaces is thus reduced to a description of right (or nonsingular) DPN pairs (Y, C), and to right DPN surfaces Y and exceptional curves on them. Here, a curve $E \subset Y$ is called **exceptional** if E is irreducible and $E^2 < 0$.

We shall need a small, elementary, and well-known fact about ramified double covers. Let $\pi : X \to Y$ be a finite morphism of degree 2 between smooth algebraic varieties. Then π is Galois with group $\mathbb{Z}/2$. Therefore, the $\mathcal{O}_Y$-algebra $\pi_*\mathcal{O}_X$ splits into (± 1)-eigenspaces as $\mathcal{O}_Y \oplus L$. Since π is flat, L is flat and hence invertible. The algebra structure is given by a homomorphism $L^2 \to \mathcal{O}_Y$, i.e. by a section $s \in H^0(Y, L^{-2})$. Locally, X is isomorphic to $y^2 = s(x)$. Since X is smooth, the ramification divisor $C = (s)$ must be smooth.

Vice versa, let L^{-1} be a sheaf dividing by two the sheaf $\mathcal{O}_Y(C)$ for an effective divisor C in $\operatorname{Pic} Y$ and let s be a section of $\mathcal{O}_Y(C)$ with $(s) = C$. Then s defines an algebra structure on $\mathcal{A} = \mathcal{O}_Y \oplus L$, and $\pi : X := \operatorname{Spec} \mathcal{A} \to Y$ is a double cover ramified in C. The representation of $\mathcal{A}$ as a quotient of $\oplus_{d \geq 0} L^d$ gives an embedding of X into a total space of the line bundle L^{-1} and a section of $\pi^* L^{-1}$ ramified along $\pi^{-1}(C)$ with multiplicity one. Hence $\pi^{-1}(C) \sim \pi^* L^{-1}$.

Let (Y, C) be a right DPN pair. Since $C \in |-2K_Y|$, there exists a double cover $\pi : X \to Y$ defined by $L^{-1} = -K_Y$, branched along C. By the above, we have $\pi^*(-K_Y) \sim \pi^{-1}(C)$.

Let ω_Y be a rational 2-dimensional differential form on Y with the divisor (ω_Y) whose components do not contain components of C. Then $(\omega_Y) \sim K_Y$, and the divisor $(\pi^*\omega_Y) = \pi^*(\omega_Y) + \pi^{-1}(C) \sim \pi^*(\omega_Y) + \pi^*(-K_Y) \sim 0$. Thus, $K_X = 0$. Then X is either a K3 surface (i. e. $q(X) = 0$) or an

Abelian surface (i. e. $q(X) = 2$), e.g. see [Shaf65]. Let X be an Abelian surface. Then C is not empty (otherwise, Y is an Enriques surface and then X is a K3 surface, [Shaf65]), and Y is rational. It follows that there exists a non-zero regular 1-dimensional differential form ω_1 on X such that $\theta^*(\omega_1) = \omega_1$ for the involution θ of the double cover π. Then $\omega_1 = \pi^*\widetilde{\omega}_1$ where $\widetilde{\omega}_1$ is a regular 1-dimensional differential form on Y. It contradicts $q(Y) = 0$. This proves that X is a K3 surface.

Let ω_X be a non-zero regular 2-dimensional differential form on X. If $\theta^*(\omega_X) = \omega_X$, then $\omega_X = \pi^*(\omega_Y)$ where ω_Y is a regular 2-dimensional differential form on Y. This contradicts the fact that Y is an Enriques or rational surface (e.g. see [Shaf65]). Thus, $\theta^*(\omega_X) = -\omega_X$, and then θ is a **non-symplectic involution** of the K3 surface X.

Vice versa, assume that (X, θ) is a K3 surface with a non-symplectic involution. Then the set X^θ of fixed points of the involution is a nonsingular curve (otherwise, θ is symplectic, i. e. $\theta^*(\omega_X) = \omega_X$ for any regular 2-dimensional differential form on X). It follows (reversing arguments above) that the pair $\left(Y = X/\{1, \theta\},\ C = \pi(X^\theta)\right)$ is a right DPN pair where $\pi : X \to Y$ is the quotient morphism.

Thus, a description of right DPN pairs (Y, C) is reduced to a description of K3 surfaces with a non-symplectic involution (X, θ).

2.2. Reminder of basic facts about K3 surfaces

Here we remind basic results about K3 surfaces that we use. We follow [Shaf65], [PS-Sh71], [Kul77] and also [Nik80a, Nik83, Nik84b]. Of course, all these results are well-known.

Let X be an algebraic K3 surface. We recall that this means that X is a projective non-singular algebraic surface, the canonical class $K_X = 0$ (i.e. there exists a non-zero regular 2-dimensional differential form ω_X on X with zero divisor), and $q(X) = \dim \Gamma(X, \Omega^1) = 0$ (i.e. X has no non-zero regular 1-dimensional differential forms). From definition, ω_X is unique up to multiplication by $\lambda \in \mathbb{C}$, $\lambda \neq 0$.

Let $F \subset X$ be an irreducible algebraic curve. By genus formula,

$$p_a(F) = \frac{F^2 + (F \cdot K_X)}{2} + 1 = \frac{F^2}{2} + 1 \geq 0. \tag{10}$$

It follows that $F^2 \equiv 0 \mod 2$, $F^2 \geq -2$, and F is non-singular rational if $F^2 = -2$. In particular, any exceptional curve F on X (i. e. F is irreducible and $F^2 < 0$) is non-singular rational with $F^2 = -2$.

By Riemann-Roch Theorem, for any divisor $D \subset X$ we have

$$l(D) + l(K_X - D) = h^1(D) + \frac{D \cdot (D - K_X)}{2} + \chi(\mathcal{O}_X),$$

which gives for a K3 surface X that

$$l(D) + l(-D) = h^1(D) + \frac{D^2}{2} + 2 \geq \frac{D^2}{2} + 2. \tag{11}$$

It follows that one of $\pm D$ is effective if $D^2 \geq -2$.

All algebraic curves on X up to linear equivalence generate the Picard lattice S_X of X. For K3 surfaces linear equivalence is equivalent to numerical, and $S_X \subset H^2(X, \mathbb{Z})$ where $H^2(X, \mathbb{Z})$ is an even unimodular lattice of the signature $(3, 19)$. Here, "even" means that x^2 is even for any $x \in H^2(X, \mathbb{Z})$. Unimodular means that for a basis $\{e_i\}$ of $H^2(X, \mathbb{Z})$ the determinant $\det(e_i \cdot e_j) = \pm 1$. Such even unimodular lattice is unique up to an isomorphism, see e.g. [Ser70]. By Hodge Index Theorem, the Picard lattice S_X is hyperbolic, i. e. it has signature $(1, \rho - 1)$ where $\rho = \operatorname{rk} S_X$. Let

$$V(S_X) = \{x \in S_X \otimes \mathbb{R} \mid x^2 > 0\}. \tag{12}$$

Since S_X is hyperbolic, $V(S_X)$ is an open cone which has two convex halves. One of these halves $V^+(X)$ is distinguished by the fact that it contains the ray $\mathbb{R}^+ h$ of a polarization h (i. e. a hyperplane section) of X, where $\mathbb{R}^+$ denotes the set of all non-negative real numbers.

Let

$$\mathrm{NEF}(X) = \{x \in S_X \otimes \mathbb{R} \mid x \cdot C \geq 0 \ \forall \text{ effective curve } C \subset X\} \tag{13}$$

be the **nef cone** of X. Since S_X is hyperbolic, for any irreducible curve C with $C^2 \geq 0$ we have that $C \in \overline{V^+(X)}$, and $C \cdot V^+(X) > 0$. It follows that

$$\mathrm{NEF}(X) = \{x \in \overline{V^+(X)} \mid x \cdot P(X) \geq 0\} \tag{14}$$

where $P(X) \subset S_X$ denotes the set of all divisor classes of irreducible non-singular rational (i. e. all exceptional) curves on X.

Let $h \in \mathrm{NEF}(X)$ be a hyperplane section. By Riemann-Roch Theorem above, $f \in S_X$ with $f^2 = -2$ is effective if and only if $h \cdot f > 0$. It follows that $\mathrm{NEF}(X)$ is a fundamental chamber (in $V^+(X)$) for the group $W^{(2)}(S_X)$ generated by reflections in all elements $f \in S_X$ with $f^2 = -2$. Each such f gives a reflection $s_f \in O(S_X)$ where

$$s_f(x) = x + (x \cdot f)f, \ \ x \in S_X, \tag{15}$$

in particular, $s_f(f) = -f$ and s_f is identical on $f^\perp$.

Since all $F \in P(X)$ have $F^2 = -2$, the nef cone $\mathrm{NEF}(X)$ is locally finite in $V^+(X)$, all its faces of codimension one are orthogonal to elements of $P(X)$. *This gives a one-to-one correspondence between the faces of codimension one of* $\mathrm{NEF}(X)$ *and the elements of* $P(X)$. Indeed, let γ be a codimension one face of $\mathrm{NEF}(X)$. Assume $F \in P(X)$ is orthogonal to γ, i. e. γ belongs to the edge of the half-space $F \cdot x \geq 0$, $x \in S_X \otimes \mathbb{R}$, containing

$\mathrm{NEF}(X)$. Such $F \in S_X$ with $F^2 = -2$ is obviously unique because any element $f \in S_X$ which is orthogonal to γ is evidently λF, $\lambda \in \mathbb{R}^+$. We have $(\lambda F)^2 = \lambda^2 F^2 = \lambda^2(-2)$, and F is distinguished by the condition $F^2 = -2$. In such a way, all faces of codimension one of $\mathrm{NEF}(X)$ give a subset $P(X)' \subset P(X)$ of elements of $P(X)$ which are orthogonal to codimension one faces of $\mathrm{NEF}(X)$. Let us show that $P(X)' = P(X)$. Obviously it will be enough to show that for any $E \in P(X)$, the orthogonal projection of $\mathrm{NEF}(X)$ into hyperplane $E^\perp$ belongs to $\mathrm{NEF}(X)$. The projection is given by the formula $H \mapsto \widetilde{H} = H + (H \cdot E)E/2$ for $H \in \mathrm{NEF}(X)$. Let us show that $\widetilde{H} \in \mathrm{NEF}(X)$. Let C be an irreducible curve on X. If $C \neq E$, then $C \cdot \widetilde{H} = C \cdot H + (H \cdot E)(C \cdot E)/2 \geq 0$ because H is nef and C is different from E. If $C = E$, then $C \cdot \widetilde{H} = E \cdot \widetilde{H} = E \cdot H + (H \cdot E)(E^2)/2 = 0 \geq 0$. Thus, $\widetilde{H} \in \mathrm{NEF}(X)$.

Therefore, we obtain a group-theoretic description of the nef cone of X and all exceptional curves of X: $\mathrm{NEF}(X)$ is the fundamental chamber for the reflection group $W^{(2)}(S_X)$ acting on $V^+(X)$, this chamber is distinguished by the condition that it contains a hyperplane section of X. The set $P(X)$ of all exceptional curves on X consists of all elements $f \in S_X$ which have $f^2 = -2$ and which are orthogonal to codimension one faces of $\mathrm{NEF}(X)$ and directed outwards (i. e. $f \cdot \mathrm{NEF}(X) \geq 0$).

It is more convenient to work with the corresponding hyperbolic space

$$\mathcal{L}(X) = V^+(X)/\mathbb{R}^+. \tag{16}$$

Elements of this space are rays $\mathbb{R}^+x$, where $x \in S_X \otimes \mathbb{R}$, $x^2 > 0$ and $x \cdot h > 0$. Each element $\beta \in S_X \otimes \mathbb{R}$ with square $\beta^2 < 0$ defines a **half-space**

$$\mathcal{H}^+_\beta = \{\mathbb{R}^+x \in \mathcal{L}(X) \mid \beta \cdot x \geq 0\}, \tag{17}$$

so that β is perpendicular to the bounding **hyperplane**

$$\mathcal{H}_\beta = \{\mathbb{R}^+x \in \mathcal{L}(X) \mid \beta \cdot x = 0\}, \tag{18}$$

and faces outward. The set

$$\mathcal{M}(X) = \mathrm{NEF}(X)/\mathbb{R}^+ = \bigcap_{\substack{f \in S_X,\, f^2=-2 \\ f \text{ is effective}}} \mathcal{H}^+_f = \bigcap_{f \in P(X)} \mathcal{H}^+_f \tag{19}$$

is a locally finite convex polytope in $\mathcal{L}(X)$. The set $P(\mathcal{M}(X))$ of vectors with square -2, perpendicular to the facets of $\mathcal{M}(X)$ and directed outward, is exactly the set $P(X)$ of divisor classes of exceptional curves on X. Moreover, $\mathcal{M}(X)$ admits a description in terms of groups. Let $O'(S_X)$ be the subgroup of index two of the full automorphism group $O(S_X)$ of the lattice S_X which consists of automorphisms preserving the half-cone $V^+(X)$.

Let $W^{(2)}(S_X) \subset O'(S_X)$ be the subgroup of $O'(S_X)$ generated by reflections s_f with respect to all elements $f \in S_X$ with square (-2). The action of the group $W^{(2)}(S_X)$, as well as $O'(S_X)$, on $\mathcal{L}(X)$ is discrete. $W^{(2)}(S_X)$ is the group generated by reflections in all hyperplanes $\mathcal{H}_f$, $f \in S_X$ and $f^2 = -2$. The set $\mathcal{M}(X)$ is a fundamental chamber for this group, i.e. $W^{(2)}(S_X)(\mathcal{M}(X))$ defines a decomposition of $\mathcal{L}(X)$ into polytopes which are congruent to $\mathcal{M}(X)$, and $W^{(2)}(S_X)$ acts transitively and without fixed elements on this decomposition (cf. [PS-Sh71, Vin85]). The fundamental chamber $\mathcal{M}(X)$ is distinguished from other fundamental chambers by the fact that it contains the ray $\mathbb{R}^+h$ of polarization.

By Hodge decomposition, we have the direct sum

$$H^2(X,\mathbb{Z})\otimes\mathbb{C} = H^2(X,\mathbb{C}) = H^{2,0}(X) + H^{1,1}(X) + H^{0,2}(X) \tag{20}$$

where $H^{2,0}(X) = \mathbb{C}\omega_X$, $H^{0,2}(X) = \overline{H^{2,0}(X)}$ and $H^{1,1}(X) = (H^{2,0}(X) + H^{0,2}(X))^{\perp}$. Then the Picard lattice of X is

$$S_X = H^2(X,\mathbb{Z}) \cap H^{1,1}(X) = \{x \in H^2(X,\mathbb{Z}) \mid x\cdot H^{2,0}(X) = 0\}. \tag{21}$$

The triplet

$$\left(H^2(X,\mathbb{Z}), H^{2,0}(X), \mathcal{M}(X)\right) \tag{22}$$

is called the **periods of X**.

An **isomorphism**

(23)
$$\phi : \left(H^2(X,\mathbb{Z}), H^{2,0}(X), \mathcal{M}(X)\right) \to \left(H^2(X',\mathbb{Z}), H^{2,0}(X'), \mathcal{M}(X')\right)$$

of periods of two K3 surfaces means an isomorphism $\phi : H^2(X,\mathbb{Z}) \to H^2(X',\mathbb{Z})$ of cohomology lattices (i. e. modules with pairing) such that $\phi(H^{2,0}(X)) = H^{2,0}(X')$, $\phi(\mathcal{M}(X)) = \mathcal{M}(X')$ for the corresponding induced maps which we denote by the same letter ϕ. By *Global Torelli Theorem for K3 surfaces* [PS-Sh71], ϕ is defined by a unique isomorphism $f : X' \to X$ of the K3 surfaces: $\phi = f^*$.

As an application of the Global Torelli Theorem, let us consider the description of $\mathrm{Aut}(X)$ from [PS-Sh71]. By Serre duality, $h^0(\mathcal{T}_X) = h^2(\Omega^1_X) = h^{1,2} = 0$. Thus, X has no regular vector-fields. It follows, that $\mathrm{Aut}(X)$ acts on S_X with only a finite kernel. Let

$$\mathrm{Sym}(\mathcal{M}(X)) = \{\phi \in O'(S_X) \mid \phi(\mathcal{M}(X)) = \mathcal{M}(X)\} \tag{24}$$

be the symmetry group of the fundamental chamber $\mathcal{M}(X)$. Let us denote by $\mathrm{Sym}(\mathcal{M}(X))^0$ its subgroup of finite index which consists of all symmetries which are identical on the discriminant group $(S_X)^*/S_X$. Elements $\phi \in \mathrm{Sym}(\mathcal{M}(X))^0$ can be extended to automorphisms of $H^2(X,\mathbb{Z})$ which are identical on the transcendental lattice $T_X = (S_X)^{\perp} \subset H^2(X,\mathbb{Z})$ (see Propositions A.3, A.4 in Appendix). We denote this extension by the same letter ϕ since it is unique. We have $H^{2,0}(X) \subset T_X \otimes \mathbb{C}$ since $H^{2,0}(X) \cdot S_X = 0$. Thus, $\phi(H^{2,0}(X)) = H^{2,0}(X)$, and ϕ is an automorphism of periods of X. Thus, $\phi = f^*$ where $f \in \mathrm{Aut}(X)$. Thus, the natural contragradient representation

$$\mathrm{Aut}(X) \to \mathrm{Sym}(\mathcal{M}(X)) \tag{25}$$

has a finite kernel and a finite cokernel. It follows that the groups $\mathrm{Aut}(X) \approx \mathrm{Sym}(\mathcal{M}(X))$ are naturally isomorphic up to finite groups. Since we have a natural isomorphism $\mathrm{Sym}(\mathcal{M}(X)) \cong O'(S_X)/W^{(2)}(S_X)$, we also obtain that

$$\mathrm{Aut}(X) \approx O'(S_X)/W^{(2)}(S_X). \tag{26}$$

In particular, $\mathrm{Aut}(X)$ is finite if and only if $[O(S_X) : W^{(2)}(S_X)] < \infty$. See [Nik83], [Nik84a], [Nik87] about the enumeration of all these cases.

Periods $(H^2(X,\mathbb{Z}), H^{2,0}(X), \mathcal{M}(X))$ of a K3 surface X satisfy *the Riemann relations:* $H^{2,0}(X) \cdot H^{2,0}(X) = 0$ and $\omega_X \cdot \overline{\omega_X} > 0$ for $0 \neq \omega_X \in H^{2,0}(X)$ (shortly we will be able to write $H^{2,0}(X) \cdot \overline{H^{2,0}(X)} > 0$).

Abstract K3 periods is a triplet

$$(L_{K3}, H^{2,0}, \mathcal{M}) \tag{27}$$

where L_{K3} is an even unimodular lattice of signature $(3,19)$, $H^{2,0} \subset L_{K3} \otimes \mathbb{C}$ is a one dimensional complex linear subspace satisfying $H^{2,0} \cdot H^{2,0} = 0$, $H^{2,0} \cdot \overline{H^{2,0}} > 0$, and $\mathcal{M}$ is a fundamental chamber of $W^{(2)}(M) \subset \mathcal{L}(M)$ where $M = \{x \in L_{K3} \mid x \cdot H^{2,0} = 0\}$ is an abstract Picard lattice. By the *surjectivity of Torelli map for K3 surfaces* [Kul77], any abstract K3 periods are isomorphic to periods of an algebraic K3 surface.

As an application of Global Torelli Theorem and Surjectivity of Torelli map for K3 surfaces, let us describe *moduli spaces of K3 surfaces with condition on Picard lattice.* For details see [Nik80a] and for real case [Nik84b].

Let M be an even (i. e. x^2 is even for any $x \in M$) hyperbolic (i. e. of signature $(1, \mathrm{rk}\, M - 1)$) lattice. Like for S_X above, we consider the light cone

$$V(M) = \{x \in M \otimes \mathbb{R} \mid x^2 > 0\} \tag{28}$$

of M, and we choose one of its half $V^+(M)$ defining the corresponding hyperbolic space $\mathcal{L}(M) = V^+(M)/\mathbb{R}^+$. We choose a fundamental chamber $\mathcal{M}(M) \subset \mathcal{L}(M)$ for the reflection group $W^{(2)}(M)$ generated by reflections in all elements $f \in M$ with $f^2 = -2$. Note that the group $\pm W^{(2)}(M)$ acts transitively on all these additional data $(V^+(M), \mathcal{M}(M))$ which shows that they are defined by the lattice M itself (i. e. by its isomorphism class), and we can fix these additional data $(V^+(M), \mathcal{M}(M))$ without loss of generality.

We consider K3 surfaces X such that a primitive sublattice $M \subset S_X$ is fixed, $V^+(X) \cap (M \otimes \mathbb{R}) = V^+(M)$, $\mathcal{M}(X) \cap \mathcal{L}(M) \neq \emptyset$, and $\mathcal{M}(X) \cap \mathcal{L}(M) \subset \mathcal{M}(M)$. (This is one of the weakest possible conditions of degeneration.) Such a K3 surface X is called a **K3 surface with the condition M on Picard lattice**. A **general K3 surface** X with the condition M on Picard lattice (i. e. with moduli or periods general enough) has $S_X = M$, as we will show later. Then $S_X = M$, $V^+(X) = V^+(M)$, and $\mathcal{M}(X) = \mathcal{M}(M)$. One can consider this condition as a marking of elements of the Picard lattice S_X by elements of the standard lattice M.

Let $(X, M \subset S_X)$ be a K3 surface with the condition M on the Picard lattice. Then $M \subset S_X \subset H^2(X, \mathbb{Z})$ defines a primitive sublattice $M \subset H^2(X, \mathbb{Z})$. Depending on the isomorphism class of this primitive sublattice, we obtain different irreducible components of moduli of K3 surfaces with the condition M on Picard lattice.

We fix a primitive embedding $M \subset L_{K3}$. We consider marked K3 surfaces $(X, M \subset S_X)$ with the condition M on Picard lattice and the class $M \subset L_{K3}$ of the condition M on cohomology. Here marking means an isomorphism $\xi : H^2(X, \mathbb{Z}) \cong L_{K3}$ of lattices such that $\xi|M$ is identity. Taking

$$(29) \qquad \left(L_{K3}, H^{2,0} = \xi(H^{2,0}(X)), \mathcal{M} = \xi(\mathcal{M}(X))\right)$$

we obtain periods of a marked K3 surface $(X, M \subset S_X, \xi)$ with condition M on Picard lattice. By the surjectivity of Torelli map, any abstract periods

$$(L_{K3}, H^{2,0}, \mathcal{M})$$

where $H^{2,0} \cdot M = 0$, $\mathcal{M} \cap \mathcal{L}(M) \neq \emptyset$, and $\mathcal{M} \cap \mathcal{L}(M) \subset \mathcal{M}(M)$ correspond to a marked K3 surface with the condition M on Picard lattice. Let us denote by $\widetilde{\Omega}_{M \subset L_{K3}}$ the space of all these abstract periods. It is called the **period domain** of K3 surfaces $(X, M \subset S_X)$ with the condition M on Picard lattice and with the type $M \subset L_{K3}$ of this condition on cohomology. Let

$$(30) \qquad \Omega_{M \subset L_{K3}} = \{H^{2,0} = \mathbb{C}\omega \subset L_{K3} \otimes \mathbb{C} \mid \omega \cdot M = 0,\ \omega^2 = 0\ and\ \omega \cdot \overline{\omega} > 0\}.$$

We have the natural projection $p : \widetilde{\Omega}_{M \subset L_{K3}} \to \Omega_{M \subset L_{K3}}$ forgetting $\mathcal{M}$. The space $\Omega_{M \subset L_{K3}}$ is an open subset of a projective quadric of dimension $\operatorname{rk} L_{K3} - \operatorname{rk} M - 2 = 20 - \operatorname{rk} M$. It follows that for a general K3 surface X with the condition M on Picard lattice we have $S_X = M$. Indeed, if $\operatorname{rk} S_X > \operatorname{rk} M$ for all K3 surfaces X with the condition $M \subset L_{K3}$, then, since $H^{2,0} \cdot \xi(S_X) = 0$, periods $H^{2,0}$ define a quadric of smaller dimension $20 - \operatorname{rk} S_X < 20 - \operatorname{rk} M$ which leads to a contradiction. It also follows that the forgetful map $p : \widetilde{\Omega}_{M \subset L_{K3}} \to \Omega_{M \subset L_{K3}}$ is an isomorphism in general points: e.g. it is isomorphism in all points with $S_X = M$. In fact, p gives an étale covering which makes $\widetilde{\Omega}_{M\subset L_{K3}}$ non-Hausdorff in special points (see Burns and Rapoport [BR75] about construction and use of this covering).

Considerations above also show that an even hyperbolic lattice M is isomorphic to a Picard lattice S_X of some K3 surface X if and only if M has a primitive embedding $M \subset L_{K3}$. In particular, such an embedding always exists if $\operatorname{rk} M \leq \operatorname{rk} L_{K3}/2 = 11$ (see [Nik80b]): any even hyperbolic lattice M of $\operatorname{rk} M \leq 11$ is Picard lattice of some K3 surface. See other sufficient and necessary conditions in Theorems A.5 and Corollary A.6 of Appendix.

The period space $\Omega_{M\subset L_{K3}}$ is a Hermitian symmetric domain of type IV in the classification by É. Cartan. The domains $\Omega_{M\subset L_{K3}}$ and hence also $\widetilde{\Omega}_{M\subset L_{K3}}$ have two connected components which are complex conjugate. Indeed, $H^{2,0} \subset L_{K3} \otimes \mathbb{C}$ is equivalent to an oriented positive definite real subspace $(H^{2,0} + \overline{H^{2,0}}) \cap (L_{K3} \otimes \mathbb{R}) \subset L_{K3} \otimes \mathbb{R}$ which is orthogonal to $M \subset L_{K3}$. Let us consider the orthogonal complement $T = M^{\perp}$ in L_{K3} and the automorphism group $O(2, 20 - \operatorname{rk} M)$ of $T \otimes \mathbb{R}$. Then

$$\Omega_{M\subset L_{K3}} = O(2, 20 - \operatorname{rk} M)/\left(SO(2) \times O(20 - \operatorname{rk} M)\right) \tag{31}$$

has two connected components since $SO(2) \times O(20 - \operatorname{rk} M)$ has index two in the maximal compact subgroup $O(2) \times O(20 - \operatorname{rk} M)$ of $O(2, 20 - \operatorname{rk} M)$. The number of connected components of $O(2, 20 - \operatorname{rk} M)$ and $O(2) \times O(20 - \operatorname{rk} M)$ coincide.

Let

$$O(M \subset L_{K3}) = \{\phi \in O(L_{K3}) \mid \phi|M = \text{identity}\} \tag{32}$$

be the automorphism group of the period domain $\widetilde{\Omega}_{M\subset L_{K3}}$. By Global Torelli Theorem, the corresponding K3 surfaces are isomorphic if and only if their periods are conjugate by this group. This group is discrete. Thus

$$\operatorname{Mod}_{M\subset L_{K3}} = \widetilde{\Omega}_{M\subset L_{K3}}/O(M \subset L_{K3}) \tag{33}$$

gives the coarse **moduli space of K3 surfaces with the condition M on Picard lattice and with the type $M \subset L_{K3}$ of the embedding in cohomology.** Usually $O(M \subset L_{K3})$ contains an automorphism which permutes the two connected components of periods (equivalently it has the spinor norm -1, i. e. it does not belong to a connected component of $SO(2) \times O(20 - \operatorname{rk} M)$ of $O(2, 20 - \operatorname{rk} M)$ above). Then the moduli space (33) is irreducible.

Two primitive embeddings $a : M \subset L_{K3}$, $b : M \subset L_{K3}$ give the same moduli space (33), if they are conjugate by an automorphism of the lattice L_{K3}, i. e. they are equivalent. Taking disjoint union of moduli spaces $\operatorname{Mod}_{M\subset L_{K3}}$ for all equivalence classes $M \subset L_{K3}$ of primitive embeddings of lattices, we obtain the **moduli space**

$$\text{Mod}_M = \bigsqcup_{\text{class of } M \subset L_{K3}} \text{Mod}_{M \subset L_{K3}} \tag{34}$$

of K3 surfaces with the condition M on Picard lattice. If the primitive embedding $M \subset L_{K3}$ is unique up to isomorphisms, and if $O(M \subset L_{K3})$ has an automorphism of spinor norm -1, then the moduli space Mod_M is irreducible. We remark that the same results about components of moduli of K3 surfaces with conditions on Picard lattice can be obtained using only Global Torelli Theorem and local surjectivity of Torelli map for K3 surfaces (see the paper [Nik80a] which had been written before the surjectivity of Torelli map for K3 was established).

2.3. The lattice S, and the main invariants (r, a, δ), (k, g, δ)

All results of this Section were obtained in [Nik80a, Nik80b, Nik79, Nik83] (see also [Nik87]). Here we omit some technical proofs. They will be given in Section A.2 of Appendix.

Let X be an algebraic K3 surface with a non-symplectic involution θ. (We remark that existence of a non-symplectic involution on a Kähler K3 surface X implies that X is algebraic (see [Nik80a])).

For a module Q with action of θ we denote by $Q_\pm$ the ± 1 eigenspaces of θ.

The lattice (i. e. a free $\mathbb{Z}$-module with a non-degenerate symmetric bilinear form)

$$S = H^2(X, \mathbb{Z})_+$$

considered up to isomorphisms is called **the main invariant** of (X, θ). Since θ is non-symplectic, we have $H^{2,0}(X) \subset H^2(X, \mathbb{Z})_- \otimes \mathbb{C}$. It follows that $S \cdot H^{2,0}(X) = 0$. Thus, $S \subset S_X$ is a sublattice of the Picard lattice S_X of X. Let $h \in S_X$ be a polarization of X. Then $h_1 = h + \theta^* h \in S$ is also a polarization of X, and $h_1^2 > 0$. It follows that S is hyperbolic like the Picard lattice S_X. The rank $r = \text{rk}\, S$ is one of main invariants of S.

The following property of the sublattice $S \subset S_X$ is very important: *The lattice $(S_X)_-$ (i. e. the orthogonal complement to S in S_X) has no elements f with $f^2 = -2$.* Indeed, by Riemann-Roch Theorem for K3, then $\pm f$ is effective and $\theta^*(\pm f) = \mp f$, which is impossible.

Let $T = S^\perp$ be the orthogonal complement to S in $H^2(X, \mathbb{Z})$. Canonical epimorphisms $H^2(X, \mathbb{Z}) \to S^*$ and $H^2(X, \mathbb{Z}) \to T^*$ defined by intersection pairing give canonical θ-equivariant epimorphisms

$$S^*/S \cong H^2(X, \mathbb{Z})/(S \oplus T) \cong T^*/T$$

because $H^2(X, \mathbb{Z})$ is an unimodular lattice. The involution θ is $+1$ on S^*/S, and it is -1 on T^*/T. It follows that the groups $S^*/S \cong T^*/T \cong (\mathbb{Z}/2\mathbb{Z})^a$ are 2-elementary. Only in this case multiplications by ± 1 coincide. Thus, the lattice S is 2-elementary, which means that its discriminant group $\mathfrak{A}_S = S^*/S \cong (\mathbb{Z}/2\mathbb{Z})^a$ is 2-elementary where a gives another important invariant of S.

There is one more invariant δ of S which takes values in $\{0, 1\}$. One has $\delta = 0 \iff (x^*)^2 \in \mathbb{Z}$ for every $x^* \in S^* \iff$ the **discriminant quadratic form**

$$q_S : \mathfrak{A}_S = S^*/S \to \mathbb{Q}/2\mathbb{Z}, \quad q_S(x^* + S) = (x^*)^2 + 2\mathbb{Z}$$

of S is even: it takes values in $(\mathbb{Z}/2\mathbb{Z})$. See Appendix, Section A.1 about discriminant forms of lattices.

The invariants (r, a, δ) *of* S *define the isomorphism class of a 2-elementary even hyperbolic lattice* S. See more general statement and the proof in Appendix, Section A.2 and Theorem A.9.

Thus, any two even hyperbolic 2-elementary lattices with the same invariants (r, a, δ) are isomorphic. The invariants (r, a, δ) of S are equivalent to the main invariant S, and we later call them the **main invariants** of a K3 surface X with non-symplectic involution θ.

Vice versa, let S be a hyperbolic even 2-elementary lattice having a primitive embedding to L_{K3}. Let $S \subset L_{K3}$ be one of primitive embeddings. Considering $T = S^\perp$ in L_{K3} and the diagram similar to above,

$$S^*/S \cong L_{K3}/(S \oplus T) \cong T^*/T\ ,$$

we obtain that there exists an involution θ^* of L_{K3} which is $+1$ on S, and -1 on T. Let us consider the moduli space

$$\mathrm{Mod}'_S \subset \mathrm{Mod}_S \tag{35}$$

of K3 surfaces $(X, S \subset S_X)$ with condition S on Picard lattice (see (34)) where for $(X, S \subset S_X)$ from Mod'_S we additionally assume that the orthogonal complement $S^\perp$ to S in S_X has no elements with square (-2). One can easily see that Mod'_S is Zariski open subset in Mod_S. Any general $(X, S \subset S_X)$ (i. e. when $S = S_X$) belongs to Mod'_S. Thus, the difference between Mod'_S and Mod_S is in complex codimension one, and they have the same irreducible components. By Global Torelli Theorem, the action of θ^* on L_{K3} can be lifted to a non-symplectic involution θ on X with $H^2(X, \mathbb{Z})_+ = S$. Thus, the moduli space Mod'_S in (35) can be considered as **moduli space of K3 surfaces with non-symplectic involution** and the main invariant S. Since S is defined by the main invariants (r, a, δ), it can also be denoted as

$$\mathrm{Mod}_{(r,a,\delta)} = \mathrm{Mod}'_S \tag{36}$$

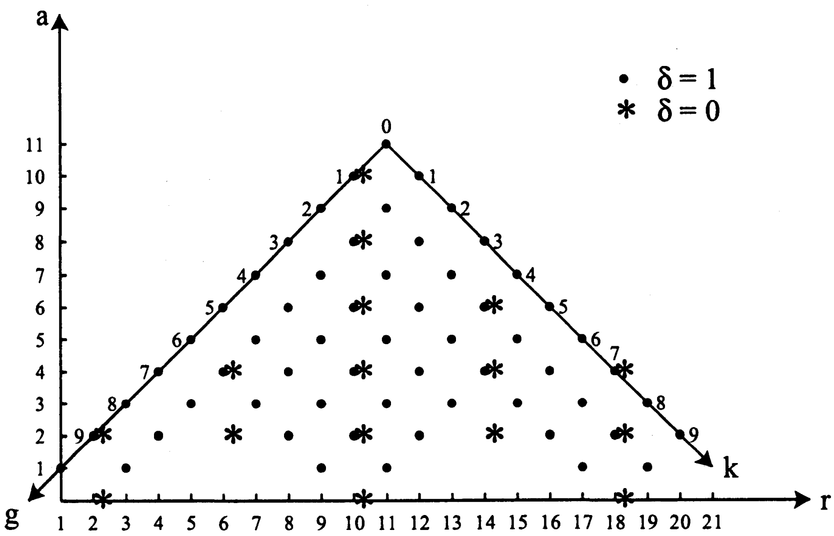


FIGURE 1. All possible main invariants (r, a, δ)

and can be considered as moduli space of K3 surfaces with non-symplectic involution and the main invariants (r, a, δ). Any even hyperbolic 2-elementary lattice S has a unique primitive embedding to L_{K3} (up to isomorphisms) if it exists. Moreover, the group $O(S \subset L_{K3})$ always has an automorphism of spinor norm -1. Thus, *the moduli space* $\mathrm{Mod}_{(r,a,\delta)}$ *is irreducible.*

Evidently, to classify all possible main invariants S (equivalently (r, a, δ)) one just needs to classify all even hyperbolic 2-elementary lattices S having a primitive embedding $S \subset L_{K3}$. All such possibilities for (r, a, δ) (equivalently, $(k = (r-a)/2, g = (22-r-a)/2, \delta)$, see below) are known and are shown on Figure 1.

The triple (r, a, δ) admits an interpretation in terms of $X^\theta = C$. If $(r, a, \delta) \neq (10, 8, 0)$ or $(10, 10, 0)$ then

$$X^\theta = C = C_g + E_1 + \cdots + E_k,$$

where C_g is a nonsingular irreducible curve of genus g, and $E_1, \dots E_k$ are nonsingular irreducible rational curves, the curves are disjoint to each other,

$$g = (22 - r - a)/2, \quad k = (r - a)/2$$

(we shall formally use the same formulae for g and k even in cases $(r, a, \delta) = (10, 8, 0)$ or $(10, 10, 0)$). If $(r, a, \delta) = (10, 8, 0)$ then $X^\theta = C = C_1^{(1)} + C_1^{(2)}$, where $C_1^{(i)}$ are elliptic (genus 1) curves. If $(r, a, \delta) = (10, 10, 0)$ then $X^\theta = C = \emptyset$, i.e. in this case Y is an Enriques surface. One has

(37)

$$\delta = 0 \iff X^\theta \sim 0 \mod 2 \text{ in } S_X \text{ (equivalently in } H_2(X, \mathbb{Z})) \iff$$

there exist signs $(\pm)_i$ for which

$$\frac{1}{4}\sum_i (\pm)_i cl(C^{(i)}) \in S_Y = H^2(Y,\mathbb{Z}), \tag{38}$$

where $C^{(i)}$ go over all irreducible components of C. Signs $(\pm)_i$ for $\delta = 0$ are defined uniquely up to a simultaneous change. They define a new natural orientation (different from the complex one) of the components of C; a positive sign gives the complex orientation and a negative sign the opposite orientation.

The main invariants S, equivalently (r, a, δ) (or (k, g, δ)) of K3 surfaces with non-symplectic involution, and the corresponding DPN pairs and DPN surfaces play a crucial role in our classification.

2.4. Exceptional curves on (X, θ) and Y

A description of exceptional curves on a DPN surface Y can also be reduced to the K3 surface X with a non-symplectic involution θ considered above.

Let (X, θ) be a K3 surface with a non-symplectic involution and $(Y = X/\{1, \theta\}, C = \pi(X^\theta))$ the corresponding DPN pair. If $E \subset Y$ is an exceptional curve, then the curve $F = \pi^*(E)_{red}$ is either an irreducible curve with negative square on the K3 surface X, or $F = F_1 + \theta(F_1)$, where

$$F^2 = F_1^2 + \theta(F_1)^2 + 2F_1 \cdot \theta(F_1) = 2E^2 < 0.$$

The curves F_1 and $\theta(F_1)$ are irreducible and have negative square (i.e. equal to (-2), see Section 2.2). Using this, in an obvious way we get the following four possibilities for E and F (see Fig. 2)

- I $E^2 = -4$, E is a component of C; respectively F is a component of X^θ, and $F^2 = -2$.
- IIa $E^2 = -1$, $E \cdot C = 2$ and E intersects C transversally at two points; respectively F is irreducible and $F^2 = -2$.
- IIb $E^2 = -1$, $E \cdot C = 2$ and E is tangent to C at one point; respectively $F = F_1 + \theta(F_1)$, where $(F_1)^2 = -2$ and $F_1 \cdot \theta(F_1) = 1$.
- III $E^2 = -2$, $E \cap C = \emptyset$; respectively $F = F_1 + \theta(F_1)$, where $(F_1)^2 = -2$ and $F_1 \cdot \theta(F_1) = 0$.

If Y is an Enriques surface, we let S_Y be the Picard lattice of Y modulo torsion. Let $P(Y) \subset S_Y$ be the subset of divisor classes of all exceptional curves E on Y, and $P(X)_+ \subset S = (S_X)_+$ be the set of divisor classes of all $F = \pi^*(E)_{red}$. We call them **exceptional classes of the pair (X, θ).**

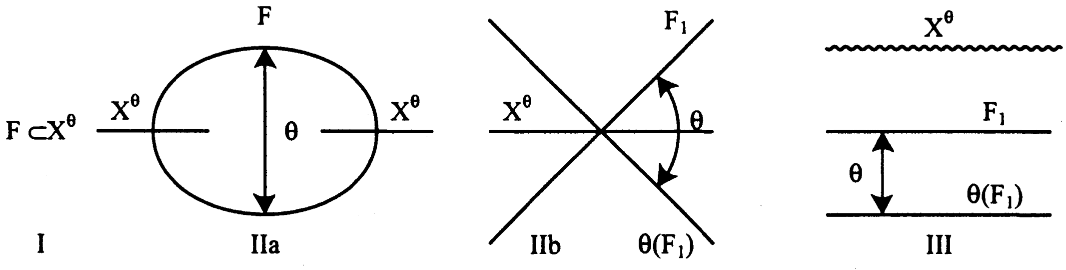


FIGURE 2. Pictures of exceptional curves.

By what we said above, $P(Y)$ and $P(X)_+$ are divided into subsets:

$$(39)\quad P(Y) = P(Y)_I \bigsqcup P(Y)_{IIa} \bigsqcup P(Y)_{IIb} \bigsqcup P(Y)_{III}.$$

$$(40)\quad P(X)_+ = P(X)_{+I} \bigsqcup P(X)_{+IIa} \bigsqcup P(X)_{+IIb} \bigsqcup P(X)_{+III}.$$

By projection formula,

$$\pi^*(\mathrm{NEF}(Y)) = \mathrm{NEF}(X) \cap (S \otimes \mathbb{R}) = \mathrm{NEF}(X)_+ .$$

In the same way as for K3 surfaces X in Section 2.2, we have

$$(41)\qquad \mathrm{NEF}(Y) = \{y \in \overline{V^+(Y)} \mid y \cdot P(Y) \geq 0\},$$

a locally finite polyhedron in $V^+(Y)$ whose facets are orthogonal and numerated by elements of $P(Y)$. Since $\pi^*(\mathrm{NEF}(Y)) = \mathrm{NEF}(X) \cap (S \otimes \mathbb{R}) = \mathrm{NEF}(X)_+$, we obtain that

$$(42)\quad \mathrm{NEF}(X)_+ = \mathrm{NEF}(X) \cap (S \otimes \mathbb{R}) = \{x \in \overline{V^+(S)} \mid x \cdot P(X)_+ \geq 0\}$$

is a locally finite polyhedron whose facets are orthogonal and numerated by elements of $P(X)_+$. Here $V^+(S) = V^+(X) \cap (S \otimes \mathbb{R})$.

As for K3 surfaces in Section 2.2, we can interpret the above results using hyperbolic spaces. Since the lattice $S = (S_X)_+$ is hyperbolic, and $S \subset S_X$, we have embeddings of cones

$$V(S) \subset V(S_X), \qquad V^+(S) \subset V^+(S_X) = V^+(X)$$

and hyperbolic spaces $\mathcal{L}(S) = V^+(S)/\mathbb{R}^+ \subset V^+(X)/\mathbb{R}^+ = \mathcal{L}(X)$.

If h is a polarization of X, the set $\mathcal{L}(S)$ contains the polarization ray $\mathbb{R}^+(h + \theta^* h)$ of X. Therefore, $\mathcal{M}(X)_+ = \mathrm{NEF}(X)_+/\mathbb{R}^+ = \mathcal{M}(X) \cap \mathcal{L}(S)$ is a non-degenerate (i.e. containing a nonempty open subset of $\mathcal{L}(S)$) convex locally finite polytope in $\mathcal{L}(S)$. Since S is even, it is easy to see that $P(X)_+$ is precisely the set of primitive elements of S, perpendicular to facets of $\mathcal{M}(X)_+$ and directed outward. One has

$$P^{(2)}(X)_+ \overset{\text{Def}}{=} \{f \in P(X)_+ \mid f^2 = -2\} \tag{43}$$

$$= P(X)_{+I} \bigsqcup P(X)_{+IIa} \bigsqcup P(X)_{+IIb}, \tag{44}$$

$$P^{(4)}(X)_+ \overset{\text{Def}}{=} \{f \in P(X)_+ \mid f^2 = -4\} = P(X)_{+III}. \tag{45}$$

Moreover, $\mathcal{M}(X)_+$, like $\mathcal{M}(X)$ for K3 surfaces in Section 2.2, admits a description in terms of groups.

Indeed, by Section 2.2

$$\mathcal{M}(X)_+ = \{\mathbb{R}^+ x \in \mathcal{L}(S) \mid x \cdot f \geq 0\} \tag{46}$$

for any effective $f \in S_X$ with $f^2 = -2$. Let us write $f = f_+^* + f_-^*$ where $f_+^* \in S^*$ and $f_-^* \in (S_X)_-^*$. We have $2f_+^* = f + \theta^*(f) \in S$ and $2f_- = f - \theta^*(f) \in (S_X)_-$. It follows that $f = (f_+ + f_-)/2$ where $f_+ \in S$ and $f_- \in (S_X)_-$. If $f_+^2 \geq 0$, then $f_+ \cdot V^+(S) \geq 0$, and f does nothing in (46). Thus, in (46) we can assume that $f_+^2 < 0$. Since $(S_X)_-$ is negative definite and the lattice S_X is even, we then obtain that $f_+ \in \Delta_+^{(2)} \cup \Delta_+^{(4)}$, defined below.

Let

$$\begin{aligned}
\Delta_\pm^{(4)} &= \{f_\pm \in (S_X)_\pm \mid f_\pm^2 = -4, \text{ and } \exists f_\mp \in (S_X)_\mp, \\
&\qquad \text{for which } f_\mp^2 = -4 \text{ and } (f_\pm + f_\mp)/2 \in S_X\}; \\
\Delta_+^{(2)} &= \Delta^{(2)}(S) = \{f_+ \in S \mid f_+^2 = -2\}; \\
\Delta_{+t}^{(2)} &= \{f_+ \in \Delta^{(2)}(S) \mid \exists f_- \in (S_X)_-, \\
&\qquad \text{for which } f_-^2 = -6 \text{ and } (f_+ + f_-)/2 \in S_X\}; \\
\Delta_-^{(6)} &= \{f_- \in (S_X)_- \mid f_-^2 = -6 \text{ and } \exists f_+ \in \Delta_{+t}^{(2)}, \\
&\qquad \text{for which } (f_+ + f_-)/2 \in S_X\}.
\end{aligned}$$

If $f_\pm \in \Delta_\pm^{(4)}$, then $f_\pm \cdot (S_X)_\pm \equiv 0 \mod 2$. Hence, $f_\pm \in \Delta_\pm^{(4)}$ are roots of $(S_X)_\pm$, and there exists a reflection $s_{f_\pm} \in O'((S_X)_\pm)$ with respect to element $f_\pm$:

$$s_{f_\pm}(x) = x + \frac{(x \cdot f_\pm)}{2} f_\pm, \quad x \in (S_X)_\pm.$$

One has a very important property:

$$s_{f_\pm}(\Delta_\pm^{(2)} \cup \Delta_\pm^{(4)}) = \Delta_\pm^{(2)} \cup \Delta_\pm^{(4)} \ \forall f_\pm \in \Delta_\pm^{(2)} \cup \Delta_\pm^{(4)} \tag{47}$$

where we formally put $\Delta_-^{(2)} = \emptyset$ because the lattice $(S_X)_-$ has no elements f_- with $f_-^2 = -2$ (see the previous section).

Let us prove (47). Assume $f_+ \in \Delta_+^{(2)}$. The reflection $s_{f_+} \in O(S_X)$ and $s_{f_+}((S_X)_\pm) = (S_X)_\pm$. This implies (47) for such s_{f_+}. Assume $f_+ \in \Delta_+^{(4)}$. Then there exists $f_- \in \Delta_-^{(4)}$ such that $\alpha_1 = (f_+ + f_-)/2 \in S_X$. The element $\alpha_2 = (f_+ + f_-)/2 - f_- = (f_+ - f_-)/2$ also belongs to S_X. We have $\alpha_1^2 = \alpha_2^2 = -2$. Thus, the reflections s_{α_1} and s_{α_2} belong to $O(S_X)$. It follows that $s = s_{\alpha_2}s_{\alpha_1} \in O(S_X)$. On the other hand, a simple calculation shows that $s((S_X)_\pm) = (S_X)_\pm$, and s in $(S_X)_\pm$ coincides with the reflection $s_{f_\pm}$. It follows that

$$s_{f_+}(\Delta_+^{(2)} \cup \Delta_+^{(4)}) = s(\Delta_+^{(2)} \cup \Delta_+^{(4)}) = \Delta_+^{(2)} \cup \Delta_+^{(4)}.$$

For $f_- \in \Delta_-^{(2)} \cup \Delta_-^{(4)}$ the arguments are the same. This simple but very important trick had been first used by Dolgachev [Dol84] for Enriques surfaces.

By (47), reflections with respect to all the elements of $\Delta^{(2)}(S) \cup \Delta_+^{(4)} = \Delta_+^{(2,4)}(S)$ generate a group $W_+^{(2,4)} \subset O'(S)$ which geometrically is the group generated by reflections in the hyperplanes of $\mathcal{L}(S)$ which are orthogonal to $\Delta_+^{(2,4)}(S)$, any reflection in a hyperplane of $\mathcal{L}(S)$ from this group is reflection in an element of $\Delta_+^{(2,4)}(S)$. It follows (by exactly the same considerations as for the K3 surface X in Section 2.2) that $\mathcal{M}(X)_+$ is a fundamental chamber for $W_+^{(2,4)}$. Thus,

$$P(X)_+ = P(\mathcal{M}(X)_+)$$

is the set of primitive elements of S, which are orthogonal to facets of $\mathcal{M}(X)_+$ and directed outward. We obtain the description of $P(X)_+$ and $P(Y)$ using the reflection group $W_+^{(2,4)}$.

Denote by

$$A(X,\theta) = \{\phi \in O'(S) \mid \phi(\mathcal{M}(X)_+) = \mathcal{M}(X)_+\}$$

the subgroup of automorphisms of $\mathcal{M}(X)_+$ in $O'(S)$ and by $\mathrm{Aut}(X,\theta)$ the normalizer of the involution θ in $\mathrm{Aut}\,X$. The action of $\mathrm{Aut}(X,\theta)$ on S_X and S defines a contravariant representation

$$\mathrm{Aut}(X,\theta) \to A(X,\theta). \tag{48}$$

Like for K3 surfaces X in Section 2.2, Global Torelli theorem for K3 surfaces [PS-Sh71] implies that this representation has a finite kernel and cokernel. Therefore, it defines an isomorphism up to finite groups: $\mathrm{Aut}(X,\theta) \approx A(X,\theta)$.

2.4.1. Computing $P(X)_+$

First, we consider calculation of the fundamental chamber $\mathcal{M}^{(2)} \subset \mathcal{L}(S)$ of $W^{(2)}(S)$.

For that, it is important to consider a larger group $W^{(2,4)}(S)$ generated by reflections in all elements of $\Delta^{(2)}(S)$ and all elements of

$$\Delta^{(4)}(S) = \{f \in S \mid f^2 = -4 \text{ and } f \cdot S \equiv 0 \mod 2\} \tag{49}$$

of all roots with square (-4) of the lattice S. Both sets $\Delta^{(2)}(S)$ and $\Delta^{(4)}(S)$ are invariant with respect to $W^{(2,4)}(S)$. It follows that every reflection from $W^{(2,4)}(S)$ gives a hyperplane $\mathcal{H}_f$ where $f \in \Delta^{(2)}(S) \cup \Delta^{(4)}(S)$. The subgroup $W^{(2)}(S) \lhd W^{(2,4)}(S)$ is normal, and any reflection from $W^{(2)}(S)$ is reflection in an element of $\Delta^{(2)}(S)$. Similarly, the subgroup $W^{(4)}(S) \lhd W^{(2,4)}(S)$, generated by reflections in $\Delta^{(4)}(S)$, is normal and any reflection from $W^{(4)}(S)$ is reflection in an element of $\Delta^{(4)}(S)$.

This implies the following description of a fundamental chamber $\mathcal{M}^{(2)}$ of $W^{(2)}(S)$. Let $\mathcal{M}^{(2,4)} \subset \mathcal{L}(S)$ be a fundamental chamber of $W^{(2,4)}(S)$. It will be extremely important for our further considerations. Let $P^{(2)}(\mathcal{M}^{(2,4)})$ and $P^{(4)}(\mathcal{M}^{(2,4)})$ be elements of $\Delta^{(2)}(S)$ and $\Delta^{(4)}(S)$ respectively directed outwards and orthogonal to $\mathcal{M}^{(2,4)}$ (i. e. to facets of $\mathcal{M}^{(2,4)}$).

Proposition 2.2. *Let $W^{(4)}(\mathcal{M}^{(2)})$ be the group generated by reflections in all elements of $P^{(4)}(\mathcal{M}^{(2,4)})$.*

Then the fundamental chamber $\mathcal{M}^{(2)}$ of $W^{(2)}(S)$ containing $\mathcal{M}^{(2,4)}$ is equal to

$$\mathcal{M}^{(2)} = W^{(4)}(\mathcal{M}^{(2)})(\mathcal{M}^{(2,4)}),$$
$$P(\mathcal{M}^{(2)}) = W^{(4)}(\mathcal{M}^{(2)})(P^{(2)}(\mathcal{M}^{(2,4)})).$$

Moreover,

$$W^{(4)}(\mathcal{M}^{(2)}) = \{w \in W^{(4)}(S) \mid w(\mathcal{M}^{(2)}) = \mathcal{M}^{(2)}\}.$$

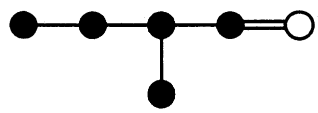
Reflections which are contained in $W^{(4)}(\mathcal{M}^{(2)})$ are exactly the reflections in elements

$$\Delta^{(4)}(\mathcal{M}^{(2)}) = \{f \in \Delta^{(4)}(S) \mid \mathcal{H}_f \text{ intersects the interior of } \mathcal{M}^{(2)}\}.$$

Proof. This easily follows from the facts that $W^{(2)}(S) \lhd W^{(2,4)}(S)$ and $W^{(4)}(S) \lhd W^{(2,4)}(S)$ are normal subgroups, and $\Delta^{(2)}(S) \cap \Delta^{(4)}(S) = \emptyset$. We leave details to the reader. □

Example 1. Let us consider the hyperbolic 2-elementary lattice $S = \langle 2\rangle \oplus 5\langle -2\rangle$ with the invariants $(r, a, \delta) = (6, 6, 1)$. Here and in what follows we denote by $\langle A\rangle$ the integral lattice given by an integral symmetric matrix A in some its basis. We denote by $\oplus$ the orthogonal sum of lattices.

The Dynkin diagram of $W^{(2,4)}(S)$ (which is equivalent to the Gram matrix of all elements of $P(\mathcal{M}^{(2,4)})$) is



(see [Vin85]). Here black vertices correspond to elements from $P^{(4)}(\mathcal{M}^{(2,4)})$ and white vertices correspond to elements from $P^{(2)}(\mathcal{M}^{(2,4)})$ (see Section 3.1 below about edges). From the diagram, one can see that $W^{(4)}(\mathcal{M}^{(2)})$ is the Weyl group of the root system D_5, the $\Delta^{(4)}(\mathcal{M}^{(2)})$ is the root system D_5, the set $P(\mathcal{M}^{(2)}) = P^{(2)}(\mathcal{M}^{(2)})$ is the orbit of the Weyl group of D_5 applied to the unique element of $P^{(2)}(\mathcal{M}^{(2,4)})$ which corresponds to the white vertex. Calculations show that $P(\mathcal{M}^{(2)})$ consists of 16 elements, and it is not easy to draw their Dynkin (or Gram) diagram, but it is completely defined by the diagram above which has only 6 vertices.

Now let us consider a subset $\Delta_+^{(4)} \subset \Delta^{(4)}(S)$ and the subgroup $W_+^{(2,4)}$ of reflections generated by this subset and by the set $\Delta^{(2)}(S)$. As in our case (47), we shall assume that the set $\Delta_+^{(4)}$ is $W_+^{(2,4)}$-invariant. Then each reflection from $W_+^{(2,4)}$ is a reflection in a hyperplane $\mathcal{H}_f$, $f \in \Delta^{(2)}(S) \cup \Delta_+^{(4)}$. As before, $W^{(2)}(S) \lhd W_+^{(2,4)}$ is a normal subgroup. We denote by $W_+^{(4)}$ the group generated by reflections in $\Delta_+^{(4)}$, it is normal in $W_+^{(2,4)}$ as well. Thus, for a fundamental chamber $\mathcal{M}_+^{(2,4)} \subset \mathcal{M}^{(2)}$ of $W_+^{(2,4)}$ we can similarly define $P^{(4)}(\mathcal{M}_+^{(2,4)})$, $P^{(2)}(\mathcal{M}_+^{(2,4)})$ (which are the sets of all elements in $\Delta_+^{(4)}$ and $\Delta^{(2)}(S)$ respectively which are orthogonal to $\mathcal{M}_+^{(2,4)}$), the group $W_+^{(4)}(\mathcal{M}^{(2)})$ generated by reflections in $P^{(4)}(\mathcal{M}_+^{(2,4)})$, the set

$$\Delta_+^{(4)}(\mathcal{M}^{(2)}) = W_+^{(4)}(\mathcal{M}^{(2)})\left(P^{(4)}(\mathcal{M}_+^{(2,4)})\right).$$

We get similar statements to Proposition 2.2:

$$W_+^{(4)}(\mathcal{M}^{(2)}) = \{w \in W_+^{(4)} \mid w(\mathcal{M}^{(2)}) = \mathcal{M}^{(2)}\}, \tag{50}$$

$$\Delta_+^{(4)}(\mathcal{M}^{(2)}) = \{f \in \Delta_+^{(4)} \mid \mathcal{H}_f \text{ intersects the interior of } \mathcal{M}^{(2)}\}, \tag{51}$$

the group $W_+^{(4)}(\mathcal{M}^{(2)})$ is generated by reflections in the $\Delta_+^{(4)}(\mathcal{M}^{(2)})$, and every reflection from $W_+^{(4)}(\mathcal{M}^{(2)})$ is a reflection in a hyperplane $\mathcal{H}_f$, $f \in \Delta_+^{(4)}(\mathcal{M}^{(2)})$.

Obviously, the fundamental chamber $\mathcal{M}_+^{(2,4)} \subset \mathcal{M}^{(2)}$ for $W_+^{(2,4)}$ is the fundamental chamber of the group $W_+^{(4)}(\mathcal{M}^{(2)})$ considered as a group acting on $\mathcal{M}^{(2)}$.

Let us show how one can calculate a fundamental chamber $\mathcal{M}_+^{(2,4)}$ of $W_+^{(2,4)}$ contained in the fixed fundamental chamber $\mathcal{M}^{(2)}$ of $W^{(2)}(S)$.

Proposition 2.3. *We have:*

$$P(\mathcal{M}^{(2)}) = W_+^{(4)}(\mathcal{M}^{(2)})(P^{(2)}(\mathcal{M}_+^{(2,4)}))$$

and

$$P^{(2)}(\mathcal{M}_+^{(2,4)}) = \{f \in P(\mathcal{M}^{(2)}) \mid f \cdot P^{(4)}(\mathcal{M}_+^{(2,4)}) \geq 0\}.$$

Proof. The first statement is analogous to Proposition 2.2. We denote the right hand side of the proving second equality as $P^{(2)}$. Since $P^{(2)} \subset P(\mathcal{M}^{(2)})$ and $\mathcal{M}^{(2)}$ has acute angles, $f \cdot f' \geq 0$ for any two different elements $f, f' \in P^{(2)} \cup P^{(4)}(\mathcal{M}_+^{(2,4)})$. It follows that $P(\mathcal{M}_+^{(2,4)}) \subset P^{(2)} \cup P^{(4)}(\mathcal{M}_+^{(2,4)})$ because the fundamental chamber $\mathcal{M}_+^{(2,4)}$ must have acute angles. Then

$$\bigcap_{f \in P^{(2)} \cup P^{(4)}(\mathcal{M}_+^{(2,4)})} \mathcal{H}_f^+ \subset \mathcal{M}_+^{(2,4)}$$

where the left hand side is not empty. Indeed, it contains the non-empty subset

$$\bigcap_{f \in P(\mathcal{M}^{(2)}) \cup P^{(4)}(\mathcal{M}_+^{(2,4)})} \mathcal{H}_f^+ = \mathcal{M}^{(2)} \bigcap \left(\bigcap_{f \in P^{(4)}(\mathcal{M}_+^{(2,4)})} \mathcal{H}_f^+ \right) \supset \mathcal{M}_+^{(2,4)}.$$

It follows (see Proposition 3.1 in [Vin85]) that

$$P(\mathcal{M}_+^{(2,4)}) = P^{(2)} \cup P^{(4)}(\mathcal{M}_+^{(2,4)})$$

because for all $f \neq f' \in P^{(2)} \cup P^{(4)}(\mathcal{M}_+^{(2,4)})$ we have $f \cdot f' \geq 0$. □

Propositions 2.2 and 2.3 imply the result which will be very important in further considerations.

Theorem 2.4. *Let $\mathcal{M}^{(2,4)}$ be a fundamental chamber of $W^{(2,4)}(S)$ in $\mathcal{L}(S)$, and $W^{(4)}(\mathcal{M}^{(2)})$ the group generated by reflections in all elements of $P^{(4)}(\mathcal{M}^{(2,4)})$, and $\Delta^{(4)}(\mathcal{M}^{(2)}) = W^{(4)}(\mathcal{M}^{(2)})(P^{(4)}(\mathcal{M}^{(2,4)}))$.*

Then

(1) Subsets $\Delta_+^{(4)} \subset \Delta^{(4)}(S)$ which are invariant for the group $W_+^{(2,4)}$ generated by reflections in all elements of $\Delta^{(2)}(S) \cup \Delta_+^{(4)}$ are in one-to-one

correspondence with subsets $\Delta_+^{(4)}(\mathcal{M}^{(2)}) \subset \Delta^{(4)}(\mathcal{M}^{(2)})$ *which are invariant for the group* $W_+^{(4)}(\mathcal{M}^{(2)})$ *generated by reflections in all elements of* $\Delta_+^{(4)}(\mathcal{M}^{(2)})$*. This correspondence is given by*

$$\Delta_+^{(4)}(\mathcal{M}^{(2)}) = \Delta_+^{(4)} \cap \Delta^{(4)}(\mathcal{M}^{(2)});\ \ \Delta_+^{(4)} = W^{(2)}(S)(\Delta_+^{(4)}(\mathcal{M}^{(2)})).$$

(2) The fundamental chamber $\mathcal{M}^{(2)}$ *of* $W^{(2)}(S)$ *containing* $\mathcal{M}_+^{(2,4)}$ *is* $\mathcal{M}^{(2)} = W_+^{(4)}(\mathcal{M}^{(2)})(\mathcal{M}_+^{(2,4)})$*. Moreover,*

$$P(\mathcal{M}^{(2)}) = W_+^{(4)}(\mathcal{M}^{(2)})(P^{(2)}(\mathcal{M}_+^{(2,4)})).$$

(3) Under the one-to-one correspondence in (1), any fundamental chamber $\mathcal{M}_+^{(2,4)} \subset \mathcal{M}^{(2)}$ *of* $W_+^{(2,4)}$ *can be obtained as follows: Let* $\mathcal{M}_+^{(4)}(\mathcal{M}^{(2)})$ *be a fundamental chamber for* $W_+^{(4)}(\mathcal{M}^{(2)})$*. Then*

$$\mathcal{M}_+^{(2,4)} = \mathcal{M}^{(2)} \cap \mathcal{M}_+^{(4)}(\mathcal{M}^{(2)}) \text{ and } P^{(4)}(\mathcal{M}_+^{(2,4)}) = P^{(4)}(\mathcal{M}_+^{(4)}(\mathcal{M}^{(2)})),$$

$$P^{(2)}(\mathcal{M}_+^{(2,4)}) = \{f \in W_+^{(4)}(\mathcal{M}^{(2)})(P^{(2)}(\mathcal{M}_+^{(2,4)})) \mid f \cdot P^{(4)}(\mathcal{M}_+^{(2,4)}) \geq 0\}.$$

Proof. Only the statement (1) requires some clarification. Assume that $\Delta_+^{(4)}(\mathcal{M}^{(2)}) \subset \Delta^{(4)}(\mathcal{M}^{(2)})$ is invariant with respect to the subgroup $W_+^{(4)}(\mathcal{M}^{(2)})$ generated by reflections in $\Delta_+^{(4)}(\mathcal{M}^{(2)})$. Note that $\mathcal{M}^{(2)}$ is invariant with respect to $W_+^{(4)}(\mathcal{M}^{(2)})$. It follows that the fundamental chamber $\mathcal{M}_+^{(2,4)}$ for $W_+^{(4)}(\mathcal{M}^{(2)})$ acting on $\mathcal{M}^{(2)}$ will be the fundamental chamber for the group $W_+^{(2,4)}$ generated by reflections in all elements of $\Delta^{(2)}(S)$ and $\Delta_+^{(4)}(\mathcal{M}^{(2)})$. It follows that $\Delta_+^{(4)} = W^{(2)}(S)(\Delta_+^{(4)}(\mathcal{M}^{(2)}))$ is invariant with respect to $W_+^{(2,4)}$. It follows that $\Delta_+^{(4)}(\mathcal{M}^{(2)}) = \Delta_+^{(4)} \cap \Delta^{(4)}(\mathcal{M}^{(2)})$.

The remaining statements are obvious. □

In Chapter 3 we apply this theorem to describe DPN surfaces of elliptic type.

2.5. The root invariant of a pair (X, θ)

To describe the group $W_+^{(2,4)}$ and sets $P(X)_{+III}$, and $P(X)_{+IIa}$, $P(X)_{+IIb}$, one should add to the main invariants (r, a, δ) (equivalently (k, g, δ)) of (X, θ) the so-called root invariants. We describe them below. The root invariants for DPN surfaces had been introduced and considered in [Nik84a] and [Nik87].

Everywhere below we follow Appendix, Section A.1 about lattices and discriminant forms of lattices.

Let M be a lattice (i. e. a non-degenerate integral symmetric bilinear form). Following [Nik80b], $M(k)$ denotes a lattice obtained from M by multiplying the form of M by $k \in \mathbb{Q}$.

Let $K(2)$ be the sublattice of $(S_X)_-$ generated by $\Delta_-^{(4)} \subset (S_X)_-$. Since $\Delta_-^{(4)} \cdot (S_X)_- \equiv 0 \mod 2$, the lattice $K = K(2)(1/2)$ is integral and is generated by its subset $\Delta_-^{(4)} \subset K$ of elements with square (-2) defining in K a root system, since reflections with respect to elements of $\Delta_-^{(4)}$ send $\Delta_-^{(4)}$ to itself. If follows that the lattice K is isomorphic to the orthogonal sum of root lattices A_n, D_m and E_k corresponding to the root systems A_n, D_m, E_k (or their Dynkin diagrams), and $\Delta_-^{(4)} = \Delta^{(2)}(K)$ is the set of all elements of K with square (-2). Equivalently, $\Delta_-^{(4)} = \Delta^{(4)}(K(2))$ is the set of all elements with square (-4) of $K(2)$. Moreover, we have a natural homomorphism of groups

$$\xi : Q = \frac{1}{2}K(2)/K(2) \to \mathfrak{A}_S = S^*/S \tag{52}$$

such that $\xi(\frac{1}{2}f_- + K(2)) = \frac{1}{2}f_+ + S$, if $f_\mp \in \Delta_\mp^{(4)}$ and $(f_- + f_+)/2 \in S_X$. This defines a homomorphism of finite quadratic forms $\xi : q_{K(2)}|Q \to -q_S$ with values in $\frac{1}{2}\mathbb{Z}/2\mathbb{Z} \subset \mathbb{Q}/2\mathbb{Z}$. (Here $q_M : \mathfrak{A}_M = M^*/M \to \mathbb{Q}/2\mathbb{Z}$ denotes the discriminant quadratic form of an even lattice M.) The homomorphism ξ is equivalent to the homomorphism (which we denote by the same letter ξ) of the finite quadratic forms

$$\xi : K \mod 2 \to -q_S, \tag{53}$$

by the natural isomorphism $\frac{1}{2}K(2)/K(2) \cong K/2K$, where $q_{K(2)}|Q$ is replaced by the finite quadratic form $\frac{1}{2}x^2 \mod 2$ for $x \in K$.

We define the **root invariant** of the pair (X, θ) or the corresponding DPN pair (Y, C) as the equivalence class of the triplet

$$R(X,\theta) = (K(2), \Delta_-^{(4)}, \xi) \cong (K(2), K(2)^{(4)}, \xi) \cong (K, \Delta^{(2)}(K), \xi), \tag{54}$$

up to isomorphisms of lattices K and automorphisms of the lattice S. Clearly, similarly we can introduce abstract root invariants, without any relation to K3 surfaces with involutions and DPN pairs; see beginning of Section 2.7 below.

We have the following statement from [Nik80b].

Lemma 2.5. *Let S be an even hyperbolic 2-elementary lattice.*
Then the natural homomorphism $O(S) \to O(q_S)$ is surjective.

Proof. We remind the proof from [Nik80b]. If $\operatorname{rk} S \geq 3$, this follows from the general theorem 1.14.2 in [Nik80b]. If $\operatorname{rk} S = 2$, then $S \cong U =$

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $U(2)$, $\langle 2\rangle \oplus \langle -2\rangle$. If $\operatorname{rk} S = 1$, then $S = \langle 2\rangle$. For all these lattices one can check the statement directly. See Appendix, Theorems A.7, A.9 for more details. □

Lemma 2.6. *Let S be an even hyperbolic 2-elementary lattice and $\operatorname{rk} S \geq 2$. Then every $x \in \mathfrak{A}_S$ with $q_S(x) = n/2 \mod 2$, $n \in \mathbb{Z}$, can be represented as $x = u/2 \mod S$ where $u \in S$ and $u^2 = 2n$.*

Proof. If $\operatorname{rk} S = 2$, then $S \cong U$, $U(2)$, $\langle 2\rangle \oplus \langle -2\rangle$, and the statement can be checked directly. Assume that the statement is valid for $\operatorname{rk} S \leq k$ where $k \geq 2$. Let $\operatorname{rk} S = k + 1$. By Theorem 1.12.2 from [Nik80b] about existence of an even lattice with a given discriminant quadratic form (see Appendix, Theorem A.1), we get that $S = S' \oplus T$ where S' is a hyperbolic 2-elementary lattice of the rank $\operatorname{rk} S' \geq 2$, and T is a negative definite 2-elementary lattice of the $\operatorname{rk} T \geq 1$. Let $x = y \oplus z, x \in \mathfrak{A}_S$, $y \in \mathfrak{A}_T$, and assume $z = u/2 \mod T$ where $u \in T$ and $u^2 = 2m$, $m \in \mathbb{Z}$. By the induction assumption, there exists $v \in S'$ with $y = v/2 \mod S'$ and $v^2 = 2n - 2m$ since $q_{S'}(y) = q(x) - q(z) = (n - m)/2 \mod 2$. □

Lemma 2.7. *Let $q : A \to \mathbb{Q}/2\mathbb{Z}$ be a non-degenerate quadratic form on a finite 2-elementary group A and $\phi : H_1 \cong H_2$ be an isomorphism of two subgroups in A which preserves $q|H_1$ and $q|H_2$. Assume that the characteristic element a_q of q on A either does not belong to both subgroups H_1 and H_2 or belongs to both of them. In the second case we additionally assume that $\phi(a_q) = a_q$.*

Then ϕ can be extended to an automorphism of q.

Proof. See Proposition 1.9.1 in [Nik84b] (we repeated the proof in Appendix, Proposition A.11). We remind that $a_q \in A$ is the **characteristic element** of q, if $q(x) \equiv b_q(x, a_q) \mod 1$ for any $x \in A$. Here b_q is the bilinear form of q. This defines the characteristic element a_q uniquely. □

Lemmas 2.5 — 2.7 imply

Proposition 2.8. *The root invariant $R(X,\theta)$ of (X,θ) (or (Y,C)) is equivalent to the triplet*

$$R(X,\theta) = (K(2); H; \alpha, \overline{a}) \cong (K; H; \alpha, \overline{a}).$$

Here $H = Ker\,\xi$ is an isotropic for $q_{K(2)}$ subgroup in Q (equivalently in $K \mod 2$); $\alpha = 0$, if $\xi(Q) = \xi(K \mod 2)$ contains the characteristic element a_{q_S} of the quadratic form q_S, and $\alpha = 1$ otherwise; if $\alpha = 0$, the element $\overline{a} = \xi^{-1}(a_{q_S}) + H \in Q/H$; if $\alpha = 1$, the element $\overline{a}$ is not defined.

The root invariant $R(X,\theta)$ is important because it defines

$$\Delta_+^{(4)} = \{f_+ \in S \mid f_+^2 = -4,\ f_+/2 \mod S \in$$

$$\xi(\frac{1}{2}\Delta^{(4)}(K(2)) \mod K(2)) = \xi(\Delta^{(2)}(K) \mod 2K)\},$$

and $W_+^{(2,4)}$, $\mathcal{M}(X)_+$, $P(\mathcal{M}(X)_+) = P(X)_+$, up to the action of $O(S)$. Moreover, for $f_+ \in P(X)_+^{(4)}$, the root invariant defines the decomposition $f_+ = f + \theta^*(f)$, $f \in P(X)$, uniquely up to permutation of f and $\theta^*(f)$. More precisely, we have the following. Let $f_- \in \Delta^{(4)}(K(2))$ and $\xi(f_-/2 \mod K(2)) = f_+/2 \mod S$. Then $f = (f_+ \pm f_-)/2$, $\theta(f) = (f_+ \mp f_-)/2$. Indeed, if $f_-' \in \Delta^{(4)}(K(2))$ satisfies the same conditions, then $(f_- + f_-')/2 \in K(2)$. In $K(2)$, if $f_-' \neq \pm f_-$ then either $f_- \cdot f_-' = 0$ or $f_- \cdot f_-' = \pm 2$. If $f_- \cdot f_-' = 0$ then $((f_- + f_-')/2)^2 = -2$, and we get a contradiction since $(S_X)_-$ does not have elements with the square (-2). If $f_- \cdot f_-' = \pm 2$ then $f_- \cdot (f_- + f_-')/2 = -2 \pm 1$, and we get a contradiction since $f_- \cdot (S_X)_- \equiv 0 \mod 2$. Thus, $f_-' = \pm f_-$, and the pair of elements f and $\theta^*(f)$ is defined uniquely.

Similarly one can define a **generalized root invariant**

$$R_{gen}(X,\theta) = (K_{gen}(2),\ \Delta_-^{(4)} \cup \Delta_-^{(6)},\ \xi_{gen}) \cong$$

$$(K_{gen}, \Delta_-^{(2)} \cup \Delta_-^{(3)}, \xi_{gen}),$$

where for $f_- \in \Delta_-^{(6)}$ one has

$$\xi_{gen}(f_-/2 \mod K_{gen}(2)) = f_+/2 \mod S$$

where $f_+ \in \Delta_{+t}^{(2)}(S)$ and $(f_+ + f_-)/2 \in S_X$. Here $K_{gen}(2) \subset (S_X)_-$ is generated by $\Delta_-^{(4)} \cup \Delta_-^{(6)}$.

Using Lemmas 2.5 — 2.7, one can similarly prove that it is equivalent to the tuple

$$R_{gen}(X,\theta) = (K_{gen}(2), \Delta_-^{(4)} \cup \Delta_-^{(6)};\ H_{gen};\ \alpha_{gen},\ \overline{a}_{gen}) \cong \tag{55}$$

$$(K_{gen}, \Delta_-^{(2)} \cup \Delta_-^{(3)};\ H_{gen};\ \alpha_{gen},\ \overline{a}_{gen}).$$

It is defined similarly to the root invariant.

Importance of the generalized root invariant is that it contains the root invariant $R(X,\theta)$. Thus, it defines $W_+^{(2,4)}$, $\mathcal{M}(X)_+$ and also $P(\mathcal{M}(X)_+) = P(X)_+$, up to the action of $O(S)$. But, it also defines

$$\Delta_{+t}^{(2)} = \{f_+ \in S \mid (f_+)^2 = -2,\ f_+/2 \mod S \in$$

$$\xi_{gen}(\frac{1}{2}\Delta_-^{(6)} \mod K_{gen}(2)) = \xi(\Delta_-^{(3)} \mod 2K_{gen})\},$$

and then it defines

$$P(X)_{+IIb} = P^{(2)}(X)_{+t} = \{f_+ \in P(X)_+ \mid f_+ \in \Delta^{(2)}_{+t}\}.$$

Thus, using root invariants, we know how to find $P(X)_{+III} = P^{(4)}(X)_+$, $P(X)_{+IIb} = P^{(2)}(X)_{+t}$, and hence, we know $P(X)_{+I} \cup P(X)_{+IIa} = P^{(2)}(X)_+ - P^{(2)}(X)_{+t}$. To distinguish $P(X)_{+I}$ and $P(X)_{+IIa}$, it is sufficient to know $P(X)_{+I}$.

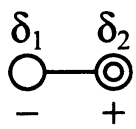
2.6. Finding the locus X^θ

Here we show how one can find $P(X)_{+I}$. This is based on the following considerations (similar to [Nik83]):

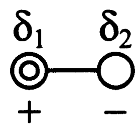
1) Since $W^{(2)}(S) \lhd W_+^{(2,4)}$ is a normal subgroup, the fundamental chamber $\mathcal{M}_+^{(2,4)}$ is contained in one fundamental chamber $\mathcal{M}^{(2)}$ of $W^{(2)}(S)$; we have $\mathcal{M}_+^{(2,4)} \subset \mathcal{M}^{(2)}$. One can consider replacing $\mathcal{M}_+^{(2,4)}$ by $\mathcal{M}^{(2)}$ as a deformation of a pair (X, θ) to a general pair $(\widetilde{X}, \widetilde{\theta})$ having $S_{\widetilde{X}} = S$, $\mathcal{M}(\widetilde{X}) = \mathcal{M}^{(2)}$ and $P(\widetilde{X}) = P(\mathcal{M}^{(2)})$. See Section 2.3 on corresponding results about moduli.

The divisor classes of fixed points of the involution do not change under this deformation, thus $P(X)_{+I} = P(\widetilde{X})_{+I}$. In particular, $P(X)_{+I}$ does not change when a root invariant changes (with fixed main invariants (r, a, δ) equivalent to the lattice S).

2) Let δ_1, δ_2 belong to $P^{(2)}(X)_+$ and $\delta_1 \cdot \delta_2 = 1$, i. e. the curves D_1, D_2 corresponding to them intersect transversally. Then one of δ_1, δ_2 belongs to $P(X)_{+I}$, and another to $P(X)_{+II} = P(X)_{+IIa} \cup P(X)_{+IIb} = P(X)_{+IIa}$ for the general case we consider. See the diagrams below where an element of $P(X)_{+I}$ is denoted by a double transparent vertex, and an element of $P(X)_{+II}$ by a single transparent vertex.

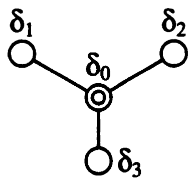


or



Indeed, the intersection point $D_1 \cap D_2$ is a fixed point of θ, tangent directions of D_1 and D_2 at this point are eigenvectors of θ_*. We know that they have eigenvalues $+1$ and -1.

3) If the Gram diagram of elements δ_0, δ_1, δ_2 and δ_3 from $P^{(2)}(X)_+$ has the form as shown



then $\delta_0 \in P(X)_{+I}$, and δ_1, δ_2, $\delta_3 \in P(X)_{+II} = P(X)_{+IIa}$ (for the general case we consider). Indeed, the rational curve corresponding to δ_0 has three different fixed points of θ, and hence belongs to X^θ.

4) If $\delta \in P(X)_{+III} = P^{(4)}(X)_+$ and $\delta_1 \in P(X)_{+I}$, then $\delta_1 \cdot \delta = 0$. This is obvious from the definition of $P(X)_{+III}$.

Considering all possible lattices S, it is not difficult to see that statements 1) — 3) are sufficient for finding $P(X)_{+I}$ and the divisor class of the irreducible component C_g of the curve X^θ of fixed points. The statement 4) simplifies these considerations, if some elements of $P^{(4)}(X)_+$ are known.

2.7. Conditions for the existence of root invariants

Assume that the main invariants (r, a, δ) (equivalently (k, g, δ)) of (X, θ) are known and fixed. Here we want to give conditions which are necessary and sufficient for the existence of a pair (X, θ) with a given root or generalized root invariant. We consider the root invariant. Similarly, one can consider the generalized root invariant.

Assume that $(K, \Delta^{(2)}(K), \xi)$ is the root invariant of a pair (X, θ).

Then the conditions 1 and 2 below must be satisfied:

Condition 1. *The lattice*

$$K_H = [K; x/2 \textit{ where } x + 2K \in H] \tag{56}$$

does not have elements with the square (-1). *Equivalently, the lattice*

$$K_H(2) = [K(2); x/2 \textit{ where } x/2 + K(2) \in H]$$

does not have elements with square (-2). *We remind that* $H = \operatorname{Ker} \xi$.

Indeed, the lattice $K_H(2) \subset (S_X)_-$, but the lattice $(S_X)_-$ does not have elements with square (-2).

Condition 2. $\operatorname{rk} S + \operatorname{rk} K = r + \operatorname{rk} K \leq 20$.

Indeed, $S \oplus K(2) \subset S_X$ and $\operatorname{rk} S_X \leq 20$.

A pair (X, θ) (or the corresponding DPN pair (Y, C), or right DPN surface Y) is called **standard**, if $K_H(2)$ is a primitive sublattice of $(S_X)_-$, and the primitive sublattice $[S \oplus K(2)]_{\rm pr}$ in S_X generated by $S \oplus K(2)$ is defined by the homomorphism ξ, i. e. it is equal to

$$(57)\qquad \begin{aligned} M = &[S \oplus K(2); \{a + b \mid \forall a \in S^*,\ \forall b \in K(2)/2, \\ &\text{such that } \xi(b + K(2)) = a + S\}]. \end{aligned}$$

Clearly, $M \subset [S \oplus K(2)]_{\rm pr}$ is always a sublattice of finite index.

Let $l(\mathfrak{A})$ be the **minimal number of generators** of a finite Abelian group $\mathfrak{A}$. Let $\mathfrak{A}_M = M^*/M$ be the **discriminant group** of a lattice M.

Let us consider an **abstract root invariant** $(K(2), \xi)$. This means that K is a negative definite lattice generated by its elements with square (-2), and $K(2)$ is obtained by multiplying the form of K by 2. The map

$$\xi : q_{K(2)}|Q = \frac{1}{2}K(2)/K(2) \to -q_S$$

is a homomorphism of finite quadratic forms. We assume that for each $f_- \in \Delta^{(4)}(K(2))$ there exists $f_+ \in \Delta^{(4)}(S)$ such that $\xi(f_-/2 + K(2)) = f_+/2 + S$ (by Lemma 2.6, this condition is always satisfied). As above, we denote $H = \operatorname{Ker} \xi$.

Proposition 2.9. *A standard pair (X, θ) with a given root invariant (K, ξ) satisfying Conditions 1 and 2 does exist, if additionally*

$$(58)\qquad r + a + 2l(H) < 22$$

and

$$(59)\qquad r + \operatorname{rk} K + l(\mathfrak{A}_{K_p}) < 22$$

for any prime $p > 2$. Here $K_p = K \otimes \mathbb{Z}_p$, where $\mathbb{Z}_p$ is the ring of p-adic integers.

Proof. By Global Torelli Theorem [PS-Sh71] and surjectivity of Torelli Map [Kul77] for K3 (see Section 2.2), the pair (X, θ) does exist, if there exists a primitive embedding of the lattice M described in (57) into an even unimodular lattice $L_{K3} \cong H^2(X, \mathbb{Z})$ of the signature $(3, 19)$ (see the proof of Proposition 2.10 below). By Corollary 1.12.3 in [Nik80b] (see Appendix, Corollary A.6), such a primitive embedding does exist, if $\operatorname{rk} M + l(\mathfrak{A}_{M_p}) < 22$ for all prime $p \geq 2$.

If $p > 2$, then $\operatorname{rk} M + l(\mathfrak{A}_{M_p}) = r + rkK + l(\mathfrak{A}_{K_p}) < 22$ by (59) (here we recall that the lattice S is 2-elementary).

Assume that $p = 2$. Let Γ_ξ be the graph of ξ. Then $\mathfrak{A}_M = (\Gamma_\xi)^\perp/\Gamma_\xi$ for the discriminant form $q_S \oplus q_{K(2)}$. Let $Q = H \oplus Q'$ (see (52)) where Q' is a complementary subgroup, and $\xi' = \xi|Q'$. Then $\Gamma_{\xi'} \subset \Gamma_\xi$, moreover $\Gamma_{\xi'} \subset \mathfrak{A}_S \oplus \mathfrak{A}_{K(2)} = \mathfrak{A}$ is a 2-elementary subgroup, and $\Gamma_{\xi'} \cap 2\,\mathfrak{A} = \{0\}$ since $2\,\mathfrak{A} = \{0\} \oplus 2\,\mathfrak{A}_{K(2)}$ and ξ' is injective. Let $\mathfrak{A}^{(2)}$ be the kernel of multiplication by 2 in $\mathfrak{A}$, and $q = (q_S \oplus q_{K(2)})|\,\mathfrak{A}^{(2)}$. It is easy to see that the kernel $\operatorname{Ker} q = \mathfrak{A}^{(2)} \cap 2\,\mathfrak{A}$. Since $\Gamma_{\xi'} \cap 2\,\mathfrak{A} = \{0\}$, then $\Gamma_{\xi'} \cap \operatorname{Ker} q = \{0\}$. Let $\mathfrak{A}_1^{(2)}$ be a subgroup in $\mathfrak{A}^{(2)}$ which is complementary to $\operatorname{Ker} q$ and contains $\Gamma_{\xi'}$. Then $q_S \oplus q_{K(2)} = q_1 \oplus q_2$ where $q_1 = q_S \oplus q_{K(2)}|\,\mathfrak{A}_1^{(2)}$ and q_2 is the orthogonal complement to q_1 (since q_1 is non-degenerate). The subgroup $\Gamma_{\xi'}$ is isotropic for the non-degenerate 2-elementary form q_1 and has rank $\operatorname{rk} K - \operatorname{rk} H$. It follows that

$$l((\Gamma_{\xi'})^{\perp}_{q_1}/\Gamma_{\xi'}) = l(\mathfrak{A}_1^{(2)}) - 2l(\Gamma_{\xi'}),$$

and then

$$l(\mathfrak{A}_{M_2}) \le l(\mathfrak{A}) - 2l(\Gamma_{\xi'}) = a + \operatorname{rk} K - 2(\operatorname{rk} K - l(H)).$$

This implies that

$$\operatorname{rk} M + l(\mathfrak{A}_{M_2}) \le r + a + 2l(H) < 22$$

by (58). □

Finally, in general, by Global Torelli Theorem [PS-Sh71] and surjectivity of Torelli Map [Kul77] for K3 (see Section 2.2), we have the following necessary and sufficient conditions of existence of a pair (X, θ) with a root invariant $(K(2), \xi)$. It even takes into consideration the more delicate invariant which is the isomorphism class of embedding of lattices $M \subset H^2(X, \mathbb{Z}) \cong L_{K3}$.

Proposition 2.10. *There exists a K3 pair (X, θ) with a root invariant $(K(2), \xi)$ and the isomorphism class of embedding $\phi : M \subset L_{K3}$ of lattices (see (57)), if and only if*

1) $\phi(S) \subset L_{K3}$ is a primitive sublattice;

2) the primitive sublattice $\phi(K(2))_{\mathrm{pr}} \subset L_{K3}$ generated by $\phi(K(2))$ in L_{K3} does not have elements with square (-2);

3) we have:

(60)
$$\begin{aligned}\phi(\Delta^{(4)}(K(2))) =\{f_- \in \phi(K(2))_{\mathrm{pr}} \mid f_-^2 = -4,\ \textit{and}\ \exists f_+ \in S\\ \textit{such that}\ f_+^2 = -4\ \textit{and}\ (\phi(f_+) + f_-)/2 \in L_{K3}\}.\end{aligned}$$

Here, the right hand side always contains the left hand side. We remind that $\Delta^{(4)}(K(2))$ is the set of all elements in $K(2)$ with square (-4).

Proof. Using (57), we construct an even lattice M which contains $S \oplus K(2) \subset M$ as a sublattice of finite index. It contains $S \subset M$ as a primitive sublattice, and its primitive sublattice generated by $K(2)$ is $K(2)_H$ where $H = \operatorname{Ker} \xi$.

Let $\phi : M \to L_{K3}$ be an embedding of lattices. If ϕ corresponds to a K3 pair (X, θ) with the root invariant $(K(2), \xi)$, then conditions 1), 2) and

3) must be satisfied. Now we assume that they are valid for the abstract embedding $\phi : M \to L_{K3}$ of lattices we consider.

Then $\phi(S) \subset L_{K3}$ is a primitive sublattice. To simplify notation, we identify $S = \phi(S) \subset L_{K3}$ and $K(2) = \phi(K(2))$. Since S is 2-elementary, there exists an involution α on L_{K3} with $(L_{K3})_+ = S$ and $(L_{K3})_- = S^\perp$. Then $\alpha = -id$ on $K(2)$. We denote by $\widetilde{M}$ the primitive sublattice in L_{K3} generated by $\phi(M) = M$.

Assume that $f \in \widetilde{M}$ satisfies $f^2 = -2$, $f = f_-^* + f_+^*$ where $f_-^* \in (K(2)_{\text{pr}})^*$, $f_+^* \in S^*$ and $(f_+^*)^2 < 0$. Since $2f_-^* = f_- = f - \alpha(f) \in K(2)_{\text{pr}}$, $2f_+^* = f_+ = f + \alpha(f) \in S$, $K(2)$ is negative definite and satisfies 2), it follows that either $f = (f_- + f_+)/2$ where $f_- = 0$ and $f = f_+/2 \in \Delta^{(2)}(S)$, or $f = (f_- + f_+)/2$ where $f_- \in K(2)^{(4)}$, $f_+ \in \Delta^{(4)}(S)$, or $f = (f_- + f_+)/2$ where $(f_-)^2 = -6$ and $f_+ \in \Delta^{(2)}(S)$.

It follows that there exists $h_+ \in S$ with $(h_+)^2 > 0$ such that $h_+ \cdot f \neq 0$ for any $f \in \Delta^{(2)}(\widetilde{M})$.

By surjectivity of Torelli map for K3 surfaces [Kul77], we can assume that there exists a K3 surface X with $H^2(X, \mathbb{Z}) = L_{K3}$, $S_X = \widetilde{M}$ and a polarization h_+. The involution α preserves periods of X. By Global Torelli Theorem for K3 [PS-Sh71], $\alpha = \theta^*$ corresponds to an automorphism θ of X. The automorphism θ is non-symplectic because $H^2(X, \mathbb{Z})_+ = (S_X)_+ = \widetilde{M}_+ = S$ is hyperbolic. By 3), the root invariant of (X, θ) is $(K(2), \xi)$. See Sections 2.2 and 2.3 about the used results on K3 surfaces. □

We remark that from the proof above we can even describe the moduli $\mathrm{Mod}_{(S,K(2),\xi,\phi)}$ of K3 surfaces with a non-symplectic involution θ having the main invariant S, the root invariant $(K(2), \xi)$ and the embedding $\phi : M \to L_{K3}$ of the corresponding lattice M which satisfies conditions of Proposition 2.10. As in the proof we denote by $\widetilde{M} \supset M$ the overlattice of M of finite index such that $\phi(\widetilde{M}) \subset L_{K3}$ is the primitive sublattice in L_{K3} generated by $\phi(M)$.

We consider a fundamental chamber $\mathcal{M}(\widetilde{M})$ for $W^{(2)}(\widetilde{M})$ such that $\mathcal{M}(\widetilde{M}) \cap \mathcal{L}(S) \neq \emptyset$. Then $\mathcal{M}(\widetilde{M}) \cap \mathcal{L}(S)$ defines a unique $\mathcal{M}(S)$ containing $\mathcal{M}(\widetilde{M}) \cap \mathcal{L}(S)$. Up to isomorphisms of the pair $S \subset \widetilde{M}$ there exists only finite number of such $\mathcal{M}(\widetilde{M})$. We have

$$\mathrm{Mod}_{(S,K(2),\xi,\phi)} = \cup_{class\ of\ \mathcal{M}(\widetilde{M})} \mathrm{Mod}_{(S,K(2),\xi,\phi,\mathcal{M}(\widetilde{M}))} \tag{61}$$

where

$$\mathrm{Mod}_{(S,K(2),\xi,\phi,\mathcal{M}(\widetilde{M}))} \subset \mathrm{Mod}_{\phi:\widetilde{M}\subset L_{K3}} \cap \mathrm{Mod}'_{\phi:S\subset L_{K3}} \tag{62}$$

consists of K3 surfaces $(X, \widetilde{M} \subset S_X)$ with the condition $\widetilde{M}$ and $\mathcal{M}(\widetilde{M})$ on the Picard lattice and the class $\phi : \widetilde{M} \subset L_{K3}$ of the embedding on cohomology; moreover X has a non-symplectic involution θ with the main invariant S (i. e. $(X,\theta)\in \mathrm{Mod}'_{\phi:S\subset L_{K3}}$) and (X,θ) has the root invariant $(K(2),\xi)$. See (33), (35). A general such a K3 surface X has $S_X=\widetilde{M}$, and the dimension of moduli is equal to

$$\dim \mathrm{Mod}_{(S,K(2),\xi,\phi)} = 20-\mathrm{rk}\, S-\mathrm{rk}\, K(2). \tag{63}$$

Taking union over different classes of embeddings $\phi: M\subset L_{K3}$ (their number is obviously finite), we obtain the moduli space of K3 surfaces X with a non-symplectic involution θ, and the main invariant S, and the root invariant $(K(2),\xi)$.

Proposition 2.10 implies the following result important for us.

Corollary 2.11. *Let $(K(2),\xi)$ be the root invariant of a pair (X,θ) and $K'(2)\subset K(2)$ a primitive sublattice of $K(2)$ generated by its elements $\Delta^{(4)}(K'(2))$ with the square (-4).*

Then the pair $(K'(2),\xi'=\xi|Q'=\frac{1}{2}K'(2)/K'(2))$ is also the root invariant of some K3 pair (X',θ').

If the pair (X,θ) is standard, the pair (X',θ') also can be taken standard.

Corollary 2.11 shows that to describe all possible root invariants of pairs (X,θ), it is enough to describe all possible root invariants of extremal pairs. Here a *pair* (X',θ') is called **extremal**, if its root invariant $R(X',\theta')=(K'(2),\xi')$ cannot be obtained using Corollary 2.11 from the root invariant $R(X,\theta)=(K(2),\xi)$ of any other pair (X,θ) with $\mathrm{rk}\,K(2)>\mathrm{rk}\,K'(2)$.

2.8. Three types of non-symplectic involutions of K3 surfaces

It is natural to divide non-symplectic involutions (X,θ) of K3 and the corresponding DPN surfaces in three types:

Elliptic type: $X^\theta\cong C\cong C_g+E_1+\cdots+E_k$ where C_g is an irreducible curve of genus $g\ge 2$ (equivalently, $(C_g)^2>0$), and $E_1,\dots,E_k$ are non-singular irreducible rational curves. By Section 2.3, this is equivalent to $r+a\le 18$ and $(r,a,\delta)\ne(10,8,0)$. Then $\mathrm{Aut}(X,\theta)$ is finite because $(C_g)^2>0$, see [Nik79] , [Nik83] and Section 3.1 below.

Parabolic type: Either $X^\theta\cong C\cong C_1+E_1+\cdots+E_k$ (using the same notation), or $X^\theta\cong C\cong C_1^{(1)}+C_1^{(2)}$ is a union of two elliptic (i. e. of genus 1) curves. By Section 2.3, this is equivalent to either $r+a=20$ and $(r,a,\delta)\ne(10,10,0)$, or $(r,a,\delta)=(10,8,0)$. Then $\mathrm{Aut}(X,\theta)$ is Abelian up to finite index and usually non-finite, see [Nik79], [Nik83]. Here $(C_1)^2=0$.

Hyperbolic type: Either $X^\theta \cong C \cong E_0 + E_1 + \cdots + E_k$ is a union of non-singular irreducible rational curves, or $X^\theta = \emptyset$. By Section 2.3, this is equivalent to either $r + a = 22$, or $(r, a, \delta) = (10, 10, 0)$. Then $\mathrm{Aut}(X, \theta)$ is usually non-Abelian up to finite index, see [Nik79], [Nik83]. Here $C_0 = E_0$ has $C_0^2 = -2$, if $X^\theta \neq \emptyset$.

Thus, pairs (X, θ) of elliptic type are the simplest, and we describe them completely in Chapter 3. On the other hand, for classification of log del Pezzo surfaces of index ≤ 2 we need only these pairs.