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Log del Pezzo surfaces of index ≤ 2 and Smooth Divisor Theorem

1.1. Basic definitions and notation

Let Z be a normal algebraic surface, and K_Z be a canonical Weil divisor on it. The surface Z is called **$\mathbb{Q}$-Gorenstein** if a certain positive multiple of K_Z is Cartier, and **$\mathbb{Q}$-factorial** if this is true for any Weil divisor D. These properties are local: one has to require all singularities to be $\mathbb{Q}$-Gorenstein, respectively $\mathbb{Q}$-factorial.

Let us denote by $Z^1(Z)$ and $\mathrm{Div}(Z)$ the groups of Weil and Cartier divisors on Z. Assume that Z is $\mathbb{Q}$-factorial. Then the groups $Z^1(Z) \otimes \mathbb{Q}$ and $\mathrm{Div}(Z) \otimes \mathbb{Q}$ of $\mathbb{Q}$-Cartier divisors and $\mathbb{Q}$-Weil divisors coincide. The intersection form defines natural pairings

$$\mathrm{Div}(Z) \otimes \mathbb{Q} \times \mathrm{Div}(Z) \otimes \mathbb{Q} \to \mathbb{Q},$$

$$\mathrm{Div}(Z) \otimes \mathbb{R} \times \mathrm{Div}(Z) \otimes \mathbb{R} \to \mathbb{R}.$$

Quotient groups modulo kernels of these pairings are denoted $N_{\mathbb{Q}}(Z)$ and $N_{\mathbb{R}}(Z)$ respectively; if the surface Z is projective, they are finite-dimensional linear spaces. The **Kleiman–Mori cone** is a convex cone $\overline{\mathrm{NE}}(Z)$ in $N_{\mathbb{R}}(Z)$, the closure of the cone generated by the classes of effective curves.

Let D be a $\mathbb{Q}$-Cartier divisor on Z. We will say that D is ample if some positive multiple is an ample Cartier divisor in the usual sense. By *Kleiman's criterion* [Kle66], for this to hold it is necessary and sufficient that D defines a strictly positive linear function on $\overline{\mathrm{NE}}(Z) - \{0\}$.

One says that the **surface Z has only log terminal singularities** if it is $\mathbb{Q}$-Gorenstein and for one (and then any) resolution of singularities $\pi : Y \to Z$, in a natural formula $K_Y = \pi^* K_Z + \sum \alpha_i F_i$, where F_i are irreducible divisors and $\alpha_i \in \mathbb{Q}$, one has $\alpha_i > -1$. The least common multiple of denominators of α_i is called the **index** of Z.

It is known that two-dimensional log terminal singularities in characteristic zero are exactly the quotient singularities [Kaw84]. A self-contained and characteristic-free classification in terms of dual graphs of resolutions is given in [Ale92]. Log terminal singularities are rational and $\mathbb{Q}$-factorial. We can now formulate the following:

Definition 1.1. A normal complete surface Z is called a **log del Pezzo surface** if it has only log terminal singularities and the anticanonical divisor $-K_Z$ is ample. It has index $\leq k$ if all of its singularities are of index $\leq k$.

We will use the following notation. If D is a $\mathbb{Q}$-Weil divisor, $D = \sum c_i C_i$, $c_i \in \mathbb{Q}$, then $\ulcorner D \urcorner$ will denote the round-up $\sum \ulcorner c_i \urcorner D_i$, and $\{D\} = \sum \{c_i\} C_i$ the fractional part. A divisor D is **nef** if for any curve C one has $D \cdot C \geq 0$; D is **big and nef** if in addition $D^2 > 0$.

Below we will frequently use the following generalization of Kodaira's vanishing theorem. The two-dimensional case is due to Miyaoka [Miy80] and does not require the normal-crossing condition. The higher-dimensional case is due to Kawamata [Kaw82] and Viehweg [Vie82].

Theorem 1.2 (Generalized Kodaira's Vanishing theorem). *Let Y be a smooth surface and let D be a $\mathbb{Q}$-divisor on Y such that*

(1) $\operatorname{supp}\{D\}$ *is a divisor with normal crossings;*
(2) D *is big and nef.*

Then $H^i(K_Y + \ulcorner D \urcorner) = 0$ for $i > 0$.

1.2. Log terminal singularities of index 2

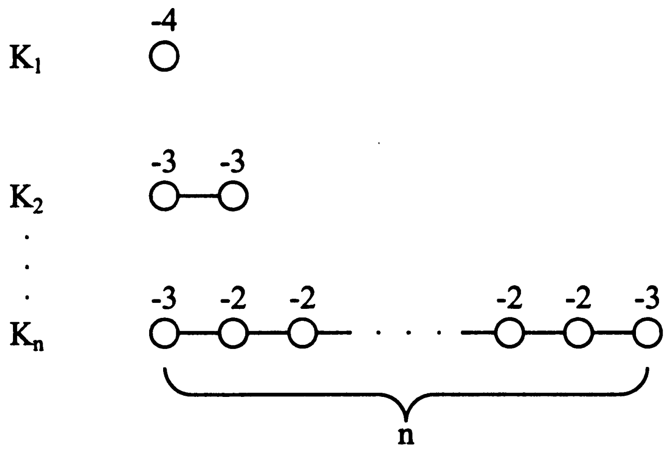
Let (Z, p) be a two-dimensional log terminal singularity of index ≤ 2, and $\pi : \widetilde{Z} \to Z$ be its minimal resolution. We have $K_{\widetilde{Z}} = \pi^* K_Z + \sum \alpha_i F_i$, where $-1 < \alpha_i \leq 0$ and $F_i^2 \leq -2$. Therefore, for each i one has $\alpha_i = -1/2$ or 0. One can rewrite the set of equations $K_{\widetilde{Z}} \cdot F_i = -F_i^2 - 2$ in a matrix form:

$$M \cdot (\alpha_1, \dots, \alpha_n)^t = (-F_1^2 - 2, \dots, -F_n^2 - 2)^t,$$

where $M = (F_i \cdot F_j)$ is the intersection matrix. By a basic theorem of Mumford [Mum61], M is negative definite and, in particular, nondegenerate. All the entries of the inverse matrix M^{-1} are strictly negative [Art62].

Let us give some easy consequences of this formula.

(1) If for some i_0, $\alpha_{i_0} = 0$ then all $\alpha_i = 0$, and the singularity (Z, p) is Du Val, of type A_n, D_n, E_6, E_7 or E_8.
(2) If all $\alpha_i = -1/2$ then we get the following list of singularities:



In these graphs every curve F_i corresponds to a vertex with weight F_i^2, two vertices are connected by an edge if $F_i \cdot F_j = 1$ and are not connected if $F_i \cdot F_j = 0$.

1.3. Basic facts about log del Pezzo surfaces

Lemma 1.3. *All log del Pezzo surfaces Z are rational.*

Proof. Let $\pi : \widetilde{Z} \to Z$ be the minimal resolution of singularities, $K_{\widetilde{Z}} = \pi^* K_Z + \sum \alpha_i F_i$, $-1 < \alpha_i \le 0$. Then $\ulcorner -\pi^* K_Z \urcorner = -K_{\widetilde{Z}}$, and so

$$h^1(O_{\widetilde{Z}}) = h^1(K_{\widetilde{Z}} + \ulcorner -\pi^* K_Z \urcorner) = 0$$

by Theorem 1.2.

Also, $h^0(nK_{\widetilde{Z}}) = 0$ for any positive integer n since $-\pi_* K_{\widetilde{Z}} = -K_Z$ is an effective nonzero $\mathbb{Q}$-Weil divisor. Therefore, by Castelnuovo criterion the surface $\widetilde{Z}$, and hence also Z, are rational. □

Lemma 1.4. *In the above notation, if $\widetilde{Z} \neq \mathbb{P}^2$ or $\mathbb{F}_n$ then the Kleiman–Mori cone of the surface $\widetilde{Z}$ is generated by the curves F_i and exceptional curves of the 1st kind. The number of these curves is finite. There are no other irreducible curves with negative self-intersection number (i. e.* **exceptional curves***) on $\widetilde{Z}$.*

Moreover, in this statement the minimal resolution $\widetilde{Z}$ can be replaced by any resolution of singularities $\pi : Z' \to Z$ such that $\alpha_i \le 0$, where $K_{Z'} = \pi^* K_Z + \sum \alpha_i F_i$ (for example, by the right resolution of Z, see Section 1.5 below).

Proof. Let us show that on the surface $\widetilde{Z}$ (or Z') there exists a $\mathbb{Q}$-divisor Δ with $\Delta \ge 0$, $[\Delta] = 0$ and such that the divisor $-(K_{\widetilde{Z}} + \Delta)$ is ample.

Choose $E = \sum \beta_i F_i$ so that $Z \cdot F_i > 0$. Since the matrix $(F_i \cdot F_j)$ is negative definite and $F_i \cdot F_j \geq 0$ for $i \neq j$, one has $\beta_i < 0$. Let us show that for a small $0 < \epsilon \ll 1$ the divisor $T = -\pi^* K_Z + \epsilon E$ is ample. Since $(-K_Z)^2 > 0$, we may assume that $T^2 > 0$. Now, let us check that, if the positive number ϵ is sufficiently small, then $C \cdot T > 0$ for any irreducible curve C on Z.

If $C^2 \geq 0$, this follows from the fact that the intersection form on $N_{\mathbb{R}}(\widetilde{Z})$ is hyperbolic. If $C = F_i$ then $F_i \cdot T = \epsilon F_i \cdot E > 0$.

If $C^2 < 0$ and $C \neq F_i$ then

$$C \cdot K_{\widetilde{Z}} = C \cdot (\pi^* K_{\widetilde{Z}} + \sum \alpha_i F_i) < 0.$$

On the other hand, $p_a(C) = \dfrac{C^2 + C \cdot K_{\widetilde{Z}}}{2} + 1 \geq 0$. So, $C^2 < 0$ and $C \cdot K_{\widetilde{Z}} < 0$ imply that $C^2 = -1$ and $p_a(C) = 0$, i.e. C is an **exceptional curve of the 1st kind.**

If n is the index of Z then $C \cdot (-\pi^* K_Z) \in (1/n)\mathbb{Z}$. On the other hand, $0 < -C \cdot \pi^* K_Z = 1 + \sum \alpha_i F_i \cdot C \leq 1$. Hence, there are only finitely many possibilities for $(-\pi^* K_Z) \cdot C$ and $\sum \alpha_i F_i \cdot C$, and for ϵ small enough, $C \cdot T > 0$.

By Kleiman's criterion, this implies that T is ample. Since the degree of the (-1)-curves with respect to T is bounded, there are only finitely many of them.

One has $-\pi^* K_Z + \epsilon E = -\left(K_{\widetilde{Z}} + \sum(-\alpha_i - \epsilon\beta_i)F_i\right)$. Therefore, $\Delta \geq 0$, and for $\epsilon \ll 1$, we have $[\Delta] = 0$, since $-\alpha_i < 1$.

Now, by Cone theorem [Kaw84, Thm.4.5], $\overline{\mathrm{NE}}(\widetilde{Z}) = \sum R_j$, where R_j are "good extremal rays". The rays generated by the curves F_i and exceptional curves of the 1st kind are obviously extremal. On the other hand, let R_j be a "good extremal ray", generated by an irreducible curve C. If $C \notin \{F_1, \ldots, F_k\}$ then $C \cdot K_{\widetilde{Z}} = C \cdot (\pi^* K_Z + \sum \alpha_i F_i) < 0$. Hence, by [Mor82] the curve C is an exceptional curve of the 1st kind, unless $Z \simeq \mathbb{P}^2$ or $\mathbb{F}_n$. □

1.4. Smooth Divisor Theorem

Theorem 1.5. *Let Z be a log del Pezzo surface of index ≤ 2. Then the linear system $|-2K_Z|$ is nonempty, has no fixed components and contains a nonsingular element $D \in |-2K_Z|$.*

Proof. Let $\pi : \widetilde{Z} \to Z$ be the minimal resolution of singularities. It is sufficient to prove the statement for the linear system $|-\pi^*(2K_Z)|$ on $\widetilde{Z}$. We have $2K_{\widetilde{Z}} = \pi^*(2K_Z) - \sum a_i F_i$, and all $a_i = 0$ or 1 $(a_i = -2\alpha_i)$.

1. *Nonemptiness.*

$$-\pi^*(2K_Z) = K_{\widetilde{Z}} + (-3K_{\widetilde{Z}} - \sum a_i F_i) = K_{\widetilde{Z}} + \ulcorner D \urcorner,$$

where $D = \dfrac{3}{2}(-2K_{\widetilde{Z}} - \sum a_i F_i) = -\pi^*(3K_Z)$ is big and nef. Hence, by Vanishing Theorem 1.2, $H^i(-\pi^*(2K_Z)) = 0$ for $i > 0$ and $h^0(-\pi^*(2K_Z)) = \chi(-\pi^*(2K_Z)) = 3K_{\widetilde{Z}}^2 + 1 > 0$.

2. *Nonexistence of fixed components.* Let E be the fixed part, so that $|-\pi^*(2K_Z) - E|$ is a movable linear system. Then

$$\begin{aligned} &h^0(-\pi^*(2K_Z)) = h^0(-\pi^*(2K_Z) - E), \\ &-\pi^*(2K_Z) - E = K_{\widetilde{Z}} + (-3K_{\widetilde{Z}} - \sum a_i F_i - E) = K_{\widetilde{Z}} + \ulcorner D \urcorner, \\ &D = \frac{3}{2}(-2K_{\widetilde{Z}} - \sum a_i F_i) - E = (-\pi^*(2K_Z) - E) + (-\pi^* K_Z). \end{aligned}$$

The first of these divisors is movable and the second is big and nef, so the sum is big and nef. By Vanishing Theorem 1.2, we have

$$h^i(-\pi^*(2K_Z) - E) = 0, \quad i > 0,$$
$$\chi(-\pi^*(2K_Z)) = \chi(-\pi^*(2K_Z) - E),$$

(8) $$2\chi(-\pi^*(2K_Z)) - 2\chi(-\pi^*(2K_Z) - E) = E \cdot (-2\pi^*(2K_Z) - K_{\widetilde{Z}} - E).$$

Let us show that this expression (8) is not equal to zero. Suppose

$$-\pi^* K_Z \cdot (K_{\widetilde{Z}} + E) = -\pi^* K_Z \cdot E - K_{\widetilde{Z}}^2 < 0.$$

Then the divisor $K_{\widetilde{Z}} + E$ cannot be effective. Therefore,

$$\chi(-E) = h^0(-E) - h^1(-E) + h^0(K_{\widetilde{Z}} + E) \leq 0.$$

Hence, $E \cdot (K_{\widetilde{Z}} + E) = 2\chi(-E) - 2 < 0$, and the expression (8) is strictly positive. So, we can assume that $-\pi^* K_Z \cdot E \geq K_{\widetilde{Z}}^2$. Let us write

$$E = \beta(-\pi^* K_Z) + F, \quad F \in (\pi^* K_Z)^\perp$$

in $N_{\mathbb{Q}}(\widetilde{Z})$. One has $\beta \leq 2$ since $-\pi^* K_Z \cdot (-\pi^*(2K_Z) - E) \geq 0$. Then

$$E \cdot (-2\pi^*(2K_Z) - K_{\widetilde{Z}} - E) = (5 - \beta)\beta K_{\widetilde{Z}}^2 - F \cdot (\sum \alpha_i F_i + F).$$

The first term in this sum is ≥ 3 since $\beta K_{\widetilde{Z}}^2 = -\pi^* K_Z \cdot E \geq K_{\widetilde{Z}}^2$ and $K_{\widetilde{Z}}^2 = \chi(\pi^*(2K_Z)) - 1$ is a positive integer. The second term achieves the minimum for $F = -\dfrac{1}{2}\sum \alpha_i F_i$ and equals $-\dfrac{m}{4}$, where m is the number of non-Du Val singularities. Therefore, all that remains to be shown is that the surface Z has fewer than 12 non-Du Val singularities.

By Lemma 1.3, the surface $\widetilde{Z}$ is rational. By Noether's formula, $(K_{\widetilde{Z}})^2 + \operatorname{rk}\operatorname{Pic}\widetilde{Z} = 10$. By Lemma 1.6 below, $(K_{\widetilde{Z}})^2 \geq 0$. Hence Z has no more than $\operatorname{rk}\operatorname{Pic}\widetilde{Z} - 1 \leq 9$ singular points.

Lemma 1.6. *Let Z be a log del Pezzo surface of index ≤ 2 and $\pi : \widetilde{Z} \to Z$ be its minimal resolution of singularities. Then $K_{\widetilde{Z}}^2 \geq 0$.*

Proof. One has $K_{\widetilde{Z}} = \pi^* K_Z + \sum \alpha_i F_i$. Denote $\overline{K} = \pi^* K_Z - \sum \alpha_i F_i$. Let us show that $-\overline{K}$ is nef. By Lemma 1.4, one has to show that $-\overline{K} \cdot F_i \geq 0$ and $-\overline{K} \cdot C \geq 0$ if C is an exceptional curve of the 1st kind. We have $-\overline{K} \cdot F_i = K_{\widetilde{Z}} \cdot F_i = -F_i^2 - 2 \geq 0$ since the resolution π is minimal. Next,

$$-\pi^* K_Z \cdot C = -K_{\widetilde{Z}} \cdot C + \sum \alpha_i F_i \cdot C = 1 + \sum \alpha_i F_i \cdot C > 0.$$

Since this number is a half-integer,

$$-\overline{K} \cdot C = 1 + 2\sum \alpha_i F_i \cdot C \geq 0.$$

So, $-\overline{K}$ is nef and $K_{\widetilde{Z}}^2 = \overline{K}^2 \geq 0$. Finally, if $\widetilde{Z} = \mathbb{P}^2$ or $\mathbb{F}_n$ then $K_{\widetilde{Z}}^2 = 9$ or 8 respectively. □

3. *Existence of a smooth element.* Assume that all divisors in the linear system $|-\pi^*(2K_Z)|$ are singular. Then there exists a base point P, and for a general element $D \in |-\pi^*(2K_Z)|$ the multiplicity of D at P is $k \geq 2$. This point does not lie on F_i since $-\pi^*(2K_Z) \cdot F_i = 0$. Let $\epsilon : Y \to \widetilde{Z}$ be the blowup at P, $f = \pi\epsilon : Y \to Z$, and let L be the exceptional divisor of ϵ. We have: $h^0(-f^*(2K_Z)) = h^0(-f^*(2K_Z) - L)$, the linear system $|-f^*(2K_Z) - kL|$ is movable, and

$$\begin{aligned}
&2K_Y = f^*(2K_Z) - \sum a_i F_i + 2L,\\
&-f^*(2K_Z) = K_Y + (-3K_Y - \sum a_i F_i + 2L) = K_Y + \ulcorner D \urcorner,\\
&D = \frac{3}{2}(-2K_Y - \sum a_i F_i + L) = \frac{3}{2}(-f^*(2K_Z) - L) =\\
&\frac{3}{2}[(-f^*(2K_Z) - kL) + (k-1)L].
\end{aligned}$$

The divisor D is nef since for any irreducible curve $C \neq L$, $C \cdot D \geq 0$, and also $D \cdot L = 3/2$. It is big since $(-f^*(2K_Z) - L)^2 = 4K_Z^2 - 1 > 0$. Now,

$$-f^*(2K_Z) - L = K_Y + (-3K_Y - \sum a_i F_i + L) = K_Y + \ulcorner D \urcorner,$$

$$D = \frac{3}{2}(-2K_Y - \sum a_i F_i + \frac{2}{3}L) = \frac{3}{2}((-f^*(2K_Z) - kL) + (k - \frac{4}{3})L).$$

The latter divisor D is nef since for any irreducible curve $C \neq L$, $C \cdot D \geq 0$, and also $D \cdot L = 2$; D is big since

$$(-f^*(2K_Z) - \frac{4}{3}L)^2 = 4K_Z^2 - \frac{16}{9} > 0.$$

Now, again by Vanishing Theorem 1.2,

$$h^i(-f^*(2K_Z)) = h^i(-f^*(2K_Z) - L) = 0 \quad \text{for } i > 0,$$

and one must have $\chi(-f^*(2K_Z)) = \chi(-f^*(2K_Z) - L)$. But

$$\chi(-f^*(2K_Z)) - \chi(-f^*(2K_Z) - L) =$$

$$= \frac{1}{2}L \cdot (-2f^*(2K_Z) - K_Y - L) = 1 + L \cdot (-f^*(2K_Z)) > 0.$$

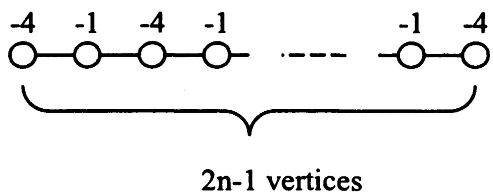
The contradiction thus obtained completes the proof of the theorem. □

Remark 1.7. In the same way, parts 1 and 3 can be proved for a log del Pezzo surface of arbitrary index n and the linear system $-\pi^*(nK_Z)$. Part 2 is easy to prove under the assumption that $\pi(E)$ passes only through (some of) the Du Val singularities.

1.5. Reduction to DPN surfaces of elliptic type

Let Z be a log del Pezzo surface of index ≤ 2. Consider the resolution of singularities $f : Y \to Z$ for which every Du Val singularity is resolved by inserting the usual tree of (-2)-curves, and the singularity K_n by the following chain:

(9)



The latter resolution is obtained by blowing up all intersection points of exceptional curves on the minimal resolution of K_n points, see their diagrams in Section 1.2. In contrast to the minimal resolution, we will call this the **right resolution** of singularities. Consider a smooth element $C_g \in |-2K_Z|$. It does not pass through singularities of the surface Z. If we identify the curve C_g with its image under the morphism f, then it is easy to see from the formulae of Section 1.2 that $-f^*2K_Z$ is linearly equivalent to C_g, and $-2K_Y$ with the disjoint union of C_g and curves in the above diagrams which have self-intersection -4. Moreover, it is easy to compute the genus of the curve C_g, and it equals $g = K_Z^2 + 1 \geq 2$. This shows that the surface Y is a right DPN surface of elliptic type in the sense of the next Chapter (see Sections 2.1 and 2.8).

Vice versa, the results of Chapters 2 and 3 will imply (see Chapter 4) that a right DPN surface Y of elliptic type admits a unique contraction of exceptional curves $f : Y \to Z$ to a log del Pezzo surface of index ≤ 2.

In this way, the classification of log del Pezzo surfaces of index ≤ 2 is reduced to classification of right DPN surfaces of elliptic type.