CHAPTER IV

Divisors on bundles

We calculate σ-decompositions of pseudo-effective divisors defined over varieties given by toric construction or defined over varieties admitting projective bundle structure. In §**1**, we recall some basics on toric varieties, extracting from the book [**110**], and we prove the existence of Zariski-decomposition for pseudo-effective $\mathbb{R}$-divisors on toric varieties. The notion of toric bundles is introduced in §**2**: a toric bundle is a fiber bundle of a toric variety whose transition group is the open torus. We give a counterexample to the Zariski-decomposition conjecture by constructing a divisor on such a toric bundle. We also consider projective bundles over curves in §**3**. We prove the existence of Zariski-decomposition for pseudo-effective $\mathbb{R}$-divisors on the bundles. The content of the preprint [**106**] is written in §**4**, where we study the relation between the stability of a vector bundle $\mathcal{E}$ and the pseudo-effectivity of the normalized tautological divisor $\Lambda_{\mathcal{E}}$. For example, the vector bundles with $\Lambda_{\mathcal{E}}$ being nef are characterized by semi-stability, Bogomolov's inequality, and projectively flat metrics. We shall classify and list the A-semi-stable vector bundles of rank two for an ample divisor A such that $\Lambda_{\mathcal{E}}$ is not nef but pseudo-effective. In particular, we can show that $\Lambda_{\mathcal{E}}$ for the tangent bundle $\mathcal{E}$ of any K3 surface is not pseudo-effective.

§1. Toric varieties

§1.a. Fans. We begin with recalling the notion of toric varieties. Let N be a free abelian group of finite rank and let M be the dual $\mathsf{N}^\vee = \mathrm{Hom}(\mathsf{N}, \mathbb{Z})$. We denote the natural pairing $\mathsf{M} \times \mathsf{N} \to \mathbb{Z}$ by $\langle\ \ ,\ \ \rangle$. For subsets $\mathcal{S}$ and $\mathcal{S}'$ of $\mathsf{N}_\mathbb{R} = \mathsf{N} \otimes \mathbb{R}$ and for a subset $R \subset \mathbb{R}$, we set

$$\mathcal{S} + \mathcal{S}' = \{n + n' \mid n \in \mathcal{S},\, n' \in \mathcal{S}'\}, \quad R\mathcal{S} = \{rn \mid n \in \mathcal{S},\, r \in R\},$$

$$\mathcal{S}^\vee = \{m \in \mathsf{M}_\mathbb{R} \mid \langle m, n\rangle \geq 0 \text{ for } n \in \mathcal{S}\}, \quad \mathcal{S}^\perp = \{m \in \mathsf{M}_\mathbb{R} \mid \langle m, n\rangle = 0 \text{ for } n \in \mathcal{S}\}.$$

A subset $\boldsymbol{\sigma} \subset \mathsf{N}_\mathbb{R}$ is called a *convex cone* if $\mathbb{R}_{\geq 0}\boldsymbol{\sigma} = \boldsymbol{\sigma}$ and $\boldsymbol{\sigma} + \boldsymbol{\sigma} = \boldsymbol{\sigma}$. If $\boldsymbol{\sigma} = \sum_{x \in \mathcal{S}} \mathbb{R}_{\geq 0} x$ for a subset $\mathcal{S} \subset \mathsf{N}_\mathbb{R}$, then we say that $\mathcal{S}$ generates the convex cone $\boldsymbol{\sigma}$. The set $\boldsymbol{\sigma}^\vee$ for a convex cone $\boldsymbol{\sigma}$ is a closed convex cone of $\mathsf{M}_\mathbb{R} = \mathsf{M} \otimes \mathbb{R}$, which is called the *dual cone* of $\boldsymbol{\sigma}$. It is well-known that $\boldsymbol{\sigma} = (\boldsymbol{\sigma}^\vee)^\vee$ for a closed convex cone $\boldsymbol{\sigma}$. The dimension of a convex cone $\boldsymbol{\sigma}$ is defined as that of the vector subspace $\mathsf{N}_{\mathbb{R},\boldsymbol{\sigma}} = \boldsymbol{\sigma} + (-\boldsymbol{\sigma})$. The quotient vector space $\mathsf{N}_\mathbb{R}(\boldsymbol{\sigma}) = \mathsf{N}_\mathbb{R}/\mathsf{N}_{\mathbb{R},\boldsymbol{\sigma}}$ is dual to the vector space $\boldsymbol{\sigma}^\perp$. The vector subspace $(\boldsymbol{\sigma}^\vee)^\perp \subset \mathsf{N}_\mathbb{R}$ is the maximum vector subspace contained in $\boldsymbol{\sigma}$. If $(\boldsymbol{\sigma}^\vee)^\perp = 0$, then $\boldsymbol{\sigma}$ is called *strictly convex*. A *face*

$\boldsymbol{\tau} \prec \boldsymbol{\sigma}$ is a subset of the form $m^{\perp} \cap \boldsymbol{\sigma}$ for some element $m \in \boldsymbol{\sigma}^{\vee}$. The relative interior of $\boldsymbol{\sigma}$ is denoted by $\operatorname{Int} \boldsymbol{\sigma}$, which is just the complement of the union of proper faces of $\boldsymbol{\sigma}$. A real-valued function $h \colon \boldsymbol{\sigma} \to \mathbb{R}$ is called *upper convex* if $h(x+y) \geq h(x)+h(y)$ and $h(rx) = rh(x)$ hold for any x, $y \in \boldsymbol{\sigma}$, $r \in \mathbb{R}_{\geq 0}$. A real-valued function h on $\boldsymbol{\sigma}$ is called *lower convex* if $-h$ is upper convex.

A convex cone $\boldsymbol{\sigma}$ generated by a finite subset of $\mathsf{N}_{\mathbb{R}}$ is called a *convex polyhedral cone*. The dual cone of a convex polyhedral cone is also convex polyhedral. A convex cone $\boldsymbol{\sigma}$ generated by a finite subset of N is called a *convex rational polyhedral cone* (with respect to N).

Let $\boldsymbol{\sigma}$ be a convex rational polyhedral cone. We define $\mathsf{N}_{\boldsymbol{\sigma}}$ to be the submodule $(\boldsymbol{\sigma} + (-\boldsymbol{\sigma})) \cap \mathsf{N}$ and $\mathsf{N}(\boldsymbol{\sigma})$ to be the quotient $\mathsf{N}/\mathsf{N}_{\boldsymbol{\sigma}}$. Then $\mathsf{N}_{\boldsymbol{\sigma},\mathbb{R}} = \mathsf{N}_{\boldsymbol{\sigma}} \otimes \mathbb{R} = \mathsf{N}_{\mathbb{R},\boldsymbol{\sigma}}$, $\mathsf{N}(\boldsymbol{\sigma})_{\mathbb{R}} = \mathsf{N}(\boldsymbol{\sigma}) \otimes \mathbb{R} = \mathsf{N}_{\mathbb{R}}(\boldsymbol{\sigma})$, and $\boldsymbol{\sigma}^{\perp} \simeq \operatorname{Hom}(\mathsf{N}(\boldsymbol{\sigma}), \mathbb{R})$. The submodule $\mathsf{M}(\boldsymbol{\sigma}) := \boldsymbol{\sigma}^{\perp} \cap \mathsf{M}$ is isomorphic to $\operatorname{Hom}(\mathsf{N}(\boldsymbol{\sigma}), \mathbb{Z})$. The intersection $\boldsymbol{\sigma}^{\vee} \cap \mathsf{M}$ is a finitely generated semi-group, which is known as Gordan's lemma. If $\boldsymbol{\sigma}$ is strictly convex, then $\boldsymbol{\sigma}^{\vee} \cap \mathsf{M}$ generates the abelian group M.

A *fan* $\boldsymbol{\Sigma}$ of N is a set of strictly convex rational polyhedral cones of $\mathsf{N}_{\mathbb{R}}$ with respect to N satisfying the following conditions:

(1) If $\boldsymbol{\sigma} \in \boldsymbol{\Sigma}$ and $\boldsymbol{\tau} \prec \boldsymbol{\sigma}$, then $\boldsymbol{\tau} \in \boldsymbol{\Sigma}$;
(2) If $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2 \in \boldsymbol{\Sigma}$, then $\boldsymbol{\sigma}_1 \cap \boldsymbol{\sigma}_2 \prec \boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_1 \cap \boldsymbol{\sigma}_2 \prec \boldsymbol{\sigma}_2$.

A fan always contains the zero cone $\mathbf{0} = \{0\}$. For a strictly convex rational polyhedral cone $\boldsymbol{\sigma}$, the set of its faces is a fan, which is denoted by the same symbol $\boldsymbol{\sigma}$. Let $\boldsymbol{\Sigma}$ be a fan of N. The union $\bigcup \boldsymbol{\sigma}$ of all $\boldsymbol{\sigma} \in \boldsymbol{\Sigma}$ is called the *support* of $\boldsymbol{\Sigma}$ and is denoted by $|\boldsymbol{\Sigma}|$. The intersection of N and the vector subspace of $\mathsf{N}_{\mathbb{R}}$ generated by $|\boldsymbol{\Sigma}|$ is denoted by $\mathsf{N}_{\boldsymbol{\Sigma}}$. The quotient $\mathsf{N}/\mathsf{N}_{\boldsymbol{\Sigma}}$ is denoted by $\mathsf{N}(\boldsymbol{\Sigma})$. If $\boldsymbol{\Sigma}$ is a finite set, then $\boldsymbol{\Sigma}$ is called *finite*. A finite fan with $|\boldsymbol{\Sigma}| = \mathsf{N}_{\mathbb{R}}$ is called *complete*. Let N' be another free abelian group of finite rank and let $\boldsymbol{\Sigma}'$ be a fan of N'. A homomorphism $\phi \colon \mathsf{N} \to \mathsf{N}'$ of abelian groups is called compatible with $\boldsymbol{\Sigma}$ and $\boldsymbol{\Sigma}'$, and is regarded as a morphism $(\mathsf{N}, \boldsymbol{\Sigma}) \to (\mathsf{N}', \boldsymbol{\Sigma}')$ of fans if the following condition is satisfied: For any $\boldsymbol{\sigma} \in \boldsymbol{\Sigma}$, there is a cone $\boldsymbol{\sigma'} \in \boldsymbol{\Sigma}'$ such that $\phi(\boldsymbol{\sigma}) \subset \boldsymbol{\sigma'}$. If the following condition is satisfied in addition, then $\boldsymbol{\Sigma}$ is called *proper* over $\boldsymbol{\Sigma}'$ and ϕ is called *proper*: For any $\boldsymbol{\sigma'} \in \boldsymbol{\Sigma}'$,

$$\boldsymbol{\Sigma}_{\boldsymbol{\sigma'}} := \{\boldsymbol{\sigma} \in \boldsymbol{\Sigma} \mid \phi(\boldsymbol{\sigma}) \subset \boldsymbol{\sigma'}\}$$

is a finite fan with $|\boldsymbol{\Sigma}_{\boldsymbol{\sigma'}}| = \phi^{-1}(\boldsymbol{\sigma'})$. If $\mathsf{N}' = \mathsf{N}$, ϕ is the identity, and $|\boldsymbol{\Sigma}'| = |\boldsymbol{\Sigma}|$, then $\boldsymbol{\Sigma}'$ is called a *subdivision* of $\boldsymbol{\Sigma}$. If $\boldsymbol{\Sigma}'$ is proper over $\boldsymbol{\Sigma}$, then it is called a *proper subdivision* or a *locally finite subdivision* of $\boldsymbol{\Sigma}$.

Let $\boldsymbol{\sigma} \subset \mathsf{N}_{\mathbb{R}}$ be a strictly convex rational polyhedral cone. The *affine toric variety* $\mathsf{T}_{\mathsf{N}}(\boldsymbol{\sigma})$ is defined as the affine scheme over $\mathbb{C}$ associated with the semi-group ring $\mathbb{C}[\boldsymbol{\sigma}^{\vee} \cap \mathsf{M}]$. The associated analytic space $\mathsf{T}_{\mathsf{N}}(\boldsymbol{\sigma})^{\mathrm{an}} = \operatorname{Specan} \mathbb{C}[\boldsymbol{\sigma}^{\vee} \cap \mathsf{M}]$ is denoted by $\mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma})$. For a face $\boldsymbol{\tau} \prec \boldsymbol{\sigma}$, an open immersion $\mathsf{T}_{\mathsf{N}}(\boldsymbol{\tau}) \subset \mathsf{T}_{\mathsf{N}}(\boldsymbol{\sigma})$ is defined by the inclusion $\boldsymbol{\sigma}^{\vee} \cap \mathsf{M} \subset \boldsymbol{\tau}^{\vee} \cap \mathsf{M}$. We set $\mathsf{T}_{\mathsf{N}} = \mathsf{T}_{\mathsf{N}}(\mathbf{0})$ for the zero cone $\mathbf{0}$, which is an algebraic torus. The associated analytic space $\mathbb{T}_{\mathsf{N}} := \mathsf{T}_{\mathsf{N}}^{\mathrm{an}}$ is isomorphic to $\mathsf{N} \otimes \mathbb{C}^{\star}$. The *toric variety* $\mathsf{T}_{\mathsf{N}}(\boldsymbol{\Sigma})$ associated with a fan $\boldsymbol{\Sigma}$ is defined as the natural union of $\mathsf{T}_\mathsf{N}(\boldsymbol{\sigma})$ for $\boldsymbol{\sigma} \in \boldsymbol{\Sigma}$. This is a separated scheme locally of finite type over $\operatorname{Spec}\mathbb{C}$. The associated analytic space is denoted by $\mathbb{T}_\mathsf{N}(\boldsymbol{\Sigma})$. There are an action of T_N on $\mathsf{T}_\mathsf{N}(\boldsymbol{\Sigma})$ and an equivariant open immersion $\mathsf{T}_\mathsf{N} \subset \mathsf{T}_\mathsf{N}(\boldsymbol{\Sigma})$. Toric varieties are normal.

For a strictly convex rational polyhedral cone $\boldsymbol{\sigma} \subset \mathsf{N}_\mathbb{R}$, there is a natural surjective $\mathbb{C}$-algebra homomorphism $\mathbb{C}[\boldsymbol{\sigma}^\vee \cap \mathsf{M}] \twoheadrightarrow \mathbb{C}[\boldsymbol{\sigma}^\perp \cap \mathsf{M}]$ given by

$$\boldsymbol{\sigma}^\vee \cap \mathsf{M} \ni m \mapsto \begin{cases} m, & \text{if } m \in \boldsymbol{\sigma}^\perp, \\ 0, & \text{otherwise.} \end{cases}$$

This induces a closed immersion

$$\mathsf{T}_{\mathsf{N}(\boldsymbol{\sigma})} \hookrightarrow \mathsf{T}_\mathsf{N}(\boldsymbol{\sigma}).$$

The left hand side is an *orbit* of T_N and is denoted by $\mathsf{O}_{\boldsymbol{\sigma}}$. In fact, for the composite $\pi_{\boldsymbol{\sigma}} \colon \mathsf{T}_\mathsf{N} \to \mathsf{T}_{\mathsf{N}(\boldsymbol{\sigma})} \hookrightarrow \mathsf{T}_\mathsf{N}(\boldsymbol{\sigma})$, we have

$$\pi_{\boldsymbol{\sigma}}(t) = t \cdot \pi_{\boldsymbol{\sigma}}(\mathbf{1}) = \pi_{\boldsymbol{\sigma}}(\mathbf{1}) \cdot t$$

for $t \in \mathsf{T}_\mathsf{N}$ and for the unit $\mathbf{1}$ of T_N, where $\cdot$ indicates the left and right actions of T_N on $\mathsf{T}_\mathsf{N}(\boldsymbol{\sigma})$. For a face $\boldsymbol{\tau} \prec \boldsymbol{\sigma}$, let $\boldsymbol{\sigma}/\boldsymbol{\tau}$ be the image of $\boldsymbol{\sigma}$ under $\mathsf{N}_\mathbb{R} \to \mathsf{N}(\boldsymbol{\tau})_\mathbb{R}$, which is also a strictly convex rational polyhedral cone with respect to $\mathsf{N}(\boldsymbol{\tau})$. Then $(\boldsymbol{\sigma}/\boldsymbol{\tau})^\vee \cap \mathsf{M}(\boldsymbol{\tau})$ is identified with $\boldsymbol{\sigma}^\vee \cap \boldsymbol{\tau}^\perp \cap \mathsf{M}$. The Zariski-closure of $\mathsf{O}_{\boldsymbol{\tau}}$ in $\mathsf{T}_\mathsf{N}(\boldsymbol{\sigma})$ is isomorphic to $\mathsf{T}_{\mathsf{N}(\boldsymbol{\tau})}(\boldsymbol{\sigma}/\boldsymbol{\tau})$ by a natural surjective homomorphism $\mathbb{C}[\boldsymbol{\sigma}^\vee \cap \mathsf{M}] \twoheadrightarrow \mathbb{C}[\boldsymbol{\sigma}^\vee \cap \boldsymbol{\tau}^\perp \cap \mathsf{M}]$ given by

$$\boldsymbol{\sigma}^\vee \cap \mathsf{M} \ni m \mapsto \begin{cases} m, & \text{if } m \in \boldsymbol{\tau}^\perp, \\ 0, & \text{otherwise.} \end{cases}$$

For a fan $\boldsymbol{\Sigma}$ of N and for a cone $\boldsymbol{\sigma} \in \boldsymbol{\Sigma}$, the set

$$\boldsymbol{\Sigma}/\boldsymbol{\sigma} := \{\boldsymbol{\sigma}'/\boldsymbol{\sigma} \mid \boldsymbol{\sigma} \prec \boldsymbol{\sigma}' \in \boldsymbol{\Sigma}\}$$

is a fan of $\mathsf{N}(\boldsymbol{\sigma})$. Then the Zariski-closure $\mathsf{V}(\boldsymbol{\sigma})$ of $\mathsf{O}_{\boldsymbol{\sigma}}$ in $\mathsf{T}_\mathsf{N}(\boldsymbol{\Sigma})$ is isomorphic to $\mathsf{T}_{\mathsf{N}(\boldsymbol{\sigma})}(\boldsymbol{\Sigma}/\boldsymbol{\sigma})$. If $\boldsymbol{\sigma} \in \boldsymbol{\Sigma}$ is not a proper face of another cone in $\boldsymbol{\Sigma}$, then it is called a *maximal cone*. In this case, $\mathsf{O}_{\boldsymbol{\sigma}} = \mathsf{V}(\boldsymbol{\sigma})$.

An element $m \in \mathsf{M}$ is regarded as a nowhere-vanishing regular function on T_N, which is denoted by $\mathsf{e}(m)$. It is also a rational function on the toric variety $\mathsf{T}_\mathsf{N}(\boldsymbol{\Sigma})$ associated with a fan $\boldsymbol{\Sigma}$ of N. An integral primitive vector $v \in \mathsf{N}$ is called a *vertex* of $\boldsymbol{\Sigma}$ if $\mathbb{R}_{\geq 0} v \in \boldsymbol{\Sigma}$. The set of vertices of $\boldsymbol{\Sigma}$ is denoted by $\operatorname{Ver}(\boldsymbol{\Sigma})$ or $\operatorname{Ver}(\mathsf{N}, \boldsymbol{\Sigma})$. For $v \in \operatorname{Ver}(\boldsymbol{\Sigma})$, let $\boldsymbol{\Gamma}_v$ be the prime divisor $\mathsf{V}(\mathbb{R}_{\geq 0} v)$. Then the principal divisor $\operatorname{div}(\mathsf{e}(m))$ is written by

$$\sum\nolimits_{v \in \operatorname{Ver}(\boldsymbol{\Sigma})} \langle m, v \rangle \boldsymbol{\Gamma}_v$$

as a Weil divisor. Since $\operatorname{div} \circ \mathsf{e}$ is a group homomorphism $\mathsf{M} \to \operatorname{Div}(\mathsf{T}_\mathsf{N}(\boldsymbol{\Sigma}))$, the principal $\mathbb{R}$-divisor $\operatorname{div}(\mathsf{e}(m'))$ is also defined for $m' \in \mathsf{M}_\mathbb{R}$; if $m' = \sum r_i m_i$, then

$$\operatorname{div}(\mathsf{e}(m')) = \sum r_i \operatorname{div}(\mathsf{e}(m_i)),$$

where $r_i \in \mathbb{R}$, $m_i \in \mathsf{M}$.

Remark (1) $\mathsf{T}_{\mathsf{N}}(\boldsymbol{\sigma})$ is non-singular if and only if the set $\mathrm{Ver}(\mathsf{N}, \boldsymbol{\sigma})$ is a basis of the free abelian group $\mathsf{N}_{\boldsymbol{\sigma}}$. Similarly, $\mathsf{T}_{\mathsf{N}}(\boldsymbol{\sigma})$ has only quotient singularities if and only if $\mathrm{Ver}(\mathsf{N}, \boldsymbol{\sigma})$ is a basis of the $\mathbb{Q}$-vector space $\mathsf{N}_{\boldsymbol{\sigma}} \otimes \mathbb{Q}$. A fan $\boldsymbol{\Sigma}$ is called *non-singular* if $\mathsf{T}_{\mathsf{N}}(\boldsymbol{\Sigma})$ is non-singular.

(2) Let $\phi\colon (\mathsf{N}, \boldsymbol{\Sigma}) \to (\mathsf{N}', \boldsymbol{\Sigma}')$ be a morphism into another free abelian group N' of finite rank with a fan $\boldsymbol{\Sigma}'$. Then it induces a morphism $\phi_*\colon \mathsf{T}_{\mathsf{N}}(\boldsymbol{\Sigma}) \to \mathsf{T}_{\mathsf{N}'}(\boldsymbol{\Sigma}')$ which is equivariant under the homomorphism $\mathsf{T}_{\mathsf{N}} \to \mathsf{T}_{\mathsf{N}'}$. If ϕ is proper, then ϕ_* is proper.

(3) There is a proper subdivision $\boldsymbol{\Sigma}'$ of $\boldsymbol{\Sigma}$ such that $\boldsymbol{\Sigma}'$ is non-singular. In particular, $\mathsf{T}_{\mathsf{N}}(\boldsymbol{\Sigma}') \to \mathsf{T}_{\mathsf{N}}(\boldsymbol{\Sigma})$ is a proper birational morphism from a non-singular variety.

(4) If $\boldsymbol{\Sigma}$ is a finite fan such that $|\boldsymbol{\Sigma}|$ is a convex cone, then the toric variety $\mathsf{X} = \mathsf{T}_{\mathsf{N}}(\boldsymbol{\Sigma})$ is proper over an affine toric variety. The vanishing $\mathrm{H}^p(\mathsf{X}, \mathcal{O}_{\mathsf{X}}) = 0$ for $p > 0$ holds, which is shown in a general form in [**62**, Chapter I, §3] and [**9**, §7] (cf. [**110**, §2.2]). In particular, toric varieties have only rational singularities.

1.1. Lemma *Let $\phi\colon (\mathsf{N}, \boldsymbol{\Sigma}) \to (\mathsf{L}, \boldsymbol{\Lambda})$ be a morphism of fans and let $\mathsf{f} = \phi_*\colon \mathsf{T}_{\mathsf{N}}(\boldsymbol{\Sigma}) \to \mathsf{T}_{\mathsf{L}}(\boldsymbol{\Lambda})$ be the associated morphism of toric varieties. Then*

$$\mathsf{f}^{-1}\mathsf{T}_{\mathsf{L}}(\boldsymbol{\lambda}) \simeq \mathsf{T}_{\mathsf{N}}(\boldsymbol{\Sigma}_{\boldsymbol{\lambda}})$$

for $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$. Moreover,

$$\mathsf{f}^{-1}\mathsf{O}_{\boldsymbol{\lambda}} = \bigsqcup\nolimits_{\phi(\boldsymbol{\sigma}) \subset \boldsymbol{\lambda},\, \phi(\boldsymbol{\sigma}) \cap \mathrm{Int}\, \boldsymbol{\lambda} \neq \emptyset} \mathsf{O}_{\boldsymbol{\sigma}}.$$

If f is proper, then $\mathsf{f}^{-1}(\mathsf{V}(\boldsymbol{\lambda}))$ is set-theoretically the union

$$\bigcup\nolimits_{\phi(\boldsymbol{\sigma}) \subset \boldsymbol{\lambda},\, \phi(\boldsymbol{\sigma}) \cap \mathrm{Int}\, \boldsymbol{\lambda} \neq \emptyset} \mathsf{V}(\boldsymbol{\sigma}).$$

Proof. The first isomorphism is derived from the definition of f, which is given by the gluing of natural morphisms $\mathsf{T}_{\mathsf{N}}(\boldsymbol{\sigma}) \to \mathsf{T}_{\mathsf{L}}(\boldsymbol{\lambda})$ for $\boldsymbol{\sigma} \subset \phi^{-1}(\boldsymbol{\lambda})$.

For a cone $\boldsymbol{\sigma} \in \boldsymbol{\Sigma}$, let $\boldsymbol{\lambda}_1 \in \boldsymbol{\Lambda}$ be the minimum cone containing $\phi(\boldsymbol{\sigma})$. Then $\boldsymbol{\lambda}_1 = \boldsymbol{\lambda}$ if and only if $\phi(\boldsymbol{\sigma}) \subset \boldsymbol{\lambda}$ and $\phi(\boldsymbol{\sigma}) \cap \mathrm{Int}\, \boldsymbol{\lambda} \neq \emptyset$. The transpose $\phi^\vee\colon \mathsf{L}^\vee \to \mathsf{N}^\vee = \mathsf{M}$ induces $\boldsymbol{\lambda}_1^\perp \cap \mathsf{L}^\vee \to \boldsymbol{\sigma}^\perp \cap \mathsf{M}$. Hence $\mathsf{f}(\mathsf{O}_{\boldsymbol{\sigma}}) \subset \mathsf{O}_{\boldsymbol{\lambda}_1}$. By considering the orbit decomposition of $\mathsf{f}^{-1}\mathsf{O}_{\boldsymbol{\lambda}}$, we have the equality for $\mathsf{f}^{-1}\mathsf{O}_{\boldsymbol{\lambda}}$. In the proper case, taking the closure, we have the equality for $\mathsf{f}^{-1}(\mathsf{V}(\boldsymbol{\lambda}))$, since f is a closed map. □

An element $0 \neq a \in \mathsf{N}$ defines a 1-parameter subgroup $\mathsf{T}_{\mathbb{Z}a} \subset \mathsf{T}_{\mathsf{N}}$. If $a \in |\boldsymbol{\Sigma}|$, then we have a morphism $\phi_a\colon (\mathbb{Z}, \mathbb{R}_{\geq 0}) \to (\mathsf{N}, \boldsymbol{\Sigma})$ of fans by $\phi_a(1) = a$. The induced morphism $\mathsf{f}_a = \phi_{a*}\colon \mathsf{T}_{\mathbb{Z}}(\mathbb{R}_{\geq 0}) \simeq \mathbb{A}^1 \to \mathsf{T}_{\mathsf{N}}(\boldsymbol{\Sigma})$ of toric varieties is an extension of $\mathsf{T}_{\mathbb{Z}a} \subset \mathsf{T}_{\mathsf{N}}$. Let $\boldsymbol{\sigma} \in \boldsymbol{\Sigma}$ be the minimum cone containing a. Then $\mathsf{f}_a(0) = \pi_{\boldsymbol{\sigma}}(\mathbf{1}) \in \mathsf{O}_{\boldsymbol{\sigma}}$ for the origin $0 \in \mathbb{A}^1$, where $\pi_{\boldsymbol{\sigma}}$ is the composite $\mathsf{T}_{\mathsf{N}} \to \mathsf{T}_{\mathsf{N}(\boldsymbol{\sigma})} \hookrightarrow \mathsf{T}_{\mathsf{N}}(\boldsymbol{\sigma})$. Thus $\lim_{t\to 0} \mathsf{f}_a(t) \cdot P = \pi_{\boldsymbol{\sigma}}(P)$ for any point $P \in \mathsf{T}_{\mathsf{N}}$. If $P \in \mathsf{O}_{\boldsymbol{\tau}}$ for some face $\boldsymbol{\tau} \prec \boldsymbol{\sigma}$, then $\lim_{t\to 0} \mathsf{f}_a(t) \cdot P = \pi_{\boldsymbol{\sigma}/\boldsymbol{\tau}}(P)$, where $\pi_{\boldsymbol{\sigma}/\boldsymbol{\tau}}$ is the composite $\mathsf{T}_{\mathsf{N}(\boldsymbol{\tau})} \to \mathsf{T}_{\mathsf{N}(\boldsymbol{\sigma})} \simeq \mathsf{O}_{\boldsymbol{\sigma}} \subset \mathsf{T}_{\mathsf{N}(\boldsymbol{\tau})}(\boldsymbol{\sigma}/\boldsymbol{\tau})$. Suppose that $P \in \mathsf{O}_{\boldsymbol{\tau}}$ for $\boldsymbol{\tau} \in \boldsymbol{\Sigma}$ with $\boldsymbol{\tau} \not\subset \boldsymbol{\sigma}$ and that $a' := a$

mod $\mathsf{N}_{\boldsymbol{\tau}} \in \mathsf{N}(\boldsymbol{\tau})$ is contained in $|\boldsymbol{\Sigma}/\boldsymbol{\tau}|$. Let $\boldsymbol{\sigma}'/\boldsymbol{\tau} \in \boldsymbol{\Sigma}/\boldsymbol{\tau}$ be the minimum cone containing a'. Then $\lim_{t\to 0} \mathsf{f}_a(t) \cdot P = \pi_{\boldsymbol{\sigma}'/\boldsymbol{\tau}}(P)$.

1.2. Lemma *A complete subvariety of* $\mathsf{X} = \mathsf{T}_{\mathsf{N}}(\boldsymbol{\Sigma})$ *of dimension* $k < \dim \mathsf{N}_{\boldsymbol{\Sigma}}$ *is rationally equivalent to a complete effective algebraic k-cycle supported on the union of* $\mathsf{V}(\boldsymbol{\tau})$ *with* $\dim |\boldsymbol{\Sigma}/\boldsymbol{\tau}| = k$.

Proof. Let V be such a complete subvariety of X. Then V is contracted to a point by $\mathsf{X} \to \mathsf{T}_{\mathsf{N}(\boldsymbol{\Sigma})}$. Thus we may assume that $|\boldsymbol{\Sigma}|$ generates $\mathsf{N}_{\mathbb{R}}$. We consider the action of the 1-parameter subgroup $\mathsf{T}_{\mathbb{Z}a}$ for $0 \neq a \in \mathsf{N} \cap |\boldsymbol{\Sigma}|$. Let $\mathsf{f}_a \colon \mathbb{A}^1 \to \mathsf{X}$ be the morphism defined above. The action of $\mathsf{T}_{\mathbb{Z}a}$ on X extends to a rational map $\psi \colon \mathbb{A}^1 \times \mathsf{X} \dashrightarrow \mathsf{X}$. It is a morphism over $\mathbb{A}^1 \times \mathsf{T}_{\mathsf{N}}$, where $\psi(t, P) = \mathsf{f}_a(t) \cdot P$. We have a toric variety Y and a proper birational morphism $\mu \colon \mathsf{Y} \to \mathbb{A}^1 \times \mathsf{X}$ of toric varieties such that $\varphi = \psi \circ \mu \colon \mathsf{Y} \to \mathsf{X}$ is a morphism. Let $\mathcal{V}$ be the proper transform of $\mathbb{A}^1 \times \mathsf{V}$ in Y. Then the projection $p \colon \mathcal{V} \to \mathbb{A}^1$ is a proper flat morphism. In particular, the image of $(p, \varphi) \colon \mathcal{V} \to \mathbb{A}^1 \times \mathsf{X}$ is also proper and flat over $\mathbb{A}^1$. For the fiber $\mathcal{V}_t = p^{-1}(t)$, the image $\varphi(\mathcal{V}_t)$ is just V multiplied by $\mathsf{f}_a(t)$ for $t \neq 0$. The push-forward $\varphi_* \mathcal{V}_0$ is a complete effective algebraic k-cycle rationally equivalent to V. Here, any prime component of $\varphi_* \mathcal{V}_0$ is preserved by the action of $\mathsf{T}_{\mathbb{Z}a}$. We set $a_1 = a$ and choose elements $a_2, \ldots a_l \in \mathsf{N} \cap |\boldsymbol{\Sigma}|$ such that $\sum_{i=1}^{l} \mathbb{Z}a_i \subset \mathsf{N}$ is a finite index subgroup, where $l = \operatorname{rank} \mathsf{N}$. Applying the same limit argument for a_2 to prime components of $\varphi_* \mathcal{V}_0$, we have a new complete effective algebraic k-cycle which is preserved by the actions of $\mathsf{T}_{\mathbb{Z}a_1}$ and $\mathsf{T}_{\mathbb{Z}a_2}$. Applying the same argument successively, we finally have a complete effective algebraic k-cycle V_* such that V_* is rationally equivalent to V and that $\operatorname{Supp} \mathsf{V}_*$ is preserved by the action of T_{N}. Hence $\operatorname{Supp} \mathsf{V}_*$ is written as the union of some orbits $\mathsf{O}_{\boldsymbol{\tau}}$, where $\dim \mathsf{O}_{\boldsymbol{\tau}} \leq k < l$. Thus we are done. □

Remark Let $\boldsymbol{\tau}$ be a cone in $\boldsymbol{\Sigma}$. In our notation, $\mathsf{N}(\boldsymbol{\tau})_{\boldsymbol{\Sigma}/\boldsymbol{\tau}}$ is the intersection of $\mathsf{N}(\boldsymbol{\tau})$ and the vector subspace of $\mathsf{N}(\boldsymbol{\tau})_{\mathbb{R}}$ generated by $|\boldsymbol{\Sigma}/\boldsymbol{\tau}|$, and $\mathsf{N}(\boldsymbol{\tau})(\boldsymbol{\Sigma}/\boldsymbol{\tau})$ is the quotient $\mathsf{N}(\boldsymbol{\tau})/\mathsf{N}(\boldsymbol{\tau})_{\boldsymbol{\Sigma}/\boldsymbol{\tau}}$. We have an isomorphism

$$\mathsf{V}(\boldsymbol{\tau}) = \mathsf{T}_{\mathsf{N}(\boldsymbol{\tau})}(\boldsymbol{\Sigma}/\boldsymbol{\tau}) \simeq \mathsf{T}_{\mathsf{N}(\boldsymbol{\tau})_{\boldsymbol{\Sigma}/\boldsymbol{\tau}}}(\boldsymbol{\Sigma}/\boldsymbol{\tau}) \times \mathsf{T}_{\mathsf{N}(\boldsymbol{\tau})(\boldsymbol{\Sigma}/\boldsymbol{\tau})}.$$

Thus any complete subvariety of $\mathsf{V}(\boldsymbol{\tau})$ of dimension equal to $\dim |\boldsymbol{\Sigma}/\boldsymbol{\tau}|$ is a fiber of the projection $\mathsf{V}(\boldsymbol{\tau}) \to \mathsf{T}_{\mathsf{N}(\boldsymbol{\tau})(\boldsymbol{\Sigma}/\boldsymbol{\tau})}$.

§1.b. Support functions. Let $\boldsymbol{\Sigma}$ be a finite fan of N. A $\boldsymbol{\Sigma}$*-linear support function* h is a continuous function $h \colon |\boldsymbol{\Sigma}| \to \mathbb{R}$ that is linear on every $\boldsymbol{\sigma} \in \boldsymbol{\Sigma}$. For a subset $\mathfrak{K} \subset \mathbb{R}$, let $\mathrm{SF}_{\mathsf{N}}(\boldsymbol{\Sigma}, \mathfrak{K})$ be the set of $\boldsymbol{\Sigma}$-linear support functions h with $h(\mathsf{N} \cap |\boldsymbol{\Sigma}|) \subset \mathfrak{K}$. Then $\mathrm{SF}_{\mathsf{N}}(\boldsymbol{\Sigma}, \mathbb{Z}) \otimes \mathbb{Q} \simeq \mathrm{SF}_{\mathsf{N}}(\boldsymbol{\Sigma}, \mathbb{Q})$ and $\mathrm{SF}_{\mathsf{N}}(\boldsymbol{\Sigma}, \mathbb{Q}) \otimes \mathbb{R} \simeq \mathrm{SF}_{\mathsf{N}}(\boldsymbol{\Sigma}, \mathbb{R})$. In fact, in the vector space $\mathrm{Map}(\mathrm{Ver}(\mathsf{N}, \boldsymbol{\Sigma}), \mathbb{R}) = \prod_{v \in \mathrm{Ver}(\mathsf{N}, \boldsymbol{\Sigma})} \mathbb{R}$, the subspace $\mathrm{SF}_{\mathsf{N}}(\boldsymbol{\Sigma}, \mathbb{R})$ is determined by a finite number of relations defined over $\mathbb{Q}$.

A $\boldsymbol{\Sigma}$*-convex support function* h is a continuous function $h \colon |\boldsymbol{\Sigma}| \to \mathbb{R}$ satisfying the following conditions:

(1) The restriction $h|_{\boldsymbol{\sigma}}$ to $\boldsymbol{\sigma} \in \boldsymbol{\Sigma}$ is upper convex for any $\boldsymbol{\sigma} \in \boldsymbol{\Sigma}$;

(2) For any $\boldsymbol{\sigma} \in \boldsymbol{\Sigma}$, there is a finite fan $\boldsymbol{\Lambda}_{\boldsymbol{\sigma}}$ of N with $|\boldsymbol{\Lambda}_{\boldsymbol{\sigma}}| = \boldsymbol{\sigma}$ such that $h|_{\boldsymbol{\sigma}}$ is $\boldsymbol{\Lambda}_{\boldsymbol{\sigma}}$-linear.

For a subset $\mathfrak{K} \subset \mathbb{R}$, the set of $\boldsymbol{\Sigma}$-convex support functions h with $h(|\boldsymbol{\Sigma}| \cap \mathsf{N}) \subset \mathfrak{K}$ is denoted by $\mathrm{SFC}_{\mathsf{N}}(\boldsymbol{\Sigma}, \mathfrak{K})$. Functions contained in $\mathrm{SFC}_{\mathsf{N}}(\boldsymbol{\Sigma}, \mathbb{Z})$ and $\mathrm{SFC}_{\mathsf{N}}(\boldsymbol{\Sigma}, \mathbb{Q})$ are called integral and rational, respectively.

For $h \in \mathrm{SFC}_{\mathsf{N}}(\boldsymbol{\Sigma}, \mathbb{R})$ and for a closed convex cone $C \subset |\boldsymbol{\Sigma}|$, we define

$$\Box_h(C) := \{m \in \mathsf{M}_{\mathbb{R}} \mid \langle m, x\rangle \geq h(x) \text{ for any } x \in C\},$$
$$\triangle_h(C) := \sum\nolimits_{x \in C} \mathbb{R}_{\geq 0}(x, h(x)) + \mathbb{R}_{\geq 0}(0, -1) \subset \mathsf{N}_{\mathbb{R}} \times \mathbb{R}.$$

Then $\Box_h(C)$ is a convex set and $\triangle_h(C)$ is a closed convex cone, since $\boldsymbol{\Sigma}$ is finite and h is $\boldsymbol{\Sigma}$-convex. If C is a convex polyhedral cone, then $\triangle_h(C)$ is so. The dual cone of $\triangle_h(C)$ is written by

$$C^{\vee} \times \{0\} \cup \mathbb{R}_{\geq 0}(\Box_h(C) \times \{-1\}).$$

In particular, $\Box_h(C) = \emptyset$ if and only if $\triangle_h(C) \ni (0, 1)$. When $\Box_h(C) \neq \emptyset$, we define a function by

$$h^{\dagger}_C(x) := \inf\{\langle m, x\rangle \mid m \in \Box_h(C)\}. \tag{IV-1}$$

Then $h^{\dagger}_C(x) \geq h(x)$ for $x \in C$. Since $\triangle_h(C) = (\triangle_h(C)^{\vee})^{\vee}$,

$$h^{\dagger}_C(x) = \max\{r \in \mathbb{R} \mid (x, r) \in \triangle_h(C)\} \tag{IV-2}$$

for $x \in C$.

1.3. Lemma *The following conditions are equivalent*:

(1) *h is upper convex on C*;
(2) $\triangle_h(C) = \{(x, r) \in C \times \mathbb{R} \mid h(x) \geq r\}$;
(3) $\Box_h(C) \neq \emptyset$ *and* $h^{\dagger}_C(x) = h(x)$ *for* $x \in C$.

Proof. (1) $\Rightarrow$ (2): The right hand side is a convex cone contained in the left. On the other hand, $(x, h(x))$ is contained in the right for $x \in C$. Thus the equality holds.

(2) $\Rightarrow$ (3): We infer $(0, 1) \notin \triangle_h(C)$, which implies $\Box_h(C) \neq \emptyset$. The equality $h^{\dagger}_C = h$ on C follows directly from the equality (IV-2).

(3) $\Rightarrow$ (1): By the definition (IV-1), we infer that $h^{\dagger}_C$ is upper convex on C. Thus we are done. $\Box$

1.4. Lemma (1) *If C' is a face of C, then*

$$\triangle_h(C') = \triangle_h(C) \cap (C' \times \mathbb{R}).$$

In particular, $h^{\dagger}_{C'}(x) = h^{\dagger}_C(x)$ *for* $x \in C'$ *provided that* $\Box_h(C) \neq \emptyset$.

(2) $\Box_h(C) \neq \emptyset$ *if and only if* $\Box_h((C^{\vee})^{\perp}) \neq \emptyset$.

PROOF. (1) Let (x,t) be an element of the right hand side. Then $x = \sum r_i x_i$ and $t \le \sum r_i h(x_i)$ for finitely many vectors $x_i \in C$ and for real numbers $r_i > 0$. The face C' is written as $l^{\perp} \cap C$ for some $l \in C^{\vee}$. Then $\langle l, x\rangle = 0$ implies that $x_i \in C'$ for any i. In particular, $(x,t) \in \triangle_h(C')$. Thus we have the equality.

(2) follows from (1) and from that $\Box_h(C) = \emptyset$ if and only if $(0,1) \in \triangle_h(C)$. □

1.5. Lemma *Suppose that* $h \in \mathrm{SFC}_{\mathsf{N}}(\boldsymbol{\Sigma}, \mathfrak{K})$ *for* $\mathfrak{K} = \mathbb{Q}$ *or* $\mathbb{R}$. *Then there is a finite subdivision* $\boldsymbol{\Sigma}'$ *of* $\boldsymbol{\Sigma}$ *such that* $h \in \mathrm{SF}_{\mathsf{N}}(\boldsymbol{\Sigma}', \mathfrak{K})$.

PROOF. For a cone $\boldsymbol{\sigma} \in \boldsymbol{\Sigma}$, let $\boldsymbol{\Lambda}_{\boldsymbol{\sigma}}$ be a fan with $|\boldsymbol{\Lambda}_{\boldsymbol{\sigma}}| = \boldsymbol{\sigma}$ such that $h|_{\boldsymbol{\sigma}} \in \mathrm{SF}_{\mathsf{N}}(\boldsymbol{\sigma}, \mathfrak{K})$. Any one-dimensional face of the convex polyhedral cone $\triangle_h(\boldsymbol{\sigma})$ except $\mathbb{R}_{\ge 0}(0,-1)$ is written by $\mathbb{R}_{\ge 0}(v, h(v))$ for some $v \in \mathrm{Ver}(\boldsymbol{\Lambda}_{\boldsymbol{\sigma}})$. Therefore, the image $\boldsymbol{\sigma}_{\boldsymbol{\lambda}}$ of a face $\boldsymbol{\lambda}$ of $\triangle_h(\boldsymbol{\sigma})$ under the first projection $\mathsf{N}_{\mathbb{R}} \times \mathbb{R} \to \mathsf{N}_{\mathbb{R}}$ is a convex *rational* polyhedral cone with respect to N. The function h is linear on $\boldsymbol{\sigma}_{\boldsymbol{\lambda}}$. There is a finite subdivision $\boldsymbol{\Sigma}'$ of $\boldsymbol{\Sigma}$ such that $\boldsymbol{\sigma}_{\boldsymbol{\lambda}}$ is a union of cones belonging to $\boldsymbol{\Sigma}'$ for any $\boldsymbol{\sigma} \in \boldsymbol{\Sigma}$ and $\boldsymbol{\lambda} \prec \triangle_h(\boldsymbol{\sigma})$. Here, $h \in \mathrm{SF}_{\mathsf{N}}(\boldsymbol{\Sigma}', \mathfrak{K})$. □

Remark Among the finite subdivisions of **1.5**, we can find the maximum: There exists a finite subdivision $\boldsymbol{\Sigma}^{\sharp}$ of $\boldsymbol{\Sigma}$ satisfying $h \in \mathrm{SF}_{\mathsf{N}}(\boldsymbol{\Sigma}^{\sharp}, \mathfrak{K})$ such that $\boldsymbol{\Sigma}' \precsim \boldsymbol{\Sigma}^{\sharp}$ for any finite subdivision $\boldsymbol{\Sigma}'$ satisfying $h \in \mathrm{SF}_{\mathsf{N}}(\boldsymbol{\Sigma}', \mathfrak{K})$. This is shown by **1.15** below, for example.

1.6. Lemma *Let* $g\colon \mathrm{Ver}(\boldsymbol{\Sigma}) \to \mathfrak{K}$ *is a map for* $\mathfrak{K} = \mathbb{Z}$, $\mathbb{Q}$ *or* $\mathbb{R}$. *Then there exists a unique function* $h \in \mathrm{SFC}_{\mathsf{N}}(\boldsymbol{\Sigma}, \mathfrak{K})$ *satisfying the following conditions*:

(1) $g(v) = h(v)$ *for* $v \in \mathrm{Ver}(\boldsymbol{\Sigma})$;
(2) *If* $h' \in \mathrm{SFC}_{\mathsf{N}}(\boldsymbol{\Sigma}, \mathfrak{K})$ *satisfies* $h'(v) \ge g(v)$ *for any* $v \in \mathrm{Ver}(\boldsymbol{\Sigma})$, *then* $h'(x) \ge h(x)$ *for any* $x \in |\boldsymbol{\Sigma}|$.

The function h is called the convex interpolation of g in [**62**, Chapter I, §2].

PROOF. First, we consider the case $\mathfrak{K} \supset \mathbb{Q}$. For $\boldsymbol{\sigma} \in \boldsymbol{\Sigma}$ and $x \in \boldsymbol{\sigma}$, we set

$$\triangle(\boldsymbol{\sigma}) := \sum\nolimits_{v \in \mathrm{Ver}(\boldsymbol{\sigma})} \mathbb{R}_{\ge 0}(v, g(v)) + \mathbb{R}_{\ge 0}(0,-1), \quad \text{and}$$
$$h^0_{\boldsymbol{\sigma}}(x) := \max\{r \in \mathbb{R} \mid (x,r) \in \triangle(\boldsymbol{\sigma})\}.$$

Then $h^0_{\boldsymbol{\sigma}} \in \mathrm{SFC}_{\mathsf{N}}(\boldsymbol{\sigma}, \mathfrak{K})$. If $\boldsymbol{\tau} \prec \boldsymbol{\sigma}$, then $\triangle(\boldsymbol{\tau}) = \triangle(\boldsymbol{\sigma}) \cap (\boldsymbol{\tau} \times \mathbb{R})$ by the same argument as in **1.4**. Thus $h^0_{\boldsymbol{\tau}}(x) = h^0_{\boldsymbol{\sigma}}(x)$ for any $x \in \boldsymbol{\tau}$. In particular, we have a function $h^0 \in \mathrm{SFC}_{\mathsf{N}}(\boldsymbol{\Sigma}, \mathfrak{K})$ such that $h^0|_{\boldsymbol{\sigma}} = h^0_{\boldsymbol{\sigma}}$ for any $\boldsymbol{\sigma} \in \boldsymbol{\Sigma}$ and $h^0(v) = g(v)$ for $v \in \mathrm{Ver}(\boldsymbol{\Sigma})$. The function h^0 satisfies the second required condition for h by **1.3**.

Next, we consider the case $\mathfrak{K} = \mathbb{Z}$. If $\boldsymbol{\Sigma}$ is non-singular, then $h^0 \in \mathrm{SFC}_{\mathsf{N}}(\boldsymbol{\Sigma}, \mathbb{Q})$ is integral. Otherwise, let us consider a non-singular finite subdivision $\boldsymbol{\Sigma}^{\sharp}$ of $\boldsymbol{\Sigma}$. We set $g^{\sharp}\colon \mathrm{Ver}(\boldsymbol{\Sigma}^{\sharp}) \to \mathbb{Z}$ by $g^{\sharp}(v) = \ulcorner h^0(v) \urcorner$. Let h be the function in $\mathrm{SFC}_{\mathsf{N}}(\boldsymbol{\Sigma}^{\sharp}, \mathbb{Q})$ satisfying the required condition for $g^{\sharp}$. Then h is integral. Thus h is the convex interpolation of g. □

Let X be the toric variety $\mathsf{T}_{\mathsf{N}}(\boldsymbol{\Sigma})$ associated with the fan $\boldsymbol{\Sigma}$ and let $j\colon \mathsf{T}_{\mathsf{N}} \hookrightarrow \mathsf{X}$ be the open immersion.

For $h \in \mathrm{SFC}_{\mathsf{N}}(\mathbf{\Sigma}, \mathbb{Z})$, we define a coherent $\mathcal{O}_{\mathsf{X}}$-submodule $\mathcal{F}_h$ of $j_*\mathcal{O}_{\mathsf{T}_{\mathsf{N}}}$ by

$$\mathrm{H}^0(\mathsf{T}_{\mathsf{N}}(\boldsymbol{\sigma}), \mathcal{F}_h) = \bigoplus_{m \in \Box_h(\boldsymbol{\sigma}) \cap \mathsf{M}} \mathsf{e}(m) \subset \mathbb{C}[\mathsf{M}]$$

for $\boldsymbol{\sigma} \in \mathbf{\Sigma}$. The subsheaf is invariant under the action of T_{N}. Conversely, any T_{N}-invariant coherent $\mathcal{O}_{\mathsf{X}}$-submodule of $j_*\mathcal{O}_{\mathsf{T}_{\mathsf{N}}}$, which is complete, is written as $\mathcal{F}_h$ for some $h \in \mathrm{SFC}_{\mathsf{N}}(\mathbf{\Sigma}, \mathbb{Z})$ (cf. [**62**, Chapter I, §2]). Here, $h \in \mathrm{SF}_{\mathsf{N}}(\mathbf{\Sigma}, \mathbb{Z})$ if and only if $\mathcal{F}_h$ is invertible. If $h' \in \mathrm{SFC}_{\mathsf{N}}(\mathbf{\Sigma}, \mathbb{Z})$ is the convex interpolation of the map $\mathrm{Ver}(\mathbf{\Sigma}) \ni v \mapsto h(v) \in \mathbb{Z}$, then $\mathcal{F}_{h'}$ is the double-dual of $\mathcal{F}_h$.

For $h \in \mathrm{SFC}_{\mathsf{N}}(\mathbf{\Sigma}, \mathbb{R})$, we define an $\mathbb{R}$-divisor of X by

$$\mathsf{D}_h := \sum\nolimits_{v \in \mathrm{Ver}(\mathbf{\Sigma})} (-h(v)) \mathbf{\Gamma}_v.$$

The associated $\mathbb{R}$-divisor $\mathsf{D}_h^{\mathrm{an}}$ on the analytic variety $\mathbb{T}_{\mathsf{N}}(\mathbf{\Sigma})$ is denoted by D_h. For $\mathfrak{K} = \mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{R}$, any $\mathfrak{K}$-divisor of X supported in $\mathsf{X} \setminus \mathsf{T}_{\mathsf{N}}$ is expressed as D_h for some $h \in \mathrm{SFC}_{\mathsf{N}}(\mathbf{\Sigma}, \mathfrak{K})$ by **1.6**. Moreover, any $\mathfrak{K}$-divisor D of X is $\mathfrak{K}$-linearly equivalent to D_h for some $h \in \mathrm{SFC}_{\mathsf{N}}(\mathbf{\Sigma}, \mathfrak{K})$, since $\mathsf{D}|_{\mathsf{T}_{\mathsf{N}}}$ is a principal $\mathfrak{K}$-divisor. If $h' \in \mathrm{SFC}_{\mathsf{N}}(\mathbf{\Sigma}, \mathbb{Z})$ is the convex interpolation of the map $\mathrm{Ver}(\mathbf{\Sigma}) \ni v \mapsto \ulcorner h(v) \urcorner \in \mathbb{Z}$, then $\llcorner \mathsf{D}_h \lrcorner = \mathsf{D}_{h'}$ and $\mathcal{F}_{h'} = \mathcal{O}_{\mathsf{X}}(\mathsf{D}_{h'})$.

1.7. Remark Suppose that $h \in \mathrm{SF}_{\mathsf{N}}(\mathbf{\Sigma}, \mathfrak{K})$ for $\mathfrak{K} = \mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$. Then D_h is $\mathfrak{K}$-Cartier. In fact, the restriction of D_h to $\mathsf{T}_{\mathsf{N}}(\boldsymbol{\sigma})$ for $\boldsymbol{\sigma} \in \mathbf{\Sigma}$ coincides with the principal $\mathfrak{K}$-divisor $-\operatorname{div}(\mathsf{e}(l_{\boldsymbol{\sigma}}))$ for $l_{\boldsymbol{\sigma}} \in \mathsf{M}_{\mathfrak{K}}$ such that $h(x) = \langle l_{\boldsymbol{\sigma}}, x \rangle$ for $x \in \boldsymbol{\sigma}$. The choice of $l_{\boldsymbol{\sigma}}$ is unique up to $\boldsymbol{\sigma}^{\perp} \cap \mathsf{M}_{\mathfrak{K}}$. Let $h^{\boldsymbol{\sigma}}(x) = h(x) - \langle l_{\boldsymbol{\sigma}}, x \rangle$. If $\dim \boldsymbol{\sigma} = \dim |\mathbf{\Sigma}|$, then $h^{\boldsymbol{\sigma}}$ is a function defined on $|\mathbf{\Sigma}|$ which is independent of the choice of $l_{\boldsymbol{\sigma}}$. Even if $\dim \boldsymbol{\sigma} < \dim |\mathbf{\Sigma}|$, $h^{\boldsymbol{\sigma}}$ is regarded as a function defined on $|\mathbf{\Sigma}/\boldsymbol{\sigma}|$ which belongs to $\mathrm{SF}_{\mathsf{N}(\boldsymbol{\sigma})}(\mathbf{\Sigma}/\boldsymbol{\sigma}, \mathfrak{K})$. Here, the restriction of D_h to $\mathsf{V}(\boldsymbol{\sigma})$ is $\mathfrak{K}$-linearly equivalent to $\mathsf{D}_{h^{\boldsymbol{\sigma}}}$.

1.8. Remark If $\boldsymbol{\tau} = \boldsymbol{\sigma} \cap \boldsymbol{\sigma}'$ for two maximal cones $\boldsymbol{\sigma}$, $\boldsymbol{\sigma}' \in \mathbf{\Sigma}$ such that $\dim \boldsymbol{\tau} = \dim |\mathbf{\Sigma}| - 1$, then there is an isomorphism $\mathsf{V}(\boldsymbol{\tau}) \simeq \mathsf{P}^1 \times \mathsf{T}_{\mathsf{N}(\boldsymbol{\tau})}$, in which $\mathsf{V}(\boldsymbol{\sigma}/\boldsymbol{\tau}) \simeq \{0\} \times \mathsf{T}_{\mathsf{N}(\boldsymbol{\tau})}$ and $\mathsf{V}(\boldsymbol{\sigma}'/\boldsymbol{\tau}) \simeq \{\infty\} \times \mathsf{T}_{\mathsf{N}(\boldsymbol{\tau})}$. Here,

$$\mathsf{D}_{h^{\boldsymbol{\sigma}}}|_{\mathsf{V}(\boldsymbol{\tau})} = -h^{\boldsymbol{\sigma}}(v') \left(\{\infty\} \times \mathsf{T}_{\mathsf{N}(\boldsymbol{\tau})}\right)$$

for the primitive element $v' \in \mathsf{N}(\boldsymbol{\tau})$ generating the ray $\boldsymbol{\sigma}'/\boldsymbol{\tau}$. In particular, for a fiber $F \simeq \mathsf{P}^1$ of $\mathsf{V}(\boldsymbol{\tau}) \to \mathsf{T}_{\mathsf{N}(\boldsymbol{\tau})}$, we have

$$\mathsf{D}_h \cdot F = -h^{\boldsymbol{\sigma}}(y) = \langle l_{\boldsymbol{\sigma}}, y \rangle - \langle l_{\boldsymbol{\sigma}'}, y \rangle$$

for $y \in \boldsymbol{\sigma}' \cap \mathsf{N} \setminus \boldsymbol{\sigma}$ with $y \mod \mathsf{N}_{\boldsymbol{\tau}} = v'$.

Suppose that $|\mathbf{\Sigma}|$ is a convex cone. For $h \in \mathrm{SFC}_{\mathsf{N}}(\mathbf{\Sigma}, \mathbb{R})$, we write $\Box_h = \Box_h(|\mathbf{\Sigma}|)$ and $\triangle_h = \triangle_h(|\mathbf{\Sigma}|)$ for short. If $|\mathbf{\Sigma}| = \mathsf{N}_{\mathbb{R}}$, then $\Box_h$ is compact, since $-h(-e_i) \geq \langle m, e_i \rangle \geq h(e_i)$ for a basis $\{e_i\}$ of $\mathsf{N}_{\mathbb{R}}$ and for $m \in \Box_h$. If $h \in \mathrm{SFC}_{\mathsf{N}}(\mathbf{\Sigma}, \mathbb{Z})$ and $\mathcal{F}_h$ is reflexive, then $\Box_h \subset \mathsf{M}_{\mathbb{R}}$ is the set of $m \in \mathsf{M}_{\mathbb{R}}$ satisfying $\operatorname{div}(\mathsf{e}(m)) + \mathsf{D}_h \geq 0$.

The vector space $\mathrm{H}^0(\mathsf{X}, \mathcal{F}_h)$ admits an action of T_N. Since this is a subspace of $\mathrm{H}^0(\mathsf{T}_\mathsf{N}, \mathcal{O}_{\mathsf{T}_\mathsf{N}}) \simeq \mathbb{C}[\mathsf{M}]$, we have an isomorphism

$$\text{(IV-3)} \qquad \mathrm{H}^0(\mathsf{X}, \mathcal{F}_h) \simeq \bigoplus\nolimits_{m \in \Box_h \cap \mathsf{M}} \mathbb{C}\mathsf{e}(m).$$

Suppose that $h \in \mathrm{SFC}_\mathsf{N}(\boldsymbol{\Sigma}, \mathbb{R})$ is the convex interpolation of $\mathrm{Ver}(\boldsymbol{\Sigma}) \ni v \mapsto h(v) \in \mathbb{R}$ in the sense of **1.6** for $\mathfrak{K} = \mathbb{R}$. Then

$$\text{(IV-4)} \qquad \mathrm{H}^0(\mathsf{X}, \llcorner \mathsf{D}_h \lrcorner) \simeq \bigoplus\nolimits_{m \in \Box_h \cap \mathsf{M}} \mathbb{C}\mathsf{e}(m)$$

by (IV-3). Furthermore, $\Box_h \neq \emptyset$ if and only if there is an effective $\mathbb{R}$-divisor $\mathbb{R}$-linearly equivalent to D_h (cf. **1.16**-(1) below).

1.9. Lemma *Suppose that $|\boldsymbol{\Sigma}|$ is convex. Let $\boldsymbol{\sigma}$ be a maximal cone of $\boldsymbol{\Sigma}$ and let $\mathfrak{K} = \mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{R}$. For a function $h \in \mathrm{SF}_\mathsf{N}(\boldsymbol{\Sigma}, \mathfrak{K})$, let $l_{\boldsymbol{\sigma}}$ and $h^{\boldsymbol{\sigma}}$ be the same as in* **1.7**. *Then the following three conditions are equivalent*:

1. *$h^{\boldsymbol{\sigma}}(x) \leq 0$ for any $x \in |\boldsymbol{\Sigma}|$*;
2. *$\Box_h \neq \emptyset$ and $h^{\dagger}_{|\boldsymbol{\Sigma}|}(x) = h(x)$ for any $x \in \boldsymbol{\sigma}$*;
3. *There is a T_N-invariant effective $\mathfrak{K}$-divisor Δ on X such that $\Delta \cap \mathsf{V}(\boldsymbol{\sigma}) = \emptyset$ and $\Delta \sim_{\mathfrak{K}} \mathsf{D}_h$ on X.*

Proof. (1) $\Leftrightarrow$ (2): (1) is equivalent to: $l_{\boldsymbol{\sigma}} \in \Box_h$, which implies (2). For $y \in |\boldsymbol{\Sigma}| \smallsetminus \boldsymbol{\sigma}$, let us choose $x \in \mathrm{Int}\, \boldsymbol{\sigma}$ and a number $0 < t < 1$ such that $(1-t)x + ty \in \boldsymbol{\sigma}$. Since $h^{\dagger}_{|\boldsymbol{\Sigma}|}$ is upper convex, we have

$$\langle l_{\boldsymbol{\sigma}}, y \rangle = \frac{1}{t}\left(h^{\dagger}_{|\boldsymbol{\Sigma}|}((1-t)x + ty) - (1-t)h^{\dagger}_{|\boldsymbol{\Sigma}|}(x)\right) \geq h^{\dagger}_{|\boldsymbol{\Sigma}|}(y) \geq h(y)$$

under the condition of (2).

(1) $\Rightarrow$ (3): The $\mathfrak{K}$-Cartier divisor $\mathsf{D}_{h^{\boldsymbol{\sigma}}} = \mathrm{div}(\mathsf{e}(l_{\boldsymbol{\sigma}})) + \mathsf{D}_h$ is effective on X and is away from $\mathsf{V}(\boldsymbol{\sigma})$.

(3) $\Rightarrow$ (1): Δ is written by $\mathsf{D}_h + \mathrm{div}(\mathsf{e}(m))$ for some $m \in \mathsf{M}_{\mathfrak{K}}$. Then $\langle m, v \rangle = h(v)$ for $v \in \mathrm{Ver}(\boldsymbol{\sigma})$. In particular, $m = l_{\boldsymbol{\sigma}} \in \Box_h$. □

1.10. Corollary *If $|\boldsymbol{\Sigma}|$ is a convex cone, then the following conditions are equivalent for $h \in \mathrm{SF}_\mathsf{N}(\boldsymbol{\Sigma}, \mathfrak{K})$*:

1. *h is upper convex on $|\boldsymbol{\Sigma}|$*;
2. *$l_{\boldsymbol{\sigma}} \in \Box_h$ for any maximal cone $\boldsymbol{\sigma}$*;
3. *For any point $p \in \mathsf{X}$, there is an effective divisor Δ of X such that $\Delta \sim_{\mathfrak{K}} \mathsf{D}_h$ and $p \notin \Delta$*;
4. *For any two maximal cones $\boldsymbol{\sigma}$, $\boldsymbol{\sigma}' \in \boldsymbol{\Sigma}$ with $\boldsymbol{\tau} = \boldsymbol{\sigma} \cap \boldsymbol{\sigma}'$ being of codimension one, the intersection number $\mathsf{D}_h \cdot F$ is non-negative for a fiber F of $\mathsf{V}(\boldsymbol{\tau}) \to \mathsf{T}_{\mathsf{N}(\boldsymbol{\tau})}$*;
5. *For any two maximal cones $\boldsymbol{\sigma}$, $\boldsymbol{\sigma}' \in \boldsymbol{\Sigma}$ with $\boldsymbol{\sigma} \cap \boldsymbol{\sigma}'$ being of codimension one, $h^{\boldsymbol{\sigma}}(y) \leq 0$ for any $y \in \boldsymbol{\sigma}'$.*

Proof. (1) $\Leftrightarrow$ (2) is shown in **1.9**. (3) $\Rightarrow$ (4) is trivial. (4) $\Leftrightarrow$ (5) is shown in **1.8**.

(2) $\Rightarrow$ (3): Let $\mathsf{Z} \subset \mathsf{X}$ be the set of points p such that $p \in \Delta$ for any effective divisor $\Delta \sim_{\mathfrak{K}} \mathsf{D}_h$. Then Z is a Zariski-closed subset invariant under the action of T. If $\mathsf{Z} \neq \emptyset$, then $\mathsf{V}(\boldsymbol{\sigma}) \subset \mathsf{Z}$ for a maximal cone $\boldsymbol{\sigma} \in \boldsymbol{\Sigma}$. By **1.9**-(3), we have $\mathsf{Z} = \emptyset$.

(5) $\Rightarrow$ (2): Let us fix $y \in |\boldsymbol{\Sigma}| \setminus \boldsymbol{\sigma}$. We take $x \in \operatorname{Int} \boldsymbol{\sigma}$ and consider a line segment $\{x(t) = (1-t)x + ty \mid t \in [0,1]\}$. If x is in a general position, then there exist a sequence of maximal cones $\boldsymbol{\sigma}_i$ and numbers $t_i \in [0,1)$ for $0 \leq i \leq k$ such that

- $\boldsymbol{\sigma}_0 = \boldsymbol{\sigma}$, $t_0 = 0$, $y \in \boldsymbol{\sigma}_k$,
- $\boldsymbol{\sigma}_i \cap \boldsymbol{\sigma}_{i+1}$ is of codimension one for any $i < k$,
- $\{t \in [0,1] \mid x(t) \in \boldsymbol{\sigma}_i\} = [t_i, t_{i+1}]$ for $i < k$ and $x(t) \in \boldsymbol{\sigma}_k$ for $t \geq t_k$.

The function $h^{\boldsymbol{\sigma}}(x(t))$ is linear on each $[t_i, t_{i+1}]$ for $i < k$ and on $[t_k, 1]$. Thus (5) implies that $h(x(t))$ is upper convex on $[0,1]$. Hence $h^{\boldsymbol{\sigma}}(y) \leq 0$ and $l_{\boldsymbol{\sigma}} \in \Box_h$. $\Box$

Suppose still that $|\boldsymbol{\Sigma}|$ is convex. A support function $h \in \mathrm{SF}_{\mathsf{N}}(\boldsymbol{\Sigma}, \mathbb{R})$ is called *strictly upper convex* with respect to $\boldsymbol{\Sigma}$ if it is upper convex on $|\boldsymbol{\Sigma}|$ and the set

$$\{x \in |\boldsymbol{\Sigma}|\,;\, \langle m, x\rangle = h(x)\}$$

is a cone belonging to $\boldsymbol{\Sigma}$ for any $m \in \Box_h$.

1.11. Lemma *Suppose that $|\boldsymbol{\Sigma}|$ is a convex cone and let $h \in \mathrm{SF}_{\mathsf{N}}(\boldsymbol{\Sigma}, \mathbb{R})$. For a maximal cone $\boldsymbol{\sigma} \in \boldsymbol{\Sigma}$, let $l_{\boldsymbol{\sigma}}$ be the same as in* **1.7**. *Then the following conditions are equivalent*:

1. *h is strictly convex with respect to $\boldsymbol{\Sigma}$*;
2. *$l_{\boldsymbol{\sigma}} \in \Box_h$ and*
$$\{x \in |\boldsymbol{\Sigma}|\,;\, h(x) = \langle l_{\boldsymbol{\sigma}}, x\rangle\} = \boldsymbol{\sigma}$$
for any maximal cone $\boldsymbol{\sigma} \in \boldsymbol{\Sigma}$;
3. *For maximal cones $\boldsymbol{\sigma}$, $\boldsymbol{\sigma}' \in \boldsymbol{\Sigma}$ with $\boldsymbol{\sigma} \cap \boldsymbol{\sigma}'$ being of codimension one, $h(y) < \langle l_{\boldsymbol{\sigma}}, y\rangle$ for any $y \in \boldsymbol{\sigma}' \setminus \boldsymbol{\sigma}$*;
4. *For maximal cones $\boldsymbol{\sigma}$, $\boldsymbol{\sigma}' \in \boldsymbol{\Sigma}$ with $\boldsymbol{\tau} = \boldsymbol{\sigma} \cap \boldsymbol{\sigma}'$ being of codimension one, the intersection number $\mathsf{D}_h \cdot F$ is positive for a fiber F of $\mathsf{V}(\boldsymbol{\tau}) \to \mathsf{T}_{\mathsf{N}(\boldsymbol{\tau})}$.*

Proof. (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are trivial. (3) $\Leftrightarrow$ (4) is shown in **1.8**.

(3) $\Rightarrow$ (2): Let $\boldsymbol{\sigma}$ be a maximal cone of $\boldsymbol{\Sigma}$. We fix $y \in |\boldsymbol{\Sigma}| \setminus \boldsymbol{\sigma}$, take $x \in \operatorname{Int} \boldsymbol{\sigma}$, and consider the line segment $\{x(t) = (1-t)x + ty \mid t \in [0,1]\}$. By choosing x in a general position, we may assume that there exist maximal cones $\boldsymbol{\sigma}_i$ and numbers $t_i \in [0,1)$ satisfying the same condition as in the proof of **1.10**. Then $h^{\boldsymbol{\sigma}}(y) < 0$ by (3). Thus (2) follows.

(2) $\Rightarrow$ (1): For $m \in \Box_h$, the set

$$C_m = \{x \in |\boldsymbol{\Sigma}|\,;\, h(x) = \langle m, x\rangle\}$$

is a convex polyhedral cone. For a point $y \in \operatorname{Int} C_m$, let $\boldsymbol{\sigma} \in \boldsymbol{\Sigma}$ be a maximal cone containing y. Then

$$C_m \cap \boldsymbol{\sigma} = (l_{\boldsymbol{\sigma}} - m)^{\perp} \cap \boldsymbol{\sigma}$$

is a face of $\boldsymbol{\sigma}$, since $m-l_{\boldsymbol{\sigma}} \in \boldsymbol{\sigma}^{\vee}$. By (2), $l_{\boldsymbol{\sigma}}-m \in C_m^{\vee}$ and $C_m \cap \boldsymbol{\sigma} = (l_{\boldsymbol{\sigma}}-m)^{\perp} \cap C_m$ is also a face of C_m. Thus $C_m = C_m \cap \boldsymbol{\sigma} \prec \boldsymbol{\sigma}$ by $y \in \operatorname{Int} C_m$. In particular, $C_m \in \boldsymbol{\Sigma}$. □

§1.c. Relative toric situations. Let L be another free abelian group and let $\boldsymbol{\Lambda}$ be a finite fan of L. Let $\phi\colon (\mathsf{N}, \boldsymbol{\Sigma}) \to (\mathsf{L}, \boldsymbol{\Lambda})$ be a proper morphism of fans and let $\mathsf{f}\colon \mathsf{X} = \mathsf{T}_{\mathsf{N}}(\boldsymbol{\Sigma}) \to \mathsf{S} = \mathsf{T}_{\mathsf{L}}(\boldsymbol{\Lambda})$ be the induced morphism. We shall consider the relative σ-decomposition over S of the $\mathbb{R}$-Cartier divisor D_h for a function $h \in \mathrm{SF}_{\mathsf{N}}(\boldsymbol{\Sigma}, \mathbb{R})$. By **1.4**, we have

$$\triangle_h(\phi^{-1}\boldsymbol{\nu}) = \triangle_h(\phi^{-1}\boldsymbol{\lambda}) \cap (\phi^{-1}\boldsymbol{\nu} \times \mathbb{R})$$

for $\boldsymbol{\nu} \prec \boldsymbol{\lambda}$. Moreover, for any $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$, the condition $\square_h(\phi^{-1}\boldsymbol{\lambda}) \neq \emptyset$ is equivalent to $\square_h(\phi^{-1}\mathbf{0}) \neq \emptyset$ for the zero cone $\mathbf{0} \in \boldsymbol{\Lambda}$. If $\square_h(\phi^{-1}\mathbf{0}) \neq \emptyset$, then we can define a function over $|\boldsymbol{\Sigma}|$ by

$$h^{\dagger}(x) := h^{\dagger}_{\boldsymbol{\Sigma}/\boldsymbol{\Lambda}}(x) := h^{\dagger}_{\phi^{-1}\boldsymbol{\lambda}}(x)$$

for $x \in \phi^{-1}\boldsymbol{\lambda}$, which is independent of the choice of $\boldsymbol{\lambda}$ for x.

1.12. Lemma $h^{\dagger}_{\boldsymbol{\Sigma}/\boldsymbol{\Lambda}} \in \mathrm{SFC}_{\mathsf{N}}(\boldsymbol{\Sigma}, \mathbb{R})$.

Proof. For any $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$, we have

$$\square_{h^{\dagger}}(\phi^{-1}\boldsymbol{\lambda}) = \square_h(\phi^{-1}\boldsymbol{\lambda}), \quad \text{and} \quad \triangle_{h^{\dagger}}(\phi^{-1}\boldsymbol{\lambda}) = \triangle_h(\phi^{-1}\boldsymbol{\lambda}).$$

By the same argument as in **1.5**, there is a finite subdivision $\boldsymbol{\Sigma}'$ of $\boldsymbol{\Sigma}$ such that the image of any face of $\triangle_h(\phi^{-1}\boldsymbol{\lambda})$ under the first projection $\mathsf{N}_{\mathbb{R}} \times \mathbb{R} \to \mathsf{N}_{\mathbb{R}}$ is a union of some cones belonging to $\boldsymbol{\Sigma}'$. Thus $h^{\dagger} \in \mathrm{SF}_{\mathsf{N}}(\boldsymbol{\Sigma}', \mathbb{R})$. □

Remark $h^{\dagger}_{\boldsymbol{\Sigma}/\boldsymbol{\Lambda}}$ is not necessarily integral for $h \in \mathrm{SF}_{\mathsf{N}}(\boldsymbol{\Sigma}, \mathbb{Z})$.

1.13. Lemma (1) $\square_h(\phi^{-1}\mathbf{0}) \cap \mathsf{M} \neq \emptyset$ *if and only if* $\mathsf{f}_*\mathcal{O}_{\mathsf{X}}(\llcorner \mathsf{D}_h \lrcorner) \neq 0$.
(2) *If* $\mathsf{f}_*\mathcal{O}_{\mathsf{X}}(\llcorner \mathsf{D}_h \lrcorner) \neq 0$, *then* $\mathsf{D}_h - \mathsf{D}_{h^{\dagger}}$ *is identical to the* f*-fixed part of* $|\mathsf{D}_h|$.
(3) *The following conditions are equivalent to each other*:
  (a) h *is upper-convex on* $\phi^{-1}(\boldsymbol{\lambda})$ *for any* $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$;
  (b) $\square_h(\phi^{-1}\mathbf{0}) \neq \emptyset$ *and* $h^{\dagger} = h$;
  (c) *For any* $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$ *and for any maximal cone* $\boldsymbol{\sigma} \in \boldsymbol{\Sigma}_{\boldsymbol{\lambda}}$, $h^{\boldsymbol{\sigma}}(x) \leq 0$ *for* $x \in \boldsymbol{\Sigma}_{\boldsymbol{\lambda}}$, *where* $h^{\boldsymbol{\sigma}}$ *is as in* **1.7**;
  (d) D_h *is* f*-nef.*
If $h \in \mathrm{SF}_{\mathsf{N}}(\boldsymbol{\Sigma}, \mathbb{Z})$, *then these are also equivalent to*:
  (e) D_h *is* f*-free.*

Proof. (1) follows from the isomorphism (IV-4). (2) follows from (IV-4) and **1.10**. The assertion (3) is proved as follows: (a) ⇔ (b) follows from **1.3**. (e) ⇒ (d) is well-known. (d) ⇒ (b), (b) ⇔ (c), and (b) ⇔ (e) are shown in **1.10**. (c) ⇒ (d) is derived from **1.10**-(3). □

1.14. Lemma *For a support function* $h \in \mathrm{SF}_{\mathsf{N}}(\boldsymbol{\Sigma}, \mathbb{R})$, *the following conditions are equivalent*:

(1) D_h *is* f*-ample;*
(2) *For any* $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$*, for any two maximal cones* $\boldsymbol{\sigma}$*,* $\boldsymbol{\sigma}' \in \boldsymbol{\Sigma}_{\boldsymbol{\lambda}}$ *with* $\boldsymbol{\tau} = \boldsymbol{\sigma} \cap \boldsymbol{\sigma}'$ *being of codimension one, the intersection number* $\mathsf{D}_h \cdot F$ *is positive for a fiber* F *of* $\mathsf{V}(\boldsymbol{\tau}) \to \mathsf{T}_{\mathsf{N}(\boldsymbol{\tau})}$*;*
(3) h *is strictly convex on* $\boldsymbol{\Sigma}_{\boldsymbol{\lambda}}$ *for any* $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$*.*

Proof. (1) $\Rightarrow$ (2) is trivial. (2) $\Leftrightarrow$ (3) is shown in **1.11**.

(2) $\Rightarrow$ (1): First, we consider the case $h \in \mathrm{SF}_{\mathsf{N}}(\boldsymbol{\Sigma}, \mathbb{Q})$. Then $kh \in \mathrm{SF}_{\mathsf{N}}(\boldsymbol{\Sigma}, \mathbb{Z})$ for some $k > 0$ and $k\mathsf{D}_h = \mathsf{D}_{kh}$ is f-free by **1.13**-(3). Hence D_h is f-ample if and only if $\mathsf{D}_h \cdot \gamma > 0$ for any irreducible curve γ contained in a fiber of f. By **1.2**, we infer that D_h is f-ample if and only if the condition (2) is satisfied.

Next, we consider the general case. Note that $\mathrm{SF}_{\mathsf{N}}(\boldsymbol{\Sigma}, \mathbb{R}) \simeq \mathrm{SF}_{\mathsf{N}}(\boldsymbol{\Sigma}, \mathbb{Q}) \otimes \mathbb{R}$. Hence there is a support function $h_1 \in \mathrm{SF}_{\mathsf{N}}(\boldsymbol{\Sigma}, \mathbb{Q})$ such that $\mathsf{D}_{h_1} \cdot F > 0$ for any $\boldsymbol{\tau}$ in the condition (2). In particular, D_{h_1} is an f-ample $\mathbb{Q}$-Cartier divisor. Since $\boldsymbol{\Lambda}$ is finite, we can find a positive number ε such that $(\mathsf{D}_h - \varepsilon \mathsf{D}_{h_1}) \cdot F \geq 0$ for any $\boldsymbol{\tau}$. Therefore, $\mathsf{D}_h - \varepsilon \mathsf{D}_{h_1}$ is f-nef and thus D_h is an f-ample $\mathbb{R}$-Cartier divisor. □

Remark Since $\boldsymbol{\Sigma}$ is finite, there is a finite subdivision $\boldsymbol{\Sigma}'$ of $\boldsymbol{\Sigma}$ such that $\boldsymbol{\Sigma}'$ is non-singular and the composite $\mathsf{T}_{\mathsf{N}}(\boldsymbol{\Sigma}') \to \mathsf{X} \to \mathsf{S}$ is projective (cf.[**9**], [**110**]). This is a toric version of relative Chow's lemma.

1.15. Lemma *Let* h *be a function in* $\mathrm{SF}_{\mathsf{N}}(\boldsymbol{\Sigma}, \mathfrak{K})$ *for* $\mathfrak{K} = \mathbb{Z}$*,* $\mathbb{Q}$*, or* $\mathbb{R}$*. Suppose that* h *is upper convex on* $\phi^{-1}\boldsymbol{\lambda}$ *for any* $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$*. Then there exist a free abelian group* $\mathsf{N}_\flat$*, homomorphisms* $\mu \colon \mathsf{N} \to \mathsf{N}_\flat$*,* $\nu \colon \mathsf{N}_\flat \to \mathsf{L}$*, a fan* $\boldsymbol{\Sigma}_\flat$ *of* $\mathsf{N}_\flat$*, and a support function* $h_\flat \in \mathrm{SF}_{\mathsf{N}_\flat}(\boldsymbol{\Sigma}_\flat, \mathfrak{K})$ *such that*

(1) μ *is surjective and* $\nu \circ \mu = \phi$*,*
(2) $(\mathsf{N}, \boldsymbol{\Sigma}) \to (\mathsf{N}_\flat, \boldsymbol{\Sigma}_\flat)$ *and* $(\mathsf{N}_\flat, \boldsymbol{\Sigma}_\flat) \to (\mathsf{L}, \boldsymbol{\Lambda})$ *are morphisms of fans,*
(3) *the function* $h(x) - h_\flat(\mu(x))$ *is linear on* $x \in |\boldsymbol{\Sigma}|$*,*
(4) $h_\flat$ *is strictly convex on* $(\boldsymbol{\Sigma}_\flat)_{\boldsymbol{\lambda}} = \{\boldsymbol{\sigma}_\flat \in \boldsymbol{\Sigma}_\flat \mid \nu(\boldsymbol{\sigma}_\flat) \subset \boldsymbol{\lambda}\}$ *for any* $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$*.*

In particular, D_h *is* $\mathfrak{K}$*-linearly equivalent to the pullback of the relatively ample* $\mathfrak{K}$*-divisor* $\mathsf{D}_{h_\flat}$ *of* $\mathsf{T}_{\mathsf{N}_\flat}(\boldsymbol{\Sigma}_\flat)$ *over* S*.*

Proof. We set

$$V_h = \{x \in |\boldsymbol{\Sigma}|\,;\, \phi(x) = 0 \text{ and } h(-x) = -h(x)\},$$
$$C_{\boldsymbol{\lambda},m} = \{x \in |\boldsymbol{\Sigma}_{\boldsymbol{\lambda}}|\,;\, \langle m, x\rangle = h(x)\}$$

for $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$ and $m \in \square_h(\phi^{-1}\boldsymbol{\lambda})$. Then $C_{\boldsymbol{\lambda},m}$ is a convex cone, since

$$h(x+y) \geq h(x) + h(y) = \langle m, x+y\rangle \geq h(x+y)$$

for x, $y \in C_{\boldsymbol{\lambda},m}$. If x, $-x \in C_{\boldsymbol{\lambda},m}$, then $x \in V_h$, since $\boldsymbol{\lambda}$ is strictly convex. If $x \in V_h$, then $x \in C_{\boldsymbol{\lambda},m}$ for any $\boldsymbol{\lambda}$, m by $-h(-x) \geq \langle m, x\rangle \geq h(x)$. Therefore, for any $\boldsymbol{\lambda}$ and m, V_h is the maximum vector subspace of $\mathsf{N}_{\mathbb{R}}$ contained in the convex cone $C_{\boldsymbol{\lambda},m}$.

Let $\mathsf{N}_\flat$ be the image of the natural homomorphism $\mu \colon \mathsf{N} \to \mathsf{N}_{\mathbb{R}}/V_h$. Then $\mu(C_{\boldsymbol{\lambda},m})$ is a strictly convex rational polyhedral cone and the set

$$\boldsymbol{\Sigma}_\flat = \{\mu(C_{\boldsymbol{\lambda},m}) \mid \boldsymbol{\lambda} \in \boldsymbol{\Lambda}, m \in \square_h(\phi^{-1}\boldsymbol{\lambda})\}$$

is a fan of $\mathsf{N}_\flat$. Here, the support of $(\mathbf{\Sigma}_\flat)_{\boldsymbol{\lambda}}$ coincides with $\nu^{-1}\boldsymbol{\lambda}$ for the induced homomorphism $\nu\colon \mathsf{N}_\flat \to \mathsf{L}$. We choose a maximal cone $\boldsymbol{\sigma} \in \mathbf{\Sigma_0}$ and $l_{\boldsymbol{\sigma}} \in \square_h(\phi^{-1}\mathbf{0}) \cap \mathfrak{K}$ satisfying $h(x) = \langle l_{\boldsymbol{\sigma}}, x\rangle$ for $x \in \boldsymbol{\sigma}$. We define $h_\flat \in \mathrm{SF}_{\mathsf{N}}(\mathbf{\Sigma}, \mathfrak{K})$ by $h_\flat(x) := h(x) - \langle l_{\boldsymbol{\sigma}}, x\rangle$. Then $h_\flat$ descends to a support function belonging to $\mathrm{SF}_{\mathsf{N}_\flat}(\mathbf{\Sigma}_\flat, \mathfrak{K})$. Thus $h_\flat$ is strictly convex on $(\mathbf{\Sigma}_\flat)_{\boldsymbol{\lambda}}$ for any $\boldsymbol{\lambda} \in \mathbf{\Lambda}$. □

1.16. Lemma *Let h be a $\mathbf{\Sigma}$-linear support function.*

(1) D_h *is* f*-pseudo-effective if and only if* $\square_h(\phi^{-1}\mathbf{0}) \neq \emptyset$.

(2) *Suppose that* D_h *is* f*-pseudo-effective. Then*

$$\sigma_{\mathbf{\Gamma}_v}(\mathsf{D}_h; \mathsf{X}/\mathsf{S}) = h^{\dagger}_{\mathbf{\Sigma}/\mathbf{\Lambda}}(v) - h(v)$$

for $v \in \mathrm{Ver}(\mathbf{\Sigma})$*. In particular,* D_h *is* f*-movable if and only if* $h^{\dagger}_{\mathbf{\Sigma}/\mathbf{\Lambda}}(v) = h(v)$ *for any* $v \in \mathrm{Ver}(\mathbf{\Sigma})$.

Proof. By taking a finite subdivision of $\mathbf{\Sigma}$, we may assume from the first that X is non-singular and there is a function $a \in \mathrm{SF}_{\mathsf{N}}(\mathbf{\Sigma}, \mathbb{Z})$ with $\mathsf{A} = \mathsf{D}_a$ being f-ample.

(1) For $\boldsymbol{\lambda} \in \mathbf{\Lambda}$, let us denote $\mathsf{S}_{\boldsymbol{\lambda}} = \mathsf{T}_{\mathsf{L}}(\boldsymbol{\lambda})$ and $\mathsf{X}_{\boldsymbol{\lambda}} = \mathsf{T}_{\mathsf{N}}(\mathbf{\Sigma}_{\boldsymbol{\lambda}}) = \mathsf{f}^{-1}\mathsf{S}_{\boldsymbol{\lambda}}$. If $m \in \square_h(\phi^{-1}\boldsymbol{\lambda})$, then $\mathrm{div}(\mathsf{e}(m)) + \mathsf{D}_h \geq 0$ over $\mathsf{X}_{\boldsymbol{\lambda}}$. Hence if $\square_h(\phi^{-1}\mathbf{0}) \neq \emptyset$, then D_h restricted to $\mathsf{X}_{\boldsymbol{\lambda}}$ is $\mathbb{R}$-linearly equivalent to an effective $\mathbb{R}$-divisor for any $\boldsymbol{\lambda} \in \mathbf{\Lambda}$. Thus one implication follows. Next, suppose that $\square_h(\phi^{-1}\mathbf{0}) = \emptyset$. This is equivalent to $\triangle_h(\phi^{-1}\mathbf{0}) \ni (0, 1)$, i.e.,

$$(0, 1) = \sum\nolimits_{v \in \mathrm{Ver}(\mathbf{\Sigma_0})} r_v(v, h(v))$$

for some $r_v \in \mathbb{R}_{\geq 0}$. If $m \in \mathsf{M} \cap \square_{l(kh+a)}(\phi^{-1}\mathbf{0})$ for some k, $l \in \mathbb{N}$, then $\langle m, v\rangle \geq lkh(v) + la(v)$ for all $v \in \mathrm{Ver}(\mathbf{\Sigma_0})$. Thus

$$0 = \frac{1}{l}\sum r_v\langle m, v\rangle \geq \sum(kr_v h(v) + a(v)) = k + \sum a(v).$$

In particular, if $k \gg 0$, then no effective $\mathbb{R}$-divisor on $\mathsf{X}_{\mathbf{0}} = \mathsf{f}^{-1}\mathsf{T}_{\mathsf{L}}$ is linearly equivalent to $l(k\mathsf{D}_h + \mathsf{A})$ for any $l \in \mathbb{N}$, by (IV-4). Thus the other implication follows.

(2) Let us fix a vertex $v \in \mathrm{Ver}(\mathbf{\Sigma})$. For $\boldsymbol{\lambda} \in \mathbf{\Lambda}$ with $\phi(v) \in \boldsymbol{\lambda}$, we have

$$\begin{aligned}\inf\{\mathrm{mult}_{\mathbf{\Gamma}_v} \Delta \mid 0 \leq \Delta \sim_{\mathbb{R}} \mathsf{D}_h|_{\mathsf{X}_{\boldsymbol{\lambda}}}\} &= \inf\{\langle m, v\rangle - h(v) \mid m \in \square_h(\phi^{-1}\boldsymbol{\lambda})\} \\ &= h^{\dagger}_{\mathbf{\Sigma}/\mathbf{\Lambda}}(v) - h(v),\end{aligned}$$

by (IV-4). Hence, if D_h is f-big, then $h^{\dagger}_{\mathbf{\Sigma}/\mathbf{\Lambda}}(v) - h(v) = \sigma_{\mathbf{\Gamma}_v}(\mathsf{D}_h; \mathsf{X}/\mathsf{S})$. In general, $\sigma_{\mathbf{\Gamma}_v}(\mathsf{D}_h; \mathsf{X}/\mathsf{S}) \leq h^{\dagger}_{\mathbf{\Sigma}/\mathbf{\Lambda}}(v) - h(v)$ holds. In order to show the equality in general case, we may assume $\sigma_{\mathbf{\Gamma}_v}(\mathsf{D}_h; \mathsf{X}/\mathsf{S}) = 0$, by replacing D_h with $\mathsf{D}_h - \sigma_{\mathbf{\Gamma}_v}(\mathsf{D}_h; \mathsf{X}/\mathsf{S})\mathbf{\Gamma}_v$. We shall derive a contradiction from the assumption: $h^{\dagger}_{\mathbf{\Sigma}/\mathbf{\Lambda}}(v) > h(v)$. Then there exist vertices $v_i \in \mathrm{Ver}(\mathbf{\Sigma}_{\boldsymbol{\lambda}})$ and real numbers $r_i > 0$ such that $v = \sum r_i v_i$ and $\sum r_i h(v_i) > h(v)$. However $(h + \varepsilon a)^{\dagger}_{\mathbf{\Sigma}/\mathbf{\Lambda}}(v) = (h + \varepsilon a)(v)$ for any $\varepsilon > 0$, since

$\mathsf{D}_{h+\varepsilon a} = \mathsf{D}_h + \varepsilon \mathsf{A}$ is f-big. Hence

$$h(v) + \varepsilon a(v) \geq \sum r_i(h(v_i) + \varepsilon a(v_i)) = \sum r_i h(v_i) + \varepsilon \sum r_i a(v_i).$$

Taking $\varepsilon \to 0$, we have a contradiction. □

1.17. Theorem (cf. [**57**]) *Let* $\mathsf{f}\colon \mathsf{X} = \mathsf{T}_\mathsf{N}(\boldsymbol{\Sigma}) \to \mathsf{S} = \mathsf{T}_\mathsf{L}(\boldsymbol{\Lambda})$ *be the morphism induced from a proper morphism* $\phi\colon (\mathsf{N}, \boldsymbol{\Sigma}) \to (\mathsf{L}, \boldsymbol{\Lambda})$ *of finite fans. Then any* f*-pseudo-effective* $\mathbb{R}$*-Cartier divisor of* X *admits a relative Zariski-decomposition over* S*.*

PROOF. We may assume that X is non-singular and is projective over S. We have only to consider the $\mathbb{R}$-divisor D_h for $h \in \mathrm{SF}_\mathsf{N}(\boldsymbol{\Sigma}, \mathbb{R})$ with $\square_h(\phi^{-1}\mathbf{0}) \neq \emptyset$. There is a finite subdivision $\boldsymbol{\Sigma}'$ of $\boldsymbol{\Sigma}$ with $h^\dagger = h^\dagger_{\boldsymbol{\Sigma}/\boldsymbol{\Lambda}} \in \mathrm{SF}_\mathsf{N}(\boldsymbol{\Sigma}', \mathbb{R})$. We may assume that $\mathsf{X}' = \mathsf{T}_\mathsf{N}(\boldsymbol{\Sigma}')$ is non-singular and is projective over S. Let $\mu\colon \mathsf{X}' \to \mathsf{X}$ be the induced projective birational morphism. Then the effective $\mathbb{R}$-divisor $\mu^*\mathsf{D}_h - \mathsf{D}_{h^\dagger}$ is the negative part of the relative σ-decomposition of $\mu^*\mathsf{D}_h$ over S by **1.16**-(2). This is a relative Zariski-decomposition over S since the positive part $\mathsf{D}_{h^\dagger}$ is relatively nef by **1.13**-(3). □

1.18. Theorem *Let* $f\colon X \to Y$ *be a proper surjective morphism of normal complex analytic varieties. Suppose that, for any point* $y \in Y$*, there exist an open neighborhood* $\mathcal{Y}$*, a proper morphism* $(\mathsf{N}, \boldsymbol{\Sigma}) \to (\mathsf{L}, \boldsymbol{\Lambda})$ *of finite fans, and a smooth morphism* $\mathcal{Y} \to \mathbb{T}_\mathsf{L}(\boldsymbol{\Lambda})$ *such that*

$$f^{-1}\mathcal{Y} \simeq \mathbb{T}_\mathsf{N}(\boldsymbol{\Sigma}) \times_{\mathbb{T}_\mathsf{L}(\boldsymbol{\Lambda})} \mathcal{Y}$$

over $\mathcal{Y}$*. Then any* f*-pseudo-effective* $\mathbb{R}$*-Cartier divisor of* X *admits a relative Zariski-decomposition over* Y*.*

PROOF. Let D be an f-pseudo-effective $\mathbb{R}$-Cartier divisor on X. For a point $y \in Y$, let $\mathcal{X} = f^{-1}\mathcal{Y}$ for the open neighborhood $\mathcal{Y}$ above. We have the vanishing $\mathrm{R}^i f_*\mathcal{O}_X = 0$ for $i > 0$ and an isomorphism

$$\mathrm{R}^1 f_*\mathcal{O}_X^\star \simeq \mathrm{R}^2 f_*\mathbb{Z}_X.$$

Hence we may assume that there exist an $\mathbb{R}$-Cartier divisor E of $\mathcal{Y}$ and a support function $h \in \mathrm{SF}_\mathsf{N}(\boldsymbol{\Sigma}, \mathbb{R})$ such that $D|_\mathcal{X} \sim_\mathbb{R} f^*E + p_1^*D_h$ for the first projection $p_1\colon \mathcal{X} \to \mathbb{T}_\mathsf{N}(\boldsymbol{\Sigma})$. By **1.17**, there exists a bimeromorphic morphism $\mu\colon \mathcal{X}' \to \mathcal{X}$ such that the positive part P of the relative σ-decomposition of $\mu^*(D|_\mathcal{X})$ is relatively nef over $\mathcal{Y}$. By **1.15**, we may assume that the $\mathbb{R}$-divisor P is relatively ample over $\mathcal{X}$. Then μ and P are uniquely determined up to isomorphisms. Gluing $\mathcal{X}'$ and P for such neighborhoods $\mathcal{Y}$, we obtain a bimeromorphic morphism $g\colon X' \to X$ such that the positive part of the relative σ-decomposition of g^*D is relatively nef over Y and is relatively ample over X. □

§2. Toric bundles

§2.a. Definition of toric bundles. We shall give a relative version of the notion of toric variety (cf. [**125**]). Let M and N be the same free abelian groups as before.

2.1. Definition Let S be a complex analytic space and let

$$\mathcal{L}\colon \mathsf{M} \ni m \longmapsto \mathcal{L}^m \in \operatorname{Pic}(S)$$

be a group homomorphism. For a subset $\mathcal{S} \subset \mathsf{M}$, we set

$$\mathcal{L}[\mathcal{S}] := \bigoplus\nolimits_{m\in\mathcal{S}} \mathcal{L}^m.$$

For a strictly convex rational polyhedral cone $\boldsymbol{\sigma} \subset \mathsf{N}_{\mathbb{R}}$, the *affine toric bundle* over S of type $(\mathsf{N}, \boldsymbol{\sigma}, \mathcal{L})$ is defined by

$$\mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma}, \mathcal{L}) = \operatorname{Specan}_S \mathcal{L}[\boldsymbol{\sigma}^\vee \cap \mathsf{M}].$$

Similarly, for a fan $\boldsymbol{\Sigma}$ of N, the *toric bundle* $\mathbb{T}_{\mathsf{N}}(\boldsymbol{\Sigma}, \mathcal{L})$ of type $(\mathsf{N}, \boldsymbol{\Sigma}, \mathcal{L})$ is defined as the natural union of $\mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma}, \mathcal{L})$ for $\boldsymbol{\sigma} \in \boldsymbol{\Sigma}$.

Remark $\mathcal{L}$ is regarded as an element of $\mathsf{N} \otimes \operatorname{Pic}(S) = \mathrm{H}^1(S, \mathsf{N} \otimes \mathcal{O}_S^\star)$, in which $\mathsf{N} \otimes \mathcal{O}_S^\star$ is regarded as the sheaf of germs of holomorphic mappings $S \to \mathbb{T}_{\mathsf{N}}$. By the action of $\mathbb{T}_{\mathsf{N}}$ on $\mathbb{T}_{\mathsf{N}}(\boldsymbol{\Sigma})$, $\mathbb{T}_{\mathsf{N}}(\boldsymbol{\Sigma}, \mathcal{L}) \to S$ is the fiber bundle obtained from $\mathbb{T}_{\mathsf{N}}(\boldsymbol{\Sigma}) \times S \to S$ by the twist by $\mathcal{L}$. The cohomology class in $H^1(S, \mathsf{N} \otimes \mathcal{O}_S^\star)$ attached to the principal fiber bundle $\mathbb{T}_{\mathsf{N}}(\mathbf{0}, \mathcal{L}) \to S$ is $-\mathcal{L}$.

There is a natural surjective $\mathcal{O}_S$-algebra homomorphism $\mathcal{L}[\boldsymbol{\sigma}^\vee \cap \mathsf{M}] \twoheadrightarrow \mathcal{L}[\boldsymbol{\sigma}^\perp \cap \mathsf{M}]$ such that the kernel is $\mathcal{L}[(\boldsymbol{\sigma}^\vee \smallsetminus \boldsymbol{\sigma}^\perp) \cap \mathsf{M}]$. This induces a closed immersion

$$\mathbb{T}_{\mathsf{N}(\boldsymbol{\sigma})}(\mathbf{0}, \mathcal{L}) \hookrightarrow \mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma}, \mathcal{L}).$$

The left hand side is fiberwise an orbit of $\mathbb{T}_{\mathsf{N}}$ and is denoted by $\mathbb{O}_{\boldsymbol{\sigma}}(\mathcal{L})$. For a face $\boldsymbol{\tau} \prec \boldsymbol{\sigma}$, the closure of $\mathbb{O}_{\boldsymbol{\tau}}(\mathcal{L})$ in $\mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma}, \mathcal{L})$ is isomorphic to $\mathbb{T}_{\mathsf{N}(\boldsymbol{\tau})}(\boldsymbol{\sigma}/\boldsymbol{\tau}, \mathcal{L})$ by the natural surjective homomorphism

$$\mathcal{L}[\boldsymbol{\sigma}^\vee \cap \mathsf{M}] \twoheadrightarrow \mathcal{L}[\boldsymbol{\sigma}^\vee \cap \boldsymbol{\tau}^\perp \cap \mathsf{M}].$$

The closure $\mathbb{V}(\boldsymbol{\sigma}, \mathcal{L})$ of $\mathbb{O}_{\boldsymbol{\sigma}}(\mathcal{L})$ in $\mathbb{T}_{\mathsf{N}}(\boldsymbol{\Sigma}, \mathcal{L})$ is isomorphic to $\mathbb{T}_{\mathsf{N}(\boldsymbol{\sigma})}(\boldsymbol{\Sigma}/\boldsymbol{\sigma}, \mathcal{L})$.

Suppose that S is a normal complex analytic variety. Let $p\colon Y \to S$ be the morphism $\mathbb{T}_{\mathsf{N}}(\boldsymbol{\Sigma}, \mathcal{L}) \to S$. An element $m \in \mathsf{M}$ defines a meromorphic section $\mathsf{e}(m)$ of $p^*\mathcal{L}^{-m}$ by the natural embedding

$$\mathcal{O}_S \simeq \mathcal{L}^{-m} \otimes \mathcal{L}^m \hookrightarrow \mathcal{L}^{-m} \otimes \mathcal{L}[\mathsf{M}].$$

For a vertex $v \in \operatorname{Ver}(\boldsymbol{\Sigma})$, let Γ_v be the prime divisor $\mathbb{V}(\mathbb{R}_{\geq 0}v, \mathcal{L})$. The divisor $\operatorname{div}(\mathsf{e}(m))$ associated with the meromorphic section $\mathsf{e}(m)$ of $p^*\mathcal{L}^{-m}$ is written by

$$\sum\nolimits_{v\in\operatorname{Ver}(\boldsymbol{\Sigma})} \langle m, v\rangle \Gamma_v$$

as a Weil divisor. In particular,

$$\mathcal{O}_Y\left(\sum\nolimits_{v\in\operatorname{Ver}(\boldsymbol{\Sigma})} \langle m, v\rangle \Gamma_v\right) \simeq p^*\mathcal{L}^{-m}.$$

Even for $m \in \mathsf{M}_{\mathbb{R}}$, we can define $\operatorname{div}(\mathsf{e}(m))$ to be an $\mathbb{R}$-Cartier divisor by the linearity of $\operatorname{div} \circ \mathsf{e} \colon \mathsf{M} \to \operatorname{CDiv}(Y, \mathbb{R})$. Similarly, we denote by $\mathcal{L}^m$ the image of m under $\mathcal{L} \otimes \mathbb{R} \colon \mathsf{M}_{\mathbb{R}} \to \operatorname{Pic}(S, \mathbb{R})$. Then $\operatorname{div}(\mathsf{e}(m)) \sim_{\mathbb{R}} f^*\mathcal{L}^{-m}$ for $m \in \mathsf{M}_{\mathbb{R}}$. For $h \in \mathrm{SFC}_{\mathsf{N}}(\mathbf{\Sigma}, \mathbb{R})$, we define

$$D_h = \sum\nolimits_{v \in \operatorname{Ver}(\mathbf{\Sigma})} (-h(v))\Gamma_v.$$

If $h \in \mathrm{SF}_{\mathsf{N}}(\mathbf{\Sigma}, \mathbb{R})$, then D_h is $\mathbb{R}$-Cartier.

Remark We can consider a kind of differential form:

$$\mathrm{d} \log \mathsf{e}(m) = \mathsf{e}(m)^{-1}\, \mathrm{d}\, \mathsf{e}(m)$$

for $m \in \mathsf{M}$. It is not a well-defined meromorphic 1-form on $Y = \mathbb{T}_{\mathsf{N}}(\mathbf{\Sigma}, \mathcal{L})$. Suppose that $\mathbf{\Sigma}$ is a non-singular fan and S is non-singular. Let B be the normal crossing divisor $Y \setminus \mathbb{T}_{\mathsf{N}}(\mathbf{0}, \mathcal{L})$. Then $\mathrm{d} \log \mathsf{e}(m)$ is regard as a global section of the sheaf $\Omega^1_{Y/S}(\log B)$ of germs of relative logarithmic 1-forms. Moreover, we have an isomorphism

$$\mathsf{M} \otimes \mathcal{O}_Y \simeq \Omega^1_{Y/S}(\log B).$$

In particular, $K_Y + B \sim p^*K_S$.

2.2. Proposition *Let Y be a toric bundle $\mathbb{T}_{\mathsf{N}}(\mathbf{\Sigma}, \mathcal{L})$ over a complex analytic space S and let X be a toric bundle $\mathbb{T}_{\mathsf{N}_0}(\mathbf{\Sigma}_0, \mathcal{L}_0)$ over Y. Let $p \colon Y \to S$ and $\pi \colon X \to Y$ be the structure morphisms. Assume that $\mathcal{L}_0 \colon \mathsf{M}_0 = \operatorname{Hom}(\mathsf{N}_0, \mathbb{Z}) \to \operatorname{Pic}(Y)$ is the composite of a homomorphism $\mathsf{M}_0 \to \mathrm{SF}_{\mathsf{N}}(\mathbf{\Sigma}, \mathbb{Z}) \oplus \operatorname{Pic}(S)$ and the natural homomorphism $\mathrm{SF}_{\mathsf{N}}(\mathbf{\Sigma}, \mathbb{Z}) \oplus \operatorname{Pic}(S) \ni (h, \mathcal{M}) \mapsto \mathcal{O}_Y(D_h) \otimes p^*\mathcal{M} \in \operatorname{Pic}(Y)$. Then X is isomorphic to a toric bundle $\mathbb{T}_{\mathsf{N}_0 \oplus \mathsf{N}}(\widetilde{\mathbf{\Sigma}}, \widetilde{\mathcal{L}})$ over S and π is induced from the second projection $\mathsf{N}_0 \oplus \mathsf{N} \to \mathsf{N}$.*

Proof. The homomorphism $\mathsf{M}_0 \to \mathrm{SF}_{\mathsf{N}}(\mathbf{\Sigma}, \mathbb{Z}) \oplus \operatorname{Pic}(S)$ is defined by an element $\mathbf{h} \in \mathrm{SF}_{\mathsf{N}}(\mathbf{\Sigma}, \mathbb{Z}) \otimes \mathsf{N}_0$ and by a homomorphism $\mathcal{L}_1 \colon \mathsf{M}_0 \to \operatorname{Pic}(S)$. Here $\mathbf{h}$ is regarded as a continuous function $|\mathbf{\Sigma}| \to (\mathsf{N}_0)_{\mathbb{R}} = \mathsf{N}_0 \otimes \mathbb{R}$ such that the restriction $\mathbf{h}|_{\boldsymbol{\sigma}}$ to a cone $\boldsymbol{\sigma} \in \mathbf{\Sigma}$ is linear and is induced from a homomorphism $\mathsf{N}_{\boldsymbol{\sigma}} \to \mathsf{N}_0$. For $m_0 \in \mathsf{M}_0$, we write by $\langle m_0, \mathbf{h} \rangle$ the support function $x \mapsto \langle m_0, \mathbf{h}(x) \rangle$. Then

$$\mathcal{L}_0^{m_0} = \mathcal{O}_Y(D_{\langle m_0, \mathbf{h} \rangle}) \otimes p^*\mathcal{L}_1^{m_0}.$$

For $\boldsymbol{\sigma} \in \mathbf{\Sigma}$, we can take a homomorphism $\psi_{\boldsymbol{\sigma}} \colon \mathsf{M}_0 \to \mathsf{M}$ such that the composite $\mathsf{M}_0 \to \mathsf{M} \to \mathsf{M}_{\boldsymbol{\sigma}}$ is dual to the homomorphism $\mathsf{N}_{\boldsymbol{\sigma}} \to \mathsf{N}_0$ above defined by $\mathbf{h}$. Then $\langle m_0, \mathbf{h}(x) \rangle = \langle \psi_{\boldsymbol{\sigma}}(m_0), x \rangle$ for $x \in \boldsymbol{\sigma}$. In particular,

$$\square_{\langle m_0, \mathbf{h} \rangle}(\boldsymbol{\sigma}) = \{ m \in \mathsf{M}_{\mathbb{R}} \mid \langle m, x \rangle \geq \langle m_0, \mathbf{h}(x) \rangle \text{ for } x \in \boldsymbol{\sigma} \} = \psi^{\vee}_{\boldsymbol{\sigma}}(m_0) + \boldsymbol{\sigma}^{\vee}.$$

For cones $\boldsymbol{\sigma}_0 \in \mathbf{\Sigma}_0$ and $\boldsymbol{\sigma} \in \mathbf{\Sigma}$, let $Y_{\boldsymbol{\sigma}} \subset Y$ be the open subset $\mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma}, \mathcal{L})$ and let $X_{\boldsymbol{\sigma}_0, \boldsymbol{\sigma}} \subset \pi^{-1} Y_{\boldsymbol{\sigma}}$ be the open subset $\mathbb{T}_{\mathsf{N}_0}(\boldsymbol{\sigma}_0, \mathcal{L}_0)$ over $Y_{\boldsymbol{\sigma}}$. Then $Y_{\boldsymbol{\sigma}} \simeq \operatorname{Specan}_S \mathcal{L}[\boldsymbol{\sigma}^{\vee} \cap \mathsf{M}]$ and the invertible sheaf $\mathcal{O}_{Y_{\boldsymbol{\sigma}}}(D_h)$ for $h \in \mathrm{SF}_{\mathsf{N}}(\mathbf{\Sigma})$ is associated with the $\mathcal{L}[\boldsymbol{\sigma}^{\vee} \cap \mathsf{M}]$-module $\mathcal{L}[\square_h(\boldsymbol{\sigma}) \cap \mathsf{M}]$. Similarly, $X_{\boldsymbol{\sigma}_0, \boldsymbol{\sigma}} \simeq \operatorname{Specan}_{Y_{\boldsymbol{\sigma}}} \mathcal{L}_0[\boldsymbol{\sigma}_0^{\vee} \cap \mathsf{M}_0]$. Therefore, $X_{\boldsymbol{\sigma}_0, \boldsymbol{\sigma}} \simeq \operatorname{Specan}_S \mathcal{A}_{\boldsymbol{\sigma}_0, \boldsymbol{\sigma}}$ for the subalgebra

$$\mathcal{A}_{\boldsymbol{\sigma}_0, \boldsymbol{\sigma}} = \bigoplus\nolimits_{m_0 \in \mathsf{M}_0 \cap \boldsymbol{\sigma}_0^{\vee},\, m \in \square_{\langle m_0, \mathbf{h} \rangle}(\boldsymbol{\sigma})} \mathcal{L}_1^{m_0} \otimes \mathcal{L}^m \subset \widetilde{\mathcal{L}}[\mathsf{M}_0 \oplus \mathsf{M}],$$

where $\widetilde{\mathcal{L}} := \mathcal{L}_1 \oplus \mathcal{L} \in (\mathsf{N}_0 \oplus \mathsf{N}) \otimes \mathrm{Pic}(S)$. For the cone

$$C(\boldsymbol{\sigma}_0, \boldsymbol{\sigma}; \mathbf{h}) := \{(x_0, x) \in (\mathsf{N}_0)_{\mathbb{R}} \oplus \mathsf{N}_{\mathbb{R}} \mid x_0 + \mathbf{h}(x) \in \boldsymbol{\sigma}_0,\ x \in \boldsymbol{\sigma}\},$$

we have an isomorphism $X_{\boldsymbol{\sigma}_0, \boldsymbol{\sigma}} \simeq \mathbb{T}_{\mathsf{N}_0 \oplus \mathsf{N}}(C(\boldsymbol{\sigma}_0, \boldsymbol{\sigma}; \mathbf{h}), \widetilde{\mathcal{L}})$ over S, since

$$\{(m_0, m) \in \mathsf{M}_0 \oplus \mathsf{M} \mid m_0 \in \boldsymbol{\sigma}_0^{\vee},\ m \in \Box_{\langle m_0, \mathbf{h}\rangle}(\boldsymbol{\sigma})\} = C(\boldsymbol{\sigma}_0, \boldsymbol{\sigma}; \mathbf{h})^{\vee} \cap (\mathsf{M}_0 \oplus \mathsf{M}).$$

The structure morphism $\pi\colon X_{\boldsymbol{\sigma}_0, \boldsymbol{\sigma}} \to Y_{\boldsymbol{\sigma}}$ is interpreted as a morphism of toric bundles over S which is induced from the second projection $\mathsf{N}_0 \oplus \mathsf{N} \to \mathsf{N}$.

For faces $\boldsymbol{\tau}_0 \prec \boldsymbol{\sigma}_0$ and $\boldsymbol{\tau} \prec \boldsymbol{\sigma}$, the cone $C(\boldsymbol{\tau}_0, \boldsymbol{\tau}; \mathbf{h})$ is a face of $C(\boldsymbol{\sigma}_0, \boldsymbol{\sigma}; \mathbf{h})$ and the open immersion $X_{\boldsymbol{\tau}_0, \boldsymbol{\tau}} \subset X_{\boldsymbol{\sigma}_0, \boldsymbol{\sigma}}$ is induced from the open immersion as toric bundles over S. For other cones $\boldsymbol{\sigma}'_0 \in \boldsymbol{\Sigma}_0$ and $\boldsymbol{\sigma}' \in \boldsymbol{\Sigma}$, we have $C(\boldsymbol{\sigma}_0, \boldsymbol{\sigma}; \mathbf{h}) \cap C(\boldsymbol{\sigma}'_0, \boldsymbol{\sigma}'; \mathbf{h}) = C(\boldsymbol{\sigma}_0 \cap \boldsymbol{\sigma}'_0, \boldsymbol{\sigma}' \cap \boldsymbol{\sigma}; \mathbf{h})$. Thus

$$\boldsymbol{\Sigma}_{\mathbf{h}} := \{C(\boldsymbol{\sigma}_0, \boldsymbol{\sigma}; \mathbf{h}) \mid \boldsymbol{\sigma}_0 \in \boldsymbol{\Sigma}_0,\ \boldsymbol{\sigma} \in \boldsymbol{\Sigma}\}$$

is a fan of $\mathsf{N}_0 \oplus \mathsf{N}$ and $X \simeq \mathbb{T}_{\mathsf{N}_0 \oplus \mathsf{N}}(\boldsymbol{\Sigma}_{\mathbf{h}}, \widetilde{\mathcal{L}})$ over S. □

§2.b. Pseudo-effective divisors on toric bundles. Suppose that $\boldsymbol{\Sigma}$ is a complete fan and that S is a normal complex analytic variety. Let $p\colon Y \to S$ be the structure morphism of the toric bundle $Y = \mathbb{T}_{\mathsf{N}}(\boldsymbol{\Sigma}, \mathcal{L})$.

2.3. Lemma (1) *For a line bundle $\mathcal{M}$ of Y, there exist a line bundle $\mathcal{N}$ of S and a support function $h \in \mathrm{SF}_{\mathsf{N}}(\boldsymbol{\Sigma}, \mathbb{Z})$ such that $\mathcal{M} \simeq p^*\mathcal{N} \otimes \mathcal{O}_Y(D_h)$. In particular, there is an isomorphism*

$$p_*\mathcal{M} \simeq \mathcal{N} \otimes \mathcal{L}[\Box_h \cap \mathsf{M}].$$

(2) *For an $\mathbb{R}$-Cartier divisor D of Y, there exists a support function $h \in \mathrm{SF}_{\mathsf{N}}(\boldsymbol{\Sigma}, \mathbb{R})$ such that $D \sim_{\mathbb{R}} p^*\Xi + D_h$ for some $\Xi \in \mathrm{Pic}(S, \mathbb{R})$.*

Proof. From the vanishing $\mathrm{R}^i\, p_*\mathcal{O}_Y = 0$ for $i > 0$, we have exact sequences

$$0 \to \mathrm{Pic}(S) \to \mathrm{Pic}(Y) \to \mathrm{H}^0(S, \mathrm{R}^2\, p_*\mathbb{Z}_Y),$$
$$0 \to \mathrm{Pic}(S, \mathbb{R}) \to \mathrm{Pic}(Y, \mathbb{R}) \to \mathrm{H}^0(S, \mathrm{R}^2\, p_*\mathbb{R}_Y).$$

On the toric variety $\mathbb{T}_{\mathsf{N}}(\boldsymbol{\Sigma})$, any line bundle is associated with the Cartier divisor D_h for some $h \in \mathrm{SF}_{\mathsf{N}}(\boldsymbol{\Sigma}, \mathbb{Z})$, and any $\mathbb{R}$-Cartier divisor is $\mathbb{R}$-linearly equivalent to D_h for some $h \in \mathrm{SF}_{\mathsf{N}}(\boldsymbol{\Sigma}, \mathbb{R})$. Thus, in (1), $\mathcal{M} \otimes \mathcal{O}_Y(-D_h)$ restricted to a fiber of p is numerically trivial for some $h \in \mathrm{SF}_{\mathsf{N}}(\boldsymbol{\Sigma}, \mathbb{Z})$, and hence $\mathcal{M} \simeq p^*\mathcal{N} \otimes \mathcal{O}_Y(D_h)$ for a line bundle $\mathcal{N}$ of S. Similarly, in (2), $D - D_h$ is p-numerically trivial for some $h \in \mathrm{SF}_{\mathsf{N}}(\boldsymbol{\Sigma}, \mathbb{R})$. Hence $D - D_h \sim_{\mathbb{R}} p^*\Xi$ for some $\Xi \in \mathrm{Pic}(S, \mathbb{R})$. Note that there is an isomorphism $p_*\mathcal{O}_Y(D_h) \simeq \mathcal{L}[\Box_h \cap \mathsf{M}]$ by (IV-4), since p is proper. □

For $h \in \mathrm{SF}_{\mathsf{N}}(\boldsymbol{\Sigma}, \mathbb{R})$, we write $h^{\dagger} = h^{\dagger}_{\mathsf{N}_{\mathbb{R}}}$ for short. Let $\mathcal{M}$ be an invertible sheaf of Y such that $\mathcal{M} \simeq f^*\mathcal{N} \otimes \mathcal{O}_Y(D_h)$ for some $\mathcal{N} \in \mathrm{Pic}(S)$ and $h \in \mathrm{SF}_{\mathsf{N}}(\boldsymbol{\Sigma}, \mathbb{Z})$. Then the following conditions are mutually equivalent by **1.13**:

(1) h is upper convex on $\mathsf{N}_{\mathbb{R}}$; (2) $\Box_h \neq \emptyset$ and $h^{\dagger} = h$;
(3) $\mathcal{M}$ is p-free; (4) $\mathcal{M}$ is p-nef.

Furthermore, $\mathcal{M}$ is p-ample if and only if h is strictly upper convex with respect to $\mathbf{\Sigma}$ by **1.14**. Let D be an $\mathbb{R}$-Cartier divisor of Y such that $D \sim_{\mathbb{R}} f^*E + D_h$ for some $\mathbb{R}$-Cartier divisor E of S and for $h \in \mathrm{SF}_{\mathsf{N}}(\mathbf{\Sigma}, \mathbb{R})$. Then the following conditions are mutually equivalent by **1.16**:

(1) $\square_h \neq \emptyset$ and $h = h^\dagger$; (2) h is upper convex; (3) D is p-nef.

If D is p-pseudo-effective, then $\sigma_{\Gamma_v}(D; Y/S) = h^\dagger(v) - h(v)$ for $v \in \mathrm{Ver}(\mathbf{\Sigma})$ by **1.16**.

Suppose that S is a normal projective variety. We study the (absolute) σ-decomposition for a pseudo-effective $\mathbb{R}$-Cartier divisor of $Y = \mathbb{T}_{\mathsf{N}}(\mathbf{\Sigma}, \mathcal{L})$. For an $\mathbb{R}$-Cartier divisor E of S and for a support function $h \in \mathrm{SF}_{\mathsf{N}}(\mathbf{\Sigma}, \mathbb{R})$, we define

$$\square_{\mathrm{PE}}(E, h) := \{m \in \square_h \mid E + \mathcal{L}^m \text{ is pseudo-effective}\},$$
$$\square_{\mathrm{Nef}}(E, h) := \{m \in \square_h \mid E + \mathcal{L}^m \text{ is nef}\}.$$

These are compact convex subsets of $\mathsf{M}_{\mathbb{R}}$.

2.4. Proposition *Suppose that S is a normal projective variety. Let $D = p^*E + D_h$ be an $\mathbb{R}$-Cartier divisor of $Y = \mathbb{T}_{\mathsf{N}}(\mathbf{\Sigma}, \mathcal{L})$ for $h \in \mathrm{SF}_{\mathsf{N}}(\mathbf{\Sigma}, \mathbb{R})$.*

(1) *D is pseudo-effective if and only if $\square_{\mathrm{PE}}(E, h) \neq \emptyset$.*

(2) *The following conditions are equivalent to each other*:

(a) *D is nef*;

(b) *$l_{\boldsymbol{\sigma}} \in \square_{\mathrm{Nef}}(E, h)$ for any maximal cone $\boldsymbol{\sigma} \in \mathbf{\Sigma}$, where $l_{\boldsymbol{\sigma}} \in \mathsf{M}_{\mathbb{R}}$ is defined by $h(x) = \langle l_{\boldsymbol{\sigma}}, x\rangle$ for $x \in \boldsymbol{\sigma}$* (cf. **1.7**);

(c) *$\square_{\mathrm{Nef}}(E, h) \neq \emptyset$ and, for any $x \in \mathsf{N}_{\mathbb{R}}$,*

$$h(x) = \min\{\langle m, x\rangle \mid m \in \square_{\mathrm{Nef}}(E, h)\}.$$

(3) *Suppose that D is pseudo-effective. Then*

$$\sigma_{p^{-1}\Theta}(D) = \min\{\sigma_\Theta(E + \mathcal{L}^m) \mid m \in \square_{\mathrm{PE}}(E, h)\},$$
$$\sigma_{\Gamma_v}(D) = \min\{\langle m, v\rangle \mid m \in \square_{\mathrm{PE}}(E, h)\} - h(v),$$

for any prime divisor $\Theta \subset S$ and for any $v \in \mathrm{Ver}(\mathbf{\Sigma})$.

(4) *Suppose that D is pseudo-effective. Then D is movable if and only if $\sigma_{p^{-1}\Theta}(D) = \sigma_{\Gamma_v}(D) = 0$ for any prime divisor $\Theta \subset S$ and for any $v \in \mathrm{Ver}(\mathbf{\Sigma})$.*

(5) *Suppose that D is pseudo-effective. Then D is numerically movable if and only if*

$$\{m \in \square_h \mid (E + \mathcal{L}^m)|_\Theta \text{ is pseudo-effective } \} \neq \emptyset, \quad \text{and}$$
$$\{m \in \square_{\mathrm{PE}}(E, h) \mid h(v) = \langle m, v\rangle\} \neq \emptyset,$$

for any prime divisor $\Theta \subset S$ and for any $v \in \mathrm{Ver}(\mathbf{\Sigma})$.

Proof. The image $c \in \mathsf{N} \otimes \mathrm{N}^1(S)$ of $\mathcal{L} \in \mathsf{N} \otimes \mathrm{Pic}(S)$ satisfies $\langle m, c\rangle = c_1(\mathcal{L}^m) \in \mathrm{N}^1(S)$ for $m \in \mathsf{M}_{\mathbb{R}}$. Let us consider the set

$$\Omega := \{(e, h, m) \in \mathrm{N}^1(S) \times \mathrm{SF}_{\mathsf{N}}(\mathbf{\Sigma}, \mathbb{R}) \times \mathsf{M}_{\mathbb{R}} \mid m \in \square_h,\ e + \langle m, c\rangle \in \mathrm{PE}(S)\}.$$

Then $\pi\colon \Omega \to \mathrm{N}^1(S) \times \mathrm{SF}_{\mathsf{N}}(\mathbf{\Sigma}, \mathbb{R})$ is proper, since $\square_h$ is compact for $h \in \mathrm{SF}_{\mathsf{N}}(\mathbf{\Sigma})$. In particular, $\pi(\Omega)$ is closed. Let us consider

$$\varphi\colon\ \mathrm{N}^1(S) \times \mathrm{SF}_{\mathsf{N}}(\mathbf{\Sigma}, \mathbb{R}) \ni (e, h) \mapsto p^*e + c_1(D_h) \in \mathrm{N}^1(Y).$$

Then (1) means that $\varphi^{-1}(\mathrm{PE}(Y)) = \pi(\Omega)$. We note the following $\mathbb{R}$-equivalence relation for $m \in \mathsf{M}_{\mathbb{R}}$:

$$D_h + p^*E \sim_{\mathbb{R}} \mathrm{div}(\mathsf{e}(m)) + D_h + p^*(E + \mathcal{L}^m). \tag{IV-5}$$

Thus $\varphi^{-1}(\mathrm{PE}(Y)) \supset \pi(\Omega)$. In the proof, we may assume that S and Y are non-singular and Y is projective over S.

(1) It is enough to show $\varphi^{-1}(\mathrm{Big}(Y) \cap \mathrm{NS}(Y)_{\mathbb{Q}}) \subset \pi(\Omega)$. Thus we may assume that D is a big $\mathbb{Q}$-divisor. In particular, E is a $\mathbb{Q}$-divisor and h is rational. Then kD are kE is Cartier and $\mathrm{H}^0(Y, kD) \neq 0$ for some $k \in \mathbb{N}$. In particular, $\mathrm{H}^0(S, \mathcal{L}^m + kE) \neq 0$ for some $m \in \mathsf{M} \cap k\square_h$ by (IV-4). Hence $(c_1(E), h) \in \pi(\Omega)$.

(2) (a) $\Rightarrow$ (b): Let $\boldsymbol{\sigma} \in \mathbf{\Sigma}$ be a maximal cone. Then $\mathbb{V}(\boldsymbol{\sigma}, \mathcal{L})$ is a section of $p\colon Y \to S$ and $h^{\boldsymbol{\sigma}}(x) = h(x) - \langle l_{\boldsymbol{\sigma}}, x\rangle \leq 0$ for any $x \in \mathsf{N}_{\mathbb{R}}$, since D_h is p-nef. Note that $D_{h^{\boldsymbol{\sigma}}} \cap \mathbb{V}(\boldsymbol{\sigma}, \mathcal{L}) = \emptyset$ and $D_{h^{\boldsymbol{\sigma}}} = D_h + \mathrm{div}(\mathsf{e}(l_{\boldsymbol{\sigma}}))$. Therefore, $D_h|_{\mathbb{V}(\boldsymbol{\sigma}, \mathcal{L})}$ is $\mathbb{R}$-linearly equivalent to $\mathcal{L}^{l_{\boldsymbol{\sigma}}}$. Thus $E + \mathcal{L}^{l_{\boldsymbol{\sigma}}}$ is nef and $l_{\boldsymbol{\sigma}} \in \square_{\mathrm{Nef}}(E, h)$.

(b) $\Rightarrow$ (c): For any $y \in \mathsf{N}_{\mathbb{R}}$, there is a maximal cone $\boldsymbol{\sigma} \in \mathbf{\Sigma}$ containing $y \in \boldsymbol{\sigma}$. Then $h(y) = \langle l_{\boldsymbol{\sigma}}, y\rangle = \min\{\langle m, y\rangle \mid m \in \square_{\mathrm{Nef}}(E, h)\}$.

(c) $\Rightarrow$ (b): h is upper-convex by the expression. For a vector $x_0 \in \boldsymbol{\sigma}$, there is an $m_0 \in \square_{\mathrm{Nef}}(E, h)$ such that $h(x_0) = \langle l_{\boldsymbol{\sigma}}, x_0\rangle = \langle m_0, x_0\rangle$. Since $m_0 - l_{\boldsymbol{\sigma}} \in \boldsymbol{\sigma}^{\vee}$, we infer that $m_0 = l_{\boldsymbol{\sigma}} \in \square_{\mathrm{Nef}}(E, h)$.

(b) $\Rightarrow$ (a): Let W be the intersection of the supports of effective $\mathbb{R}$-Cartier divisors $D_h + \mathrm{div}(\mathsf{e}(m))$ for $m \in \square_{\mathrm{Nef}}(E, h)$. Then W is written as the union of $\mathbb{V}(\boldsymbol{\sigma}, \mathcal{L})$ for suitable cones $\boldsymbol{\sigma} \in \mathbf{\Sigma}$. In particular, if $W \neq \emptyset$, then $W \supset \mathbb{V}(\boldsymbol{\sigma}, \mathcal{L})$ for a maximal cone $\boldsymbol{\sigma}$. Thus $W = \emptyset$ and D is nef.

(3) If $f\colon \Omega \to \mathbb{R}$ is a lower semi-continuous function, then

$$\tilde{f}(e, h) := \inf\{f(e, h, m) \mid (e, h, m) \in \Omega\} = \min\{f(e, h, m) \mid (e, h, m) \in \Omega\},$$

which gives rise to a lower semi-continuous function on $\pi(\Omega)$. For a prime divisor $\Theta \subset S$, σ_Θ is lower semi-continuous on $\mathrm{PE}(S)$. For a vertex $v \in \mathrm{Ver}(\mathbf{\Sigma})$, $m \mapsto \langle m, v\rangle$ is linear. Hence

$$r(E, h, \Theta) := \min\{\sigma_\Theta(E + \mathcal{L}^m) \mid m \in \square_{\mathrm{PE}}(E, h)\},$$
$$r(E, h, v) := \min\{\langle m, v\rangle \mid m \in \square_{\mathrm{PE}}(E, h)\} - h(v)$$

are well-defined, and $(E, h) \mapsto r(E, h, \Theta)$ and $(E, h) \mapsto r(E, h, v)$ are lower semi-continuous on $\pi(\Omega)$.

If $m \in \square_{\mathrm{PE}}(E, h)$, then

$$\sigma_{p^{-1}\Theta}(D) \leq \sigma_{p^{-1}\Theta}(p^*(E + \mathcal{L}^m)) = \sigma_\Theta(E + \mathcal{L}^m),$$
$$\sigma_{\Gamma_v}(D) \leq \mathrm{mult}_{\Gamma_v}(\mathrm{div}(\mathsf{e}(m)) + D_h) = \langle m, v\rangle - h(v),$$

by (IV-5), since $\mathrm{div}(\mathsf{e}(m)) + D_h$ is an effective $\mathbb{R}$-divisor containing no fiber of p. Thus $\sigma_{p^{-1}\Theta}(D) \leq r(E, h, \Theta)$ and $\sigma_{\Gamma_v}(D) \leq r(E, h, v)$.

Suppose that D is a big $\mathbb{Q}$-divisor. Then E is a $\mathbb{Q}$-divisor and h is rational. By (IV-4) and (IV-5), we infer that any effective $\mathbb{Q}$-divisor $\mathbb{Q}$-linearly equivalent to D is written by $\mathrm{div}(\mathsf{e}(m)) + D_h + p^*\Delta$ for some $m \in \square_h \cap \mathsf{M}_{\mathbb{Q}}$ and for some effective $\mathbb{Q}$-divisor $\Delta \sim_{\mathbb{Q}} E + \mathcal{L}^m$. Thus $\sigma_{p^{-1}\Theta}(D) = r(E, h, \Theta)$ and $\sigma_{\Gamma_v}(D) = r(E, h, v)$.

By the lower semi-continuity, the expected equalities also hold for any pseudo-effective $\mathbb{R}$-divisor $D = p^*E + D_h$.

(4) Let $\Gamma \subset Y$ be a prime divisor with $\sigma_\Gamma(D) > 0$. This is stable under the action of $\mathbb{T}_{\mathsf{N}}$. Therefore, $\Gamma = p^{-1}\Theta$ for a prime divisor $\Theta \subset S$ or $\Gamma = \Gamma_v$ for a vertex $v \in \mathrm{Ver}(\mathbf{\Sigma})$. Thus we have the equivalence.

(5) If $D|_\Gamma$ is not pseudo-effective for a prime divisor $\Gamma \subset Y$, then $\Gamma = \Gamma_v$ for a vertex $v \in \mathrm{Ver}(\mathbf{\Sigma})$ or $\Gamma = p^{-1}\Theta$ for a prime divisor $\Theta \subset S$. In case $\Gamma = \Gamma_v$, we choose $l_v \in \mathsf{M}_{\mathbb{R}}$ satisfying $h(v) = \langle l_v, v\rangle$ and let $h^v \in \mathrm{SF}_{\mathsf{N}(v)}(\mathbf{\Sigma}/\mathbb{R}_{\geq 0}v, \mathbb{R})$ be the function defined by $h^v(x) = h(x) - \langle l_v, x\rangle$. Since $D_{h^v} \sim_{\mathbb{R}} D_h + p^*\mathcal{L}^{-l_v}$, the restriction $D|_{\Gamma_v}$ is pseudo-effective if and only if $\square_{\mathrm{PE}}(E + \mathcal{L}^{l_v}, h_v) \cap v^\perp \neq \emptyset$ by (1). This is equivalent to the existence of $m \in \square_{\mathrm{PE}}(E, h)$ with $h(v) = \langle m, v\rangle$. In case $\Gamma = p^{-1}\Theta$, we note that Γ is a toric bundle over Θ. By considering the normalization of Θ, we infer from (1) that $D|_{p^{-1}\Theta}$ is pseudo-effective if and only if $(E + \mathcal{L}^m)|_\Theta$ is pseudo-effective for some $m \in \square_h$. Thus we are done. □

2.5. Theorem *Let S be a non-singular projective variety such that*

(1) $\mathrm{PE}(S) \subset \mathrm{N}^1(S) = \mathrm{NS}(S) \otimes \mathbb{R}$ *is a convex rational polyhedral cone with respect to* $\mathrm{NS}(S)$*, and*
(2) $\mathrm{Nef}(S) = \mathrm{PE}(S)$.

Then any pseudo-effective $\mathbb{R}$-Cartier divisor of a projective toric bundle $\mathbb{T}_{\mathsf{N}}(\mathbf{\Sigma}, \mathcal{L})$ over S admits a Zariski-decomposition.

Proof. We may assume that $Y = \mathbb{T}_{\mathsf{N}}(\mathbf{\Sigma}, \mathcal{L})$ is non-singular and projective. Then a pseudo-effective $\mathbb{R}$-divisor D of Y is $\mathbb{R}$-linearly equivalent to $p^*E + D_h$ for an $\mathbb{R}$-divisor E of S and for an $h \in \mathrm{SF}_{\mathsf{N}}(\mathbf{\Sigma}, \mathbb{R})$ such that $\square_{\mathrm{PE}}(E, h) \neq \emptyset$. By assumption,

$$\mathrm{PE}(S) = \{\xi \in \mathrm{N}^1(S) \mid \xi \cdot \gamma_i \geq 0 \ (1 \leq i \leq k)\}$$

for some 1-cycles $\gamma_1, \gamma_2, \ldots, \gamma_k$ of S. Let $c\colon \mathsf{M} \to \mathrm{N}^1(S)$ be the homomorphism defined by $c(m) = c_1(\mathcal{L}^m)$ and let $c^\vee\colon \mathrm{N}_1(S) \to \mathsf{N}_{\mathbb{R}}$ be its dual. Both c and $c^\vee$ are defined over $\mathbb{Q}$. Then the cone $\mathbb{R}_{\geq 0}(\square_{\mathrm{PE}}(E, h) \times \{-1\})$ is the dual cone of

$$\triangle(E, h) = \triangle_h + \sum\nolimits_{i=1}^{k} \mathbb{R}_{\geq 0}(c^\vee(\gamma_i), -E \cdot \gamma_i).$$

For $x \in \mathsf{N}_{\mathbb{R}}$, let us define

$$h^\ddagger(x) = \min\{\langle m, x\rangle \mid m \in \square_{\mathrm{PE}}(E, h)\}.$$

Then $h^\ddagger(x) \geq h(x)$ and $\square_{\mathrm{PE}}(E, h) = \square_{\mathrm{PE}}(E, h^\ddagger)$. Moreover, $h^\ddagger \in \mathrm{SFC}_{\mathsf{N}}(\mathbf{\Sigma}, \mathbb{R})$, since the image of any face of $\triangle(E, h)$ under the first projection $\mathsf{N}_{\mathbb{R}} \times \mathbb{R} \to \mathsf{N}_{\mathbb{R}}$ is a *rational* polyhedral cone. Let $\mathbf{\Sigma}'$ be a finite subdivision of $\mathbf{\Sigma}$ such that $h^\ddagger \in$

$\mathrm{SF}_{\mathsf{N}}(\boldsymbol{\Sigma}', \mathbb{R})$ and let $\mu\colon Y' = \mathbb{T}_{\mathsf{N}}(\boldsymbol{\Sigma}', \mathcal{L}) \to Y$ be the associated proper bimeromorphic morphism. Then

$$N_\sigma(\mu^* D) = \sum\nolimits_{v \in \mathrm{Ver}(\boldsymbol{\Sigma}')} (h^\ddagger(v) - h(v))\Gamma_v$$

by **2.4**-(3). Here $P_\sigma(\mu^* D) \sim_{\mathbb{R}} p^* E + D_{h^\ddagger}$, which is nef by **2.4**-(2). $\square$

§2.c. Examples of toric bundles. Let S be a non-singular projective variety and let $L_1, L_2, \ldots, L_r$ be divisors of S. Let $p\colon \mathbb{P} = \mathbb{P}(\mathcal{E}) \to S$ be the projective bundle associated with $\mathcal{E} = \bigoplus_{i=1}^r \mathcal{O}_S(L_i)$. This is described as a toric bundle $\mathbb{T}_{\mathsf{N}}(\boldsymbol{\Sigma}, \mathcal{L})$ as follows:

(1) N is of rank $r-1$ with a basis $e_1, e_2, \ldots, e_{r-1}$;

(2)
$$\mathcal{L} = \sum\nolimits_{i=1}^{r-1} e_i \otimes \mathcal{O}_S(L_i - L_r) \in \mathsf{N} \otimes \mathrm{Pic}(S);$$

(3) We set $e_r = -\sum_{i=1}^{r-1} e_i \in \mathsf{N}$. The fan $\boldsymbol{\Sigma}$ consists of the faces of the $(r-1)$-dimensional cones

$$\boldsymbol{\sigma}_i := \sum\nolimits_{1 \le j \le r,\, j \ne i} \mathbb{R}_{\ge 0}\, e_j \quad (1 \le i \le r).$$

Let $h\colon \mathsf{N}_{\mathbb{R}} \to \mathbb{R}$ be the function defined by

$$h\left(\sum\nolimits_{j=1}^{r-1} x_j e_j\right) = \begin{cases} x_i, & \text{if } x \in \boldsymbol{\sigma}_i \text{ for } i < r; \\ 0, & \text{if } x \in \boldsymbol{\sigma}_r. \end{cases}$$

Then $h \in \mathrm{SF}_{\mathsf{N}}(\boldsymbol{\Sigma}, \mathbb{Z})$. In fact, $h(x) = \min\{\langle l_i, x\rangle \mid 1 \le i \le r\}$ for the dual basis $(l_1, l_2, \ldots, l_{r-1})$ of M to $(e_1, e_2, \ldots, e_{r-1})$ and $l_r = 0$. Note that $h(e_i) = 0$ for $i < r$, and $h(e_r) = -1$, where $\mathrm{Ver}(\boldsymbol{\Sigma}) = \{e_1, e_2, \ldots, e_r\}$. In particular, D_h is just the prime divisor Γ_{e_r} and hence $D_h \sim H - p^* L_r$ for the tautological divisor $H = H_{\mathcal{E}}$. We consider the standard convex polytope

$$\square := \left\{\boldsymbol{s} = (s_1, s_2, \ldots, s_r) \in [0,1]^r \;\middle|\; \sum\nolimits_{i=1}^r s_i = 1\right\},$$

where $[0,1] = \{r \in \mathbb{R} \mid 0 \le r \le 1\}$. For $\boldsymbol{s} \in \square$, an $\mathbb{R}$-divisor Δ of S, and for a real number $b \ge 0$, we define

$$\Delta(\boldsymbol{s}) := \Delta + b\left(\sum\nolimits_{i=1}^r s_i L_i\right),$$
$$\square_{\mathrm{PE}}(\Delta, L_\bullet, b) := \{\boldsymbol{s} \in \square \mid \Delta(\boldsymbol{s}) \text{ is pseudo-effective}\}.$$

If we identify $\mathsf{M}_{\mathbb{R}} \simeq \mathbb{R}^{r-1}$ by the dual basis to $(e_1, e_2, \ldots, e_{r-1})$, then

$$\square_h = \left\{(m_1, m_2, \ldots, m_{r-1}) \in \mathbb{R}^{r-1}_{\ge 0} \;\middle|\; \sum\nolimits_{i=1}^{r-1} m_i \le 1\right\},$$

and hence $\square_{\mathrm{PE}}(bL_r + \Delta, bh)$ is identified with the set of vectors $(m_1, m_2, \ldots, m_{r-1}) \in \mathbb{R}^{r-1}_{\ge 0}$ such that $\sum_{i=1}^{r-1} m_i \le b$ and

$$\Delta + \sum\nolimits_{i=1}^{r-1} m_i L_i + \left(b - \sum\nolimits_{i=1}^{r-1} m_i\right) L_r \in \mathrm{PE}(S).$$

Thus, if $b > 0$, there is an identification $\square_{\mathrm{PE}}(\Delta + bL_r, bh) \leftrightarrow \square_{\mathrm{PE}}(\Delta, L_\bullet, b)$ by

$$s_i = m_i/b \quad \text{for} \quad i < r, \quad \text{and} \quad s_r = 1 - \frac{1}{b}\sum\nolimits_{i=1}^{r-1} m_i.$$

2.6. Lemma *Let D be an $\mathbb{R}$-divisor of $\mathbb{P}$ numerically equivalent to $p^*\Delta + bH$ for an $\mathbb{R}$-divisor Δ of S and $b \in \mathbb{R}$.*

1. *D is pseudo-effective if and only if $b \geq 0$ and $\square_{\mathrm{PE}}(\Delta, L_\bullet, b) \neq \emptyset$.*
2. *D is nef if and only if $b \geq 0$ and $\Delta + bL_i$ is nef for any $1 \leq i \leq r$.*
3. *D is movable if and only if $b \geq 0$ and the following two conditions are both satisfied:*
   (a) *For any prime divisor $\Theta \subset S$, there is a vector $\boldsymbol{s} \in \square_{\mathrm{PE}}(\Delta, L_\bullet, b)$ such that $\sigma_\Theta(\Delta(\boldsymbol{s})) = 0$;*
   (b) *For any $1 \leq j \leq r$, a vector $\boldsymbol{s} = (s_1, s_2, \ldots, s_r)$ with $s_j = 0$ is contained in $\square_{\mathrm{PE}}(\Delta, L_\bullet, b)$.*
4. *D is numerically movable if and only if $b \geq 0$, and the condition* (b) *above and the following condition are satisfied: For any prime divisor $\Theta \subset S$, there is a vector $\boldsymbol{s} \in \square$ such that $\Delta(\boldsymbol{s})|_\Theta$ is pseudo-effective.*

PROOF. (1) D is numerically equivalent to $bD_h + p^*(bL_r + \Delta)$. This is p-pseudo-effective if and only if $b \geq 0$. Hence (1) follows from **2.4**-(1) and from the identification $\square_{\mathrm{PE}}(\Delta + bL_r, bh) \leftrightarrow \square_{\mathrm{PE}}(\Delta, L_\bullet, b)$.

(2) A maximal cone of $\boldsymbol{\Sigma}$ is one of $\boldsymbol{\sigma}_i$ for $1 \leq i \leq r$. For $l_1, l_2, \ldots, l_r \in \mathsf{M}$ introduced above, we set $h^{(i)}(x) := h(x) - \langle l_i, x\rangle$. Then D is nef if and only if $\Delta + bL_r$ and $\Delta + bL_r + \mathcal{L}^{bl_i} = \Delta + bL_i$ for $i < r$ are all nef, by **2.4**-(2).

(3) follows from by **2.4**-(3), since

$$\sigma_{\Gamma_{e_i}}(D) = \min\{bs_i \mid \boldsymbol{s} \in \square_{\mathrm{PE}}(\Delta, L_\bullet, b)\} \quad \text{for} \quad 1 \leq i \leq r,$$
$$\sigma_{p^{-1}\Theta}(D) = \min\{\sigma_\Theta(\Delta(\boldsymbol{s})) \mid \boldsymbol{s} \in \square_{\mathrm{PE}}(\Delta, L_\bullet, b)\}.$$

(4) follows from **2.4**-(5). □

We consider the special case: $r = 2$. We may assume $L_2 = 0$ and may write $L = L_1$. Then $\mathcal{E} = \mathcal{O}_S(L) \oplus \mathcal{O}_S$, $\mathbb{P} = \mathbb{T}_{\mathsf{N}}(\boldsymbol{\Sigma}, \mathcal{L})$ for $\mathsf{N} = \mathbb{Z}$, $\boldsymbol{\Sigma} = \{\{0\}, [0, +\infty), (-\infty, 0]\}$, and $\mathcal{L}^m = \mathcal{O}_S(mL)$ for $m \in \mathbb{Z}$. The support function $h \in \mathrm{SF}_{\mathsf{N}}(\boldsymbol{\Sigma}, \mathbb{R})$ is written by $h(x) = \min\{0, x\}$, $\square_h = [0, 1] \subset \mathbb{R} = \mathsf{M}_{\mathbb{R}}$, and $D_h \sim H$ for the tautological divisor $H = H_{\mathcal{E}}$ of $\mathbb{P}$. The prime divisors Γ_1 and Γ_{-1} corresponding to the vertices in $\mathrm{Ver}(\boldsymbol{\Sigma}) = \{1, -1\}$ are sections of p. Here, $\Gamma_1 = \mathrm{div}(\mathsf{e}(1)) + D_h \sim -p^*L + H$ and $\Gamma_{-1} = D_h$. Let D be an $\mathbb{R}$-divisor of $\mathbb{P}$. Then $D \sim_{\mathbb{R}} p^*E + bH$ for some $\mathbb{R}$-divisor E of S and for some $b \in \mathbb{R}$. By **2.6**-(1), D is pseudo-effective if and only if $b \geq 0$ and $E + mL$ is pseudo-effective for some $0 \leq m \leq b$. By **2.6**-(2), in case $b \geq 0$, D is nef if and only if E and $E + bL$ are both nef. If $\mathrm{Nef}(S) = \mathrm{PE}(S)$, then any numerically movable $\mathbb{R}$-divisor D is nef, since $D|_{\Gamma_1} \sim_{\mathbb{R}} E$ and $D|_{\Gamma_{-1}} \sim_{\mathbb{R}} E + bL$. Therefore, we have proved the following:

2.7. Corollary *In the situation of* **2.6***, suppose that every effective divisor of S is nef and $r = 2$. Then $P_\nu(D)$ is nef for a pseudo-effective $\mathbb{R}$-divisor D of $\mathbb{P}$.*

2.8. Example In the situation above where $r = 2$, $L_1 = L$, $L_2 = 0$, suppose that there is an infinite sequence $\{E_n\}_{n=1}^{\infty}$ of $\mathbb{R}$-divisors of S such that

(1) $c_1(E_n) \in \mathrm{PE}(S)$ for any n,
(2) $\lim_{n\to\infty} c_1(E_n) = c_1(L)$,
(3) $E_n - tL \notin \mathrm{PE}(S)$ for any n and $t > 0$.

We fix a number $0 < \alpha < 1$ and consider pseudo-effective $\mathbb{R}$-divisors $D_n^{\alpha} = p^*E_n + \alpha\Gamma_1$. Then $D_n^{\alpha} \sim_{\mathbb{R}} p^*(E_n - \alpha L) + \alpha H$. Thus $D_n^{\alpha}|_{\Gamma_1} \sim_{\mathbb{R}} E_n - \alpha L$ is not pseudo-effective. If $(D_n^{\alpha} - r\Gamma_1)|_{\Gamma_1}$ is pseudo-effective, then $r \geq \alpha$. Hence

$$\nu_{\Gamma_1}(D_n^{\alpha}) = \sigma_{\Gamma_1}(D_n^{\alpha}) = \alpha.$$

We set $D_{\infty}^{\alpha} := p^*L + \alpha\Gamma_1$. Then $\sigma_{\Gamma_1}(D_{\infty}^{\alpha}) = 0$ by $D_{\infty}^{\alpha} \sim_{\mathbb{R}} p^*((1-\alpha)L) + \alpha\Gamma_{-1}$. Thus the function σ_{Γ_1} is not continuous on $\mathrm{PE}(\mathbb{P})$, since $c_1(D_{\infty}^{\alpha}) = \lim_{n\to\infty} c_1(D_n^{\alpha})$. If we choose S, L, and $P_n = c_1(E_n)$ as follows, then they satisfy the condition above: Let S be the product $E \times E$ for an elliptic curve E without complex multiplication and let L be a fiber of the first projection. Since $\mathrm{PE}(S) = \mathrm{Nef}(S)$ is a cone isometric to

$$\{(x, y, z) \in \mathbb{R}^3 \mid z^2 \geq x^2 + y^2,\ z \geq 0\},$$

we can find a sequence $\{P_n\}$ of points of $\mathrm{PE}(S)$ such that $P_n - tc_1(L) \notin \mathrm{PE}(S)$ for any $t > 0$ and $c_1(L) = \lim_{n\to\infty} P_n$.

2.9. Lemma *In the situation of the $\mathbb{P}^1$-bundle above, assume that* $\dim S = 2$*, L is nef, and that E is a non-singular irreducible curve of S with $E^2 < 0$. Then the $\mathbb{R}$-divisor $D = p^*E + bH$ with $b \geq 0$ admits a Zariski-decomposition.*

Proof. By taking the σ-decomposition of D, we may assume that D is movable. Thus E is pseudo-effective and $E + bL$ is nef by **2.6**-(3), since L is nef. Note that D is big. From the equivalence relations

$$D \sim_{\mathbb{R}} b\Gamma_{-1} + p^*E \sim_{\mathbb{R}} b\Gamma_1 + p^*(E + bL),$$

we infer that $\mathrm{NBs}(D)$ coincides with the non-singular complete intersection $V := \Gamma_1 \cap p^{-1}E$. Let $\psi\colon Z \to \mathbb{P}$ be the blowing-up along the ideal sheaf

$$\mathcal{J} := \mathcal{O}_{\mathbb{P}}(-m_1\Gamma_1) + \mathcal{O}_{\mathbb{P}}(-m_2 p^*E),$$

where m_1 and m_2 are positive integers satisfying $m_2E^2 = -m_1(L \cdot E)$. Then the exceptional set $G_0 := \psi^{-1}(V)$ is isomorphic to the $\mathbb{P}^1$-bundle

$$\mathbb{P}_V(\mathcal{O}_V(-m_1\Gamma_1) \oplus \mathcal{O}_V(-m_2p^*E)) \simeq \mathbb{P}_E(\mathcal{O}_E(m_1L) \oplus \mathcal{O}_E(-m_2E)).$$

Let $\nu\colon W \to Z$ be the normalization and let $\rho\colon W \to X$ be the composite. Then W has only quotient singularities and $G = \nu^{-1}G_0$ is isomorphic to G_0 by construction. The prime divisor G is $\mathbb{Q}$-Cartier and $\mathcal{O}_W(-kG) \simeq \rho^*\mathcal{J}/(\mathrm{tor})$ for some $k \in \mathbb{N}$. Let r be the minimum positive number with $(\rho^*D - rG)|_G$ being pseudo-effective. Then $(\rho^*D - rG)|_G$ is nef but not big, since G is the $\mathbb{P}^1$-bundle associated with a semi-stable vector bundle over the curve E. Thus $\rho^*D - rG$ is nef, since $\mathrm{NBs}(\rho^*D) \subset G$. Let $\mu\colon Y \to W$ be a birational morphism from a non-singular projective variety. Then $(\mu^*\rho^*D - r\mu^*G)|_{\Gamma}$ is not big for any prime component Γ of μ^*G. Thus $P_{\sigma}(\mu^*\rho^*D) = \mu^*(\rho^*D - rG)$ by **III.3.7**. $\square$

Next, we consider a special case of $\mathbb{P}^2$-bundles in order to obtain a counterexample to the existence of Zariski-decomposition.

In the description of the projective bundle $\mathbb{P}(\mathcal{E}) = \mathbb{T}_{\mathsf{N}}(\mathbf{\Sigma}, \mathcal{L})$, we assume $r = 3$, $L_3 = 0$, i.e., $\mathcal{E} = \mathcal{O}_S(L_1) \oplus \mathcal{O}_S(L_2) \oplus \mathcal{O}_S$. For the support function $h \in \mathrm{SF}_{\mathsf{N}}(\mathbf{\Sigma}, \mathbb{Z})$, we know $D_h = \Gamma_{e_3} \sim H$ for the tautological divisor $H = H_{\mathcal{E}}$. For an $\mathbb{R}$-divisor Δ of S, $\square_{\mathrm{PE}}(\Delta, h)$ is identified with

$$\Omega := \{(x, y) \in \mathbb{R}^2_{\geq 0} \mid x + y \leq 1,\ \Delta + xL_1 + yL_2 \text{ is pseudo-effective}\}.$$

We assume the following condition for S, L_1, L_2, and Δ:

(1) $\square_{\mathrm{PE}}(\Delta, h) = \square_{\mathrm{Nef}}(\Delta, h)$;
(2) L_1, L_2, $\Delta + L_1$, and $\Delta + L_2$ are ample;
(3) $\alpha := \inf\{x + y \mid (x, y) \in \Omega\} > 0$ and there exists a unique point $P_0 = (x_0, y_0) \in \Omega$ with $x_0 + y_0 = \alpha$;
(4) Ω is not locally polyhedral at P_0; In other words, if $(z, u) \in \mathbb{R}^2$ satisfies $zx_0 + uy_0 \leq zx + uy$ for any $(x, y) \in \Omega$, then $zx_0 + uy_0 < zx + uy$ for any $(x, y) \in \Omega \setminus \{P_0\}$.

Example Let S be an abelian surface of the Picard number $\rho(S) = 3$. For example, $S = E \times E$ for an elliptic curve E without complex multiplication. Then $\mathrm{PE}(S) = \mathrm{Nef}(S) \subset \mathrm{N}^1(S)$ is a cone isometric to

$$\mathcal{C} = \{(x, y, z) \in \mathbb{R}^3 \mid z^2 \geq x^2 + y^2,\ z \geq 0\}.$$

For points $\Delta = (-1, -1, 0) \notin \mathcal{C}$, $L_1 = (1, 0, a)$, $L_2 = (0, 1, a)$ for $a > 1$, the set

$$\{(x, y) \in \mathbb{R}^2 \mid \Delta + xL_1 + yL_2 \in \mathcal{C}\}$$

is written by

$$\{(x, y) \mid a^2(x + y)^2 \geq (x - 1)^2 + (y - 1)^2,\ x + y \geq 0\}.$$

Thus S, L_1, L_2, and Δ satisfy the condition above.

2.10. Theorem *If S, L_1, L_2, and Δ satisfy the condition above, then the $\mathbb{R}$-divisor $B = p^*\Delta + H$ on $\mathbb{P}(\mathcal{E})$ admits no Zariski-decompositions.*

Proof. We may assume that $\Omega^0 := \{(x, y) \in \Omega \mid y \geq y_0\}$ is not locally polyhedral at $P_0 = (x_0, y_0)$. In other words, if z, $u \in \mathbb{R}$ with $z \geq u \geq 0$ satisfies $zx + uy \geq zx_0 + uy_0$ for any $(x, y) \in \Omega$, then $zx + uy > zx_0 + uy_0$ for any $(x, y) \in \Omega^0 \setminus \{P_0\}$.

Let us consider the function on $\mathsf{N}_{\mathbb{R}}$ defined by

$$h^{\ddagger}(\boldsymbol{x}) = \min\{\langle m, \boldsymbol{x}\rangle \mid m \in \square_{\mathrm{Nef}}(\Delta, h)\}.$$

Then $h^{\ddagger}(ze_1 + ue_2) = \min\{xz + yu \mid (x, y) \in \Omega\}$ for $(z, u) \in \mathbb{R}^2$. Here, note that $h^{\ddagger} \notin \mathrm{SFC}_{\mathsf{N}}(\mathbf{\Sigma}, \mathbb{R})$, since Ω is not locally polyhedral at P_0. We have $h^{\ddagger}(e_1) = h^{\ddagger}(e_2) = 0$, and $h^{\ddagger}(e_3) = -1$. Thus B is movable by **2.6**-(3). For the maximal cones $\boldsymbol{\sigma}_i = \sum_{j \neq i} \mathbb{R}_{\geq 0} e_j$, we have $h^{\ddagger}|_{\boldsymbol{\sigma}_1} = h|_{\boldsymbol{\sigma}_1}$ and $h^{\ddagger}|_{\boldsymbol{\sigma}_2} = h|_{\boldsymbol{\sigma}_2}$, but $h^{\ddagger}|_{\boldsymbol{\sigma}_3} \neq 0$; for example, $h^{\ddagger}(e_1 + e_2) = \alpha > 0$. Hence $\mathrm{NBs}(B)$ is just the section $\mathbb{V}(\boldsymbol{\sigma}_3, \mathcal{L}) = \Gamma_{e_1} \cap \Gamma_{e_2}$, since $\mathrm{NBs}(B)$ is stable under the action of $\mathbb{T}_{\mathsf{N}}$. The blowing-up of $\mathbb{P}$

along $\mathbb{V}(\boldsymbol{\sigma}_3, \mathcal{L})$ corresponds to the subdivision $\boldsymbol{\Sigma}^{[1]}$ of $\boldsymbol{\Sigma}$ such that $\mathrm{Ver}(\boldsymbol{\Sigma}^{[1]}) = \{e_1, e_2, e_3, -e_3 = e_1 + e_2\}$. Let $\mu_1 \colon \mathbb{P}^{[1]} = \mathbb{T}_{\mathsf{N}}(\boldsymbol{\Sigma}^{[1]}, \mathcal{L}) \to \mathbb{P}$ be the blowing-up. We denote the structure morphism $\mathbb{P}^{[1]} \to S$ by the same p. For the exceptional divisor $\Gamma_{e_1+e_2} = \mathbb{V}(\mathbb{R}_{\geq 0}(e_1 + e_2), \mathcal{L}) \subset \mathbb{P}^{[1]}$, we have

$$\sigma_{\Gamma_{e_1+e_2}}(\mu_1^* B) = \sigma_{\Gamma_{e_1+e_2}}(p^*\Delta + D_h) = h^{\ddagger}(e_1 + e_2) = \alpha,$$

by **2.6**-(3). Thus $P_\sigma(\mu_1^* B) = p^*\Delta + D_{h_1}$ for the support function $h_1 \in \mathrm{SF}_{\mathsf{N}}(\boldsymbol{\Sigma}^{[1]}, \mathbb{R})$ such that $h_1(v) = h^{\ddagger}(v)$ for any $v \in \mathrm{Ver}(\boldsymbol{\Sigma}^{[1]})$. Then $h^{\ddagger}(\boldsymbol{x}) \geq h_1(\boldsymbol{x}) \geq h(\boldsymbol{x})$ for any $\boldsymbol{x} \in \mathsf{N}_{\mathbb{R}}$ and $\square_{\mathrm{PE}}(\Delta, h_1) = \square_{\mathrm{PE}}(\Delta, h)$. If $h^{\ddagger}(2e_1 + e_2) = h_1(2e_1 + e_2)$, then $h^{\ddagger}(\boldsymbol{x}) = h_1(\boldsymbol{x})$ for any $\boldsymbol{x} \in \mathbb{R}_{\geq 0} e_1 + \mathbb{R}_{\geq 0}(e_1 + e_2)$; it contradicts the assumption: Ω^0 is not locally polyhedral at P_0. Thus $h^{\ddagger}(2e_1+e_2) > h_1(2e_1+e_2)$ and the section $\mathbb{V}(\mathbb{R}_{\geq 0}e_1 + \mathbb{R}_{\geq 0}(e_1+e_2), \mathcal{L})$ of $\mathbb{P}^{[1]} \to S$ is a connected component of $\mathrm{NBs}(P_\sigma(\mu_1^* B))$. Let $\mathbb{P}^{[2]} \to \mathbb{P}^{[1]}$ be the blowing-up along the section, which corresponds to a subdivision $\boldsymbol{\Sigma}^{[2]}$ of $\boldsymbol{\Sigma}^{[1]}$ such that $\mathrm{Ver}(\boldsymbol{\Sigma}^{[2]}) = \mathrm{Ver}(\boldsymbol{\Sigma}^{[1]}) \cup \{2e_1 + e_2\}$. For the composite $\mu_2 \colon \mathbb{P}^{[2]} \to \mathbb{P}$ and for the projection $p \colon \mathbb{P}^{[2]} \to S$, we have $P_\sigma(\mu_2^* B) = p^*\Delta + D_{h_2}$ for $h_2 \in \mathrm{SF}_{\mathsf{N}}(\boldsymbol{\Sigma}^{[2]}, \mathbb{R})$ defined by $h_2(v) = h^{\ddagger}(v)$ for any $v \in \mathrm{Ver}(\boldsymbol{\Sigma}^{[2]})$. Here, $h^{\ddagger}(\boldsymbol{x}) \geq h_2(\boldsymbol{x})$ for $\boldsymbol{x} \in \mathsf{N}_{\mathbb{R}}$ and $h^{\ddagger}(3e_1 + 2e_2) > h_2(3e_1 + 2e_2)$ by the same reason above. In particular, the section $\mathbb{V}(\mathbb{R}_{\geq 0}(e_1 + e_2) + \mathbb{R}_{\geq 0}(2e_1 + e_2), \mathcal{L})$ of $p \colon \mathbb{P}^{[2]} \to S$ is a connected component of $\mathrm{NBs}(P_\sigma(\mu_2^* B))$. In this way, we can construct a non-singular subdivision $\boldsymbol{\Sigma}^{[n]}$ of $\boldsymbol{\Sigma}$ such that

$$\mathrm{Ver}(\boldsymbol{\Sigma}^{[n]}) = \mathrm{Ver}(\boldsymbol{\Sigma}) \cup \{e_1 + e_2, 2e_1 + e_2, \ldots, ne_1 + (n-1)e_2\}$$

for $n \geq 2$. Then, for the toric bundle $p \colon \mathbb{P}^{[n]} := \mathbb{T}_{\mathsf{N}}(\boldsymbol{\Sigma}^{[n]}, \mathcal{L}) \to S$, the induced birational morphism $\mathbb{P}^{[n+1]} \to \mathbb{P}^{[n]}$ is just the blowing up along the section

$$\mathbb{V}(\mathbb{R}_{\geq 0}(e_1 + e_2) + \mathbb{R}_{\geq 0}(ne_1 + (n-1)e_2), \mathcal{L})$$

of $p \colon \mathbb{P}^{[n]} \to S$, which is a connected component of $\mathrm{NBs}(P_\sigma(\mu_n^* B))$ for the birational morphism $\mu_n \colon \mathbb{P}^{[n]} \to \mathbb{P}$. Thus we are reduced to the following:

2.11. Lemma *Let*

$$\cdots \to X_n \xrightarrow{\mu_n} X_{n-1} \to \cdots \to X_1 \xrightarrow{\mu_1} X_0$$

be an infinite sequence of blowups in which centers $V_n \subset X_{n-1}$ are non-singular subvarieties of codimension two for any $n \geq 1$. Let E_n be the exceptional divisor $\mu_n^{-1}(V_n)$. Assume that there exist a sequence of pseudo-effective $\mathbb{R}$-divisors D_n on X_n satisfying the following conditions:

(1) $\mu_n(V_{n+1}) = V_n$;
(2) $\sigma_{V_n}(D_{n-1}) > 0$;
(3) $D_n = \mu_n^* D_{n-1} - \sigma_{V_n}(D_{n-1}) E_n$.

Then D_0 admits no Zariski-decompositions.

Proof. Assume the contrary. Let $f \colon Y \to X_0$ be a birational morphism with $P_\sigma(f^* D_0)$ being nef. We may assume that f is a succession of blowups with non-singular centers. Suppose that the image V_1' of the composite $E_1 \subset X_1 \dashrightarrow Y$

is not a divisor. Since $\operatorname{codim} V_1 = 2$, f is an isomorphism over a general point of V_1. On the other hand, $V_1' \subset \operatorname{Supp} N_\sigma(f^*D_0)$ and the divisor $N_\sigma(f^*D_0)$ is f-exceptional, since $N_\sigma(D_0) = 0$. This is a contradiction. Therefore V_1' is a prime divisor and is the proper transform of E_1. Furthermore, there is a Zariski-closed subset $S_1 \subset X_0$ such that $V_1 \not\subset S_1$ and $Y \cdots\rightarrow X_1$ is a morphism over $X_0 \setminus S_1$. The birational mapping $Y \cdots\rightarrow X_1$ is considered as a succession of blowups with non-singular centers over $X_0 \setminus S_1$. There is a birational morphism $\nu_1 \colon Y_1 \to Y$ from a non-singular projective variety such that $f_1 \colon Y_1 \cdots\rightarrow X_1$ is a morphism and ν_1 is an isomorphism over $X_0 \setminus S_1$. Note that $P_\sigma(f_1^*D_1) = \nu_1^* P_\sigma(f^*D_0)$. Let V_2' be the image of the composite $E_2 \subset X_2 \cdots\rightarrow Y_1$. By the same argument as above, V_2' is a divisor and is the proper transform of E_2. Since ν_1 is isomorphic outside S_1, E_2 is not exceptional for the birational mapping $X_2 \cdots\rightarrow Y$. Furthermore, there is a Zariski-closed subset $S_2 \subset X_1$ such that $\mu_1^{-1}(S_1) \subset S_2$, $V_2 \not\subset S_2$, and the birational mapping $Y_2 \cdots\rightarrow X_2$ is a morphism over $X_1 \setminus S_2$. There is also a birational morphism $\nu_2 \colon Y_2 \to Y_1$ from a non-singular projective variety such that $f_2 \colon Y_2 \cdots\rightarrow X_2$ is a morphism and ν_2 is an isomorphism over $X_1 \setminus S_2$. By continuing the same arguments, we infer that the divisor E_n is not exceptional for the birational mapping $X_n \cdots\rightarrow Y$ for any $n \geq 1$. This is a contradiction, since $f \colon Y \to X_0$ has only finitely many exceptional divisors. $\square$

§2.d. Explicit toric blowing-up. Let S be an n-dimensional complex analytic manifold and let $B_1, B_2, \ldots, B_r$ for $r \leq n$ be non-singular prime divisors such that $B = \sum B_i$ is simple normal crossing. Let $p \colon \mathbb{V} = \mathbb{V}(\mathcal{E}) \to S$ be the geometric vector bundle associated with $\mathcal{E} = \oplus_{i=1}^r \mathcal{O}_S(B_i)$. This is also considered as a toric bundle as follows: let $\mathsf{N}^\natural = \sum_{i=1}^r \mathbb{Z} e_i$ be a free abelian group with a base $(e_1, e_2, \ldots, e_r)$, $\boldsymbol{\sigma}_\natural = \sum_{i=1}^r \mathbb{R}_{\geq 0} e_i$, and let

$$\mathcal{L}_\natural = \sum\nolimits_{i=1}^r e_i \otimes \mathcal{O}_S(-B_i) \in \mathsf{N} \otimes \operatorname{Pic}(S).$$

Then $\mathbb{V} \simeq \mathbb{T}_{\mathsf{N}^\natural}(\boldsymbol{\sigma}_\natural, \mathcal{L}_\natural)$. Let $\mathsf{M}^\natural$ be the dual $\mathsf{N}^{\natural\vee}$. The prime divisor Γ_{e_i} corresponding to a vertex $e_i \in \operatorname{Ver}(\boldsymbol{\sigma}_\natural)$ is the geometric vector bundle associated with the kernel of the projection $\mathcal{E} \to \mathcal{O}_S(B_i)$. Let us consider the section $T \subset \mathbb{V}$ of p determined by the surjective ring homomorphism

$$\operatorname{Sym}(\mathcal{E}^\vee) = \mathcal{L}[\boldsymbol{\sigma}_\natural^\vee \cap \mathsf{M}^\natural] \twoheadrightarrow \mathcal{O}_S$$

induced from the natural injections $\mathcal{O}_S(-B_i) \subset \mathcal{O}_S$ (cf. Chapter **II**, §**1.b**). By the identification $T \simeq S$, we have $B_i = \Gamma_{e_i}|_T$. If $U \subset S$ is an open subset over which $\mathcal{O}_S(B_i)$ are trivial line bundles, then the composite

$$U \simeq p^{-1}U \cap T \subset p^{-1}U \simeq \mathbb{C}^r \times U \to \mathbb{C}^r$$

is a smooth morphism and the pullback of the i-th coordinate hyperplane is $B_i \cap U$. Let $\boldsymbol{\Lambda}$ be a finite subdivision of $\boldsymbol{\sigma}_\natural$. Then we have a bimeromorphic morphism $f \colon \mathbb{T}_{\mathsf{N}^\natural}(\boldsymbol{\Lambda}, \mathcal{L}_\natural) \to \mathbb{V}$ of toric bundles over S. Let us consider $S_{\boldsymbol{\Lambda}} := f^{-1}(T)$. Then $S_{\boldsymbol{\Lambda}}$ is a normal variety and the bimeromorphic morphism $f \colon S_{\boldsymbol{\Lambda}} \to S$ satisfies the condition of **1.18**, since $f^{-1}U$ is smooth over the toric variety $\mathbb{T}_{\mathsf{N}^\natural}(\boldsymbol{\Lambda})$ for the open subset U above. Note that f is isomorphic over $S \setminus B$.

2.12. Definition The bimeromorphic morphism $S_{\boldsymbol{\Lambda}} \to S$ is called the *toric blowing-up* of S along the simple normal crossing divisor $B = \sum B_i$ with respect to the subdivision $\boldsymbol{\Lambda}$.

Let Z be the intersection $B_1 \cap B_2 \cap \cdots \cap B_r$ which is smooth. If $Z \neq \emptyset$, then $T \cap p^{-1}Z = \mathbb{V}(\boldsymbol{\sigma}_\natural, \mathcal{L}_\natural) \cap p^{-1}Z$ and

$$S_{\boldsymbol{\Lambda}} \times_S Z = \bigcup\nolimits_{\boldsymbol{\lambda} \in \boldsymbol{\Lambda},\ \boldsymbol{\lambda} \cap \operatorname{Int} \boldsymbol{\sigma}_\natural \neq \emptyset} \mathbb{V}(\boldsymbol{\lambda}, \mathcal{L}_\natural|_Z)$$

by **1.1**. Here $\mathbb{V}(\boldsymbol{\lambda}, \mathcal{L}_\natural|_Z) \simeq \mathbb{T}_{\mathsf{N}^\natural(\boldsymbol{\lambda})}(\boldsymbol{\Lambda}/\boldsymbol{\lambda}, \mathcal{L}_\natural|_Z)$ and $\boldsymbol{\Lambda}/\boldsymbol{\lambda}$ is a complete fan.

2.13. Proposition *Let S be the toric bundle $\mathbb{T}_{\mathsf{N}}(\boldsymbol{\Sigma}, \mathcal{L})$ over non-singular variety Z for a non-singular fan $\boldsymbol{\Sigma}$ of a free abelian group N of rank l and for some $\mathcal{L} \in \mathsf{N}_0 \otimes \operatorname{Pic}(Z)$. Let us fix mutually distinct vertices $v_1, v_2, \ldots, v_r \in \operatorname{Ver}(\boldsymbol{\Sigma})$ for $r \leq l$ and set $B_i = \Gamma_{v_i} = \mathbb{V}(\mathbb{R}_{\geq 0}v_i, \mathcal{L}) \subset S$. Let $f\colon S_{\boldsymbol{\Lambda}} \to S$ be the toric blowing-up along the simple normal crossing divisor $B = \sum B_i$ with respect to a finite subdivision $\boldsymbol{\Lambda}$ of $\boldsymbol{\sigma}^\natural$. Then $S_{\boldsymbol{\Lambda}}$ is isomorphic to the toric bundle $\mathbb{T}_{\mathsf{N}}(\boldsymbol{\Sigma}_1, \mathcal{L})$ over Z for a finite subdivision $\boldsymbol{\Sigma}_1$ of $\boldsymbol{\Sigma}$ and f is interpreted as the morphism of toric bundles over Z associated with the subdivision.*

Proof. By **2.2**, the toric bundle $\mathbb{T}_{\mathsf{N}^\natural}(\boldsymbol{\Lambda}, \mathcal{L}_\natural)$ over S is isomorphic to the toric bundle $\mathbb{T}_{\mathsf{N}^\natural \oplus \mathsf{N}}(\boldsymbol{\Sigma}_{\mathbf{h}}, \widetilde{\mathcal{L}})$ over Z for $\widetilde{\mathcal{L}} = 0 \oplus \mathcal{L} \in (\mathsf{N}^\natural \oplus \mathsf{N}) \otimes \operatorname{Pic}(Z)$ and $\mathbf{h} \in \operatorname{SF}_{\mathsf{N}}(\boldsymbol{\Sigma}, \mathbb{Z}) \otimes \mathsf{N}^\natural$ defined as follows: As a function $|\boldsymbol{\Sigma}| \to (\mathsf{N}^\natural)_{\mathbb{R}}$, $\mathbf{h}$ is defined by

$$\mathbf{h}(v) = \begin{cases} e_i, & \text{if } v = v_i \quad \text{for} \quad 1 \leq i \leq l, \\ 0, & \text{otherwise} \end{cases}$$

for $v \in \operatorname{Ver}(\boldsymbol{\Sigma})$. Here $\boldsymbol{\Sigma}_{\mathbf{h}} = \{C(\boldsymbol{\lambda}, \boldsymbol{\sigma}; \mathbf{h}) \mid \boldsymbol{\lambda} \in \boldsymbol{\Lambda}, \boldsymbol{\sigma} \in \boldsymbol{\Sigma}\}$ for

$$C(\boldsymbol{\lambda}, \boldsymbol{\sigma}; \mathbf{h}) = \{(x', x) \in (\mathsf{N}^\natural)_{\mathbb{R}} \oplus \mathsf{N}_{\mathbb{R}} \mid x' + \mathbf{h}(x) \in \boldsymbol{\lambda},\ x \in \boldsymbol{\sigma}\}.$$

Let $U_{\boldsymbol{\sigma}} \subset S$ be the open subset $\mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma}, \mathcal{L})$. Then $U_{\boldsymbol{\sigma}} \simeq \operatorname{Specan}_Z \mathcal{L}[\boldsymbol{\sigma}^\vee \cap \mathsf{M}]$. Let $V_{\boldsymbol{\lambda}, \boldsymbol{\sigma}}$ be the toric bundle $\mathbb{T}_{\mathsf{N}^\natural}(\boldsymbol{\lambda}, \mathcal{L}_\natural)$ over $U_{\boldsymbol{\sigma}}$ for a cone $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$ or for $\boldsymbol{\lambda} = \boldsymbol{\sigma}_\natural$. Then $p^{-1}U_{\boldsymbol{\sigma}} \simeq V_{\boldsymbol{\sigma}_\natural, \boldsymbol{\sigma}}$. We have an isomorphism $V_{\boldsymbol{\lambda}, \boldsymbol{\sigma}} \simeq \operatorname{Specan} \mathcal{A}_{\boldsymbol{\lambda}, \boldsymbol{\sigma}}$ for the subalgebra

$$\mathcal{A}_{\boldsymbol{\lambda}, \boldsymbol{\sigma}} = \bigoplus\nolimits_{m' \in \boldsymbol{\lambda}^\vee \cap \mathsf{M}^\natural,\ m \in \square_{\langle m', \mathbf{h} \rangle}(\boldsymbol{\sigma})} \mathcal{L}^m \subset \widetilde{\mathcal{L}}[\mathsf{M}^\natural \oplus \mathsf{M}].$$

The section $T \cap p^{-1}U_{\boldsymbol{\sigma}} \subset p^{-1}U_{\boldsymbol{\sigma}}$ is determined by a surjective homomorphism $\mathcal{A}_{\boldsymbol{\sigma}_\natural, \boldsymbol{\sigma}} \twoheadrightarrow \mathcal{L}[\boldsymbol{\sigma}^\vee \cap \mathsf{M}]$ which is induced from the summation

$$\bigoplus\nolimits_{m' \in \boldsymbol{\lambda}^\vee \cap \mathsf{M}^\natural} \mathcal{L}^m \to \mathcal{L}^m.$$

Then the fiber product of $V_{\boldsymbol{\lambda}, \boldsymbol{\sigma}}$ and T over $\mathbb{V}$ is isomorphic to $\operatorname{Specan}_Z \mathcal{B}_{\boldsymbol{\lambda}, \boldsymbol{\sigma}}$ for the $\mathcal{O}_Z$-algebra $\mathcal{B}_{\boldsymbol{\lambda}, \boldsymbol{\sigma}}$ defined as the image of a similar homomorphism $\mathcal{A}_{\boldsymbol{\lambda}, \boldsymbol{\sigma}} \to \mathcal{L}[\mathsf{M}]$.

For $m \in \mathsf{M}$, there exists an $m' \in \boldsymbol{\lambda}^{\vee} \cap \mathsf{M}^{\natural}$ with $m \in \square_{\langle m', \mathbf{h}\rangle}(\boldsymbol{\sigma})$ if and only if $m \in C_{\mathbf{h}}(\boldsymbol{\lambda}, \boldsymbol{\sigma})^{\vee} \cap \mathsf{M}$ for the cone

$$C_{\mathbf{h}}(\boldsymbol{\lambda}, \boldsymbol{\sigma}) := \boldsymbol{\sigma} \cap \mathbf{h}^{-1}(\boldsymbol{\lambda}) = \{x \in \boldsymbol{\sigma} \mid \mathbf{h}(x) \in \boldsymbol{\lambda}\}.$$

Hence, $\mathcal{B}_{\boldsymbol{\lambda}, \boldsymbol{\sigma}} \simeq \mathcal{L}[C_{\mathbf{h}}(\boldsymbol{\lambda}, \boldsymbol{\sigma})^{\vee} \cap \mathsf{M}]$. Therefore, $S_{\boldsymbol{\Lambda}} \simeq \mathbb{T}_{\mathsf{N}}(\boldsymbol{\Sigma}_1, \mathcal{L})$ for the fan

$$\boldsymbol{\Sigma}_1 = \{C_{\mathbf{h}}(\boldsymbol{\lambda}, \boldsymbol{\sigma}) \mid \boldsymbol{\lambda} \in \boldsymbol{\Lambda},\, \boldsymbol{\sigma} \in \boldsymbol{\Sigma}\}. \qquad \square$$

A function $h \in \mathrm{SF}_{\mathsf{N}^{\natural}}(\boldsymbol{\Lambda}, \mathbb{R})$ defines an $\mathbb{R}$-Cartier divisor D_h on $\mathbb{T}_{\mathsf{N}^{\natural}}(\boldsymbol{\Lambda}, \mathcal{L}_{\natural})$. We denote its restriction to $S_{\boldsymbol{\Lambda}}$ by the same symbol D_h.

Remark For $h \in \mathrm{SF}_{\mathsf{N}^{\natural}}(\boldsymbol{\Lambda}, \mathbb{Z})$, the invertible sheaf $\mathcal{O}_{\mathbb{V}}(D_h)$ is associated with the $\mathcal{L}[\boldsymbol{\sigma}_{\natural}^{\vee} \cap \mathsf{M}^{\natural}]$-module $\mathcal{L}[\square_h(\boldsymbol{\sigma}_{\natural}) \cap \mathsf{M}^{\natural}]$. Therefore, there is an isomorphism

$$f_*\mathcal{O}_{S_{\boldsymbol{\Lambda}}}(D_h) \simeq \sum_{m \in \square_h(\boldsymbol{\sigma}_{\natural}) \cap \mathsf{M}^{\natural}} \mathcal{L}_{\natural}^m = \sum_{m \in \square_h(\boldsymbol{\sigma}_{\natural}) \cap \mathsf{M}^{\natural}} \mathcal{O}_S\Big(-\sum\nolimits_{i=1}^r m_i B_i\Big) \subset j_*\mathcal{O}_{S \setminus B}$$

for the open immersion $j \colon S \setminus B \hookrightarrow S$.

Suppose that S is projective and $Z = \bigcap_{i=1}^r B_i$ is non-empty and irreducible. For $h \in \mathrm{SF}_{\mathsf{N}^{\natural}}(\boldsymbol{\Lambda}, \mathbb{R})$ and for an $\mathbb{R}$-divisor E of S, we define

$$\square_{\mathrm{Nef}}(E|_Z, h) := \{m \in \square_h(\boldsymbol{\sigma}_{\natural}) \mid (E + \mathcal{L}_{\natural}^m)|_Z \text{ is nef}\},$$

Note that h is defined only on $|\boldsymbol{\Lambda}| = \boldsymbol{\sigma}_{\natural}$.

2.14. Lemma

(1) *The following conditions are equivalent to each other*:
  (a) *The restriction* $(D_h + f^*E)|_{f^{-1}Z}$ *is nef*;
  (b) $l_{\boldsymbol{\lambda}} \in \square_{\mathrm{Nef}}(E|_Z, h)$ *for any maximal cone* $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$*, where* $l_{\boldsymbol{\lambda}} \in \mathsf{M}^{\natural}_{\mathbb{R}}$ *is defined by* $h(x) = \langle l_{\boldsymbol{\lambda}}, x\rangle$ *for* $x \in \boldsymbol{\lambda}$;
  (c) $\square_{\mathrm{Nef}}(E|_Z, h) \neq \emptyset$ *and, for any* $x \in \boldsymbol{\sigma}_{\natural}$,
$$h(x) = \inf\{\langle m, x\rangle \mid m \in \square_{\mathrm{Nef}}(E|_Z, h)\}.$$
(2) *Assume that* $E + \mathcal{L}_{\natural}^m$ *is nef for any* $m \in \boldsymbol{\sigma}_{\natural}^{\vee}$ *with* $(E + \mathcal{L}_{\natural}^m)|_Z$ *being nef. Then* $D_h + f^*E$ *is nef on* $S_{\boldsymbol{\Lambda}}$ *if the restriction* $(D_h + f^*E)|_{f^{-1}Z}$ *is nef.*

Proof. (1) The proof is similar to **2.4**-(2).

(a) $\Rightarrow$ (b): The restriction of $D_h + f^*E$ to $f^{-1}Z$ is nef if and only if its restriction to $\mathbb{T}_{\mathsf{N}^{\natural}(\boldsymbol{\lambda})}(\boldsymbol{\Lambda}/\boldsymbol{\lambda}, \mathcal{L}_{\natural}|_Z)$ is nef for any $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$ with $\boldsymbol{\lambda} \cap \mathrm{Int}\, \boldsymbol{\sigma}_{\natural} \neq \emptyset$. For such a cone $\boldsymbol{\lambda}$, let us choose $l_{\boldsymbol{\lambda}} \in \mathsf{M}^{\natural}_{\mathbb{R}}$ such that $h(x) = \langle l_{\boldsymbol{\lambda}}, x\rangle$ for any $x \in \boldsymbol{\lambda}$ and define $h^{\boldsymbol{\lambda}}(y) := h(y) - \langle l_{\boldsymbol{\lambda}}, y\rangle$ for $y \in \boldsymbol{\sigma}_{\natural}$. Then $(E + \mathcal{L}^{l_{\boldsymbol{\lambda}}})|_Z$ is nef if $\boldsymbol{\lambda}$ is a maximal cone. If $\boldsymbol{\lambda}_1$ and $\boldsymbol{\lambda}_2$ are maximal cones of $\boldsymbol{\Lambda}$ with $\dim \boldsymbol{\lambda}_1 \cap \boldsymbol{\lambda}_2 = r - 1$, then $\boldsymbol{\lambda}_1 \cap \boldsymbol{\lambda}_2 \cap \mathrm{Int}\, \boldsymbol{\sigma}_{\natural} \neq \emptyset$. By restricting $D_h + p^*E$ to $\mathsf{V}(\boldsymbol{\lambda}_1 \cap \boldsymbol{\lambda}_2, \mathcal{L})$ over S, we infer that $\langle l_{\boldsymbol{\lambda}_1}, x\rangle \geq h(x)$ for $x \in \boldsymbol{\lambda}_1 \cup \boldsymbol{\lambda}_2$. Thus $l_{\boldsymbol{\lambda}} \in \square_h(\boldsymbol{\sigma}_{\natural})$ for a maximal cone $\boldsymbol{\lambda}$ by the same argument as in the proof of **1.10**-(5) $\Rightarrow$ **1.10**-(2). Thus $l_{\boldsymbol{\lambda}} \in \square_{\mathrm{Nef}}(E|_Z, h)$.

(b) $\Leftrightarrow$ (c) is shown by the same argument as in **2.4**-(2).

(b) $\Rightarrow$ (a): Let W_Z be the intersection of the supports of effective $\mathbb{R}$-Cartier divisors $D_h + \mathrm{div}(\mathsf{e}(m))$ for all $m \in \square_{\mathrm{Nef}}(E|_Z, h)$ in the toric bundle $\mathbb{T}_{\mathsf{N}^{\natural}}(\boldsymbol{\Lambda}, \mathcal{L}_{\natural}|_Z)$

over Z. If $W_Z \neq \emptyset$, then $W_Z \supset \mathbb{V}(\boldsymbol{\lambda}, \mathcal{L}_\natural|_Z)$ for a maximal cone; this contradicts $l_{\boldsymbol{\lambda}} \in \square_{\mathrm{Nef}}(E|_Z, h)$. Hence $W_Z = \emptyset$ and hence $(D_h + f^*E)|_{f^{-1}Z}$ is nef.

(2) By assumption, if $m \in \square_{\mathrm{Nef}}(E|_Z, h)$, then $E + \mathcal{L}_\natural^m$ is nef. Let W be the intersection of the supports of effective $\mathbb{R}$-Cartier divisors $D_h + \mathrm{div}(\mathsf{e}(m))$ in $\mathbb{T}_{\mathsf{N}^\natural}(\boldsymbol{\Lambda}, \mathcal{L}_\natural)$ for all $m \in \square_{\mathrm{Nef}}(E|_Z, h)$. Suppose that $(D_h + f^*E)|_{f^{-1}Z}$ is nef. Then $W = \emptyset$ by the same argument above. Thus $D_h + f^*E$ is nef. □

2.15. Proposition *Let S be a non-singular projective variety and let B_1, B_2, ..., B_r be non-singular prime divisors such that $B = \sum_{i=1}^r B_i$ is simple normal crossing, $r < \dim S$, and $Z = \bigcap_{i=1}^r B_i$ is non-empty and irreducible. Let E be an $\mathbb{R}$-divisor of S such that*

$$\square_{\mathrm{Nef}}(E) = \left\{(m_i)_{i=1}^r \in \mathbb{R}^r \;\middle|\; E - \sum\nolimits_{i=1}^r m_i B_i \text{ is nef}\right\} \neq \emptyset.$$

Assume that $\square_{\mathrm{Nef}}(E) \subset \mathrm{N}^1(S)$ is a rational polyhedral convex set and

$$\square_{\mathrm{Nef}}(E) = \left\{(m_i) \in \mathbb{R}^r_{\geq 0} \mid (E - \sum m_i B_i)|_Z \text{ is nef}\right\}.$$

Suppose either that $\mathrm{NBs}(E) \subset Z$ or that E admits a Zariski-decomposition. Then there exist a toric blowing-up $f\colon S_{\boldsymbol{\Lambda}} \to S$ along B associated with a finite non-singular subdivision $\boldsymbol{\Lambda}$ of the first quadrant cone $\boldsymbol{\sigma}_\natural \subset (\mathsf{N}^\natural)_{\mathbb{R}}$ for the free abelian group $\mathsf{N}^\natural$ of rank r related to B and a support function $h \in \mathrm{SF}_{\mathsf{N}^\natural}(\boldsymbol{\Lambda}, \mathbb{R})$ such that $D_h + f^*E$ is nef and is the positive part of the σ-decomposition of f^*E.

Proof. For the construction of the toric blowing-up, we consider the free abelian group $\mathsf{N}^\natural$ with the basis $(e_1, e_2, \ldots, e_r)$ and the element $\mathcal{L}_\natural = \sum e_i \otimes \mathcal{O}_S(-B_i) \in \mathsf{N}^\natural \otimes \mathrm{Pic}(S)$. Let $(\delta_1, \delta_2, \ldots, \delta_r)$ be the basis of $\mathsf{M}^\natural = (\mathsf{N}^\natural)^\vee$ dual to $(e_1, e_2, \ldots, e_r)$. By the identification $(m_i) \leftrightarrow m = \sum m_i \delta_i$, we can regard $\square_{\mathrm{Nef}}(E)$ as a subset of $\mathsf{M}^\natural_{\mathbb{R}}$. We consider the following function on $\boldsymbol{\sigma}_\natural$:

$$h^\dagger(x) := \min\{\langle m, x\rangle \mid m \in \square_{\mathrm{Nef}}(E)\}.$$

Then $h^\dagger \in \mathrm{SFC}_{\mathsf{N}^\natural}(\boldsymbol{\sigma}_\natural, \mathbb{R})$. Note that $h^\dagger$ is non-negative on $\boldsymbol{\sigma}_\natural$. Let $\boldsymbol{\Lambda}$ be a non-singular finite subdivision of $\boldsymbol{\sigma}_\natural$ such that $h^\dagger \in \mathrm{SF}_{\mathsf{N}^\natural}(\boldsymbol{\Lambda}, \mathbb{R})$. Then $E^\dagger := D_{h^\dagger} + f^*E$ is nef by **2.14**, since $\square_{\mathrm{Nef}}(E) \subset \square_{\mathrm{Nef}}(E|_Z, h^\dagger)$.

The positive part $P_\sigma(f^*E)$ of the σ-decomposition is written by $D_h + f^*E$ for some $h \in \mathrm{SF}_{\mathsf{N}^\natural}(\boldsymbol{\Lambda}, \mathbb{R})$, since $\boldsymbol{\Lambda}$ is non-singular. Here,

$$h(v) = \mathrm{mult}_{\Gamma_v} N_\sigma(f^*E) = \sigma_{\Gamma_v}(f^*E) \geq 0$$

for any $v \in \mathrm{Ver}(\boldsymbol{\Lambda})$. Note that $D_h + f^*E = P_\sigma(f^*E) \geq E^\dagger$, since $E^\dagger$ is nef. In particular, $h(v) \leq h^\dagger(v)$ for any $v \in \mathrm{Ver}(\boldsymbol{\Lambda})$ and hence $h(x) \leq h^\dagger(x)$ for $x \in \boldsymbol{\sigma}_\natural$.

Let $v \in \mathrm{Ver}(\boldsymbol{\Lambda})$ be a vertex contained in $\mathrm{Int}\,\boldsymbol{\sigma}_\natural$. Then the corresponding prime divisor $\Gamma_v \subset S_{\boldsymbol{\Lambda}}$ is isomorphic to $\mathbb{V}(\boldsymbol{\Lambda}/\mathbb{R}_{\geq 0}v, \mathcal{L}_\natural|_Z)$ over Z. The restriction of $D_h + f^*E$ to Γ_v is pseudo-effective. Then, by **2.4**-(1), there is an $l_v \in \mathsf{M}^\natural_{\mathbb{R}}$ such that

(1) $h(v) = \langle l_v, x\rangle$,
(2) $\langle l_v, x\rangle \geq h(x)$ for any $x \in \bigcup_{v \in \boldsymbol{\lambda} \in \boldsymbol{\Lambda}} \boldsymbol{\lambda}$,
(3) $E + \mathcal{L}_\natural^{l_v}$ is nef.

Since $l_v \in \square_{\mathrm{Nef}}(E)$, we have $h(v) = \langle l_v, v\rangle \geq h^\dagger(v)$. Thus $h(v) = h^\dagger(v)$.

Suppose that $\mathrm{NBs}(E) \subset Z$. If a vertex $v \in \mathrm{Ver}(\mathbf{\Lambda})$ is not contained in $\mathrm{Int}\,\boldsymbol{\sigma}_\natural$, then $f(\Gamma_v) \not\subset Z$. Thus $\sigma_{\Gamma_v}(f^*E) = h(v) = 0$. Therefore $P_\sigma(f^*E) = E^\dagger$ and it gives the Zariski-decomposition.

Next suppose that there is a vertex $v \in \mathrm{Ver}(\mathbf{\Lambda})$ such that $h(v) < h^\dagger(v)$. Then $v \notin \mathrm{Int}\,\boldsymbol{\sigma}_\natural$. There is a vertex $v' \in \mathrm{Ver}(\mathsf{N}, \mathbf{\Sigma})$ contained in $\mathrm{Int}\,\boldsymbol{\sigma}_\natural$ such that $C(v, v') = \mathbb{R}_{\geq 0}v + \mathbb{R}_{\geq 0}v'$ is a two-dimensional cone contained in $\mathbf{\Lambda}$. Here $h(v') = h^\dagger(v')$. The blowing-up $\nu\colon Y \to S_{\mathbf{\Lambda}}$ along the intersection $\Gamma_v \cap \Gamma_{v'}$ corresponds to a finite subdivision $\mathbf{\Lambda}'$ of $\mathbf{\Lambda}$ in which the new vertex $w = v + v' \in \mathrm{Ver}(\mathbf{\Lambda}')$ corresponds to the exceptional divisor Γ_w. We have

$$\begin{aligned} h^\dagger(w) = h^\dagger(v) + h^\dagger(v') &= \sigma_{\Gamma_w}(\nu^*f^*E) = \mathrm{mult}_{\Gamma_w} N_\sigma(\nu^*f^*E), \\ h(w) = h(v) + h(v') &= \mathrm{mult}_{\Gamma_w} \nu^*N_\sigma(f^*E), \\ \sigma_{\Gamma_w}(\nu^*P_\sigma(f^*E)) &= h^\dagger(w) - h(w) = h^\dagger(v) - h(v) > 0. \end{aligned}$$

Next, we consider the blowing-up of Y along $\Gamma_v \cap \Gamma_w$ whose exceptional divisor corresponds to $w + v = 2v + v'$. By continuing the process, we have a sequence $Y_k \to Y_{k-1} \to \cdots \to Y_1 = Y \to S_{\mathbf{\Lambda}}$ of blowups such that the exceptional divisor of $\nu_k\colon Y_k \to Y_{k-1}$ corresponds to $w_k = kv + v'$. For the morphisms $f_i\colon Y_i \to Y \to S$, we have the following equalities:

$$\begin{aligned} h^\dagger(w_k) &= kh^\dagger(v) + h^\dagger(v') = \sigma_{\Gamma_{w_k}}(f_k^*E), \\ \sigma_{\Gamma_{w_{k-1}}}(f_{k-1}^*E) + h(v) &= \mathrm{mult}_{\Gamma_{w_k}} \nu_k^*N_\sigma(f_{k-1}^*E), \\ \sigma_{\Gamma_{w_k}}(\nu_k^*P_\sigma(f_{k-1}^*E)) &= h^\dagger(w_k) - \mathrm{mult}_{\Gamma_{w_k}} \nu_k^*N_\sigma(f_{k-1}^*E) \\ &= h^\dagger(v) - h(v) + h^\dagger(w_{k-1}) - \sigma_{\Gamma_{w_{k-1}}}(f_{k-1}^*E) > 0. \end{aligned}$$

Thus the process does not terminate. Hence, E admits no Zariski-decompositions by **2.11**. Therefore, if E admits a Zariski-decomposition, then $h^\dagger(v) = h(v)$ for any $v \in \mathrm{Ver}(\mathbf{\Lambda})$ and hence $P_\sigma(f^*E)$ is equal to the nef $\mathbb{R}$-divisor $E^\dagger$. $\square$

§3. Vector bundles over a curve

§3.a. Filtration of vector bundles.

3.1. Lemma *Let X be a complex analytic variety and let*

$$0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0$$

be an exact sequence of vector bundles on X. Let $\pi_i\colon P_i \to X$ be the projective bundle $\mathbb{P}_X(\mathcal{E}_i)$ for $i = 1, 2, 3$. For the tautological line bundle $\mathcal{O}_{\mathcal{E}_1}(1)$, let $\mathcal{F}$ be the vector bundle on P_1 determined by the commutative diagram

$$\text{(IV-6)}\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & \pi_1^*\mathcal{E}_1 & \longrightarrow & \pi_1^*\mathcal{E}_2 & \longrightarrow & \pi_1^*\mathcal{E}_3 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathcal{O}_{\mathcal{E}_1}(1) & \longrightarrow & \mathcal{F} & \longrightarrow & \pi_1^*\mathcal{E}_3 & \longrightarrow & 0, \end{array}$$

and let $q\colon P_{12} = \mathbb{P}_{P_1}(\mathcal{F}) \to P_1$ *be the natural projection. Then, there is a morphism* $\rho\colon P_{12} \to P_2$ *over* X *such that* ρ *is isomorphic to the blowing-up along* $P_3 \subset P_2$*. Moreover, the divisor* $E = \rho^{-1}P_3$ *is isomorphic to* $P_1 \times_Y P_3$ *over* P_3*, and* $\rho^*\mathcal{O}_{\mathcal{E}_2}(1) \simeq q^*\mathcal{O}_{\mathcal{F}}(1) \otimes \mathcal{O}_{P_{12}}(-E)$*.*

Proof. The diagram (IV-6) induces a surjective homomorphism $q^*\pi_1^*\mathcal{E}_2 \twoheadrightarrow \mathcal{O}_{\mathcal{F}}(1)$ defining ρ above. Let $\mathcal{I}$ be the defining ideal sheaf of P_3 in P_2. Then there is a surjective homomorphism

$$\pi_2^*\mathcal{E}_1 \twoheadrightarrow \mathcal{I}\mathcal{O}_{\mathcal{E}_2}(1) \tag{IV-7}$$

inducing $\mathcal{E}_1 \simeq \pi_{2*}(\mathcal{I}\mathcal{O}_{\mathcal{E}_2}(1))$. There is a commutative diagram

$$\begin{array}{ccccc} \rho^*\pi_2^*\mathcal{E}_1 & = & q^*\pi_1^*\mathcal{E}_1 & \longrightarrow & q^*\mathcal{O}_{\mathcal{E}_1}(1) \\ \downarrow & & & & \downarrow \\ \rho^*(\mathcal{I}\mathcal{O}_{\mathcal{E}_2}(1)) & \longrightarrow & \rho^*\mathcal{O}_{\mathcal{E}_2}(1) & = & \mathcal{O}_{\mathcal{F}}(1). \end{array}$$

Thus $\rho^*(\mathcal{I}\mathcal{O}_{\mathcal{E}_2}(1))/(\mathrm{tor})$ is isomorphic to the line bundle $q^*\mathcal{O}_{\mathcal{E}_1}(1)$. Hence $\rho^*\mathcal{I}/(\mathrm{tor})$ is the defining ideal of the Cartier divisor $E = \mathbb{P}_{P_1}(q^*\mathcal{E}_3) \simeq P_1 \times_Y P_3$ of P_{12}. Here $\mathcal{O}_{P_{12}}(-E) \otimes \rho^*\mathcal{O}_{\mathcal{E}_2}(1) \simeq q^*\mathcal{O}_{\mathcal{E}_1}(1)$ holds. Let $\mu\colon Q \to P_2$ be the blowing-up along P_3. Then there is a morphism $\varphi\colon P_{12} \to Q$ such that $\rho = \mu\circ\varphi$. There is a morphism $Q \to P_1$ over X by the pullback μ^* of (IV-7). From (IV-6), we infer that there is a morphism $Q \to P_{12}$ over P_1 which is the inverse of φ. □

Remark If $\operatorname{rank}\mathcal{E}_1 = 1$, then $P_1 \simeq X$ and $P_{12} \simeq P_2$.

Let X be a complex analytic variety and let

$$\mathcal{E}_\bullet = [0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \cdots \subset \mathcal{E}_l = \mathcal{E}]$$

be a sequence of vector subbundles of $\mathcal{E}$ on X such that $\mathrm{Gr}_i(\mathcal{E}_\bullet) = \mathcal{E}_i/\mathcal{E}_{i-1}$ is a non-zero vector bundle for $1 \le i \le l$. The number l is called the length of $\mathcal{E}_\bullet$ and is denoted by $l(\mathcal{E}_\bullet)$.

Let us consider the following functor F from the category of complex analytic spaces over X into the category of sets: for a morphism $f\colon Y \to X$, let $\varphi_i\colon f^*\mathcal{E}_i \twoheadrightarrow \mathcal{L}_i$ be surjective homomorphisms into line bundles $\mathcal{L}_i$ of Y for $1 \le i \le l$ and let $u_i\colon \mathcal{L}_i \to \mathcal{L}_{i+1}$ be homomorphisms for $1 \le i < l$ such that the diagrams

$$\begin{array}{ccc} f^*\mathcal{E}_i & \longrightarrow & f^*\mathcal{E}_{i+1} \\ {\scriptstyle\varphi_i}\downarrow & & \downarrow{\scriptstyle\varphi_{i+1}} \\ \mathcal{L}_i & \xrightarrow{u_i} & \mathcal{L}_{i+1} \end{array}$$

are all commutative. Let $F(Y/X)$ be the set of the collections $(\varphi_i, u_i)_{i=1}^l$ above modulo isomorphisms.

3.2. Lemma-Definition *The functor* F *above is representable by a projective smooth morphism over* X*. The representing morphism is denoted by*

$$\pi = \pi_l\colon \mathbb{P}_X(\mathcal{E}_\bullet) = \mathbb{P}(\mathcal{E}_\bullet) = \mathbb{P}(\mathcal{E}_1 \subset \mathcal{E}_2 \subset \cdots \subset \mathcal{E}_l) \to X.$$

Proof. We shall prove by induction on l. If $l = 1$, then F is representable by the projective bundle $\mathbb{P}_X(\mathcal{E}) = \mathbb{P}(\mathcal{E}_1)$. For the projective bundle $p_1 : \mathbb{P}(\mathcal{E}_1) \to X$, let $\mathcal{K}_1$ be the kernel of $p_1^*\mathcal{E}_1 \to \mathcal{O}_{\mathcal{E}_1}(1)$. Then $\mathcal{K}_1$ is a subbundle of $p_1^*\mathcal{E}_i$ for any i. Let $\mathcal{E}'_i$ be the quotient vector bundle $p_1^*\mathcal{E}_i/\mathcal{K}_1$. Then we have a sequence of vector bundles

$$\mathcal{O}_{\mathcal{E}_1}(1) \subset \mathcal{E}'_2 \subset \mathcal{E}'_3 \subset \cdots \subset \mathcal{E}'_l.$$

By induction, the functor F with respect to the filtration above but starting from $\mathcal{E}'_2$ is represented by

$$Q = \mathbb{P}_{\mathbb{P}(\mathcal{E}_1)}(\mathcal{E}'_2 \subset \cdots \subset \mathcal{E}'_l) \to \mathbb{P}(\mathcal{E}_1).$$

Let $((\varphi_i : f^*\mathcal{E}_i \to \mathcal{L}_i), u_i)$ be an element of $F(Y/X)$ for a morphism $f : Y \to X$ from an analytic space. Then φ_1 induces a morphism $f_1 : Y \to \mathbb{P}(\mathcal{E}_1)$ over X and φ_i induces a surjective homomorphism $f_1^*\mathcal{E}'_i \twoheadrightarrow \mathcal{L}_i$. Hence the element of $F(Y/X)$ defines a morphism $Y \to Q$ over X. Conversely, from a morphism $h : Y \to Q$, we have a morphism $f_1 : Y \to Q \to \mathbb{P}(\mathcal{E}_1)$, surjective homomorphisms $f_1^*\mathcal{E}'_i \twoheadrightarrow \mathcal{L}_i$ into line bundles for $2 \leq i \leq l$, and compatible homomorphisms $u_i : \mathcal{L}_i \to \mathcal{L}_{i+1}$ for $2 \leq i < l$. We define $\mathcal{L}_1 = f_1^*\mathcal{O}_{\mathcal{E}_1}(1)$, $\varphi_1 : f^*\mathcal{E}_1 \to \mathcal{L}_1$ to be the pullback of $p_1^*\mathcal{E}_1 \to \mathcal{O}_{\mathcal{E}_1}(1)$, φ_i to be the composite $f^*\mathcal{E}_i \to f_1^*\mathcal{E}'_i \to \mathcal{L}_i$ for $2 \leq i \leq l$, and $u_1 : \mathcal{L}_1 \to \mathcal{L}_2$ to be the composite

$$\mathcal{L}_1 = f_1^*\mathcal{O}_{\mathcal{E}_1}(1) \to f_1^*\mathcal{E}'_2 \to \mathcal{L}_2.$$

Then (φ_i, u_i) is an element of $F(Y/X)$. In this way, we infer that $Q \to X$ represents F with respect to $\mathcal{E}_\bullet$. $\square$

For $1 \leq k \leq l$, we define the following filtrations:

$$\mathcal{E}_{\bullet \leq k} = [\mathcal{E}_1 \subset \cdots \subset \mathcal{E}_k], \quad \mathcal{E}_{\bullet \geq k} = [\mathcal{E}_k \subset \cdots \subset \mathcal{E}_l].$$

Let $((\varphi_i : \pi^*\mathcal{E}_i \to \mathcal{L}_i), u_i)$ be the universal element of $F(\mathbb{P}(\mathcal{E}_\bullet)/X)$. Note that u_i are all injective. By considering (φ_i, u_i) for $i \leq k$ or $i \geq k$, we have natural morphisms $\mathbb{P}(\mathcal{E}_\bullet) \to \mathbb{P}(\mathcal{E}_{\bullet \leq k})$ and $\mathbb{P}(\mathcal{E}_\bullet) \to \mathbb{P}(\mathcal{E}_{\bullet \geq k})$. We have a Cartesian commutative diagram

$$\text{(IV-8)} \qquad \begin{array}{ccc} \mathbb{P}(\mathcal{E}_\bullet) & \longrightarrow & \mathbb{P}(\mathcal{E}_{\bullet \geq k}) \\ \big\downarrow & & \big\downarrow \\ \mathbb{P}(\mathcal{E}_{\bullet \leq k}) & \longrightarrow & \mathbb{P}(\mathcal{E}_k) \end{array}$$

for $1 \leq k \leq l$. Here vertical arrows are smooth projective morphisms by the proof of **3.2**. We infer that the horizontal arrows are bimeromorphic by **3.1**. The bimeromorphic morphism $\mathbb{P}(\mathcal{E}_1 \subset \mathcal{E}_2) \to \mathbb{P}(\mathcal{E}_2)$ is an isomorphism if and only if $\mathcal{E}_1$ is a line bundle. Thus $\mathbb{P}(\mathcal{E}_\bullet) \to \mathbb{P}(\mathcal{E}_{\bullet \geq k})$ is an isomorphism for some $k > 1$ if and only if $l = 2$ and $\mathcal{E}_1$ is a line bundle.

3.3. Lemma

(1) *The pullback of* $\mathbb{P}(\mathcal{E}_{k+1}/\mathcal{E}_k) \subset \mathbb{P}(\mathcal{E}_{k+1})$ *by the morphism* $\mathbb{P}(\mathcal{E}_{\bullet \geq k+1}) \to \mathbb{P}(\mathcal{E}_{k+1})$ *is isomorphic to*

$$\mathbb{P}(\mathcal{E}_{k+1}/\mathcal{E}_k \subset \cdots \subset \mathcal{E}_l/\mathcal{E}_k).$$

(2) *Let E_k be the pullback of $\mathbb{P}(\mathcal{E}_{k+1}/\mathcal{E}_k) \subset \mathbb{P}(\mathcal{E}_{k+1})$ by the composite $\mathbb{P}(\mathcal{E}_\bullet) \to \mathbb{P}(\mathcal{E}_{\bullet \geq k+1}) \to \mathbb{P}(\mathcal{E}_{k+1})$ for $1 \leq k \leq l-1$. Then E_k is a divisor isomorphic to*

$$\mathbb{P}(\mathcal{E}_1 \subset \cdots \subset \mathcal{E}_k) \times_X \mathbb{P}(\mathcal{E}_{k+1}/\mathcal{E}_k \subset \cdots \subset \mathcal{E}_l/\mathcal{E}_k).$$

Here, E_k is not exceptional for the bimeromorphic morphism $\mathbb{P}(\mathcal{E}_\bullet) \to \mathbb{P}(\mathcal{E}_{\bullet \geq k+1})$ if and only if $k = \operatorname{rank} \mathcal{E}_1 = 1$.

(3) *For indices $1 \leq a(1) < a(2) < \cdots < a(e) \leq l-1$, the intersection $\bigcap_{j=1}^{e} E_{a(j)}$ is isomorphic to the fiber product*

$$\prod_{j=1}^{e+1} \mathbb{P}(\mathcal{E}_{a(j-1)+1}/\mathcal{E}_{a(j-1)} \subset \cdots \subset \mathcal{E}_{a(j)}/\mathcal{E}_{a(j-1)})$$

over X, where $a(0) = 0$ and $a(e+1) = l$.

(4) *Let H_i be the pullback of the tautological divisor $H_{\mathcal{E}_i}$ by the composite $\mathbb{P}(\mathcal{E}_\bullet) \to \mathbb{P}(\mathcal{E}_{\bullet \leq i}) \to \mathbb{P}(\mathcal{E}_i)$. Then $H_{i+1} - H_i \sim E_i$ for $1 \leq i \leq l-1$.*

Proof. Let $f : Y \to X$ be an analytic space over X.

(1) Let F' be the similar functor to F with respect to the filtration $\mathcal{E}_{\bullet \geq k+1}$. Let (φ_i, u_i) be an element of $F'(Y/X)$. Then it induces a morphism into $\mathbb{P}(\mathcal{E}_{k+1}/\mathcal{E}_k) \subset \mathbb{P}(\mathcal{E}_{k+1})$ if and only if the the composite $f^*\mathcal{E}_k \to f^*\mathcal{E}_{k+1} \to \mathcal{L}_{k+1}$ is zero. Thus we have the expected isomorphism.

(2) Let (φ_i, u_i) be an element of $F(Y/X)$. Then it induces a morphism into $\mathbb{P}(\mathcal{E}_{k+1}/\mathcal{E}_k) \subset \mathbb{P}(\mathcal{E}_{k+1})$ if and only if $u_k : \mathcal{L}_k \to \mathcal{L}_{k+1}$ is zero. Thus E_k is expressed as above. This is a divisor since $\dim E_k = \dim \mathbb{P}(\mathcal{E}_k) + \dim \mathbb{P}(\mathcal{E}/\mathcal{E}_k) - \dim X = \dim \mathbb{P}(\mathcal{E}) - 1$. This is not exceptional if and only if $\mathbb{P}(\mathcal{E}_{\bullet \leq k}) \to X$ is an isomorphism. It is equivalent to: $k = \operatorname{rank} \mathcal{E}_1 = 1$.

(3) Let (φ_i, u_i) be an element of $F(Y/X)$. It induces a morphism into the intersections of $E_{a(j)}$ if and only if $u_{a(j)} = 0$ for any j. Thus the isomorphism exists.

(4) The pullback of $\mathbb{P}(\mathcal{E}_{k+1}/\mathcal{E}_k) \subset \mathbb{P}(\mathcal{E}_{k+1})$ by the morphism $\mathbb{P}(\mathcal{E}_k \subset \mathcal{E}_{k+1}) \to \mathbb{P}(\mathcal{E}_{k+1})$ is a divisor whose pullback is E_k. The linear equivalence follows from **3.1**. □

3.4. Lemma *The projective morphism $\mathbb{P}(\mathcal{E}_\bullet) \to X$ is also characterized by the following way inductively*:

⟨2⟩ *$\mathbb{P}(\mathcal{E}_1 \subset \mathcal{E}_2)$ is the blown-up of $\mathbb{P}(\mathcal{E}_2)$ along $\mathbb{P}(\mathcal{E}_2/\mathcal{E}_1)$.*

⟨3⟩ *$\mathbb{P}(\mathcal{E}_1 \subset \mathcal{E}_2 \subset \mathcal{E}_3)$ is the blown-up of $\mathbb{P}(\mathcal{E}_2 \subset \mathcal{E}_3)$ along $\mathbb{P}(\mathcal{E}_2/\mathcal{E}_1 \subset \mathcal{E}_3/\mathcal{E}_1)$.*

⋮

⟨l⟩ *$\mathbb{P}(\mathcal{E}_1 \subset \cdots \subset \mathcal{E}_l)$ is the blown-up of $\mathbb{P}(\mathcal{E}_2 \subset \cdots \subset \mathcal{E}_l)$ along $\mathbb{P}(\mathcal{E}_2/\mathcal{E}_1 \subset \cdots \subset \mathcal{E}_l/\mathcal{E}_1)$.*

Proof. By the Cartesian diagrams (IV-8) and by **3.1**, it is enough to show that the pullback of $\mathbb{P}(\mathcal{E}/\mathcal{E}_1) \subset \mathbb{P}(\mathcal{E})$ by $\mathbb{P}(\mathcal{E}_{\bullet \geq 2}) \to \mathbb{P}(\mathcal{E})$ is isomorphic to

$$\mathbb{P}(\mathcal{E}_2/\mathcal{E}_1 \subset \cdots \subset \mathcal{E}/\mathcal{E}_1).$$

This is done in **3.3**-(1). □

3.5. Lemma *Let $\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_{l+1}$ be invertible sheaf on X and set $\mathcal{E}_k = \oplus_{i=1}^k \mathcal{L}_i$ for $1 \le k \le l+1$. Then, for the filtration $\mathcal{E}_\bullet = [\mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_{l+1}]$, the variety $\mathbb{P}_X(\mathcal{E}_\bullet)$ is isomorphic to the toric bundle $\mathbb{T}_{\mathsf{N}}(\mathbf{\Sigma}, \mathcal{L})$ over X for some fan $\mathbf{\Sigma}$ of a free abelian group N of rank l with a basis $(e_1, e_2, \dots, e_l)$ and for the element*

$$\mathcal{L} = \sum\nolimits_{i=1}^{l} e_i \otimes \left(\mathcal{L}_i \otimes \mathcal{L}_{l+1}^{-1}\right) \in \mathsf{N} \otimes \operatorname{Pic}(X).$$

Proof. We may assume $l \ge 1$. If $l = 1$, then $\mathbb{P}(\mathcal{E}_\bullet)$ is a $\mathbb{P}^1$-bundle associated with $\mathcal{E}_2 = \mathcal{L}_1 \oplus \mathcal{L}_2$. Thus it is enough to take the standard fan $\mathbf{\Sigma} = \{\mathbb{R}_{\ge 0} e_1, \mathbb{R}_{\ge 0}(-e_1), \{0\}\}$. For $l \ge 2$, we shall construct the fan $\mathbf{\Sigma}$ of the abelian group N satisfying the required condition by induction on l. We consider a free abelian group N_{l+1} of rank $l+1$ containing N such that $\mathsf{N}_{l+1} = \mathsf{N} \oplus \mathbb{Z} e_{l+1}$ for a new element $e_{l+1} \in \mathsf{N}_{l+1}$. For $1 \le i \le l$, we define $\mathsf{N}_i := \sum_{1 \le j \le i} \mathbb{Z} e_j$ and $v_{i+1} := -\sum_{1 \le j \le i} e_i \in \mathsf{N}_i$. Let $\pi_i \colon \mathsf{N}_{i+1} \to \mathsf{N}_i$ be the homomorphism given by $\pi_i(e_j) = e_j$ for $j \le i$ and $\pi_i(e_{i+1}) = -v_{i+1}$. Let us consider the first quadrant cone $\boldsymbol{\sigma}_{l+1} = \sum_{i=1}^{l} \mathbb{R}_{\ge 0} e_i$ and the following cones of $\mathsf{N}_{\mathbb{R}}$ for $1 \le i \le l$:

$$\boldsymbol{\sigma}_i = \sum_{1 \le j \le l,\, i \ne j} \mathbb{R}_{\ge 0} e_j + \mathbb{R}_{\ge 0} v_{l+1}, \quad \boldsymbol{\sigma}'_i = \sum_{1 \le j \le l,\, i \ne j} \mathbb{R}_{\ge 0} e_j + \mathbb{R}_{\ge 0}(-v_{l+1}).$$

Let $\mathbf{\Sigma}^\flat$ be the fan of N consisting of all the faces of the cones $\boldsymbol{\sigma}_i$ for $1 \le i \le l+1$. Then we have an isomorphism $\mathbb{T}_{\mathsf{N}}(\mathbf{\Sigma}^\flat, \mathcal{L}) \simeq \mathbb{P}_X(\mathcal{E}_l)$. Similarly, let $\mathbf{\Sigma}^\sharp$ be the fan of N consisting of all the faces of $\boldsymbol{\sigma}_i$ and $\boldsymbol{\sigma}'_i$ for $1 \le i \le l$. Then $\mathbf{\Sigma}^\sharp$ is a finite subdivision of $\mathbf{\Sigma}^\flat$ and the associated morphism $\mathbb{T}_{\mathsf{N}}(\mathbf{\Sigma}^\sharp, \mathcal{L}) \to \mathbb{T}_{\mathsf{N}}(\mathbf{\Sigma}^\flat, \mathcal{L})$ is just the blowing up of $\mathbb{P}_X(\mathcal{E}_l)$ along the section $\mathbb{P}_X(\mathcal{E}_l / \mathcal{E}_{l-1})$. Thus $\mathbb{T}_{\mathsf{N}}(\mathbf{\Sigma}^\sharp, \mathcal{L}) \simeq \mathbb{P}(\mathcal{E}_{l-1} \subset \mathcal{E}_l)$. Here, the $\mathbb{P}^1$-bundle structure $\mathbb{T}_{\mathsf{N}}(\mathbf{\Sigma}^\sharp) \to \mathbb{T}_{\mathsf{N}_{l-1}}(\mathbf{\Sigma}^\flat_{l-1}) \simeq \mathbb{P}(\mathcal{E}_{l-1})$ is induced from $\pi_{l-1} \colon \mathsf{N} \to \mathsf{N}_{l-1}$. By induction, there exists a fan $\mathbf{\Sigma}_{l-1}$ of N_{l-1} such that $\mathbb{T}_{\mathsf{N}_{l-1}}(\mathbf{\Sigma}_{l-1}, \mathcal{L}) \simeq \mathbb{P}_X(\mathcal{E}_{\bullet \le l-1})$. The fiber product of $\mathbb{P}(\mathcal{E}_{l-1} \subset \mathcal{E}_l)$ and $\mathbb{P}(\mathcal{E}_{\bullet \le l-1})$ over $\mathbb{P}(\mathcal{E}_{l-1})$ is isomorphic to $\mathbb{P}(\mathcal{E}_\bullet)$. Thus the set

$$\mathbf{\Sigma}_l = \{\boldsymbol{\sigma} \cap \pi_{l-1}^{-1} \boldsymbol{\tau} \mid \boldsymbol{\sigma} \in \mathbf{\Sigma}^\sharp, \boldsymbol{\tau} \in \mathbf{\Sigma}_{l-1}\}$$

is a fan giving an isomorphism $\mathbb{T}_{\mathsf{N}}(\mathbf{\Sigma}_l, \mathcal{L}) \simeq \mathbb{P}_X(\mathcal{E}_\bullet)$. □

§3.b. Projective bundles over a curve. This subsection is devoted to proving the following:

3.6. Theorem *Every pseudo-effective $\mathbb{R}$-divisor of a projective bundle $\mathbb{P}_C(\mathcal{E})$ defined over a non-singular projective curve C associated with a vector bundle $\mathcal{E}$ admits a Zariski-decomposition.*

We may assume $r = \operatorname{rank} \mathcal{E} > 1$. Let $p \colon \mathbb{P}(\mathcal{E}) = \mathbb{P}_C(\mathcal{E}) \to C$ be the structure morphism of the projective bundle, $H_{\mathcal{E}}$ a tautological divisor associated with $\mathcal{E}$, and $\mathcal{O}_{\mathcal{E}}(1)$ the tautological line bundle $\mathcal{O}_{\mathbb{P}}(H_{\mathcal{E}})$. Let F be a fiber of p. Then $\mathrm{N}^1(\mathbb{P}(\mathcal{E})) = \mathbb{R}c_1(F) + \mathbb{R}c_1(H_{\mathcal{E}})$. The Harder-Narasimhan filtration:

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_l = \mathcal{E}$$

is characterized by the following two conditions:

(1) $\mathcal{E}_i/\mathcal{E}_{i-1}$ is a non-zero semi-stable vector bundle for any $1 \leq i \leq l$;

(2) $\mu(\mathcal{E}_i/\mathcal{E}_{i-1}) > \mu(\mathcal{E}_{i+1}/\mathcal{E}_i)$ for $1 \leq i \leq l-1$, where $\mu(\mathcal{E}) := \deg(\mathcal{E})/\operatorname{rank}(\mathcal{E})$.

The number l is called the *length* of the Harder–Narasimhan filtration of $\mathcal{E}$ and is denoted by $l(\mathcal{E})$. We define $\mu_{\max}(\mathcal{E}) := \mu(\mathcal{E}_1)$ and $\mu_{\min}(\mathcal{E}) := \mu(\mathcal{E}/\mathcal{E}_{l-1})$. We have only to study the Zariski-decomposition problem for the $\mathbb{R}$-divisor $D_t := H_{\mathcal{E}} - tF$ for $t \in \mathbb{R}$. We begin with the following:

3.7. Lemma *Let $\mathcal{F}^1$, $\mathcal{F}^2$, …, $\mathcal{F}^n$ be vector bundles on a non-singular projective curve C and let Z be the fiber product*

$$\mathbb{P}_C(\mathcal{F}^1) \times_C \mathbb{P}_C(\mathcal{F}^2) \times_C \cdots \times_C \mathbb{P}_C(\mathcal{F}^n).$$

For the projections $p_i\colon Z \to \mathbb{P}_C(\mathcal{F}^i)$, $\boldsymbol{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$, $t \in \mathbb{R}$, and a fiber F of $p\colon Z \to C$, let $D(\boldsymbol{y}, t)$ be the $\mathbb{R}$-divisor

$$\sum\nolimits_{i=1}^{n} y_i p_i^* H_{\mathcal{F}^i} - tF.$$

(1) *Suppose that*

$$H^0(C, \operatorname{Sym}^{a_1}(\mathcal{F}^1) \otimes \operatorname{Sym}^{a_2}(\mathcal{F}^2) \otimes \cdots \otimes \operatorname{Sym}^{a_n}(\mathcal{F}^n)) \neq 0$$

for some a_1, …, $a_n \in \mathbb{Z}_{\geq 0}$. Then

$$\sum\nolimits_{i=1}^{n} a_i \mu_{\max}(\mathcal{F}^i) \geq 0.$$

(2) *$D(\boldsymbol{y}, t)$ is pseudo-effective if and only if $\boldsymbol{y} \in \mathbb{R}^n_{\geq 0}$ and*

$$\sum\nolimits_{i=1}^{n} y_i \mu_{\max}(\mathcal{F}^i) \geq t.$$

(3) *$D(\boldsymbol{y}, t)$ is nef if and only if $\boldsymbol{y} \in \mathbb{R}^n_{\geq 0}$ and*

$$\sum\nolimits_{i=1}^{n} y_i \mu_{\min}(\mathcal{F}^i) \geq t.$$

Proof. (1) Let $\mathcal{F}^i_\bullet$ be the Harder–Narasimhan filtration of $\mathcal{F}^i$. By considering successive quotients of symmetric tensors, we can find non-negative integers b^i_k for $1 \leq i \leq n$ and for $0 \leq k \leq l(\mathcal{F}^i)$ such that

$$\sum\nolimits_{k=1}^{l(\mathcal{F}^i)} b^i_k = a_i$$

and the vector bundle

$$\mathcal{B} = \bigotimes\nolimits_{i=1}^{n} \Big(\bigotimes\nolimits_{k=1}^{l(\mathcal{F}^i)} \operatorname{Sym}^{b^i_k} \operatorname{Gr}_k(\mathcal{F}^i_\bullet) \Big)$$

admits a non-zero global section. Here $\mathcal{B}$ is semi-stable (cf. [**82**]) and hence

$$\mu(\mathcal{B}) = \sum\nolimits_{i=1}^{n} \sum\nolimits_{k=1}^{l(\mathcal{F}^i)} b^i_k \mu(\operatorname{Gr}_k(\mathcal{F}^i))$$

is non-negative. Thus

$$\sum_{i=1}^{n} a_i \mu_{\max}(\mathcal{F}^i) \geq \mu(\mathcal{B}) \geq 0.$$

(2) The $\mathbb{R}$-linear equivalence relation

$$D(\boldsymbol{y}, t) \sim_{\mathbb{R}} \sum_{i=1}^{n} y_i (H_{\mathcal{F}^i} - \mu_{\max}(\mathcal{F}^i)F) + \bigl(\sum_{i=1}^{n} y_i \mu_{\max}(\mathcal{F}^i) - t\bigr)F$$

gives one implication. In order to show the other one, we have only to consider the case where $\boldsymbol{y} \in \mathbb{Z}^n$ and $t \in \mathbb{Z}$, since the set of the first Chern classes of big $\mathbb{Q}$-divisors is dense in the pseudo-effective cone. Then we have an isomorphism

$$p_* \mathcal{O}_Z(D(\boldsymbol{y}, t)) \simeq \bigotimes_{i=1}^{n} \operatorname{Sym}^{y_i}(\mathcal{F}^i) \otimes \mathcal{O}_C(-tP),$$

where $P = p(F) \in C$. Hence, if $|D(\boldsymbol{y}, t)| \neq \emptyset$, then $\boldsymbol{y} \in \mathbb{Z}^n_{\geq 0}$ and $\sum_{i=1}^{n} y_i \mu_{\max}(\mathcal{F}^i) \geq t$ by (1). Thus we are done.

(3) The $\mathbb{R}$-linear equivalence relation

$$D(\boldsymbol{y}, t) \sim_{\mathbb{R}} \sum_{i=1}^{n} y_i (H_{\mathcal{F}^i} - \mu_{\min}(\mathcal{F}^i)F) + \bigl(\sum_{i=1}^{n} y_i \mu_{\min}(\mathcal{F}^i) - t\bigr)F$$

gives one implication. If $D(\boldsymbol{y}, t)$ is nef, then the restriction to the subspace

$$\mathbb{P}(\mathcal{F}^1/\mathcal{F}^1_{l(\mathcal{F}^1)-1}) \times_C \cdots \times_C \mathbb{P}(\mathcal{F}^n/\mathcal{F}^n_{l(\mathcal{F}^n)-1})$$

is also nef. Hence $\boldsymbol{y} \in \mathbb{R}^n_{\geq 0}$ and $\sum y_i \mu_{\min}(\mathcal{F}^i) \geq t$ by (2). Thus we are done. □

By applying **3.7** to the case $n = 1$, $\mathcal{E} = \mathcal{F}^1$, we have:

3.8. Corollary *The $\mathbb{R}$-divisor D_t is pseudo-effective if and only if $t \leq \mu_{\max}(\mathcal{E})$. It is nef if and only if $t \leq \mu_{\min}(\mathcal{E})$.*

3.9. Lemma *$H_{\mathcal{E}} - \mu(\mathcal{E}_1)F$ admits a Zariski-decomposition.*

Proof. We may assume that $\mathcal{E}$ is not semi-stable. Thus $l = l(\mathcal{E}) \geq 2$. Let $\rho\colon Y = \mathbb{P}(\mathcal{E}_1 \subset \mathcal{E}) \to \mathbb{P}(\mathcal{E})$ be the blowing-up along $\mathbb{P}(\mathcal{E}/\mathcal{E}_1)$. Then the exceptional divisor E is isomorphic to $\mathbb{P}(\mathcal{E}_1) \times_C \mathbb{P}(\mathcal{E}/\mathcal{E}_1)$ by **3.1**. Let $\pi\colon Y \to \mathbb{P}(\mathcal{E}_1)$ be the induced projective bundle structure. The restrictions of ρ and π to E are the first and the second projections, respectively. We shall calculate the ν-decomposition of $\rho^*(H_{\mathcal{E}} - \mu(\mathcal{E}_1)F)$. Since $\pi^* H_{\mathcal{E}_1} \sim \rho^* H_{\mathcal{E}} - E$, the conormal bundle $\mathcal{O}_E(-E)$ is isomorphic to $\pi^* \mathcal{O}_{\mathcal{E}_1}(1) \otimes \rho^* \mathcal{O}_{\mathcal{E}/\mathcal{E}_1}(-1)$. Therefore, by **3.7**, the restriction of $\rho^*(H_{\mathcal{E}} - \mu(\mathcal{E}_1)F) - \alpha E$ to E is pseudo-effective if and only if $0 \leq \alpha \leq 1$ and $\mu(\mathcal{E}_1) \leq \alpha\mu(\mathcal{E}_1) + (1-\alpha)\mu(\mathcal{E}_2/\mathcal{E}_1)$. Since $\mu(\mathcal{E}_1) > \mu(\mathcal{E}_2/\mathcal{E}_1)$, these inequalities hold if and only if $\alpha = 1$. Therefore $P_\nu(\rho^*(H_{\mathcal{E}} - \mu(\mathcal{E}_1)F))$ is equal to the nef $\mathbb{R}$-divisor $\pi^*(H_{\mathcal{E}_1} - \mu(\mathcal{E}_1)F)$. Thus we have a Zariski-decomposition. □

3.10. Proposition *If $l(\mathcal{E}) = 2$, then every pseudo-effective $\mathbb{R}$-divisor of $\mathbb{P}(\mathcal{E})$ admits a Zariski-decomposition.*

PROOF. D_t is pseudo-effective but not nef if and only if $\mu(\mathcal{E}/\mathcal{E}_1) < t \le \mu(\mathcal{E}_1)$. Let $\rho\colon Y \to \mathbb{P}(\mathcal{E})$ and E be the same as in **3.9**. By the same argument, the $\mathbb{R}$-divisor $(\rho^*(D_t) - \alpha E)|_E$ is pseudo-effective if and only if $t \le \alpha\mu(\mathcal{E}_1) + (1-\alpha)\mu(\mathcal{E}/\mathcal{E}_1)$. Since $\mu(\mathcal{E}/\mathcal{E}_1) < t \le \mu(\mathcal{E}_1)$, the minimum α_1 satisfying the inequality above attains the equality: $t = \alpha_1\mu(\mathcal{E}_1) + (1-\alpha_1)\mu(\mathcal{E}/\mathcal{E}_1)$. Thus $P_\nu(\rho^* D_t)$ is nef by

$$P_\nu(\rho^* D_t) \sim_{\mathbb{R}} \alpha_1\pi^*(H_{\mathcal{E}_1} - \mu(\mathcal{E}_1)F) + (1-\alpha_1)\rho^*(H_{\mathcal{E}} - \mu(\mathcal{E}/\mathcal{E}_1)F). \qquad \square$$

We assume $l \ge 3$. Let $S = \mathbb{P}(\mathcal{E}_1 \subset \cdots \subset \mathcal{E}_l) \to C$ be the projective smooth morphism defined in **3.2** for the Harder–Narasimhan filtration $\mathcal{E}_\bullet$. Let $\rho\colon S \to \mathbb{P}(\mathcal{E})$ be the induced birational morphism and let E_k for $1 \le k \le l-1$ and H_i for $1 \le i \le l$ be the divisors defined in **3.3**. Note that $E = \sum_{k=1}^{l-1} E_k$ is a simple normal crossing divisor. By **3.9**, we may assume $\mu(\mathcal{E}/\mathcal{E}_{l-1}) < t < \mu(\mathcal{E}_1)$, equivalently $D_t = H_{\mathcal{E}} - tF$ is not nef but big. Let us define $\mu_i = \mu(\mathcal{E}_i/\mathcal{E}_{i-1})$ for $1 \le i \le l = l(\mathcal{E})$ and

$$\alpha_k(t) := \max\left\{0,\ \frac{t - \mu_{k+1}}{\mu_1 - \mu_{k+1}}\right\}.$$

for $1 \le k \le l-1$. Let $\boldsymbol{\alpha}_t$ be the vector $(\alpha_1(t), \alpha_2(t), \cdots, \alpha_{l-1}(t))$. Note that $\alpha_k(t) = 0$ for $t \le \mu_{k+1}$ and $\alpha_k(t) \ge \alpha_{k'}(t)$ for $k \le k'$. We define an $\mathbb{R}$-divisor by

$$D_t(\boldsymbol{y}) = D_t(y_1, y_2, \ldots, y_{l-1}) = \rho^* H_{\mathcal{E}} - tF - \sum\nolimits_{i=1}^{l-1} y_i E_i$$

for $\boldsymbol{y} = (y_1, y_2, \ldots, y_{l-1}) \in \mathbb{R}^{l-1}$.

3.11. Lemma (1) *$N_\sigma(\rho^* D_t) = N_\nu(\rho^* D_t) = \sum_{k=1}^{l-1} \alpha_k(t) E_k$. Moreover,* $\mathrm{NBs}(\rho^* D_t) = \{s \in S \mid \sigma_s(P_\sigma(\rho^* D_t)) > 0\} \subset E$.

(2) *$D_t(\boldsymbol{y})$ is nef if and only if its restriction to $Z = \bigcap_{k=1}^{l-1} E_k$ is nef. This is also equivalent to that $\boldsymbol{y}$ is contained in the polytope*

$$\square(\mu_\bullet, t) := \Big\{\boldsymbol{y} \in \mathbb{R}^{l-1}_{\ge 0} \mid 0 \le y_1 \le y_2 \le \cdots \le y_{l-1} \le 1,\ \sum_{k=1}^{l-1}(\mu_k - \mu_{k+1}) y_k \ge t - \mu_l\Big\}.$$

PROOF. (1) We denote the total transform of $H_{\mathcal{E}}$ by H and that of F by the same symbol F on a projective variety birational to $\mathbb{P}(\mathcal{E})$. Then $H = H_l$ on S.

We introduce the following non-negative numbers:

$$\beta_j(t) := \begin{cases} \alpha_1(t), & j = 1; \\ \alpha_j(t) - \alpha_{j-1}(t), & 2 \le j \le l-1; \\ 1 - \alpha_l(t), & j = l. \end{cases}$$

Then we can write

$$\begin{aligned} D_t(\boldsymbol{\alpha}_t) &\sim_{\mathbb{R}} \sum\nolimits_{j=1}^{l} \beta_j(t) H_j - tF \\ &\sim_{\mathbb{R}} \sum\nolimits_{j=1}^{k} \beta_j(t)(H_j - \mu_1 F) + \sum\nolimits_{j=k+1}^{l} \beta_j(t)(H_j - \mu_{k+1} F) \\ &\quad + (\alpha_k(t)\mu_1 + (1 - \alpha_k(t))\mu_{k+1} - t)F \end{aligned} \tag{IV-9}$$

for $1 \le k \le l-1$. Here $H_i - \mu_1 F$ is the pullback of a pseudo-effective $\mathbb{R}$-divisor by $S \to \mathbb{P}(\mathcal{E}_i)$ for $i \le k$. Since E_k dominates $\mathbb{P}(\mathcal{E}_i)$ for $i \le k$, we have $\sigma_{E_k}(H_i - \mu_1 F) = 0$ for $i \le k$. There is a linear equivalence relation

$$H_j - \mu_{k+1} F \sim E_{j-1} + \cdots + E_{k+1} + (H_{k+1} - \mu_{k+1} F)$$

for $j > k+1$, where $H_{k+1} - \mu_{k+1} F$ is nef. Hence $\sigma_{E_k}(H_j - \mu_{k+1} F) = 0$ for $j \ge k+1$. Therefore, $D_t(\boldsymbol{\alpha}_t)$ is pseudo-effective and $\sigma_{E_k}(D_t(\boldsymbol{\alpha}_t)) = 0$ by (IV-9). Moreover, we infer $\mathrm{NBs}(\rho^* D_t) \subset E$ by (IV-9) for $k = 1$. Thus $D_t(\boldsymbol{\alpha}_t)$ is movable.

For an index $1 \le k \le l-1$, we can write

$$\text{(IV-10)} \quad D_t(\boldsymbol{y}) \sim_{\mathbb{R}} \Big(y_k(H_k - \mu_1 F) - \sum\nolimits_{j=1}^{k-1} y_j E_j\Big) + \Big((1-y_k)(H_l - \mu_{k+1} F) + \sum\nolimits_{j=k+1}^{l-1} (y_k - y_j) E_j\Big) + \big(y_k \mu_1 + (1-y_k)\mu_{k+1} - t\big) F.$$

By **3.7**, $H_i - \mu_k F$ is pseudo-effective for $i \ge k$. Let $\rho_k \colon E_k \to \mathbb{P}(\mathcal{E}_k) \times_C \mathbb{P}(\mathcal{E}/\mathcal{E}_k)$ be the natural birational morphism. Suppose that $D_t(\boldsymbol{y})|_{E_k}$ is pseudo-effective. Then its push-forward by ρ_{k*} is also pseudo-effective. Suppose first that $\mathcal{E}_{k+1}/\mathcal{E}_k$ is not a line bundle. Then $E_j|_{E_k}$ is ρ_k-exceptional for any $j \ge k+1$. Hence $y_k \le 1$ and $t \le y_k \mu_1 + (1-y_k)\mu_{k+1}$ by (IV-10) and **3.7**. Suppose next that $\mathcal{E}_{k+1}/\mathcal{E}_k$ is a line bundle. Then $E_j|_{E_k}$ is ρ_k-exceptional for any $j > k+1$. Here $H_{k+1}|_{E_k}$ is the pullback of $H_{\mathcal{E}_{k+1}/\mathcal{E}_k}$ of $\mathbb{P}(\mathcal{E}_{k+1}/\mathcal{E}_k) \simeq C$, which is numerically equivalent to $\mu_{k+1} F$. Thus the inequalities $y_{k+1} \le 1$ and $y_k \mu_1 + (1-y_k)\mu_{k+1} \ge t$ follow from (IV-10), the $\mathbb{R}$-linear equivalence relation

$$E_{k+1} \sim_{\mathbb{R}} H_l - E_{l-1} - \cdots - E_{k+2} - H_{k+1},$$

and from **3.7**.

Hence, if $D_t(\boldsymbol{y})|_{E_k}$ is pseudo-effective, then $\alpha_k(t) \le y_k$. Since $D_t(\boldsymbol{\alpha}_t)|_{E_k}$ are all pseudo-effective, we infer that $\nu_{E_k}(D_t) = \alpha_k(t)$ for any k by **III.3.12**. Therefore $N_\sigma(\rho^* D_t) = N_\nu(\rho^* D_t) = \sum \alpha_k(t) E_k$.

(2) We can write

$$D_t(\boldsymbol{y}) \sim_{\mathbb{R}} y_1(H_1 - \mu_1 F) + \sum\nolimits_{j=2}^{l-1} (y_j - y_{j-1})(H_j - \mu_j F) + (1 - y_{l-1})(H_l - \mu_l F) + \Big(y_1 \mu_1 + \sum\nolimits_{j=2}^{l-1} (y_j - y_{j-1})\mu_j + (1-y_{l-1})\mu_l - t\Big) F.$$

If $\boldsymbol{y} \in \square(\mu_\bullet, t)$, then $D_t(\boldsymbol{y})$ is nef, since $H_i - \mu_i F$ is nef for $1 \le i \le l$. Conversely suppose that $D_t(\boldsymbol{y})$ is nef. The intersection $Z = \bigcap_{k=1}^{l-1} E_k$ is isomorphic to

$$\mathbb{P}(\mathcal{E}_1) \times_C \mathbb{P}(\mathcal{E}_2/\mathcal{E}_1) \times_C \cdots \times_C \mathbb{P}(\mathcal{E}_l/\mathcal{E}_{l-1}).$$

Since $D_t(\boldsymbol{y})|_Z$ is nef, we have $\boldsymbol{y} \in \square(\mu_\bullet, t)$ by **3.7**. $\square$

Let $\mathsf{N}^{\natural}$ be a free abelian group of rank $l-1$ with a basis $(e_1^{\natural}, e_2^{\natural}, \ldots, e_{l-1}^{\natural})$ and let $(\delta_1^{\natural}, \delta_2^{\natural}, \ldots, \delta_{l-1}^{\natural})$ be the dual basis of $\mathsf{M}^{\natural} = (\mathsf{N}^{\natural})^{\vee}$. We consider

$$\mathcal{L}_{\natural} = \sum\nolimits_{k=1}^{l-1} e_i^{\natural} \otimes \mathcal{O}_S(-B_i) \in \mathsf{N}^{\natural} \otimes \mathrm{Pic}(S) \quad \text{and} \quad \boldsymbol{\sigma}_{\natural} = \sum\nolimits_{k=1}^{l-1} \mathbb{R}_{\geq 0} e_i^{\natural} \in \mathsf{N}^{\natural}_{\mathbb{R}}$$

in order to have a toric blowing up of S along E. We note that the polytope $\square(\mu_{\bullet}, t)$ is identified with the same subset

$$\square_{\mathrm{Nef}}(H - tF) = \{m \in \mathsf{M}^{\natural}_{\mathbb{R}} \mid H - tF + \mathcal{L}_{\natural}^{m} \text{ is nef}\}$$

as in **2.15** for the $\mathbb{R}$-divisor $H - tF$ by $\boldsymbol{y} \leftrightarrow \sum y_i \delta_i^{\natural}$. Here, the subset satisfies the condition of **2.15** by **3.11**-(2). Let $h^{\dagger} \in \mathrm{SFC}_{\mathsf{N}^{\natural}}(\boldsymbol{\sigma}_{\natural}, \mathbb{R})$ be the support function defined by

$$h^{\dagger}(x) = \min\{\langle m, x\rangle \mid m \in \square_{\mathrm{Nef}}(H - tF)\}$$

and let $\boldsymbol{\Lambda}$ be a finite subdivision of $\boldsymbol{\sigma}_{\natural}$ with $h^{\dagger} \in \mathrm{SF}_{\mathsf{N}^{\natural}}(\boldsymbol{\sigma}_{\natural}, \mathbb{R})$. Then, for the toric blowing up $f\colon S_{\boldsymbol{\Lambda}} \to S$ along E associated with $\boldsymbol{\Lambda}$, we have a nef $\mathbb{R}$-Cartier divisor $P^{\dagger} := D_{h^{\dagger}} + H - tF$ on $S_{\boldsymbol{\Lambda}}$. If $H - tF$ admits a Zariski-decomposition, then $P^{\dagger}$ is the positive part of a Zariski-decomposition by **2.15**.

3.12. Lemma *Suppose that the Harder–Narasimhan filtration of $\mathcal{E}$ is split*:

$$\mathcal{E}_i = \bigoplus\nolimits_{k=1}^{i} \mathcal{E}_k/\mathcal{E}_{k-1}.$$

Then $H - tF$ admits a Zariski-decomposition. In particular, $P^{\dagger}$ is the positive part of a Zariski-decomposition of $H - tF$.

Proof. Let us consider

$$Z = \mathbb{P}(\mathcal{E}_1) \times_C \mathbb{P}(\mathcal{E}_2/\mathcal{E}_1) \times_C \cdots \times_C \mathbb{P}(\mathcal{E}_l/\mathcal{E}_{l-1}) \to C$$

and the pullback $\overline{H}_i$ of the tautological divisor $H_{\mathcal{E}_i/\mathcal{E}_{i-1}}$ to Z for any i. Then there is a birational morphism

$$M = \mathbb{P}_Z(\mathcal{O}_Z(\overline{H}_1) \oplus \cdots \mathcal{O}_Z(\overline{H}_l)) \to \mathbb{P}_C(\mathcal{E}),$$

since $\mathcal{E}_{\bullet}$ is split. We know $\mathrm{Nef}(Z) = \mathrm{PE}(Z)$ and $\mathrm{Nef}(Z) \subset \mathrm{N}^1(Z)$ is a rational polyhedral cone. Therefore, every pseudo-effective $\mathbb{R}$-divisor on the toric bundle M over Z admits a Zariski-decomposition by **2.5**. □

The following proof is more explicit than above and it does not use **2.15**:

Another proof of **3.12**. The projective bundle M in the proof above is written as a toric bundle $\mathbb{T}_{\mathsf{N}}(\boldsymbol{\Sigma}, \mathcal{L})$ over Z, where N is a free abelian group of rank $l-1$ with a basis $(e_1, e_2, \ldots, e_{l-1})$, $\mathcal{L} = \sum e_i \otimes \mathcal{O}_Z(\overline{H}_i - \overline{H}_l)$, and $\boldsymbol{\Sigma}$ is a complete fan of N defined as in §**2.c**. Here $\mathrm{Ver}(\boldsymbol{\Sigma}) = \{e_1, e_2, \ldots, e_{l-1}, e_l\}$ for $e_l = -\sum_{i=1}^{l-1} e_i$. We have the support function $h \in \mathrm{SF}_{\mathsf{N}}(\boldsymbol{\Sigma}, \mathbb{Z})$ defined by $h(x) = \min(\{\langle \delta_i, x\rangle \mid 1 \leq i \leq l-1\} \cup \{0\})$, where $(\delta_1, \ldots, \delta_{l-1})$ is the dual basis to $(e_1, e_2, \ldots, e_l)$. Then $D_h = \Gamma_{e_l} \sim \lambda^* H_{\mathcal{E}} - q^*\overline{H}_l$ for the structure morphism $q\colon M \to Z$. We define

$$\mathcal{H}_i = \mathcal{O}_Z(\overline{H}_1) \oplus \mathcal{O}_Z(\overline{H}_2) \oplus \cdots \oplus \mathcal{O}_Z(\overline{H}_i)$$

for $1 \leq i \leq l$. Then we have a filtration $\mathcal{H}_\bullet = [\mathcal{H}_1 \subset \mathcal{H}_2 \subset \cdots \subset \mathcal{H}_l]$ of subbundles of $\mathcal{H}_l$. We can show that there is a birational morphism $\mathbb{P}_Z(\mathcal{H}_\bullet) \to S = \mathbb{P}_C(\mathcal{E}_\bullet)$ which is an isomorphism over an open neighborhood of $E \subset S$ and that the total transform of $E_i \subset S$ in $\mathbb{P}_Z(\mathcal{H}_\bullet)$ is just the same E_i with respect to the filtration $\mathcal{H}_\bullet$. By **3.5**, we can write $\mathbb{P}_Z(\mathcal{H}_\bullet)$ as a toric bundle $\mathbb{T}_{\mathsf{N}}(\mathbf{\Sigma}_l, \mathcal{L})$ over Z, where $\mathrm{Ver}(\mathbf{\Sigma}_l) = \{e_1, e_2, \ldots, e_l, w_1, w_2, \ldots, w_{l-1}\}$, for $w_i := \sum_{j=1}^{i} e_j$. Note that $w_1 = e_1$ and $w_{l-1} = -e_l$. Then $E_i = \Gamma_{w_i} = \mathbb{V}(\mathbb{R}_{\geq 0} w_i, \mathcal{L}) \subset \mathbb{T}_{\mathsf{N}}(\mathbf{\Sigma}_l, \mathcal{L})$. The pullback of $H - tF$ in $\mathbb{P}_C(\mathcal{E})$ to $\mathbb{P}_Z(\mathcal{H}_\bullet)$ is written by $D_h + q^*(\overline{H}_l - tF)$ for the structure morphism $q\colon \mathbb{P}_Z(\mathcal{H}_\bullet) \to Z$. We can apply the method of **2.5** to constructing the Zariski-decomposition of $D_h + q^*(\overline{H}_l - tF)$, since $\mathrm{PE}(Z) = \mathrm{Nef}(Z)$ is a polyhedral cone. Then, by **3.7**,

$$\begin{aligned}\square_{\mathrm{Nef}}(\overline{H}_l - tF, h) &= \left\{ m \in \square_h \;\middle|\; \sum_{i=1}^{l-1} m_i \overline{H}_i + \Big(1 - \sum_{i=1}^{l-1} m_i\Big)\overline{H}_l - tF \text{ is nef} \right\} \\ &= \left\{ m \in \mathbb{R}_{\geq 0}^{l-1} \;\middle|\; \sum_{i=1}^{l-1} m_i \leq 1,\ \sum_{i=1}^{l-1} m_i \mu_i + \Big(1 - \sum_{i=1}^{l-1} m_i\Big)\mu_l \geq t \right\}.\end{aligned}$$

Therefore, the dual cone $\triangle$ of $\mathbb{R}_{\geq 0}(\square_{\mathrm{Nef}}(\overline{H}_l - tF, h) \times \{-1\})$ is written by

$$\triangle = \sum_{i=1}^{l-1} \mathbb{R}_{\geq 0}(e_i, 0) + \mathbb{R}_{\geq 0}(e_l, -1) + \mathbb{R}_{\geq 0}\left(\sum_{i=1}^{l-1} (\mu_i - \mu_l) e_i, -t \right).$$

We set $h^{\ddagger}(x) = \max\{ r \in \mathbb{R} \mid (x, r) \in \triangle \}$. We shall construct a finite subdivision $\mathbf{\Sigma}^{\sharp}$ of $\mathbf{\Sigma}$ as follows: The maximal cones of $\mathbf{\Sigma}^{\sharp}$ are

$$\begin{aligned}\boldsymbol{\sigma}_i &= \sum_{1 \leq j \leq l, i \neq j} \mathbb{R}_{\geq 0} e_j, \\ \boldsymbol{\sigma}'_i &= \sum_{1 \leq j \leq l-1, i \neq j} \mathbb{R}_{\geq 0} e_j + \mathbb{R}_{\geq 0}\left(\sum_{i=1}^{l-1} (\mu_i - \mu_l) e_i \right),\end{aligned}$$

for $1 \leq i \leq l-1$. Then $h^{\ddagger} \in \mathrm{SF}_{\mathsf{N}}(\mathbf{\Sigma}^{\sharp}, \mathbb{R})$ and hence $D_{h^{\ddagger}} + q^*(\overline{H}_l - tF)$ on $\mathbb{T}_{\mathsf{N}}(\mathbf{\Sigma}^{\sharp}, \mathcal{L})$ is the positive part of the Zariski-decomposition.

On the other hand, let us consider the toric blowup $X \to \mathbb{P}(\mathcal{H}_\bullet)$ along $E = \sum E_i$ associated with a finite subdivision $\mathbf{\Lambda}$ of $\boldsymbol{\sigma}_{\natural}$. Then, by **2.13**, X is isomorphic to the toric bundle $\mathbb{T}_{\mathsf{N}}(\mathbf{\Sigma}', \mathcal{L})$ over Z for a fan $\mathbf{\Sigma}'$ defined as follows: Let us define $\mathbf{h} \in \mathrm{SF}_{\mathsf{N}}(\mathbf{\Sigma}_l, \mathbb{Z}) \otimes \mathsf{N}^{\natural}$ by

$$\mathbf{h}(v) = \begin{cases} e_i^{\natural}, & \text{if } v = w_i \quad \text{for} \quad 1 \leq i \leq l-1, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\mathbf{\Sigma}' = \{ C_{\mathbf{h}}(\boldsymbol{\lambda}, \boldsymbol{\sigma}) \mid \boldsymbol{\lambda} \in \mathbf{\Lambda}, \boldsymbol{\sigma} \in \mathbf{\Sigma}_l \}$, where $C_{\mathbf{h}}(\boldsymbol{\lambda}, \boldsymbol{\sigma}) = \boldsymbol{\sigma} \cap \mathbf{h}^{-1}(\boldsymbol{\lambda})$.

We can identify $\square_{\mathrm{Nef}}(H - tF)$ with $\square_{\mathrm{Nef}}(\overline{H}_l - tF, h)$ by

$$\boldsymbol{y} \mapsto m = y_1 \delta_1 + \sum_{i=2}^{l-1} (y_i - y_{i-1}) \delta_i.$$

The dual $\mathsf{N}_{\mathbb{R}} \to \mathsf{N}^{\natural}_{\mathbb{R}}$ of the linear transformation coincides with $\mathbf{h}$ over the cone $\boldsymbol{\sigma}_{\flat} := \sum_{i=1}^{l} \mathbb{R}_{\geq 0} w_i$. Thus

$$h^{\dagger}(\mathbf{h}(x)) = h^{\ddagger}(x)$$

for $x \in \boldsymbol{\sigma}_{\flat}$. Note that $h^{\ddagger}$ is linear on $\boldsymbol{\sigma}'_i \in \boldsymbol{\Sigma}^{\sharp}$. The set $\{\boldsymbol{\sigma}_{\flat} \cap \boldsymbol{\sigma}'_i \mid 1 \leq i \leq l-1\}$ of cones generates a finite subdivision of $\boldsymbol{\sigma}_{\flat}$. We take $\boldsymbol{\Lambda}$ to be the corresponding subdivision of $\boldsymbol{\sigma}_{\natural}$ by $\mathbf{h}$. Then $h^{\dagger} \in \mathrm{SF}_{\mathsf{N}^{\natural}}(\boldsymbol{\Lambda}, \mathbb{R})$. Let $\boldsymbol{\Sigma}'$ be the finite subdivision of $\boldsymbol{\Sigma}_l$ corresponding to $\boldsymbol{\Lambda}$. Then $\boldsymbol{\Sigma}'$ is a finite subdivision of $\boldsymbol{\Sigma}^{\sharp}$. Here, $P^{\dagger} = D_{h^{\dagger}} + H - tF$ on X is equal to $D_{h^{\ddagger}} + q^*(\overline{H}_l - tF)$. Thus $P^{\dagger}$ is the positive part of the Zariski-decomposition. □

Now we are ready to prove the main result **3.6** of §**3.b**.

Proof of **3.6**. There is a connected analytic space Θ and a sequence of vector subbundles

$$0 = \widetilde{\mathcal{E}}_0 \subset \widetilde{\mathcal{E}}_1 \subset \widetilde{\mathcal{E}}_2 \subset \cdots \subset \widetilde{\mathcal{E}}_l$$

on $C \times \Theta$ satisfying the following conditions: let $(\mathcal{E}_i)_\theta$ be the restriction of $\widetilde{\mathcal{E}}$ to $C \times \{\theta\}$.

(1) $\widetilde{\mathcal{E}}_i/\widetilde{\mathcal{E}}_{i-1} \simeq p_1^*(\mathcal{E}_i/\mathcal{E}_{i-1})$ for any $1 \leq i \leq l$ for the first projection p_1;
(2) There is a point $0 \in \Theta$ such that the sequence $(\mathcal{E}_i)_0$ is split, i.e,

$$(\mathcal{E}_i)_0 \simeq \bigoplus\nolimits_{k=1}^{i} \mathcal{E}_k/\mathcal{E}_{k-1};$$

(3) There is a point $\theta \in \Theta$ such that $(\mathcal{E}_i)_\theta = \mathcal{E}_i$ for any i.

Let $\widetilde{S} \to C \times \Theta$ be the projective smooth morphism defined by

$$\widetilde{S} = \mathbb{P}_{C \times \Theta}(\widetilde{\mathcal{E}}_1 \subset \cdots \subset \widetilde{\mathcal{E}}_l).$$

Then we have similar effective divisors $\widetilde{E}_k$ for $1 \leq k \leq l-1$. We also have the toric blowing-up $\widetilde{f} \colon \widetilde{S}_{\boldsymbol{\Lambda}} \to \widetilde{S}$ associated with the subdivision $\boldsymbol{\Lambda}$ and $\widetilde{P}^{\dagger} = D_{h^{\dagger}} + \widetilde{f}^*(H - tF)$ that is relatively nef over Θ. Let $\widetilde{\Gamma}_v$ be the prime divisor of $\widetilde{S}_{\boldsymbol{\Lambda}}$ associated with $v \in \mathrm{Ver}(\boldsymbol{\Lambda})$. Here the restrictions of $\widetilde{P}^{\dagger}$ and $\widetilde{\Gamma}_v$ to the fiber over $\theta \in \Theta$ coincide with $P^{\dagger}$ and Γ_v, respectively. The restriction of $\widetilde{P}^{\dagger}$ to the fiber over 0 is the positive part of a Zariski-decomposition by **3.12**. In particular, $P^{\dagger}$ is nef and big and the restriction of $P^{\dagger}$ to Γ_v is not big for any $v \in \mathrm{Ver}(\boldsymbol{\Lambda})$, by **III.3.7**. Again by **III.3.7**, we infer that $P^{\dagger}$ is the positive part of the Zariski-decomposition of $H - tF$. □

§4. Normalized tautological divisors

§4.a. Projectively flatness and semi-stability. We shall prove the following theorem which may be well-known. It is derived from the study of stable vector bundles and Einstein–Hermitian metrics by Narasimhan and Seshadri [**107**], Mehta and Ramanathan [**78**], [**79**], Donaldson [**12**], Uhlenbeck and Yau [**142**], and Bando and Siu [**3**].

4.1. Theorem *Let $\mathcal{E}$ be a reflexive sheaf of rank r on a non-singular complex projective variety X of dimension d. Then the following three conditions are equivalent*:

(1) *$\mathcal{E}$ is locally free and the normalized tautological divisor $\Lambda_{\mathcal{E}}$ is nef*;

(2) *$\mathcal{E}$ is A-semi-stable and*

$$\left(c_2(\mathcal{E}) - \frac{r-1}{2r}c_1^2(\mathcal{E})\right)\cdot A^{d-2} = 0$$

for an ample divisor A;

(3) *$\mathcal{E}$ is locally free and there is a filtration of vector subbundles*

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_l = \mathcal{E}$$

such that $\mathcal{E}_i/\mathcal{E}_{i-1}$ are projectively flat and the averaged first Chern classes $\mu(\mathcal{E}_i/\mathcal{E}_{i-1})$ are numerically equivalent to $\mu(\mathcal{E})$ for any i.

Here, a vector bundle $\mathcal{E}$ is called *projectively flat* if it admits a projectively flat Hermitian metric h, namely, the curvature tensor Θ_h is written by

$$\Theta_h = \omega \cdot \mathrm{id}_{\mathcal{E}}$$

for a 2-form ω, as an $\mathcal{E}nd(\mathcal{E})$-valued C^∞-2-form. We need some preparations for the proof.

Let $\mathrm{U}(r)$ be the unitary group of degree r and let $\mathrm{PU}(r)$ be the quotient group $\mathrm{U}(r)/\mathrm{U}(1)$ by the center $\mathrm{U}(1) \simeq \mathrm{S}^1$. Let $\mathcal{O}_X^\star \times \mathrm{U}(r)$ be the direct product of the sheaf $\mathcal{O}_X^\star$ of germs of holomorphic unit functions and the constant sheaf $\mathrm{U}(r)$. Let $\mathrm{GL}(r, \mathcal{O}_X)$ be the sheaf of germs of holomorphic $r \times r$ regular matrices and let $\mathcal{O}_X^\star\,\mathrm{U}(r)$ be the image of the natural homomorphism

$$\mathcal{O}_X^\star \times \mathrm{U}(r) \to \mathrm{GL}(r, \mathcal{O}_X).$$

Then we have an exact sequence:

$$1 \to \mathrm{S}^1 \to \mathcal{O}_X^\star \times \mathrm{U}(r) \to \mathcal{O}_X^\star\,\mathrm{U}(r) \to 1,$$

in which the homomorphism from S^1 is given by $s \mapsto (s^{-1}, s)$.

4.2. Lemma *The image of the homomorphism*

$$\mathrm{H}^1(X, \mathcal{O}_X^\star\,\mathrm{U}(r)) \to \mathrm{H}^1(X, \mathrm{GL}(r, \mathcal{O}_X))$$

is regarded as the set of all the isomorphism classes of vector bundles $\mathcal{E}$ of X of rank r admitting projectively flat Hermitian metrics.

PROOF. Let $(\mathcal{E}, h)$ be a projectively flat Hermitian vector bundle of rank r. Then there are an open covering $\{U_\lambda\}$ of X and positive-valued C^∞-functions a_λ on U_λ such that $a_\lambda^{-1}h$ is a flat metric on U_λ. Thus we may assume that there exist holomorphic sections

$$e_1^\lambda, e_2^\lambda, \ldots, e_r^\lambda \in \mathrm{H}^0(U_\lambda, \mathcal{E}),$$

such that, for any $1 \le i, j \le r$,

$$h(e_i^\lambda, e_j^\lambda) = a_\lambda \delta_{i,j},$$

where $\delta_{i,j}$ denotes Kronecker's δ. Let $T_{\lambda,\mu}$ be the transition matrix of $\mathcal{E}$ with respect to the frame $\{(U_\lambda, e_i^\lambda)\}$:

$$(e_1^\lambda, e_2^\lambda, \ldots, e_r^\lambda) \cdot T_{\lambda,\mu} = (e_1^\mu, e_2^\mu, \ldots, e_r^\mu).$$

Then $T_{\lambda,\mu}$ are holomorphic $r \times r$ regular matrices and satisfy

$${}^{\mathrm{t}}T_{\lambda,\mu}\overline{T_{\lambda,\mu}} = a_\mu a_\lambda^{-1} \cdot \mathrm{id}.$$

Locally on $U_\lambda \cap U_\mu$, there is a holomorphic function u such that $a_\mu a_\lambda^{-1} = |u|^2$. Thus $u^{-1}T_{\lambda,\mu}$ is unitary. Hence $T_{\lambda,\mu} \in \mathrm{H}^0(U_\lambda \cap U_\mu, \mathcal{O}_X^\star \, \mathrm{U}(r))$. Therefore $\mathcal{E} \in \mathrm{H}^1(X, \mathrm{GL}(r, \mathcal{O}_X))$ comes from $\mathrm{H}^1(X, \mathcal{O}_X^\star \, \mathrm{U}(r))$.

Next suppose that $\mathcal{E}$ is contained in the image of $\mathrm{H}^1(X, \mathcal{O}_X^\star \, \mathrm{U}(r))$. Then, for a suitable frame $\{(U_\lambda, e_i^\lambda)\}$, the corresponding transition matrix $T_{\lambda,\mu}$ is contained in $\mathrm{H}^0(U_\lambda \cap U_\mu, \mathcal{O}_X^\star \, \mathrm{U}(r))$. Thus

$${}^{\mathrm{t}}T_{\lambda,\mu}\overline{T_{\lambda,\mu}} = v_{\lambda,\mu} \cdot \mathrm{id},$$

for a positive-valued C^∞-function $v_{\lambda,\mu}$ on $U_\lambda \cap U_\mu$. By replacing the open covering $\{U_\lambda\}$ by a finer one, we may assume that there is a positive-valued C^∞-function a_λ on U_λ such that $v_{\lambda,\mu} = a_\mu a_\lambda^{-1}$. Let h_λ be the Hermitian metric of $\mathcal{E}|_{U_\lambda}$ defined by

$$h_\lambda(e_i^\lambda, e_j^\lambda) = a_\lambda \delta_{i,j}.$$

Then $h_\lambda = h_\mu$ on $U_\lambda \cap U_\mu$. Hence we have a projectively flat metric on $\mathcal{E}$. □

4.3. Corollary *A vector bundle $\mathcal{E}$ of rank r is projectively flat if and only if the associated $\mathbb{P}^{r-1}$-bundle $\pi\colon \mathbb{P}_X(\mathcal{E}) \to X$ is induced from a projective unitary representation $\pi_1(X) \to \mathrm{PU}(r)$.*

Proof. There is a commutative diagram of exact sequences:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathcal{O}_X^\star & \longrightarrow & \mathcal{O}_X^\star \, \mathrm{U}(r) & \longrightarrow & \mathrm{PU}(r) & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathcal{O}_X^\star & \longrightarrow & \mathrm{GL}(r, \mathcal{O}_X) & \longrightarrow & \mathrm{PGL}(r, \mathcal{O}_X) & \longrightarrow & 1.
\end{array}$$

Here note that $\mathcal{O}_X^\star$ is the center of both $\mathcal{O}_X^\star \, \mathrm{U}(r)$ and $\mathrm{GL}(r, \mathcal{O}_X)$. Let $\mathcal{E}$ be an element of $\mathrm{H}^1(X, \mathrm{GL}(r, \mathcal{O}_X))$ whose image in $\mathrm{H}^1(X, \mathrm{PGL}(r, \mathcal{O}_X))$ is contained in the image of $\mathrm{H}^1(X, \mathrm{PU}(r))$. Then we can check $\mathcal{E}$ comes from $\mathrm{H}^1(X, \mathcal{O}_X^\star \, \mathrm{U}(r))$ by a diagram chasing. □

4.4. Lemma *Let $Y \subset X$ be a non-singular ample divisor of a non-singular projective variety X of dimension $d \geq 3$. Let $\mathcal{E}_Y$ be a vector bundle of Y and let $\mathcal{L}$ be a line bundle of X such that $\mathcal{E}_Y$ is projectively flat and* $\det \mathcal{E}_Y \simeq \mathcal{L} \otimes \mathcal{O}_Y$. *Then there is a projectively flat vector bundle $\mathcal{E}$ of X satisfying* $\det \mathcal{E} \simeq \mathcal{L}$ *and* $\mathcal{E} \otimes \mathcal{O}_Y \simeq \mathcal{E}_Y$.

PROOF. We shall consider the following two homomorphisms:

$$\det\colon \mathcal{O}_X^\star\,\mathrm{U}(r) \to \mathcal{O}_X^\star, \quad \text{and} \quad p\colon \mathcal{O}_X^\star\,\mathrm{U}(r) \to \mathrm{PU}(r).$$

Let $\boldsymbol{\mu}_r \subset \mathbb{C}^\star$ be the group of r-th roots of unity. Then we have an exact sequence

$$1 \to \boldsymbol{\mu}_r \to \mathcal{O}_X^\star\,\mathrm{U}(r) \xrightarrow{(\det, p)} \mathcal{O}_X^\star \times \mathrm{PU}(r) \to 1,$$

which induces an exact sequence

$$\mathrm{H}^1(X, \boldsymbol{\mu}_r) \to \mathrm{H}^1(X, \mathcal{O}_X^\star\,\mathrm{U}(r)) \to \mathrm{H}^1(X, \mathcal{O}_X^\star) \times \mathrm{H}^1(X, \mathrm{PU}(r)) \to \mathrm{H}^2(X, \boldsymbol{\mu}_r).$$

By the weak Lefschetz theorem, we have isomorphisms

$$\mathrm{H}^1(X, \boldsymbol{\mu}_r) \simeq \mathrm{H}^1(Y, \boldsymbol{\mu}_r), \quad \mathrm{H}^1(X, \mathrm{PU}(r)) \simeq \mathrm{H}^1(Y, \mathrm{PU}(r))$$

and injective homomorphisms

$$\mathrm{H}^1(X, \mathcal{O}_X^\star) \hookrightarrow \mathrm{H}^1(Y, \mathcal{O}_Y^\star), \quad \mathrm{H}^2(X, \boldsymbol{\mu}_r) \hookrightarrow \mathrm{H}^2(Y, \boldsymbol{\mu}_r).$$

Thus we can find $\mathcal{E}$ by a diagram chasing. □

4.5. Lemma *Let $\mathcal{E}$ be an A-stable reflexive sheaf with $\Delta_2(\mathcal{E}) \cdot A^{d-2} = 0$ for an ample divisor A. Then $\mathcal{E}$ is a projectively flat vector bundle.*

This is proved in [**3**, Corollary 3] in the Kähler situation. But here, we give another proof by using the argument of [**79**, 5.1] which is valid only in the projective situation.

PROOF. If $\mathcal{E}$ is locally free, then it follows from works of Donaldson [**12**], Mehta–Ramanathan [**78**], [**79**] as well as Uhlenbeck–Yau [**142**]. Thus we have only to prove that $\mathcal{E}$ is locally free in the case $d \geq 3$. Let S be the complete intersection of smooth divisors $A_1, A_2, \ldots, A_{d-2}$ contained in the linear system $|mA|$ for a sufficiently large $m \in \mathbb{N}$. Then $\mathcal{E}|_S = \mathcal{E} \otimes \mathcal{O}_S$ is a locally free sheaf and it is A-stable by [**79**]. Hence $\mathcal{E}|_S$ is a projectively flat vector bundle. By **4.4**, there is a projectively flat vector bundle $\mathcal{E}'$ such that

$$\det \mathcal{E}' \simeq \det \mathcal{E}, \quad \mathcal{E}' \otimes \mathcal{O}_S \simeq \mathcal{E} \otimes \mathcal{O}_S.$$

By the argument of [**79**, 5.1], we have an isomorphism $\mathcal{E} \simeq \mathcal{E}'$. □

4.6. Proposition *Let $\mathcal{E}$ be an A-semi-stable reflexive sheaf with $\Delta_2(\mathcal{E}) \cdot A^{d-2} = 0$ for an ample divisor A. Then $\mathcal{E}$ is locally free.*

PROOF. We shall prove by induction on $\operatorname{rank} \mathcal{E}$. We may assume $\mathcal{E}$ is not A-stable by **4.5**. Then there is an exact sequence

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{G} \to 0,$$

where $\mathcal{F}$ and $\mathcal{G}$ are non-zero torsion-free sheaves satisfying $\mu_A(\mathcal{F}) = \mu_A(\mathcal{E}) = \mu_A(\mathcal{G})$. Thus $\mathcal{F}$ and the double-dual $\mathcal{G}^\wedge = \mathcal{G}^{\vee\vee}$ of $\mathcal{G}$ are also A-semi-stable sheaves. In particular, Bogomolov's inequalities

$$\Delta_2(\mathcal{F}) \cdot A^{d-2} \geq 0, \quad \Delta_2(\mathcal{G}^\wedge) \cdot A^{d-2} \geq 0$$

hold. Note that $\Delta_2(\mathcal{G}) - \Delta_2(\mathcal{G}^\wedge)$ is represented by an effective algebraic cycle of codimension two supported in $\operatorname{Supp} \mathcal{G}^\wedge/\mathcal{G}$. By the formula (II-9), we infer that

$$\Delta_2(\mathcal{G}) = \Delta_2(\mathcal{G}^\wedge), \quad \Delta_2(\mathcal{F}) \cdot A^{d-2} = \Delta_2(\mathcal{G}^\wedge) \cdot A^{d-2} = 0,$$

and $\mu(\mathcal{F}) = \mu(\mathcal{G}) = \mu(\mathcal{E})$. By the induction, $\mathcal{F}$ and $\mathcal{G}^\wedge$ are locally free. Suppose that $\mathcal{G} \neq \mathcal{G}^\wedge$. Then $\mathcal{E}$ defines a non-zero element of $\mathrm{H}^0(X, \mathcal{E}xt^1(\mathcal{G}, \mathcal{F}))$. On the other hand, we have $\mathcal{E}xt^2(\mathcal{G}^\wedge/\mathcal{G}, \mathcal{F}) = 0$, since $\operatorname{codim} \operatorname{Supp} \mathcal{G}^\wedge/\mathcal{G} \geq 3$. It implies $\mathcal{E}xt^1(\mathcal{G}, \mathcal{F}) = 0$, a contradiction. Hence $\mathcal{G} = \mathcal{G}^\wedge$ and $\mathcal{E}$ is also locally free. □

PROOF OF **4.1**. (1) ⇒ (2): Let $C \subset X$ be a smooth projective curve. Then the normalized tautological divisor of the restriction $\mathcal{E}|_C$ is also nef. Thus $\mathcal{E}|_C$ is semi-stable. Hence $\mathcal{E}$ is A-semi-stable and Bogomolov's inequality $\Delta_2(\mathcal{E}) \cdot A^{d-2} \geq 0$ holds for any ample divisor A. On the other hand,

$$0 \leq \Lambda_{\mathcal{E}}^{r+1} \cdot \pi^* A^{d-2} = -\Delta_2(\mathcal{E}) \cdot A^{d-2}.$$

Thus $\Delta_2(\mathcal{E}) = 0$ in $\mathrm{N}^2(X)$.

(2) ⇒ (3): If $\mathcal{E}$ is A-stable, then $\mathcal{E}$ is a projectively flat vector bundle by **4.5**. Otherwise, there is an exact sequence: $0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{G} \to 0$ such that $\mathcal{F}$ and $\mathcal{G}$ are non-zero torsion-free sheaf and $\mu_A(\mathcal{E}) = \mu_A(\mathcal{F}) = \mu_A(\mathcal{G})$. By the same argument as in the proof of **4.6**, we infer that $\mathcal{F}$ and $\mathcal{G}$ are also A-semi-stable vector bundles with $\Delta_2(\mathcal{F}) \cdot A^{d-2} = \Delta_2(\mathcal{G}) \cdot A^{d-2} = 0$. Thus we have a filtration satisfying the condition (3).

(3) ⇒ (1): If $\mathcal{E}$ is projectively flat, then $f^*\mathcal{E}$ is semi-stable for any morphism $f \colon C \to X$ from a non-singular projective curve. Thus if $\mathcal{E}$ has a filtration satisfying the condition (3), then $f^*\mathcal{E}$ is also semi-stable and $\Lambda_{\mathcal{E}}$ is nef. □

Concerning with the invariant ν for nef $\mathbb{R}$-divisors defined in Chapter **V**, §**2.a**, we have the following:

4.7. Corollary *If $\Lambda_{\mathcal{E}}$ is nef, then $\nu(\Lambda_{\mathcal{E}}) = r - 1$.*

§4.b. The case of vector bundles of rank two. We next consider a weaker condition: $\Lambda_{\mathcal{E}}$ is pseudo-effective. We have the following result when $\operatorname{rank} \mathcal{E} = 2$.

4.8. Theorem *Let $\mathcal{E}$ be an A-semi-stable vector bundle of rank two on a non-singular complex projective variety X of dimension $d \geq 2$ for an ample divisor A. Suppose that the normalized tautological divisor $\Lambda_{\mathcal{E}}$ is pseudo-effective. Then $\Lambda_{\mathcal{E}}$ is nef except for the following three cases*:

(A) *There exist divisors M_1, M_2 such that*

$$M_1 \cdot A^{d-1} = M_2 \cdot A^{d-1} \quad \textit{and} \quad \mathcal{E} \simeq \mathcal{O}_X(M_1) \oplus \mathcal{O}_X(M_2);$$

(B) *There exist an unramified double-covering $\tau \colon Y \to X$ and a divisor M of Y such that*

$$\mathcal{E} \simeq \tau_* \mathcal{O}_Y(M);$$

(C) *There is an exact sequence*

$$0 \to \mathcal{O}_X(L_1) \to \mathcal{E} \to \mathcal{I}\mathcal{O}_X(L_2) \to 0,$$

where $\mathcal{I}$ *is an ideal sheaf with* $\operatorname{codim} \operatorname{Supp} \mathcal{O}_X/\mathcal{I} = 2$ *and the divisor* L_1 *is numerically equivalent to* L_2.

Remark Here $\Lambda = \Lambda_{\mathcal{E}}$ is pseudo-effective in these exceptional cases. Further, Λ is nef if and only if $M_1 \approx M_2$ in the case (A), and $M \approx \sigma^* M$ for the non-trivial involution $\sigma\colon Y \to Y$ over X in the case (B); Λ is not nef in the case (C).

4.9. Corollary *If* $\mathcal{E}$ *is an* A*-stable vector bundle of rank two for an ample divisor* A *such that the normalized tautological divisor* $\Lambda_{\mathcal{E}}$ *is pseudo-effective. Then* $\Lambda_{\mathcal{E}}$ *is nef except for the case* (B) *in* **4.8**.

The idea of our proof of **4.8** is to consider the σ-decomposition of Λ. We shall prove **4.8** after discussing exceptional cases.

Let X be a non-singular projective variety of dimension d and let A be an ample divisor.

4.10. Lemma *Let* M_1, M_2 *be divisors of* X *with* $M_1 \cdot A^{d-1} = M_2 \cdot A^{d-1}$. *Then the vector bundle* $\mathcal{E} = \mathcal{O}_X(M_1) \oplus \mathcal{O}_X(M_2)$ *is* A*-semi-stable and* $|2\Lambda_{\mathcal{E}}| \neq \emptyset$. *The* $\mathbb{Q}$*-divisor* $\Lambda_{\mathcal{E}}$ *is nef if and only if* $M_1 \approx M_2$.

Proof. If $\mathcal{L} \subset \mathcal{E}$ is an invertible subsheaf, then it is a subsheaf of $\mathcal{O}_X(M_1)$ or $\mathcal{O}_X(M_2)$. Thus $\mathcal{L} \cdot A^{d-1} \leq (1/2)c_1(\mathcal{E}) \cdot A^{d-1}$. The symmetric tensor product $\operatorname{Sym}^2 \mathcal{E}$ contains $\mathcal{O}_X(M_1 + M_2) \simeq \det \mathcal{E}$ as a direct summand. Hence $|2\Lambda_{\mathcal{E}}| \neq \emptyset$. If $M_1 \approx M_2$, then $\Lambda_{\mathcal{E}}$ is nef. Conversely if $\Lambda_{\mathcal{E}}$ is nef, then $M_1 - M_2 \approx 0$ by **4.1**, since

$$\Delta_2(\mathcal{E}) = -\frac{1}{4}(M_1 - M_2)^2 = 0. \qquad \square$$

4.11. Lemma *Let* $\tau\colon Y \to X$ *be an unramified double-covering from a non-singular variety and let* M *be a divisor of* Y. *Then, for the vector bundle* $\mathcal{E} = \tau_* \mathcal{O}_Y(M)$, *there is an isomorphism*

$$\tau^* \mathcal{E} \simeq \mathcal{O}_Y(M) \oplus \mathcal{O}_Y(\sigma^* M),$$

where $\sigma\colon Y \to Y$ *is the non-trivial involution over* X. *In particular,* $\mathcal{E}$ *is semi-stable with respect to any ample divisor of* X *and* $\Lambda_{\mathcal{E}}$ *is pseudo-effective. Further,* $\Lambda_{\mathcal{E}}$ *is nef if and only if* $M \approx \sigma^* M$.

Proof. Let us consider the natural homomorphism $\phi\colon \tau^* \tau_* \mathcal{O}_Y \to \mathcal{O}_Y$. Then $\phi + \sigma^* \phi$ gives an isomorphism

$$\tau^* \tau_* \mathcal{O}_Y \simeq \mathcal{O}_Y \oplus \mathcal{O}_Y.$$

Similarly from the natural homomorphism $\varphi\colon \tau^* \tau_* \mathcal{O}_Y(M) \to \mathcal{O}_Y(M)$, we have the homomorphism

$$\varphi + \sigma^* \varphi\colon \tau^* \mathcal{E} = \tau^* \tau_* \mathcal{O}_Y(M) \to \mathcal{O}_Y(M) \oplus \mathcal{O}_Y(\sigma^* M).$$

Since $\mathcal{O}_Y(M)$ is an invertible sheaf, we infer that the homomorphism also is an isomorphism by considering it locally over X. $\square$

4.12. Lemma *Let Z be a closed subspace locally of complete intersection of X with* $\operatorname{codim} Z = 2$ *and let $\mathcal{L}$ be an invertible sheaf of X. If there exists a locally free sheaf $\mathcal{E}$ with an exact sequence*

$$0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{I}_Z\mathcal{L} \to 0, \tag{IV-11}$$

for the defining ideal sheaf $\mathcal{I}_Z$ of Z, then

$$\mathcal{E}xt^2(\mathcal{O}_Z, \mathcal{L}^{-1}) \simeq \mathcal{O}_Z. \tag{IV-12}$$

Conversely, if the isomorphism (IV-12) *exists, then there is a naturally defined cohomology class $\delta(Z, \mathcal{L}) \in \mathrm{H}^2(X, \mathcal{L}^{-1})$ such that $\delta(Z, \mathcal{L}) = 0$ if and only if there is a locally free sheaf $\mathcal{E}$ with the exact sequence* (IV-11).

Proof. Suppose that the locally free sheaf $\mathcal{E}$ exists. Then (IV-11) induces a long exact sequence

$$0 \to \mathcal{H}om(\mathcal{I}_Z\mathcal{L}, \mathcal{O}_X) \to \mathcal{H}om(\mathcal{E}, \mathcal{O}_X) \to \mathcal{H}om(\mathcal{O}_X, \mathcal{O}_X) \to \mathcal{E}xt^1(\mathcal{I}_Z\mathcal{L}, \mathcal{O}_X) \to 0.$$

Therefore

$$\mathcal{O}_Z \simeq \mathcal{E}xt^1(\mathcal{I}_Z\mathcal{L}, \mathcal{O}_X) \simeq \mathcal{E}xt^2(\mathcal{O}_Z, \mathcal{L}^{-1}).$$

Next suppose the isomorphism (IV-12) exists. The spectral sequence

$$E_2^{p,q} = \mathrm{H}^p(X, \mathcal{E}xt^q(\mathcal{I}_Z\mathcal{L}, \mathcal{O}_X)) \Longrightarrow E^{p+q} = \mathrm{Ext}^{p+q}(\mathcal{I}_Z\mathcal{L}, \mathcal{O}_X)$$

induces an exact sequence

$$0 \to \mathrm{H}^1(X, \mathcal{L}^{-1}) \to \mathrm{Ext}^1(\mathcal{I}_Z\mathcal{L}, \mathcal{O}_X) \to \mathrm{H}^0(Z, \mathcal{O}_Z) \to \mathrm{H}^2(X, \mathcal{L}^{-1}).$$

Let $\delta = \delta(Z, L)$ be the image of $1 \in \mathrm{H}^0(Z, \mathcal{O}_Z)$ under the right homomorphism. Then $\delta = 0$ if and only if there is an extension of sheaves

$$0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{I}_Z\mathcal{L} \to 0$$

such that $\mathcal{E}xt^1(\mathcal{E}, \mathcal{O}_X) = 0$. It remains to show that $\mathcal{E}$ is locally free. We may replace X by an open neighborhood of an arbitrary point. Thus we may assume that there is an exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X^{\oplus 2} \to \mathcal{I}_Z\mathcal{L} \to 0,$$

since Z is locally a complete intersection. Pulling back the sequence by $\mathcal{E} \to \mathcal{I}_Z\mathcal{L}$, we have an exact sequence

$$0 \to \mathcal{O}_X \to \widetilde{\mathcal{E}} \to \mathcal{E} \to 0,$$

which is locally split. By the snake lemma, we infer that $\widetilde{\mathcal{E}}$ is locally free. Hence $\mathcal{E}$ is locally free. □

Example Let X be a non-singular projective surface and let x be a point. Suppose that the geometric genus $p_g(X) = \dim \mathrm{H}^2(X, \mathcal{O}_X) = 0$. Then there is a locally free sheaf $\mathcal{E}$ with an exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{E} \to \mathfrak{m}_x \to 0$$

for the maximal ideal $\mathfrak{m}_x$ at x.

Let $\tau\colon Y \to X$ be a generically finite proper surjective morphism from a variety Y with only Gorenstein singularities and let $\nu\colon V \to Y$ be the normalization. By duality, there are trace maps $\nu_*\omega_V \to \omega_Y$ and $\tau_*\omega_Y \to \omega_X$. The first trace map induces an effective divisor C of V, which is called the *conductor* of Y, such that $K_V = \nu^*K_Y - C$. If $C = 0$, then ν is an isomorphism. The pullback of differential forms induces a homomorphism $\nu^*\tau^*\omega_X \to \omega_V$, which gives rise to a splitting of the composite of trace maps above. Thus there exist an effective divisor $R_{V/X}$ of V and an effective Cartier divisor $R_{Y/X}$ of Y such that

$$K_V = \nu^*\tau^*K_X + R_{V/X}, \quad K_Y = \tau^*K_X + R_{Y/X}, \quad R_{V/X} = \tau^*R_{Y/X} - C.$$

The divisors $R_{Y/X}$ and $R_{V/X}$ are called the ramification divisors of $Y \to X$ and $V \to X$, respectively.

4.13. Lemma *If $R_{Y/X} = 0$, then τ is a finite étale morphism.*

Proof. Since the ramification divisor $R_{V/X}$ is effective, the conductor C is zero. Hence Y is normal. Let $Y \to W \to X$ be the Stein factorization of τ, where we write $\mu\colon Y \to W$ and $p\colon W \to X$. Then the dualizing sheaf ω_W is the double-dual of $\mu_*\omega_Y$. Since $R_{Y/X} = 0$, we have isomorphisms $\omega_W \simeq p^*\omega_X$ and $\omega_Y \simeq \mu^*\omega_W$. Thus $W \to X$ is étale, since p is a finite morphism. In particular, W is non-singular. Consequently, the birational morphism $Y \to W$ is isomorphic. □

Proof of **4.8**. Bogomolov's inequality $\Delta_2(\mathcal{E})\cdot A^{d-2} \geq 0$ attains the equality if and only if $\Lambda = \Lambda_{\mathcal{E}}$ is nef by **4.1**. We have only to show the equality $\Delta_2(\mathcal{E})\cdot A^{d-2} = 0$ except for the three exceptional cases. Let $\Lambda = P + N$ be the σ-decomposition of the pseudo-effective divisor Λ (cf. Chapter **III**, §**1**). Then there exist an $\mathbb{R}$-divisor D of X and a real number b such that

$$N \approxeq b\Lambda + \pi^*D \quad \text{and} \quad P \approxeq (1-b)\Lambda - \pi^*D.$$

We have $P\cdot F \geq 0$ and $N\cdot F \geq 0$ for a fiber F of the $\mathbb{P}^1$-bundle $\pi\colon \mathbb{P} = \mathbb{P}_X(\mathcal{E}) \to X$. Thus $0 \leq b \leq 1$. Let $A_1, A_2, \ldots, A_{d-1}$ be general members of the linear system $|mA|$ for a sufficiently large $m \in \mathbb{N}$. Then $\mathcal{E}|_C$ is semi-stable for the non-singular curve $C = A_1 \cap A_2 \cap \cdots \cap A_{d-1}$ by [**78**]. In particular, if $(\Lambda + \pi^*E)|_{\pi^{-1}(C)}$ is pseudo-effective for an $\mathbb{R}$-divisor E of X, then $E \cdot A^{d-1} \geq 0$. Note that $N|_{\pi^{-1}(C)}$ and $P|_{\pi^{-1}(C)}$ are pseudo-effective. Thus $D \cdot A^{d-1} \geq 0$ in the case $b > 0$, and $D \cdot A^{d-1} \leq 0$ in the case $b < 1$.

First suppose that $b < 1$. Since P is movable, P^2 is regarded as a pseudo-effective $\mathbb{R}$-cycle of codimension two. Therefore

$$\pi_*(P^2) = -2(1-b)D$$

is a pseudo-effective $\mathbb{R}$-divisor. Thus $-D$ is pseudo-effective. If $b > 0$ in addition, then $D \approxeq 0$ since $D \cdot A^{d-1} = 0$. Hence $N \approxeq b\Lambda$ and $P \approxeq (1-b)\Lambda$. This is a contradiction. Therefore $b = 0$. Thus $-N \approxeq -\pi^*D$ is pseudo-effective. Hence $N = 0$ and Λ is movable. Since $\Lambda^2 = -\pi^*\Delta_2(\mathcal{E})$, we have

$$-\Delta_2(\mathcal{E}) = \pi_*(H \cdot \Lambda^2) = \pi_*((H + m\pi^*A) \cdot \Lambda^2)$$

for any integer m. If $m > 0$ is large, then $H + m\pi^*A$ is ample and thus $(H + m\pi^*A) \cdot \Lambda^2$ is pseudo-effective. Hence $-\Delta_2(\mathcal{E})$ is pseudo-effective. By Bogomolov's inequality, we have $\Delta_2(\mathcal{E}) \cdot A^{d-2} = 0$.

Next suppose that $b = 1$. Since $P \approx -\pi^*D$ is movable, so is $-D$. On the other hand, $b > 0$ implies $D \cdot A^{d-1} \geq 0$. Hence $D \approx 0$ and $P \approx 0$. Let

$$N = \sum \sigma_i \Gamma_i$$

be the prime decomposition. For each i, there are non-negative integers b_i and $\mathbb{Q}$-divisors D_i such that

$$\Gamma_i \sim_{\mathbb{Q}} b_i\Lambda + \pi^*D_i.$$

Since $\Lambda - \sigma_i\pi^*D_i$ is pseudo-effective and since $\mathcal{E}|_C$ is semi-stable, we have $D_i \cdot A^{d-1} \leq 0$. Hence $b_i > 0$. Moreover, $D_i \cdot A^{d-1} = 0$, since $D \sim_{\mathbb{Q}} \sum \sigma_i D_i \approx 0$. We consider the following three cases:

(I) $b_i \geq 2$ for some i;
(II) N has at least two irreducible components and $b_i = 1$ for any Γ_i;
(III) N has only one irreducible component Γ_1 and $b_1 = 1$.

Let Y be an irreducible component Γ_1. Then $\pi \colon Y \to X$ is a generically finite surjective morphism of degree b_1. By adjunction, we have

$$K_Y = \pi^*K_X + ((b_1 - 2)\Lambda + \pi^*D_1)|_Y.$$

Therefore $R_{Y/X} \sim ((b_1 - 2)\Lambda + \pi^*D_1)|_Y$. Since $R_{Y/X}$ is effective,

$$\pi_*(((b_1 - 2)\Lambda + \pi^*D_1)|_Y) = \pi_*(((b_1 - 2)\Lambda + \pi^*D_1) \cdot (b_1\Lambda + \pi^*D_1)) = 2(b_1 - 1)D_1$$

is an effective divisor of X.

We consider the case (I). We may assume that $b_1 \geq 2$. Then $D_1 \sim_{\mathbb{Q}} 0$, since $D_1 \cdot A^{d-1} = 0$. Hence $Y \sim_{\mathbb{Q}} b_1\Lambda$. By the definition of σ-decomposition, we have

$$\sigma_i = \sigma_{\Gamma_i}(\Lambda) = \frac{1}{b_1}\sigma_{\Gamma_i}(Y).$$

Thus N has only one irreducible component Y and $N = (1/b_1)Y$. Furthermore, $(b_1 - 2)\Lambda|_Y \sim_{\mathbb{Q}} R_{Y/X} \geq 0$. Let us choose a positive integer m such that $H + m\pi^*A$ is ample. Then

$$\pi_*((H + m\pi^*A) \cdot ((b_1 - 2)\Lambda) \cdot Y) = b_1(b_1 - 2)\pi_*(H \cdot \Lambda^2) = -b_1(b_1 - 2)\Delta_2(\mathcal{E})$$

is also a pseudo-effective cycle. Hence by Bogomolov's inequality, if $b_1 \geq 3$, then $\Delta_2(\mathcal{E}) = 0$ and hence Λ is nef by **4.1**. This is a contradiction to: $P \approx 0$. Therefore, $b_1 = 2$ and thus $R_{Y/X} = 0$. Hence $\pi \colon Y \to X$ is an étale double-covering by **4.13**. From the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}}(H - Y) \to \mathcal{O}_{\mathbb{P}}(H) \to \mathcal{O}_Y(H) \to 0,$$

we infer that $\mathcal{E} \simeq \pi_*\mathcal{O}_Y(H)$. Thus $\mathcal{E}$ is of type (B).

Next we consider the case (II). Let Γ_1, Γ_2 be two irreducible components of N. Then $\pi_*(\Gamma_1 \cdot \Gamma_2) = D_1 + D_2$, since $b_1 = b_2 = 1$. Thus $D_1 + D_2$ is effective with $(D_1 + D_2) \cdot A^{d-1} = 0$. Therefore, $D_1 + D_2 \sim 0$ and $\Gamma_1 + \Gamma_2 \sim 2\Lambda$. Hence N has only two components and $\sigma_1 = \sigma_2 = 1/2$. We infer that every component of $\Gamma_1 \cap \Gamma_2$ is contracted by π from the vanishing $\pi_*(\Gamma_1 \cdot \Gamma_2) = 0$. Therefore

$$\pi_*(H \cdot \Gamma_1 \cdot \Gamma_2) = -\Delta_2(\mathcal{E}) + D_1 \cdot D_2 = -\Delta_2(\mathcal{E}) - D_1^2$$

is an effective cycle. On the other hand,

$$R_{\Gamma_1/X} \sim (-\Lambda + \pi^* D_1)|_{\Gamma_1}.$$

Thus we have also an effective cycle

$$\pi_*(H \cdot R_{\Gamma_1/X}) = \pi_*(H \cdot (-\Lambda + \pi^* D_1) \cdot (\Lambda + \pi^* D_1)) = D_1^2 + \Delta_2(\mathcal{E}).$$

Hence $-\Delta_2(\mathcal{E}) = D_1^2$ in $\mathrm{N}^2(X)$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$. In particular, Γ_1 and Γ_2 are mutually disjoint sections of the $\mathbb{P}^1$-bundle. Therefore

$$\mathcal{E} \simeq \pi_* \mathcal{O}_{\Gamma_1}(H) \oplus \pi_* \mathcal{O}_{\Gamma_2}(H).$$

Thus this is of type (A).

Finally, we treat the case (III). For the unique component $Y = \Gamma_1$, there is a divisor L_1 such that $Y \sim H - \pi^* L_1$. Since $N = \sigma_1 Y \approx \Lambda$, we have $\sigma_1 = 1$ and $\det \mathcal{E} \approx 2L_1$. Note that

$$R = R_{Y/X} \sim (-H + \pi^*(-L_1 + \det \mathcal{E}))|_Y.$$

By applying π_* to the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}}(H - Y) \to \mathcal{O}_{\mathbb{P}}(H) \to \mathcal{O}_Y(H) \to 0,$$

we have another exact sequence

$$0 \to \mathcal{O}_X(L_1) \to \mathcal{E} \to \mathcal{I}\mathcal{O}_X(L_2) \to 0,$$

where L_2 is a divisor linearly equivalent to $\det \mathcal{E} - L_1$ and $\mathcal{I} = \pi_* \mathcal{O}_Y(-R)$. Therefore $\mathcal{E}$ is of type (C). This completes the proof. □

Concerning with the invariant κ_σ for pseudo-effective $\mathbb{R}$-divisors defined in Chapter **V**, §**2.b**, we have the following:

4.14. Corollary *If $\mathcal{E}$ is an A-semi-stable vector bundle of rank two, then*

$$\kappa_\sigma(\Lambda_{\mathcal{E}}) \le 1.$$

Proof. We may assume that $\Lambda = \Lambda_{\mathcal{E}}$ is pseudo-effective. By **4.7**, we may assume further that Λ is not nef. By the proof of **4.8**, the positive part P of the σ-decomposition of Λ is numerically trivial and hence $\Lambda \approx N$. Thus $\kappa_\sigma(\Lambda) = 0$. □

4.15. Theorem *The tautological divisor of the tangent bundle of a K3 surface is not pseudo-effective.*

Proof. For the tangent bundle $\mathcal{E} = T_X$ of a K3 surface X, $\det(\mathcal{E}) = \mathcal{O}_X$ and $c_2(\mathcal{E}) = 24$. By [**150**], $\mathcal{E}$ is A-stable for any ample divisor A. Since X is simply connected, $\Lambda_{\mathcal{E}} = H_{\mathcal{E}}$ is not pseudo-effective by **4.9**. □

Remark Kobayashi proved $\kappa(\Lambda) = -\infty$ in [**66**, Theorem C]. On the other hand, the tangent bundle is *generically semi-positive* in the sense of Miyaoka [**81**].

Problem For a K3 surface X, are there infinitely many prime divisors $\Gamma \subset \mathbb{P}_X(T_X)$ such that $H|_\Gamma$ are not pseudo-effective?

Actually, for some K3 surface X, there is a nef divisor L of $\mathbb{P}_X(T_X)$ with $H \cdot L^2 < 0$ (cf. [**112**]). For example, if X is a smooth quadric surface, then $L = H + 2\pi^*A$ is free for a hyperplane section A. In this case, $H \cdot L^2 = -8 < 0$. A general member $\Gamma \in |L|$ is a non-singular surface birational to X, with $K_\Gamma^2 = -40$. Here $H|_\Gamma$ is not pseudo-effective. In particular, the pullback of T_X in Γ is not A'-semi-stable for an ample divisor A' of Γ.

Problem Let $\mathcal{E}$ be a vector bundle of rank two on a non-singular projective surface X. Suppose that for any generically finite morphism $f : Y \to X$ from any non-singular projective surface Y and for any ample divisor A of Y, $f^*\mathcal{E}$ is A-semi-stable. Then is $\Lambda_\mathcal{E}$ nef?

If $\Lambda_\mathcal{E}$ is not nef, then it is not pseudo-effective by **4.8** and is a negative example to **III.3.4**.

4.16. Proposition *If $\mathcal{E}$ is a vector bundle of rank two on a non-singular projective surface whose normalized tautological divisor is not pseudo-effective, then $\mathcal{E}$ is A-semi-stable for some ample divisor A.*

Proof. Assume the contrary. Then there is an exact sequence

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{I}_Z\mathcal{M} \to 0$$

such that $\mathcal{I}_Z$ is the ideal sheaf of a subspace Z of $\dim Z \leq 0$ and $(\mathcal{L} - \mathcal{M}) \cdot A > 0$ for any ample divisor A. Therefore $\mathcal{L} - \mathcal{M}$ is pseudo-effective. By the formula,

$$\Lambda_\mathcal{E} = H_\mathcal{E} - \frac{1}{2}\pi^*(\mathcal{L} + \mathcal{M}) = H_\mathcal{E} - \pi^*\mathcal{L} + \frac{1}{2}\pi^*(\mathcal{L} - \mathcal{M}),$$

we infer that $\Lambda_\mathcal{E}$ is pseudo-effective. □

4.17. Corollary *Let $\mathcal{E}$ be a vector bundle of rank two of a non-singular projective surface X. If D is a pseudo-effective $\mathbb{R}$-divisor of X with $3D^2 \geq \Delta_2(\mathcal{E})$, then $\Lambda_\mathcal{E} + \pi^*D$ is pseudo-effective.*

Proof. We may assume that $\Lambda = \Lambda_\mathcal{E}$ is not pseudo-effective. By **4.16**, $\mathcal{E}$ is A-semi-stable for an ample divisor A. Thus Bogomolov's inequality $\Delta_2(\mathcal{E}) \geq 0$ holds. Let D be a $\mathbb{Q}$-divisor with $3D^2 > \Delta_2(\mathcal{E})$. It is enough to show that $\Lambda + \pi^*D$ is big. Let m be a positive integer such that $m\Lambda$ and mD are $\mathbb{Z}$-divisors. Then D is big by the Hodge index theorem and

$$\pi_*\mathcal{O}_\mathbb{P}(m(\Lambda + \pi^*D)) \simeq \pi_*\mathcal{O}_\mathbb{P}(m\Lambda) \otimes \mathcal{O}_X(mD),$$

in which $\pi_*\mathcal{O}_\mathbb{P}(m\Lambda)$ is an A-semi-stable vector bundle with trivial first Chern class. Therefore,

$$\mathrm{H}^2(X, \pi_*\mathcal{O}_\mathbb{P}(m(\Lambda + \pi^*D)))^\vee \simeq \mathrm{H}^0(X, \pi_*\mathcal{O}_\mathbb{P}(m\Lambda)^\vee \otimes \mathcal{O}_X(K_X - mD)) = 0$$

for $m \gg 0$. Note that

$$\mathrm{H}^p(\mathbb{P}, \mathcal{O}_\mathbb{P}(m(\Lambda + \pi^*D))) \simeq \mathrm{H}^p(X, \pi_*\mathcal{O}_\mathbb{P}(m(\Lambda + \pi^*D)))$$

for any $p \geq 0$. Since $(\Lambda + \pi^*D)^3 = -\Delta_2(\mathcal{E}) + 3D^2 > 0$, we have

$$\varlimsup_{m\to\infty} m^{-3}\chi(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(m(\Lambda + \pi^*D))) > 0.$$

Therefore $\Lambda + \pi^*D$ is big. □

Problem Let $\mathcal{E}$ be a vector bundle of rank two on a non-singular projective variety X. Suppose that the normalized tautological divisor $\Lambda = \Lambda_{\mathcal{E}}$ is not pseudo-effective. Describe the set

$$V(X, \mathcal{E}) := \{D \in \mathrm{N}^1(X) \mid \Lambda + \pi^*D \text{ is pseudo-effective}\}.$$

For example, if $X = \mathbb{P}^2$ and $\mathcal{E} = T_X$, then $V(X, \mathcal{E}) = \{a\ell \mid a \geq 1/2\}$, where $\ell \subset \mathbb{P}^2$ is a line.