CHAPTER I

Overview

§1. Introduction

This article is based on the three unpublished preprints [**104**], [**105**], and [**106**] of the author. The main body is taken from [**104**], while Chapter **VI** is based upon [**105**] and Chapter **IV**, §**4** upon [**106**]. The way of unifying these preprints into one article, however, was not done in a simple manner. Some parts are moved around, although most things were kept mainly along the line of [**104**]. A new chapter II is added for the purpose of helping the reader understand this unification better. The main subjects of this article are:

- Zariski-decomposition problem;
- Addition theorem;
- Numerical D-dimension;
- Invariance of plurigenera.

§1.a. Zariski-decomposition. The theory of divisors plays an important role in algebraic geometry. Let X be a normal complete algebraic variety defined over the complex number field $\mathbb{C}$ and let D be a Cartier divisor. The complete linear system $|D|$, which is a projective space parametrizing all the effective divisors linearly equivalent to D, defines a rational map $\Phi_{|D|}\colon X \cdots\to |D|^{\vee}$ into the dual projective space $|D|^{\vee}$. The D-dimension $\kappa(D) = \kappa(D, X)$ is defined as the maximum of $\dim \Phi_{|mD|}(X)$ for $m > 0$ in the case: $|lD| \neq \emptyset$ for some $l > 0$. In the other case, i.e., $|lD| = \emptyset$ for any $l > 0$, we set $\kappa(D, X) = -\infty$ by definition. We have another expression for the D-dimension:

$$\kappa(D, X) = \begin{cases} -\infty, & \text{if } R(X, D) = \mathbb{C}\ , \\ \operatorname{tr.deg} R(X, D) - 1, & \text{otherwise,} \end{cases}$$

in terms of the graded ring

$$R(X, D) = \bigoplus\nolimits_{m\geq 0} \mathrm{H}^0(X, \mathcal{O}_X(mD)).$$

The ring $R(X, D)$ is not always finitely generated as a $\mathbb{C}$-algebra. It is finitely generated if and only if there exist a birational morphism $\mu\colon Y \to X$ from a normal complete variety Y, a positive integer m, and an effective Cartier divisor F of Y such that

(1) kF is the fixed divisor $|mk\mu^*D|_{\mathrm{fix}}$ for any $k > 0$,
(2) $\operatorname{Bs}|m\mu^*D - F| = \emptyset$.

Hence, if $R(X, D)$ is finitely generated, then $\Phi_{|mD|}(X) \simeq \operatorname{Proj} R(X, D)$ for some $m > 0$.

Let $\mathbb{N}(D)$ be the set of positive integers m with $|mD| \neq \emptyset$ and let $m_0(D) = \gcd \mathbb{N}(D)$. Then $\mathbb{N}(D)$ is a semi-group and $mm_0(D) \in \mathbb{N}(D)$ for $m \gg 0$. Let F_m be the fixed divisor $|mD|_{\text{fix}}$ for $m \in \mathbb{N}(D)$. Then $F_{m+n} \leq F_m + F_n$ and the limit

$$N_s(D) := \lim_{\mathbb{N}(D) \ni m \to \infty} \frac{1}{m} F_m$$

exists as an $\mathbb{R}$-divisor. We have $mN_s(D) \leq F_m$ for $m \in \mathbb{N}(D)$. For the $\mathbb{R}$-divisor $P_s(D) := D - N_s(D)$, we have an isomorphism

$$\mathrm{H}^0(X, \mathcal{O}_X(mD)) \simeq \mathrm{H}^0(X, \mathcal{O}_X(\llcorner mP_s(D) \lrcorner))$$

for any $m > 0$, where $\llcorner\ \lrcorner$ denotes the round-down (the integral part). The decomposition $D = P_s(D) + N_s(D)$ is called the *sectional decomposition*. If $R(X, D)$ is finitely generated, then the positive part $P_s(\mu^* D)$ is a semi-ample $\mathbb{Q}$-Cartier divisor for a projective birational morphism $\mu \colon Y \to X$ from a normal projective variety. But, in general, $P_s(D)$ is not necessarily $\mathbb{Q}$-Cartier nor semi-ample.

If $\dim X = 1$, then $R(X, D)$ is always finitely generated, but it is not so if $\dim X = 2$. However, Zariski [**151**] found a similar decomposition on a non-singular projective surface X: his decomposition $D = P + N$ satisfies and is determined by the following numerical properties:

(1) P and N are $\mathbb{Q}$-divisors;
(2) N is effective and the intersection matrix $(\Gamma_i \cdot \Gamma_j)_{i,j}$ for the prime components Γ_i of N is negative-definite;
(3) $P \cdot C \geq 0$ for any irreducible curve $C \subset X$ (in other words, P is nef);
(4) $P \cdot N = 0$.

If $D - \Delta$ is nef for an effective $\mathbb{Q}$-divisor Δ, then $N \leq \Delta$ by the properties above. In particular, $N \leq N_s(D)$ and hence

$$\mathrm{H}^0(X, \mathcal{O}_X(mD)) \simeq \mathrm{H}^0(X, \mathcal{O}_X(\llcorner mP \lrcorner))$$

for $m > 0$. If $\kappa(D, X) = \dim X = 2$, then $N_s(D) = N$ and $P_s(D) = P$. The Zariski-decomposition is calculated by finitely many linear equations. The linear system $|mD|$ is almost determined by P and N. The construction of Zariski-decomposition is generalized by Fujita [**20**] to the case where D is pseudo-effective, in other words, to the case where $\kappa(mD + A, X) \geq 0$ for $m \gg 0$ and for an ample divisor A. The Zariski-decomposition of the canonical divisor K_X is related to the minimal model $X_{\min}$. The positive part P is $\mathbb{Q}$-linearly equivalent to the pullback of $K_{X_{\min}}$.

An analogy of Zariski-decomposition is expected in the study of algebraic varieties of dimension greater than two. If D satisfies $\kappa(D, X) = \dim X$, then D is called *big*. It was conjectured that, for a big divisor D on X, there exists a birational morphism $f \colon Y \to X$ from a normal projective variety such that $P_s(f^* D)$ is a nef $\mathbb{Q}$-divisor. A counterexample was given by Cutkosky [**8**], in which $P_s(f^* D)$ is not a $\mathbb{Q}$-divisor but only a nef $\mathbb{R}$-divisor. Thus the conjecture was replaced to the one in which we only require $P_s(f^*D)$ to be a nef $\mathbb{R}$-divisor. This weakened conjecture is called the *Zariski-decomposition conjecture* for a big divisor. Kawamata [**57**] showed that if the conjecture for the canonical divisor K_X is true, then the *pluricanonical ring* $R(X, K_X)$ is finitely generated.

A numerical property of a divisor D is a property of the numerical equivalence class of D. The class is regarded as the real first Chern class $c_1(D)_{\mathbb{R}}$. The sectional decomposition is determined by the $\mathbb{Q}$-linear equivalence class, not by the numerical equivalence class. Fujita considered the Zariski-decomposition $D = P + N$ in arbitrary dimension where the sum should consist of a nef $\mathbb{Q}$-divisor P and an effective $\mathbb{Q}$-divisor N where N is minimal in some sense. The precise definition is as follows [**25**]: let D be a pseudo-effective $\mathbb{Q}$-divisor on a non-singular projective variety X. The decomposition $D = P + N$ is called a *Zariski-decomposition in the sense of Fujita* if

(1) P is nef and N is effective,
(2) for any birational morphism $f\colon Y \to X$ and for any effective $\mathbb{Q}$-divisor Δ on Y with $f^*D - \Delta$ being nef, the inequality $f^*N \leq \Delta$ holds.

The decomposition depends only on the numerical equivalence class and is unique, if exists. In the paper [**25**], P, N, and Δ are required to be $\mathbb{Q}$-divisors. But as Cutkosky's example indicates, we can only require these to be $\mathbb{R}$-divisors. Now we may conjecture the existence of a birational morphism $\nu\colon Z \to X$ from a non-singular projective variety such that ν^*D admits a Zariski-decomposition in Fujita's sense with $\mathbb{Q}$-divisors replaced with $\mathbb{R}$-divisors. If D is big, then this is equivalent to the Zariski-decomposition conjecture for a big divisor mentioned before. Note that the following even weaker conjecture is still open: *a pseudo-effective divisor can be written as $\mu_*(P+N)$ for a birational morphism $\mu\colon Y \to X$, for a nef $\mathbb{R}$-divisor P and an effective $\mathbb{R}$-divisor N on Y.*

Matsuda (cf. [**77**]) tried to construct a divisor on X which should be ν_*N for the conjectural birational morphism $\nu\colon Z \to X$ above. The divisor should be written as the limit

$$\lim_{\varepsilon \downarrow 0} N_s(D + \varepsilon A)$$

for an ample divisor A. The limit depends only on the numerical equivalence class of D. The author showed that the limit really expresses an $\mathbb{R}$-divisor by proving that the number of the prime components of the limit is less than the Picard number of X. This is our starting point (cf. Chapter **III**). We denote the limit by $N_\sigma(D)$ and $D - N_\sigma(D)$ by $P_\sigma(D)$. The decomposition $D = P_\sigma(D) + N_\sigma(D)$ is called the *σ-decomposition.* This argument is valid also for pseudo-effective $\mathbb{R}$-divisors D. It is natural to pose the following version of Zariski-decomposition conjecture: *for a pseudo-effective $\mathbb{R}$-divisor D on a non-singular projective variety X, there exists a birational morphism $f\colon Y \to X$ from a non-singular projective variety such that $P_\sigma(f^*D)$ is nef.* If D is big (the notion of big is defined even for $\mathbb{R}$-divisors), then this version is equivalent to the previous versions of the Zariski decomposition conjecture we discussed. For other divisors, this version is stronger than the previous ones.

It is usually difficult to calculate the σ-decomposition of a divisor even when it is explicitly given. The author next introduced another decomposition $D = P_\nu(D) + N_\nu(D)$, called the *ν-decomposition*, that is determined by a process similar to Zariski's original decomposition. This decomposition is calculated step by step by determining the "minimum" satisfying a system of inequalities (cf. **III.3.12**). If $P_\nu(D)$ is not nef, then we choose a suitable center of blowing-up and calculate the new ν-decomposition on the blown-up. This method is, however, not so effective for getting the Zariski-decomposition (assuming its existence). But we can calculate in some special cases.

A counterexample to the Zariski-decomposition conjecture for a big divisor was found by the calculation of ν-decomposition for a special divisor on some $\mathbb{P}^2$-bundle over an abelian surface ([**103**], cf. [**104**]). This counterexample is related to Cutkosky's example. Thus we can not have a Zariski-decomposition in general.

However, the Zariski-decomposition does exist under some special circumstances: The Zariski-decomposition of a $\mathbb{Q}$-Cartier divisor of a toric variety is given by Kawamata [**57**]. We can treat also the case of $\mathbb{R}$-Cartier divisors (cf. **IV.1.17**). Here, the σ-decomposition is calculated by a combinatorial way. A toric bundle is a fiber bundle of a toric variety whose transition group is the open torus. We can calculate σ-decompositions etc. by a combinatorial way for some toric bundles. The counterexample above to the Zariski-decomposition conjecture can also be explained by the method on toric bundles (cf. **IV.2.10**).

The Zariski-decomposition conjecture for projective bundles over a non-singular projective curve associated with vector bundles was studied by the method of ν-decompositions in [**101**], where a relation between the decomposition and the Harder–Narasimhan filtration was found. If the length is less than or equal to 3, then the Zariski-decomposition is constructed, which is explained in the old version [**104**]. The general case is proved in Chapter **IV**, §**3** by the method on toric bundles.

In order to find some other counterexamples to the Zariski-decomposition conjecture, it seems to be interesting to consider the tautological line bundle associated with some special vector bundles. The *normalized tautological divisor* is a $\mathbb{Q}$-divisor whose multiple is the minus of the relative canonical divisor of the associated projective bundle, and whose degree on a fiber is one (cf. **6.4**). In the preprint [**106**], the author studied normalized tautological divisors. The content is now written into Chapter **IV**, §**4**. It includes the following results:

(1) We can determine vector bundles over a projective manifold whose normalized tautological divisor is nef (cf. **IV.4.1**) by using the Kobayashi–Hitchin correspondence;
(2) We can determine also vector bundles of rank two whose normalized tautological divisor are not nef but pseudo-effective in **IV.4.8**.
(3) The tautological line bundle of the tangent bundle of a projective K3 surface is shown to be not pseudo-effective (**IV.4.15**).

However, new counterexamples in this direction are not obtained so far.

§1.b. Numerical D-dimension. In the study of numerical properties of divisors, we may expect a numerical version of D-dimension, which has already been defined for nef $\mathbb{R}$-divisors (cf. [**114**, (4.5)], [**24**, §3], [**55**]); if D is nef, then the numerical D-dimension $\nu(D)$ is defined as the maximum $k \in \mathbb{Z}_{\geq 0}$ such that the k-times cup-product $c_1(D)^k \in \mathrm{H}^{k,k}(X, \mathbb{R})$ is not zero (cf. **II.6.3**).

Viehweg noticed the importance of the behavior of functions

$$m \longmapsto \dim \mathrm{H}^0(X, mD + A),$$

where D is a Cartier divisor, A is an ample divisor on a non-singular projective variety X (cf. §3 Problem 6 of [**139**, Open Problems]). Fujita also considered a similar object in order to define the L-dimension $\kappa(L, \mathcal{F})$ of a coherent sheaf $\mathcal{F}$ in [**23**]. Moreover, Fujita showed that $\nu(L) = \max_{\mathcal{F}} \kappa(L, \mathcal{F})$ for nef line bundles L in [**24**, (6.6)]. A candidate $\kappa_\sigma(D) = \kappa_\sigma(D, X)$ for the numerical D-dimension is defined along this line of investigation in Chapter **V**, §**2.b**.

Suppose that $\kappa(D) = \kappa(D, X) = k$ for a divisor D on a non-singular projective variety X. Then, for any subvariety $Z \subset X$ of dimension less than k, there is an effective divisor $\Delta \in |mD|$ with $Z \subset \operatorname{Supp} \Delta$ for $m \gg 0$. This is proved by the use of Iitaka fibration. Conversely this property characterizes the D-dimension $\kappa(D)$. By considering a numerical version of the property, the author defines another candidate $\kappa_\nu(D)$ for the numerical D-dimension in Chapter **V**, §**2.d**. These invariants $\kappa_\sigma(D)$ and $\kappa_\nu(D)$ enjoy the following properties:

(1) $\kappa_\sigma(D)$ and $\kappa_\nu(D)$ depend only on the numerical equivalence class of D;
(2) D is pseudo-effective $\iff \kappa_\sigma(D) \geq 0 \iff \kappa_\nu(D) \geq 0$;
(3) If $D_1 - D_2$ is pseudo-effective, then $\kappa_\sigma(D_1) \geq \kappa_\sigma(D_2)$ and $\kappa_\nu(D_1) \geq \kappa_\nu(D_2)$;
(4) If D is nef, then $\nu(D) = \kappa_\sigma(D) = \kappa_\nu(D)$;
(5) $\kappa_\sigma(h^*D) = \kappa_\sigma(D)$ and $\kappa_\nu(h^*D) = \kappa_\nu(D)$ hold for a surjective morphism $h\colon Z \to X$ from a non-singular projective variety;
(6) $\kappa(D) \leq \kappa_\sigma(D) \leq \kappa_\nu(D)$;
(7) (Easy addition) For a fiber space $f\colon X \to Y$, the inequalities

$$\kappa_\sigma(D) \leq \kappa_\sigma(D|_{X_y}) + \dim Y \quad \text{and} \quad \kappa_\nu(D) \leq \kappa_\nu(D|_{X_y}) + \dim Y$$

hold for a 'general' fiber $X_y = f^{-1}(y)$;
(8) $\kappa_\sigma(X) = \kappa_\sigma(K_X)$ and $\kappa_\nu(X) = \kappa_\nu(K_X)$ are birational invariants.

For the proof of (2), we use the Kawamata–Viehweg vanishing theorem [**51**], [**146**]. We do not understand the difference between κ_σ and κ_ν clearly. It is expected from properties of σ-decomposition that $\kappa_\sigma(D) = \kappa_\sigma(P_\sigma(D))$ and $\kappa_\nu(D) = \kappa_\nu(P_\sigma(D))$ hold for a pseudo-effective $\mathbb{R}$-divisor D. But it is still conjectural. The following three conditions are equivalent for an $\mathbb{R}$-divisor D (cf. **V.1.12**):

(1) $\kappa_\sigma(D) = 0$; (2) $\kappa_\nu(D) = 0$; (3) $P_\sigma(D)$ is numerically trivial.

In particular, D admits a Zariski-decomposition if $\kappa_\sigma(D) = 0$. The birational invariant $\kappa_\sigma(X)$ or $\kappa_\nu(X)$ should be the numerical Kodaira dimension of X. If

X admits a minimal model $X_{\min}$, then these invariants coincide with $\nu(X_{\min}) = \nu(K_{X_{\min}})$.

§1.c. Canonical divisor. Some effective results on a non-singular projective variety X are derived from special properties of the canonical divisor K_X. One example of such is the following, which is a consequence of the Kawamata–Viehweg vanishing theorem [**51**], [**146**]: *if L is a divisor such that $L - K_X$ is nef and big, then $H^i(X, L) = 0$ for $i > 0$.* The vanishing theorem is derived from the Hodge theory, which is transcendental compared to the theory of linear systems.

Another important example is the (logarithmic) ramification formula. For most birational invariants, we actually prove that they only depend on the birational equivalence class and not on the choice of a variety in the equivalence class, by using the ramification formula.

The *Kodaira dimension* $\kappa(X)$ is a birational invariant defined as $\kappa(K_X)$. The dimension $P_m(X)$ of $\mathrm{H}^0(X, mK_X)$, called the *m-genus*, is also a birational invariant for $m \in \mathbb{N}$. The behavior of $P_m(X)$ determines $\kappa(X)$. The linear systems $|mK_X|$ are called the pluricanonical systems and define the Iitaka fibration $\Phi\colon X \cdots\to Y$ satisfying $\kappa(X_y) = 0$ for a 'general' fiber X_y and $\dim Y = \kappa(X)$, up to the birational equivalence. This reduces the study of varieties X with $0 < \kappa(X) < \dim X$ to that of X with $\kappa(X) = 0$ or $\kappa(X) = \dim X$. But the reduction step is not as straightforward as one might wish, since we must take degenerate fibers into consideration. For example, the Iitaka fibration of a surface of $\kappa = 1$ is an *elliptic fibration*: a general fiber is an elliptic curve. A surface admitting an elliptic fibration is called an *elliptic surface* and the study of elliptic surfaces is one of the most important part of the classification theory of surfaces. The singular fibers of elliptic surfaces are classified and analyzed by Kodaira [**69**], [**70**]. The analysis leads to the canonical bundle formula, which expresses the canonical divisor of the surface by the canonical divisor of the base curve and some data coming from periods and from singular fibers. Iitaka posed the addition conjecture C_n (cf. [**43**], [**44**]):

$$\kappa(X) \geq \kappa(X/Y) + \kappa(Y)$$

holds for an algebraic fiber space $f\colon X \to Y$, where $n = \dim X$ and $\kappa(X/Y)$ stands for $\kappa(X_y)$ for a 'general' fiber X_y. It is considered as a weak generalization of the canonical bundle formula above. This conjecture was the central problem of the birational classification of algebraic varieties in 1970's.

In 1980's, the minimal model theory for higher dimensional varieties was born. The theory of extremal rays by Mori ([**85**], [**86**]) was the breakthrough and the minimal model program posed by Reid [**115**], Kawamata [**54**], and Shokurov [**131**] gave a new scope to the birational classification of algebraic varieties (cf. [**61**]). Here, the canonical divisor K_X plays an important role:

(1) The minimal models are allowed to have some mild singularities such as: terminal singularities, canonical singularities, and their logarithmic versions. The definitions of such singularities are related to the logarithmic ramification formula. The canonical divisor K_X or its logarithmic version

$K_X + \Delta$ is not necessarily Cartier but $\mathbb{Q}$-Cartier. Moreover, we can treat the case of $\mathbb{R}$-Cartier divisors in the logarithmic version;
(2) The contraction morphism of an extremal ray is a fiber space $f: X \to Y$ in which $-K_X$ is f-ample.

The conjectures on the existence and on the termination of flips for small extremal contractions are the main obstructions to constructing minimal models. These are solved affirmatively in dimension 3 by Mori [**89**] and Shokurov [**131**]; moreover their logarithmic versions in dimension 3 are also proved in [**74**] generalizing Shokurov's ideas [**132**]. Here, the classification of 3-dimensional terminal singularities (cf. [**113**], [**114**], [**87**]) is essential in their proofs.

The Zariski-decomposition conjecture for K_X or $K_X + \Delta$ is still expected, since it relates to the existence of flips, which says the relative pluricanonical ring is locally finitely generated. In view of the counterexample to the Zariski-decomposition conjecture for a big divisor, we must take some information specially related to K_X and not common to all general divisors into consideration. For example, let us consider:

Conjecture Let D be a pseudo-effective $\mathbb{R}$-divisor on a non-singular projective variety X such that $D - K_X$ is ample. Then D admits a Zariski-decomposition.

The affirmative answer to above implies the existence of flip by [**57**] and, conversely, the affirmative answer follows from the existence and the termination of flips. We can consider a logarithmic version by replacing K_X with $K_X + \Delta$. The following is a local version of the base-point free theorem in [**61**]:

Conjecture Let D be a pseudo-effective $\mathbb{R}$-divisor on a non-singular projective variety X and x a point not contained in $\operatorname{Supp}\langle D\rangle$. Suppose that D is nef at x (cf. **III.2.2**) and $D - K_X$ is ample. Then there is a positive integer m such that $x \notin \operatorname{Bs} | \llcorner mD \lrcorner |$.

It is interesting if these conjectures above are solved by some standard methods including the vanishing theorems above, the logarithmic ramification formula, some covering technique, duality theorems, etc.

The following abundance conjecture lies at the core of the minimal model theory: *if X is a minimal model with at most terminal singularities, then K_X is semi-ample*. Kawamata [**55**] showed that if K_X is nef and abundant, then it is semi-ample. We can generalize the notion of abundance to $\mathbb{R}$-Cartier divisors that are not necessarily nef by using the notion of κ_σ or κ_ν (cf. Chapter**V**, §**2.e**). The abundance conjecture can now be stated in this general formulation as saying K_X is abundant. This formulation is free of the statements regarding the existence of minimal models. The conjecture is true if $\dim X \leq 3$ by Miyaoka [**83**], [**84**], and by Kawamata [**59**]. Furthermore, by the use of Iitaka fibration, it is true if $\kappa(X) \geq \dim X - 3$ (cf. **4.2**). The key result for the proof in dimensional 3 is the following theorem by Miyaoka [**83**]: $\kappa(X) \geq 0$ *for a minimal model* X. This is based on the addition theorem C_3, the Riemann–Roch theorem, and the following theorem (cf, [**81**], [**74**, Chapter 9]) derived from a study of deformations along

1-foliations in positive characteristics: *the cotangent bundle* Ω^1_X *of a non-uniruled variety* X *is generically semi-positive*. New methods may be required in order to prove the abundance in higher dimension.

§1.d. Addition theorem. In the study of fiber spaces, the notion of variation of Hodge structure [**32**] is important, in which a kind of hyperbolic geometry is hidden. The direct image sheaf $f_*\omega_{X/Y}^{\otimes m}$ of the m-th power of the *relative dualizing sheaf* $\omega_{X/Y} = \omega_X \otimes \omega_Y^{-1}$ is a key object for studying Iitaka's addition conjecture C_n. Viehweg [**147**] proved that $f_*\omega_{X/Y}^{\otimes m}$ is weakly positive generalizing the work of Fujita [**21**], [**22**] and that of Kawamata [**50**]. The positivity result follows from the curvature property of variation of Hodge structure [**32**], [**126**] or from Kollár's torsion-free theorem [**71**]. Kawamata [**56**] proved that if a general fiber admits a minimal model satisfying the abundance, then Viehweg's conjecture C_n^+, which is a refinement of C_n, is true. In the proof, an infinitesimal Torelli theorem for the minimal model is used in an essential way. Kollár [**72**] proved that C_n^+ is true if a general fiber is of general type, where the study of the multiplication maps $\mathrm{H}^0(X, mK_X)^{\otimes l} \to \mathrm{H}^0(X, mlK_X)$ is essential. It is expected that if a general fiber satisfies the abundance, then C_n^+ is true, but it is still open.

The addition theorem for κ_σ:

$$\kappa_\sigma(X) \geq \kappa_\sigma(X/Y) + \kappa_\sigma(Y)$$

is obtained in Chapter **V**, §**4**. In particular, if X satisfies the abundance, then C_n is true. That is to say, the conjecture C_n is weaker than the abundance conjecture. In the proof, the notion of weak positivity is replaced by the notion of ω-sheaf which we introduce in Chapter **V**. This comes from Kollár's torsion-free theorem [**71**] whose origin is also found in the Hodge theory. The argument of Viehweg in [**147**] fits well with the notion of ω-sheaf and is naturally extended to the case of κ_σ. Furthermore, we have some addition theorems for the log-terminal pairs (X, Δ); for example, if $X \to Y$ is an algebraic fiber space, (X, Δ) is log-terminal, and Y is of general type, then

$$\kappa(K_X + \Delta) = \kappa(K_{X_y} + \Delta|_{X_y}) + \dim Y$$

for a 'general' fiber X_y (cf. **V.4.1**). As an application, we show a special abundance theorem **V.4.9**: *if* $\kappa_\sigma(K_X + \Delta) = 0$*, then* $\kappa(K_X + \Delta) = 0$.

§1.e. Invariance of plurigenera. Deformation invariance of plurigenera of compact complex analytic surfaces was proved by Iitaka [**42**]. The author [**96**] (cf. [**98**]) proved the invariance of plurigenera of algebraic varieties (under a projective deformation) assuming the minimal model program, based upon the conjectures on the existence and the termination of flips over the ambient space, and assuming the abundance of a general fiber. Siu [**130**] proved the invariance of plurigenera under a projective deformation whose general fiber is of general type. Siu's method is transcendental but requires essentially vanishing theorems similar to the Kodaira or Kawamata–Viehweg vanishing. An algebraization and a generalization of Siu's argument was written in the preprint [**105**], which now appears as Chapter **VI** with more generalization. Most statements to prove are related to the surjectivity of the restriction homomorphisms

$$\mathrm{H}^0(V, \mathcal{O}_V(m(K_V + X))) \to \mathrm{H}^0(X, \mathcal{O}_X(mK_X))$$

for non-singular varieties V with a non-singular divisor X on it. If $m = 1$, then the homomorphism is considered from the viewpoint of Hodge theory. Siu's idea for $m > 1$ can be interpreted as a technical use of the Kawamata–Viehweg vanishing. By generalizing the idea to the restriction homomorphisms

$$\mathrm{H}^0(V, \mathcal{O}_V(m(K_V + X) + A)) \to \mathrm{H}^0(X, \mathcal{O}_X(mK_X + A|_X))$$

for ample divisors A, the author succeeded in showing that the numerical Kodaira dimension κ_σ is invariant under a projective deformation (cf. **VI.4.1**). The same idea can also be used to prove that small deformations of terminal singularities are also terminal (**VI.5.3**). The case of canonical singularities was shown by Kawamata [**60**] a few months before [**105**]. Moreover, combining with some arguments on κ_σ, the invariance of plurigenera is proved for a projective deformation in which $\kappa_\sigma(F) = \kappa(F)$ holds for a general fiber F (cf. **VI.4.5**).

§1.f. Log-terminal singularities. In the study of open surfaces $S = \overline{S} \setminus D$, it is useful to consider effective $\mathbb{Q}$-divisors with multiplicity at most one supported on the boundary D (cf. [**138**], [**80**]). In the course of generalizing the minimal model program posed by Reid–Kawamata–Shokurov to the logarithmic case, Kawamata introduced the notion of *log-terminal* for pairs (X, Δ) consisting of a normal variety X and an effective $\mathbb{Q}$-divisor with $\llcorner \Delta \lrcorner = 0$. This notion works well with many properties, including the base-point free theorem, the contraction theorem, the cone theorem, etc., which are generalized to the log-terminal case. It is also important to consider the $\mathbb{Q}$-divisors Δ with components of multiplicity one. We recognize its importance very well when we use the adjunction formula $(K_X + \Gamma)|_\Gamma \sim K_\Gamma$ as an essential tool for the original theory of open varieties. Iitaka called the geometry of open varieties of dimension n by the name of geometry of varieties of dimension $n + 1/2$ in view of the adjunction. If we allow such a component of multiplicity one, however, then many related properties to the minimal model program fail to hold in general. For example, a log-canonical singularity fails to be rational. Kawamata introduced the notion of *weak log-terminal* for the purpose to overcome these failures. As is explained in [**61**], the minimal model program is extended to the weak log-terminal pairs with some delicate change of conditions.

Shokurov used the adjunction in order to prove log-flip conjectures. He called the log-terminal above by the name of *Kawamata log terminal* (klt, for short) and introduced many other notions related to log terminal (cf. [**132**], [**74**]). Among them, the notion of *divisorial log terminal* (dlt) is most useful. This coincides with Kawamata's notion of weak log-terminal in a strong sense [**134**].

Most definitions after log-terminal are not analytically local from the view point of looking at the singularities of pairs. They are not so, because, in their definition, the existence of some special birational morphism from a non-singular variety dominating the original variety is required globally. The notions that are exceptions to the comments above and thus are analytically local are: terminal, canonical, and purely log terminal. These are related to 'birational pairs' introduced by Iitaka [**45**] (cf. Chapter **II**, §**4.d**). The author looked for a good definition of log-terminal allowing components of multiplicity one from the viewpoint of singularities and introduced the notion of strongly log-canonical singularities. In the preparation of [**104**], the author found that (X, Δ) is strongly log-canonical if and only if (X, Δ) is log-canonical and $(X, 0)$ has only admissible singularities. Here the notion of admissible singularities is defined even for pairs (X, Δ) in which $K_X + \Delta$ need not to be $\mathbb{Q}$-Cartier.

We shall discuss admissible and strongly log-canonical singularities in Chapter **VII**. This chapter corresponds to the appendix of [**104**].

§2. History

The author started the study of σ-decomposition at the beginning of 1987. The most part of contents in Chapter **III**, §§**1–4** and the prototype discussions on numerical D-dimensions in Chapter **V** were obtained in 1987–1988. These were reported at the Taniguchi symposium in Katata, in August 1988 [**100**]. The relation between the Zariski-decomposition of the tautological line bundle associated with a vector bundle over a curve and the Harder–Narasimhan filtration (cf. Chapter **IV**, §**3**) is added in the preprint [**101**]. The base-point freeness statements in Chapter **V**, §**1.a** and a criterion **VII.1.1** for rationality of a singularity were discovered during the period 1991–1993. The counterexample **IV.2.10** to the existence of Zariski-decomposition for a big divisor was reported at a symposium at Hokkaido Univ. in June 1994 [**103**]. The addition theorem **V.4.1** for κ_σ was obtained in 1994. All the results obtained before May 1997 are written in the preprint [**104**]. It includes the notion of ω-sheaf, the abundance theorem in the case $\kappa_\sigma = 0$, and the existence of Zariski-decomposition on projective bundles over a curve, whose length of Harder–Narasimhan filtration is at most 3. The preprint [**105**] giving an algebraic modification and an improvement of Siu's proof [**130**] appeared in March 1998. The preprint [**106**] showing the tautological line bundle of any algebraic K3 surface to be not pseudo-effective appeared in October 1998. The argument on toric bundles in Chapter **IV** is new and was obtained in November 2001 during the preparation of this article. Moreover, in March 2002, addition and abundance theorems in [**104**] were generalized to the log-terminal case as in Chapter **V**, §**4**.

The article was submitted to MSJ Memoirs, Mathematical Society of Japan at June 2002. The author received two referee reports: one is at September 2003 and the other at February 2004. The first report suggests improvement of English writing in the preface, the first chapter, and in the abstract of each chapter. The second points out a lot of errors from a mathematical side. On the other hand, the author found other mathematical errors mainly in Chapters **IV**, **V**. The modification was finished at June 2004.

§3. Notation

We shall use the notation similar to [**61**] and [**98**]. In addition, we use the following conventions:

(1) The sets of integers, rational numbers, real numbers, and complex numbers are denoted by $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$, respectively. The set $\mathbb{N}$ of natural numbers does not include 0. For $\mathfrak{K} = \mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and for a number $a \in \mathfrak{K}$, $\mathfrak{K}_{\geq a}$ denotes the set of numbers $x \in \mathfrak{K}$ with $x \geq a$.

(2) $\mathbb{C}^\star$ denotes $\mathbb{C} \setminus \{0\}$.

(3) *Duals* are indicated by $\vee$: $M^\vee := \mathrm{Hom}(M, \mathbb{Z})$ and $\mathcal{F}^\vee := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$ for an abelian group M and for an $\mathcal{O}_X$-module $\mathcal{F}$, respectively.

(4) The expression '... for $p \gg n$' means that 'there is a number $N > n$ such that ... for any $p \geq N$.' The other symbol $\ll$ is used in the obvious way.

(5) A subset Y of a set X is called *proper* if $Y \neq X$.