6 L^p-viscosity solutions

In this section, we discuss the L^p-viscosity solution theory for uniformly elliptic PDEs:

$$F(x, Du, D^2u) = f(x) \quad \text{in } \Omega, \tag{6.1}$$

where $F : \Omega \times \mathbf{R}^n \times S^n \to \mathbf{R}$ and $f : \Omega \to \mathbf{R}$ are given. Since we will use the fact that $u + C$ (for a constant $C \in \mathbf{R}$) satisfies the same (6.1), we suppose that F does not depend on u itself. Furthermore, to compare with classical results, we prefer to have the inhomogeneous term (the right hand side of (6.1)).

The aim in this section is to obtain the a priori estimates for L^p-viscosity solutions without assuming any continuity of the mapping $x \to F(x, q, X)$, and then to establish an existence result of L^p-viscosity solutions for Dirichlet problems.

<u>*Remark.*</u> In general, without the continuity assumption of $x \to F(x, p, X)$, even if $X \to F(x, p, X)$ is uniformly elliptic, we **cannot** expect the uniqueness of L^p-viscosity solutions. Because Nadirashvili (1997) gave a counterexample of the uniqueness.

6.1 A brief history

Let us simply consider the Poisson equation in a "smooth" domain Ω with zero-Dirichlet boundary condition:

$$\begin{cases} -\triangle u = f & \text{in } \Omega, \\ \quad\; u = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.2}$$

In the literature of the regularity theory for uniformly elliptic PDEs of second-order, it is well-known that

$$\text{“if } f \in C^{\sigma}(\overline{\Omega}) \text{ for some } \sigma \in (0,1), \text{ then } u \in C^{2,\sigma}(\overline{\Omega})\text{”}. \tag{6.3}$$

Here, $C^{\sigma}(U)$ (for a set $U \subset \mathbf{R}^n$) denotes the set of functions $f : U \to \mathbf{R}$ such that

$$\sup_{x \in U} |f(x)| + \sup_{x,y \in U, x \neq y} \frac{|f(x) - f(y)|}{|x - y|^{\sigma}} < \infty.$$

Also, $C^{k,\sigma}(U)$, for an integer $k \geq 1$, denotes the set of functions $f : U \to \mathbf{R}$ so that for any multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \{0, 1, 2, \ldots\}^n$ with $|\alpha| := \sum_{i=1}^n \alpha_i \leq k$, $D^\alpha f \in C^\sigma(U)$, where

$$D^\alpha f := \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

These function spaces are called Hölder continuous spaces and the implication in (6.3) is called the **Schauder regularity (estimates)**. Since the PDE in (6.2) is linear, the regularity result (6.3) may be extended to

$$\text{“if } f \in C^{k,\sigma}(\overline{\Omega}) \text{ for some } \sigma \in (0, 1), \text{ then } u \in C^{k+2,\sigma}(\overline{\Omega})\text{”}. \tag{6.4}$$

Moreover, we obtain that (6.4) holds for the following PDE:

$$-\text{trace}(A(x)D^2u(x)) = f(x) \quad \text{in } \Omega, \tag{6.5}$$

where the coefficient $A(\cdot) \in C^\infty(\overline{\Omega}, S^n)$ satisfies that

$$\lambda|\xi|^2 \leq \langle A(x)\xi, \xi\rangle \leq \Lambda|\xi|^2 \quad \text{for } \xi \in \mathbf{R}^n \text{ and } x \in \overline{\Omega}.$$

Furthermore, we can obtain (6.4) even for linear second-order uniformly elliptic PDEs if the coefficients are smooth enough.

Besides the Schauder estimates, we know a different kind of regularity results: For a solution u of (6.5), and an integer $k \in \{0, 1, 2, \ldots\}$,

$$\text{“if } f \in W^{k,p}(\Omega) \text{ for some } p > 1, \text{ then } u \in W^{k+2,p}(\Omega)\text{”}. \tag{6.6}$$

Here, for an open set $O \subset \mathbf{R}^n$, we say $f \in L^p(O)$ if $|f|^p$ is integrable in O, and $f \in W^{k,p}(O)$ if for any multi-index α with $|\alpha| \leq k$, $D^\alpha f \in L^p(O)$. Notice that $L^p(\Omega) = W^{0,p}(\Omega)$.

This (6.6) is called the L^p **regularity (estimates)**. For a later convenience, for $p \geq 1$, we recall the standard norms of $L^p(O)$ and $W^{k,p}(O)$, respectively:

$$\|u\|_{L^p(O)} := \left(\int_O |u(x)|^p dx\right)^{1/p}, \quad \text{and} \quad \|u\|_{W^{k,p}(O)} := \sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^p(O)}.$$

In Appendix, we will use the quantity $\|u\|_{L^p(\Omega)}$ even for $p \in (0, 1)$ although this is not the “norm” (*i.e.* the triangle inequality does not hold).

We refer to [13] for the details on the Schauder and L^p regularity theory for second-order uniformly elliptic PDEs.

As is known, a difficulty occurs when we drop the smoothness of A_{ij}.

An extreme case is that we only suppose that A_{ij} are bounded (possibly discontinuous, but still satisfy the uniform ellipticity). In this case, what can we say about the regularity of "solutions" of (6.5) ?

The extreme case for PDEs in divergence form is the following:

$$-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(A_{ij}(x)\frac{\partial u}{\partial x_j}(x)\right) = f(x) \quad \text{in } \Omega. \tag{6.7}$$

De Giorgi (1957) first obtained Hölder continuity estimates on weak solutions of (6.7) in the distribution sense; for any $\phi \in C_0^\infty(\Omega)$,

$$\int_\Omega (\langle A(x)Du(x), D\phi(x)\rangle - f(x)\phi(x))\, dx = 0.$$

Here, we set

$$C_0^\infty(\Omega) := \left\{ \phi : \Omega \to \mathbf{R} \;\middle|\; \begin{array}{c} \phi(\cdot) \text{ is infinitely many times differentiable,} \\ \text{and } \mathit{supp}\ \phi \text{ is compact in } \Omega \end{array} \right\}.$$

We refer to [14] for the details of De Giorgi's proof and, a different proof by Moser (1960).

Concerning the corresponding PDE in nondivergence form, by a stochastic approach, Krylov-Safonov (1979) first showed the Hölder continuity estimates on "strong" solutions of

$$-\mathrm{trace}(A(x)D^2u(x)) = f(x) \quad \text{in } \Omega. \tag{6.8}$$

Afterward, Trudinger (1980) (see [13]) gave a purely analytic proof for it.

Since these results appeared before the viscosity solution was born, they could only deal with strong solutions, which satisfy PDEs in the *a.e.* sense.

In 1989, Caffarelli proved the same Hölder estimate for viscosity solutions of fully nonlinear second-order uniformly elliptic PDEs.

To show Hölder continuity of solutions, it is essential to prove the following "Harnack inequality" for **nonnegative** solutions. In fact, to prove the Harnack inequality, we split the proof into two parts:

weak Harnack inequality for "super"solutions

local maximum principle for "sub"solutions

$\Longrightarrow$ **Harnack inequality for "solutions"**

In section 6.4, we will show that L^p-viscosity solutions satisfy the (interior) Hölder continuous estimates.

6.2 Definition and basic facts

We first recall the definition of L^p-strong solutions of general PDEs:

$$F(x, u, Du, D^2u) = f(x) \quad \text{in } \Omega. \tag{6.9}$$

We will use the following function space:

$$W^{2,p}_{loc}(\Omega) := \{u : \Omega \to \mathbf{R} \mid \zeta u \in W^{2,p}(\Omega) \text{ for all } \zeta \in C_0^\infty(\Omega)\}.$$

Throughout this section, we suppose at least

$$p > \frac{n}{2}$$

so that $u \in W^{2,p}_{loc}(\Omega)$ has the second-order Taylor expansion at almost all points in Ω, and that $u \in C(\Omega)$.

Definition. We call $u \in C(\Omega)$ an L^p-strong subsolution (resp., supersolution, solution) of (6.9) if $u \in W^{2,p}_{loc}(\Omega)$, and

$$F(x, u(x), Du(x), D^2u(x)) \le f(x) \quad (\text{resp., } \ge f(x), \ = f(x)) \quad a.e. \text{ in } \Omega.$$

Now, we present the definition of L^p-viscosity solutions of (6.9).

Definition. We call $u \in C(\Omega)$ an L^p-viscosity subsolution (resp., supersolution) of (6.9) if for $\phi \in W^{2,p}_{loc}(\Omega)$, we have

$$\lim_{\varepsilon \to 0} ess. \inf_{B_\varepsilon(x)} \big(F(y, u(y), D\phi(y), D^2\phi(y)) - f(y) \big) \le 0$$

$$\left(\text{resp. } \lim_{\varepsilon\to 0} ess.\sup_{B_\varepsilon(x)} \big(F(y,u(y),D\phi(y),D^2\phi(y)) - f(y)\big) \geq 0\right)$$

provided that $u-\phi$ takes its local maximum (resp., minimum) at $x\in\Omega$.

We call $u\in C(\Omega)$ an L^p-viscosity solution of (6.9) if it is both an L^p-viscosity sub- and supersolution of (6.9).

<u>Remark.</u> Although we will not explicitly utilize the above definition, we recall the definition of $ess.\sup_A$ and $ess.\inf_A$ of $h:A\to\mathbf{R}$, where $A\subset\mathbf{R}^n$ is a measurable set:

$$ess.\sup_A h(y) := \inf\{M\in\mathbf{R} \mid h\leq M \ a.e. \text{ in } A\},$$

and

$$ess.\inf_A h(y) := \sup\{M\in\mathbf{R} \mid h\geq M \ a.e. \text{ in } A\}.$$

Coming back to (6.1), we give a list of assumptions on $F:\Omega\times\mathbf{R}^n\times S^n\to\mathbf{R}$:

$$\begin{cases} (1) & F(x,0,O)=0 \text{ for } x\in\Omega,\\ (2) & x\to F(x,q,X) \text{ is measurable for } (q,X)\in\mathbf{R}^n\times S^n,\\ (3) & F \text{ is uniformly elliptic.}\end{cases} \tag{6.10}$$

We recall the uniform ellipticity condition of $X\to F(x,q,X)$ with the constants $0<\lambda\leq\Lambda$ from section 3.1.2.

For the right hand side $f:\Omega\to\mathbf{R}$, we suppose that

$$f\in L^p(\Omega) \quad \text{for } p\geq n. \tag{6.11}$$

We will often suppose the Lipschitz continuity of F with respect to $q\in\mathbf{R}^n$;

$$\begin{cases} \text{there is } \mu\geq 0 \text{ such that } |F(x,q,X)-F(x,q',X)|\leq\mu|q-q'| \\ \qquad\qquad \text{for } (x,q,q',X)\in\Omega\times\mathbf{R}^n\times\mathbf{R}^n\times S^n.\end{cases} \tag{6.12}$$

<u>Remark.</u> We note that (1) in (6.10) and (6.12) imply that F has the linear growth in Du;

$$|F(x,q,O)|\leq\mu|q| \quad \text{for } x\in\Omega \text{ and } q\in\mathbf{R}^n.$$

Remark. We note that when $x \to F(x,q,X)$ and $x \to f(x)$ are continuous, the definition of L^p-viscosity subsolution (resp., supersolution) of (6.1) coincides with the standard one under assumption (6.10) and (6.12). For a proof, we refer to a paper by Caffarelli-Crandall-Kocan-Święch [5].

In this book, we only study the case of (6.11) but most of results can be extended to the case when $p > p^\star = p^\star(\Lambda, \lambda, n) \in (n/2, n)$, where $p^\star$ is the so-called Escauriaza's constant (see the references in [4]).

The following proposition is obvious but it will be very convenient to study L^p-viscosity solutions of (6.1) under assumptions (6.10), (6.11) and (6.12).

Proposition 6.1. *Assume that* (6.10), (6.11) *and* (6.12) *hold. If* $u \in C(\Omega)$ *is an* L^p*-viscosity subsolution* (*resp., supersolution*) *of* (6.1), *then it is an* L^p*-viscosity subsolution* (*resp., supersolution*) *of*

$$\mathcal{P}^-(D^2u) - \mu|Du| \le f \quad \textit{in } \Omega$$

$$\left(\textit{resp., } \mathcal{P}^+(D^2u) + \mu|Du| \ge f \quad \textit{in } \Omega\right).$$

We recall the Aleksandrov-Bakelman-Pucci (ABP for short) maximum principle, which will play an essential role in this section (and also Appendix).

To this end, we introduce the notion of "upper contact sets": For $u : O \to \mathbf{R}$, we set

$$\Gamma[u,O] := \left\{ x \in O \;\middle|\; \begin{array}{c} \text{there is } p \in \mathbf{R}^n \text{ such that} \\ u(y) \le u(x) + \langle p, y-x\rangle \text{ for all } y \in O \end{array} \right\}.$$

Proposition 6.2. (ABP maximum principle) *For* $\mu \ge 0$, *there is* $C_0 := C_0(\Lambda, \lambda, n, \mu, \mathrm{diam}(\Omega)) > 0$ *such that if for* $f \in L^n(\Omega)$, $u \in C(\overline{\Omega})$ *is an* L^n*-viscosity subsolution* (*resp., supersolution*) *of*

$$\mathcal{P}^-(D^2u) - \mu|Du| \le f \quad \textit{in } \Omega^+[u]$$

$$(\textit{resp., } \mathcal{P}^+(D^2u) + \mu|Du| \ge f \quad \textit{in } \Omega^+[-u]),$$

then

$$\max_{\overline{\Omega}} u \le \max_{\partial\Omega} u^+ + \mathrm{diam}(\Omega) C_0 \|f^+\|_{L^n(\Gamma[u,\Omega]\cap\Omega^+[u])}$$

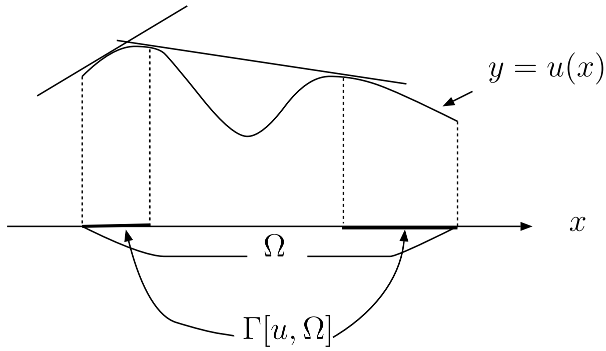

Fig 6.1

$$\left(\textit{resp.,} \quad \max_{\overline{\Omega}}(-u) \le \max_{\partial\Omega}(-u)^+ + \mathrm{diam}(\Omega) C_0 \|f^-\|_{L^n(\Gamma[-u,\Omega]\cap\Omega^+[-u])} \right),$$

where

$$\Omega^+[u] := \{x \in \Omega \mid u(x) > 0\}.$$

The next proposition is a key tool to study L^p-viscosity solutions, particularly, when f is not supposed to be continuous. The proof will be given in Appendix.

Proposition 6.3. *Assume that* (6.11) *holds for* $p \ge n$. *For any* $\mu \ge 0$, *there are an* L^p*-strong subsolution* u *and an* L^p*-strong supersolution* $v \in C(\overline{B}_1) \cap W^{2,p}_{loc}(B_1)$, *respectively, of*

$$\begin{cases} \mathcal{P}^+(D^2u) + \mu|Du| \le f & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1, \end{cases} \quad \text{and} \quad \begin{cases} \mathcal{P}^-(D^2v) - \mu|Dv| \ge f & \text{in } B_1, \\ v = 0 & \text{on } \partial B_1. \end{cases}$$

Moreover, we have the following estimates: for $w = u$ *or* $w = v$, *and small* $\delta \in (0,1)$, *there is* $\hat{C} = \hat{C}(\Lambda, \lambda, n, \mu, \delta) > 0$ *such that*

$$\|w\|_{W^{2,p}(B_\delta)} \le \hat{C}_\delta \|f\|_{L^p(B_1)}.$$

Remark. In view of the proof (Step 2) of Proposition 6.2, we see that $-C\|f^-\|_{L^n(B_1)} \le w \le C\|f^+\|_{L^n(B_1)}$ in B_1, where $w = u, v$.

6.3 Harnack inequality

In this subsection, we often use the cube $Q_r(x)$ for $r > 0$ and $x = {}^t(x_1, \ldots, x_n) \in \mathbf{R}^n$;

$$Q_r(x) := \{y = {}^t(y_1, \ldots, y_n) \mid |x_i - y_i| < r/2 \text{ for } i = 1, \ldots, n\},$$

and $Q_r := Q_r(0)$. Notice that

$$B_{r/2}(x) \subset Q_r(x) \subset B_{r\sqrt{n}/2}(x) \quad \text{for } r > 0.$$

We will prove the next two propositions in Appendix.

Proposition 6.4. (Weak Harnack inequality) *For $\mu \geq 0$, there are $p_0 = p_0(\Lambda, \lambda, n, \mu) > 0$ and $C_1 := C_1(\Lambda, \lambda, n, \mu) > 0$ such that if $u \in C(\overline{B}_{2\sqrt{n}})$ is a nonnegative L^p-viscosity supersolution of*

$$\mathcal{P}^+(D^2u) + \mu|Du| \geq 0 \quad \text{in } B_{2\sqrt{n}},$$

then we have

$$\|u\|_{L^{p_0}(Q_1)} \leq C_1 \inf_{Q_{1/2}} u.$$

Remark. Notice that p_0 might be smaller than 1.

Proposition 6.5. (Local maximum principle) *For $\mu \geq 0$ and $q > 0$, there is $C_2 = C_2(\Lambda, \lambda, n, \mu, q) > 0$ such that if $u \in C(\overline{B}_{2\sqrt{n}})$ is an L^p-viscosity subsolution of*

$$\mathcal{P}^-(D^2u) - \mu|Du| \leq 0 \quad \text{in } B_{2\sqrt{n}},$$

then we have

$$\sup_{Q_1} u \leq C_2 \|u^+\|_{L^q(Q_2)}.$$

Remark. Notice that we **do not** suppose that $u \geq 0$ in Proposition 6.5.

6.3.1 Linear growth

The next corollary is a direct consequence of Propositions 6.4 and 6.5.

Corollary 6.6. *For $\mu \geq 0$, there is $C_3 = C_3(\Lambda, \lambda, n, \mu) > 0$ such that if $u \in C(\overline{B}_{2\sqrt{n}})$ is a nonnegative L^p-viscosity sub- and supersolution of*

$$\mathcal{P}^-(D^2u) - \mu|Du| \leq 0 \quad \text{and} \quad \mathcal{P}^+(D^2u) + \mu|Du| \geq 0 \quad \text{in } B_{2\sqrt{n}},$$

respectively, then we have

$$\sup_{Q_1} u \leq C_3 \inf_{Q_1} u.$$

In order to treat inhomogeneous PDEs, we will need the following corollary:

Corollary 6.7. *For $\mu \geq 0$ and $f \in L^p(B_{3\sqrt{n}})$ with $p \geq n$, there is $C_4 = C_4(\Lambda, \lambda, n, \mu) > 0$ such that if $u \in C(\overline{B}_{3\sqrt{n}})$ is a nonnegative L^p-viscosity sub- and supersolution of*

$$\mathcal{P}^-(D^2u) - \mu|Du| \leq f \quad \text{and} \quad \mathcal{P}^+(D^2u) + \mu|Du| \geq f \quad \text{in } B_{2\sqrt{n}},$$

respectively, then we have

$$\sup_{Q_1} u \leq C_4 \left(\inf_{Q_1} u + \|f\|_{L^p(B_{3\sqrt{n}})} \right).$$

<u>*Proof.*</u> According to Proposition 6.3, we find $v,\ w \in C(\overline{B}_{3\sqrt{n}}) \cap W^{2,p}_{loc}(B_{3\sqrt{n}})$ such that

$$\begin{cases} \mathcal{P}^+(D^2v) + \mu|Dv| \leq -f^+ & a.e. \text{ in } B_{3\sqrt{n}}, \\ v = 0 & \text{on } \partial B_{3\sqrt{n}}, \end{cases}$$

and

$$\begin{cases} \mathcal{P}^-(D^2w) - \mu|Dw| \geq f^- & a.e. \text{ in } B_{3\sqrt{n}}, \\ w = 0 & \text{on } \partial B_{3\sqrt{n}}. \end{cases}$$

In view of Proposition 6.3 and its Remark, we can choose $\hat{C} = \hat{C}(\Lambda, \lambda, n, \mu) > 0$ such that

$$0 \leq -v \leq \hat{C}\|f^+\|_{L^p(B_{3\sqrt{n}})} \quad \text{in } B_{3\sqrt{n}}, \quad \|v\|_{W^{2,p}(B_{2\sqrt{n}})} \leq \hat{C}\|f^+\|_{L^p(B_{3\sqrt{n}})},$$

and

$$0 \leq w \leq \hat{C}\|f^-\|_{L^p(B_{3\sqrt{n}})} \quad \text{in } B_{3\sqrt{n}}, \quad \|w\|_{W^{2,p}(B_{2\sqrt{n}})} \leq \hat{C}\|f^-\|_{L^p(B_{3\sqrt{n}})}.$$

Since $v,\ w \in W^{2,p}(B_{2\sqrt{n}})$, it is easy to verify that $u_1 := u + v$ and $u_2 := u + w$ are, respectively, an L^p-viscosity sub- and supersolution of

$$\mathcal{P}^-(D^2u_1) - \mu|Du_1| \leq 0 \quad \text{and} \quad \mathcal{P}^+(D^2u_2) + \mu|Du_2| \geq 0 \quad \text{in } B_{2\sqrt{n}}.$$

Since $v \leq 0$ in $B_{3\sqrt{n}}$, applying Proposition 6.5 to u_1, for any $q > 0$, we find $C_2(q) > 0$ such that

$$\begin{aligned} \sup_{Q_1} u &\leq \sup_{Q_1} u_1 + \hat{C}\|f^+\|_{L^p(B_{3\sqrt{n}})} \\ &\leq C_2(q)\|(u_1)^+\|_{L^q(Q_2)} + \hat{C}\|f^+\|_{L^p(B_{3\sqrt{n}})} \\ &\leq C_2(q)\|u\|_{L^q(Q_2)} + \hat{C}\|f^+\|_{L^p(B_{3\sqrt{n}})}. \end{aligned} \tag{6.13}$$

On the other hand, applying Proposition 6.4 to u_2, there is $p_0 > 0$ such that

$$\|u\|_{L^{p_0}(Q_2)} \le \|u_2\|_{L^{p_0}(Q_2)} \le C_1 \inf_{Q_1} u_2 \le C_1 \left(\inf_{Q_1} u + \hat{C}\|f^-\|_{L^p(B_{3\sqrt{n}})} \right). \quad (6.14)$$

Therefore, combining (6.14) with (6.13) for $q = p_0$, we can find $C_4 > 0$ such that the assertion holds. □

Corollary 6.8. (Harnack inequality, final version) *Assume that* (6.10), (6.11) *and* (6.12) *hold. If* $u \in C(\Omega)$ *is an* L^p*-viscosity solution of* (6.1), *and if* $B_{3\sqrt{n}r}(x) \subset \Omega$ *for* $r \in (0,1]$, *then*

$$\sup_{Q_r(x)} u \le C_4 \left(\inf_{Q_r(x)} u + r^{2-\frac{n}{p}} \|f\|_{L^p(\Omega)} \right),$$

where $C_4 > 0$ *is the constant in Corollary 6.7.*

<u>*Proof.*</u> By translation, we may suppose that $x = 0$.

Setting $v(x) := u(rx)$ for $x \in B_{3\sqrt{n}}$, we easily see that v is an L^p-viscosity subsolution and supersolution of

$$\mathcal{P}^-(D^2 v) - \mu|Dv| \le r^2 \hat{f} \quad \text{and} \quad \mathcal{P}^+(D^2 v) + \mu|Dv| \ge -r^2 \hat{f}, \quad \text{in } B_{3\sqrt{n}},$$

respectively, where $\hat{f}(x) := f(rx)$. Note that $\|\hat{f}\|_{L^p(B_{3\sqrt{n}})} = r^{-\frac{n}{p}} \|f\|_{L^p(B_{3\sqrt{n}r})}$.

Applying Corollary 6.7 to v and then, rescaling v to u, we conclude the assertion. □

6.3.2 Quadratic growth

Here, we consider the case when $q \to F(x,q,X)$ has quadratic growth. We refer to [10] for applications where such quadratic nonlinearity appears.

We present a version of the Harnack inequality when F has a quadratic growth in Du in place of (6.12);

$$\left\{ \begin{array}{c} \text{there is } \mu \ge 0 \text{ such that } |F(x,q,X) - F(x,q',X)| \\ \le \mu(|q| + |q'|)|q - q'| \text{ for } (x,q,q',X) \in \Omega \times \mathbf{R}^n \times \mathbf{R}^n \times S^n, \end{array} \right. \quad (6.15)$$

which together with (1) of (6.10) implies that

$$|F(x,q,O)| \le \mu|q|^2 \quad \text{for } (x,q) \in \Omega \times \mathbf{R}^n.$$

The associated Harnack inequality is as follows:

Theorem 6.9. *For $\mu \geq 0$ and $f \in L^p(B_{3\sqrt{n}})$ with $p \geq n$, there is $C_5 = C_5(\Lambda, \lambda, n, \mu) > 0$ such that if $u \in C(\overline{B}_{3\sqrt{n}})$ is a nonnegative L^p-viscosity sub- and supersolution of*

$$\mathcal{P}^-(D^2u) - \mu|Du|^2 \leq f \quad \textit{and} \quad \mathcal{P}^+(D^2u) + \mu|Du|^2 \geq f \quad \textit{in } B_{3\sqrt{n}},$$

respectively, then we have

$$\sup_{Q_1} u \leq C_5 e^{\frac{2\mu}{\lambda}M} \left(\inf_{Q_1} u + \|f\|_{L^p(B_{3\sqrt{n}})} \right),$$

where $M := \sup_{B_{3\sqrt{n}}} u$.

Proof. Set $\alpha := \mu/\lambda$. Fix any $\delta \in (0,1)$.

We claim that $v := e^{\alpha u} - 1$ and $w := 1 - e^{-\alpha u}$ are, respectively, a nonnegative L^p-viscosity sub- and supersolution of

$$\mathcal{P}^-(D^2v) \leq \alpha(e^{\alpha M} + \delta)f^+ \quad \text{and} \quad \mathcal{P}^+(D^2w) \geq -\alpha(1+\delta)f^- \quad \text{in } B_{3\sqrt{n}}.$$

We shall only prove this claim for v since the other for w can be obtained similarly.

Choose $\phi \in W^{2,p}_{loc}(B_{3\sqrt{n}})$ and suppose that $u - \phi$ attains its local maximum at $x \in B_{3\sqrt{n}}$. Thus, we may suppose that $v(x) = \phi(x)$ and $v \leq \phi$ in $B_r(x)$, where $B_{2r}(x) \subset B_{3\sqrt{n}}$. Note that $0 \leq v \leq e^{\alpha M} - 1$ in $B_{3\sqrt{n}}$.

For any $\delta \in (0,1)$, in view of $W^{2,p}(B_r(x)) \subset C^{\sigma_0}(\overline{B}_r(x))$ with some $\sigma_0 \in (0,1)$, we can choose $\varepsilon_0 \in (0,r)$ such that

$$-\delta \leq \phi \leq v + \delta \quad \text{in } B_{\varepsilon_0}(x).$$

Setting $\psi(y) := \alpha^{-1} \log(\phi(y) + 1)$ for $y \in B_{\varepsilon_0}(x)$ (extending $\psi \in W^{2,p}$ in $B_{3\sqrt{n}} \setminus B_{\varepsilon_0}(x)$ if necessary), we have

$$\lim_{\varepsilon \to 0} ess \inf_{B_\varepsilon(x)} \left(\mathcal{P}^-(D^2\psi) - \mu|D\psi|^2 - f^+ \right) \leq 0.$$

Since

$$D\psi = \frac{D\phi}{\alpha(\phi+1)} \quad \text{and} \quad D^2\psi = \frac{D^2\phi}{\alpha(\phi+1)} - \frac{D\phi \otimes D\phi}{\alpha(\phi+1)^2},$$

the above inequality yields

$$\lim_{\varepsilon\to 0} ess \inf_{B_\varepsilon(x)} \left(\frac{\mathcal{P}^-(D^2\phi)}{\alpha(\phi+1)} - f^+ \right) \le 0.$$

Since $0 < 1-\delta \le \phi+1 \le e^{\alpha M}+\delta$ in $B_{\varepsilon_0}(x)$, we have

$$\lim_{\varepsilon\to 0} ess \inf_{B_\varepsilon(x)} \left(\mathcal{P}^-(D^2\phi) - \alpha(e^{\alpha M}+\delta)f^+ \right) \le 0.$$

Since $\alpha u \le v \le \alpha u e^{\alpha M}$ and $\alpha u e^{-\alpha M} \le w \le \alpha u$, using the same argument to get (6.13) and (6.14), we have

$$\begin{aligned} \sup_{Q_1} u \le \frac{1}{\alpha}\sup_{Q_1} v \quad & \le C_6 \left\{ \|v\|_{L^{p_0}(Q_2)} + (e^{\alpha M}+\delta)\|f^+\|_{L^p(B_{3\sqrt{n}})} \right\} \\ & \le C_7 \left\{ e^{2\alpha M}\|w\|_{L^{p_0}(Q_2)} + (e^{\alpha M}+\delta)\|f^+\|_{L^p(B_{3\sqrt{n}})} \right\} \\ & \le C_8 \left\{ e^{2\alpha M}\inf_{Q_1} w + (e^{2\alpha M}+\delta)\|f\|_{L^p(B_{3\sqrt{n}})} \right\} \\ & \le C_9 e^{2\alpha M} \left\{ \inf_{Q_1} u + (1+\delta)\|f\|_{L^p(B_{3\sqrt{n}})} \right\}. \end{aligned}$$

Since C_k $(k = 6, \ldots, 9)$ are independent of $\delta > 0$, sending $\delta \to 0$, we conclude the proof. □

Remark. We note that the same argument by using two different transformations for sub- and supersolutions as above can be found in [14] for uniformly elliptic PDEs in divergence form with the quadratic nonlinearity.

6.4 Hölder continuity estimates

In this subsection, we show how the Harnack inequality implies the Hölder continuity.

Theorem 6.10. *Assume that* (6.10), (6.11) *and* (6.12) *hold. For each compact set* $K \subset \Omega$, *there is* $\sigma = \sigma(\Lambda, \lambda, n, \mu, p, \mathrm{dist}(K, \partial\Omega), \|f\|_{L^p(\Omega)}) \in (0,1)$ *such that if* $u \in C(\Omega)$ *is an* L^p*-viscosity solution of* (6.1), *then there is* $\hat{C} = \hat{C}(\Lambda, \lambda, n, \mu, p, \mathrm{dist}(K, \partial\Omega), \max_{\overline{\Omega}}|u|, \|f\|_{L^p(\Omega)}) > 0$

$$|u(x) - u(y)| \le \hat{C}|x-y|^\sigma \quad \text{for } x, y \in K.$$

Remark. We notice that σ is independent of $\sup_\Omega |u|$.
In our proof below, we may relax the dependence $\max_{\overline{\Omega}} |u|$ in $\hat{C}$ by

$$\sup\{|u(x)| \mid \mathrm{dist}(x,K) < \varepsilon\} \quad \text{for small } \varepsilon > 0.$$

Proof. Setting $r_0 := \min\{1, \mathrm{dist}(K,\partial\Omega)/(3\sqrt{n})\} > 0$, we may suppose that there is $C_4 > 1$ such that if $w \in C(\Omega)$ is a **nonnegative** L^p-viscosity sub- and supersolution of

$$\mathcal{P}^-(D^2w) - \mu|Dw| \le f \quad \text{and} \quad \mathcal{P}^+(D^2w) + \mu|Dw| \ge f \quad \text{in } \Omega,$$

respectively, then we see that for any $r \in (0, r_0]$ and $x \in K$ (*i.e.* $B_{3\sqrt{n}r}(x) \subset \Omega$),

$$\sup_{Q_r(x)} w \le C_4 \left(\inf_{Q_r(x)} w + r^{2-\frac{n}{p}} \|f\|_{L^p(\Omega)} \right).$$

For simplicity, we may suppose $x = 0 \in K$.
Now, we set

$$M(r) := \sup_{Q_r} u, \quad m(r) := \inf_{Q_r} u \quad \text{and} \quad osc(r) := M(r) - m(r).$$

It is sufficient to find $C > 0$ and $\sigma \in (0,1)$ such that

$$M(r) - m(r) \le Cr^\sigma \quad \text{for small } r > 0.$$

We denote by $S(r)$ the set of all nonnegative $w \in C(\overline{B}_{3\sqrt{n}r})$, which is, respectively, an L^p-viscosity sub- and supersolution of

$$\mathcal{P}^-(D^2w) - \mu|Dw| \le |f| \quad \text{and} \quad \mathcal{P}^+(D^2w) + \mu|Dw| \ge -|f| \quad \text{in } B_{3\sqrt{n}r}.$$

Setting $v_1 := u - m(r)$ and $w_1 := M(r) - u$, we see that v_1 and w_1 belong to $S(r)$. Hence, setting $C_{10} := \max\{C_4\|f\|_{L^p(\Omega)}, C_4, 4\} > 3$, we have

$$\sup_{Q_{r/2}} v_1 \le C_{10} \left(\inf_{Q_{r/2}} v_1 + r^{2-\frac{n}{p}} \right) \quad \text{and} \quad \sup_{Q_{r/2}} w_1 \le C_{10} \left(\inf_{Q_{r/2}} w_1 + r^{2-\frac{n}{p}} \right).$$

Thus, setting $\beta := 2 - \frac{n}{p} > 0$, we have

$$M(r/2) - m(r) \le C_{10} \left(m(r/2) - m(r) + (r/2)^\beta \right),$$

$$M(r) - m(r/2) \le C_{10} \left(M(r) - M(r/2) + (r/2)^\beta \right).$$

Hence, adding these inequalities, we have

$$(C_{10}+1)(M(r/2)-m(r/2)) \le (C_{10}-1)(M(r)-m(r)) + 2C_{10}(r/2)^{\beta}.$$

Therefore, setting $\theta := (C_{10}-1)/(C_{10}+1) \in (1/2,1)$ and $C_{11} := 2C_{10}/(C_{10}+1)$, we see that

$$osc(r/2) \le \theta osc(r) + C_{11}(r/2)^{\beta}.$$

Moreover, changing $r/2^{k-1}$ for integers $k \ge 2$, we have

$$\begin{aligned} osc(r/2^k) &\le \theta^k osc(r) + C_{11} r^{\beta} \sum_{j=1}^{k} 2^{-\beta j} \\ &\le \theta^k osc(r_0) + \frac{C_{11}}{2^{\beta}-1} r^{\beta} \qquad \le C_{12}(\theta^k + r^{\beta}), \end{aligned}$$

where $C_{12} := \max\{osc(r_0), C_{11}/(2^{\beta}-1)\}$.

For $r \in (0, r_0)$, by setting $s = r^{\alpha}$, where $\alpha = \log\theta/(\log\theta - \beta\log 2) \in (0,1)$, there is a unique integer $k \ge 1$ such that

$$\frac{s}{2^k} \le r < \frac{s}{2^{k-1}},$$

which yields

$$\frac{\log(s/r)}{\log 2} \le k < \frac{\log(s/r)}{\log 2} + 1.$$

Hence, recalling $\theta \in (1/2,1)$, we have

$$osc(r) \le osc(s/2^{k-1}) \le C_{12}(\theta^k + (2s)^{\beta}) \le 2^{\beta} C_{12} \left(\theta^{(\alpha-1)\log r/\log 2} + r^{\beta\alpha}\right).$$

Setting $\sigma := (\alpha-1)\log\theta/\log 2 \in (0,1)$ (because $\theta \in (1/2,1)$), we have

$$\theta^{(\alpha-1)\log r/\log 2} = r^{\sigma} \quad \text{and} \quad r^{\beta\alpha} = r^{\sigma}.$$

Thus, setting $C_{13} := 2^{\beta} C_{12}$, we have

$$osc(r) \le C_{13} r^{\sigma}. \quad \Box \tag{6.16}$$

Remark. We note that we may derive (6.16) when $p > n/2$ by taking $\beta = 2 - \frac{n}{p} > 0$.

We shall give the corresponding Hölder continuity for PDEs with quadratic nonlinearity (6.15). Since we can use the same argument as in the proof of

Theorem 6.1 using Theorem 6.9 instead of Corollaries 6.7 and 6.8, we omit the proof of the following:

Corollary 6.11. *Assume that* (6.10), (6.11) *and* (6.15) *hold. For each compact set* $K \subset \Omega$, *there are* $\hat{C} = \hat{C}(\Lambda, \lambda, n, \mu, p, \mathrm{dist}(K, \partial\Omega), \sup_\Omega |u|) > 0$ *and* $\sigma = \sigma(\Lambda, \lambda, n, \mu, p, \mathrm{dist}(K, \partial\Omega), \sup_\Omega |u|) \in (0, 1)$ *such that if an* L^p-*viscosity solution* $u \in C(\Omega)$ *of* (6.1), *then we have*

$$|u(x) - u(y)| \le \hat{C}|x - y|^\sigma \quad \textit{for } x, y \in K.$$

Remark. Note that both of σ and $\hat{C}$ depend on $\sup_\Omega |u|$ in this quadratic case.

6.5 Existence result

For the existence of L^p-viscosity solutions of (6.1) under the Dirichlet condition, we only give an outline of proof, which was first shown in a paper by Crandall-Kocan-Lions-Święch in [7] (1999).

Theorem 6.12. *Assume that* (6.10), (6.11) *and* (6.12) *hold. Assume also that* (1) *of* (5.17) *holds.*

For given $g \in C(\partial\Omega)$, *there is an* L^p-*viscosity solution* $u \in C(\overline{\Omega})$ *of* (6.1) *such that*

$$u(x) = g(x) \quad \textit{for } x \in \partial\Omega. \tag{6.17}$$

Remark. We may relax assumption (1) of (5.17) so that the assertion holds for Ω which may have some "concave" corners. Such a condition is called "uniform exterior cone condition".

Sketch of proof.

Step1: We first solve approximate PDEs, which have to satisfy a sufficient condition in Step 3; instead of (6.1), under (6.17), we consider

$$F_k(x, Du, D^2u) = f_k \quad \text{in } \Omega, \tag{6.18}$$

where "smooth" F_k and f_k approximate F and f, respectively. In fact, F_k and f_k are given by $F * \rho_{1/k}$ and $f * \rho_{1/k}$, where $\rho_{1/k}$ is the standard mollifier with respect to x-variables. We remark that $F * \rho_{1/k}$ means the convolution of $F(\cdot, p, X)$ and $\rho_{1/k}$.

We find a viscosity solution $u_k \in C(\overline{\Omega})$ of (6.18) under (6.17) via Perron's method for instance. At this stage, we need to suppose the smoothness of $\partial\Omega$ to construct viscosity sub- and supersolutions of (6.18) with (6.17). Remember that if F and f are continuous, then the notion of L^p-viscosity solutions equals to that of the standard ones (see Proposition 2.9 in [5]).

In view of (1) of (5.17) (*i.e.* the uniform exterior sphere condition), we can construct viscosity sub- and supersolutions of (6.18) denoted by $\xi \in USC(\overline{\Omega})$ and $\eta \in LSC(\overline{\Omega})$ such that $\xi = \eta = g$ on $\partial\Omega$. To show this fact, we only note that we can modify the argument in Step 1 in section 7.3.

Step 2: We next obtain the a priori estimates for u_k so that they converge to a continuous function $u \in C(\overline{\Omega})$, which is the candidate of the original PDE.

For this purpose, after having established the L^∞ estimates via Proposition 6.2, we apply Theorem 6.10 (interior Hölder continuity) to u_k in Step 1 because (6.10)-(6.12) hold for approximate PDEs with the same constants λ, Λ, μ.

We need a careful analysis to get the equi-continuity up to the boundary $\partial\Omega$. See Step 1 in section 7.3 again.

Step 3: Finally, we verify that the limit function u is the L^p-viscosity solution via the following stability result, which is an L^p-viscosity version of Proposition 4.8.

To state the result, we introduce some notations: For $B_{2r}(x) \subset \Omega$ with $r > 0$ and $x \in \Omega$, and $\phi \in W^{2,p}(B_r(x))$, we set

$$G_k[\phi](y) := F_k(y, D\phi(y), D^2\phi(y)) - f_k(y),$$

and

$$G[\phi](y) := F(y, D\phi(y), D^2\phi(y)) - f(y)$$

for $y \in B_r(x)$.

Proposition 6.13. *Assume that F_k and F satisfy* (6.10) *and* (6.12) *with $\lambda, \Lambda > 0$ and $\mu \geq 0$. For $f, f_k \in L^p(\Omega)$ with $p \geq n$, let $u_k \in C(\Omega)$ be an L^p-viscosity subsolution* (*resp., supersolution*) *of* (6.18). *Assume also that u_k converges to u uniformly on any compact subsets of Ω as $k \to \infty$, and that for any $B_{2r}(x) \subset \Omega$ with $r > 0$ and $x \in \Omega$, and $\phi \in W^{2,p}(B_r(x))$,*

$$\lim_{k\to\infty} \|(G[\phi] - G_k[\phi])^+\|_{L^p(B_r(x))} = 0$$

$$\left(\text{resp., } \lim_{k\to\infty} \|(G[\phi] - G_k[\phi])^-\|_{L^p(B_r(x))} = 0\right).$$

Then, $u \in C(\Omega)$ is an L^p-viscosity subsolution (*resp., supersolution*) *of* (6.1).

Proof of Proposition 6.13. We only give a proof of the assertion for subsolutions.

Suppose the contrary: There are $r > 0$, $\varepsilon > 0$, $x \in \Omega$ and $\phi \in W^{2,p}(B_{2r}(x))$ such that $B_{3r}(x) \subset \Omega$, $0 = (u - \phi)(x) \geq (u - \phi)(y)$ for $y \in B_{2r}(x)$, and

$$u - \phi \leq -\varepsilon \quad \text{in } B_{2r}(x) \setminus B_r(x), \tag{6.19}$$

and

$$G[\phi](y) \geq \varepsilon \quad a.e. \text{ in } B_{2r}(x). \tag{6.20}$$

For simplicity, we shall suppose that $r = 1$ and $x = 0$.

It is sufficient to find $\phi_k \in W^{2,p}(B_2)$ such that $\lim_{k\to\infty} \sup_{B_2} |\phi_k| = 0$, and

$$G_k[\phi + \phi_k](y) \geq \varepsilon \quad a.e. \text{ in } B_2.$$

Indeed, since $u_k - (\phi + \phi_k)$ attains its maximum over B_2 at an interior point $z \in B_2$ by (6.19), the above inequality contradicts the fact that u_k is an L^p-viscosity subsolution of (6.18).

Setting $h(x) := G[\phi](x)$ and $h_k(x) := G_k[\phi](x)$, in view of Proposition 6.3, we can find $\phi_k \in C(\overline{B}_2) \cap W^{2,p}_{loc}(B_2)$ such that

$$\begin{cases} \mathcal{P}^-(D^2\phi_k) - \mu|D\phi_k| \geq (h - h_k)^+ & a.e. \text{ in } B_2, \\ \phi_k = 0 & \text{on } \partial B_2, \\ 0 \leq \phi_k \leq C\|(h - h_k)^+\|_{L^p(B_2)} & \text{in } B_2, \\ \|\phi_k\|_{W^{2,p}(B_1)} \leq C\|(h - h_k)^+\|_{L^p(B_2)}. & \end{cases}$$

We note that our assumption together with the third inequality in the above yields $\lim_{k\to\infty} \sup_{B_2} |\phi_k| = 0$.

Using (6.10), (6.12) and (6.20), we have

$$\begin{aligned} G_k[\phi + \phi_k] &\geq \mathcal{P}^-(D^2\phi_k) - \mu|D\phi_k| + h_k \\ &\geq (h - h_k)^+ + \varepsilon - (h - h_k) \\ &\geq \varepsilon \quad a.e. \text{ in } B_2. \quad \square \end{aligned}$$