5 Generalized boundary value problems

In order to obtain the uniqueness of solutions of an ODE, we have to suppose certain initial or boundary condition. In the study of PDEs, we need to impose appropriate conditions on $\partial\Omega$ for the uniqueness of solutions.

Following the standard PDE theory, we shall treat a few typical boundary conditions in this section.

Since we are mainly interested in degenerate elliptic PDEs, we **cannot** expect "solutions" to satisfy the given boundary condition on the whole of $\partial\Omega$. The simplest example is as follows: For $\Omega := (0,1)$, consider the "degenerate" elliptic PDE

$$-\frac{du}{dx} + u = 0 \quad \text{in } (0,1).$$

Note that it is impossible to find a solution u of the above such that $u(0) = u(1) = 1$.

Our plan is to propose a definition of "generalized" solutions for boundary value problems. For this purpose, we extend the notion of viscosity solutions to possibly discontinuous PDEs on $\overline{\Omega}$ while we normally consider those in Ω.

For general $G : \overline{\Omega} \times \mathbf{R} \times \mathbf{R}^n \times S^n \to \mathbf{R}$, we are concerned with

$$G(x, u, Du, D^2u) = 0 \quad \text{in } \overline{\Omega}. \tag{5.1}$$

As in section 4.3, we define

$$G_*(x,r,p,X) := \liminf_{\varepsilon\to 0} \left\{ G(y,s,q,Y) \;\middle|\; \begin{array}{c} y \in \overline{\Omega} \cap B_\varepsilon(x), |s-r| < \varepsilon, \\ |q-p| < \varepsilon, \|Y - X\| < \varepsilon \end{array} \right\},$$

$$G^*(x,r,p,X) := \limsup_{\varepsilon\to 0} \left\{ G(y,s,q,Y) \;\middle|\; \begin{array}{c} y \in \overline{\Omega} \cap B_\varepsilon(x), |s-r| < \varepsilon, \\ |q-p| < \varepsilon, \|Y - X\| < \varepsilon \end{array} \right\}.$$

$\boxed{\textbf{Definition.}}$ We call $u : \overline{\Omega} \to \mathbf{R}$ a viscosity subsolution (resp., supersolution) of (5.1) if, for any $\phi \in C^2(\overline{\Omega})$,

$$G_*(x, u^*(x), D\phi(x), D^2\phi(x)) \le 0$$

$$\big(\text{resp., } G^*(x, u_*(x), D\phi(x), D^2\phi(x)) \ge 0\big)$$

provided that $u^* - \phi$ (resp., $u_* - \phi$) attains its maximum (resp., minimum) at $\underline{x \in \overline{\Omega}}$.

We call $u : \overline{\Omega} \to \mathbf{R}$ a viscosity solution of (5.1) if it is both a viscosity sub- and supersolution of (5.1).

Our comparison principle in this setting is as follows:

"Comparison principle in this setting"

$$\boxed{\left.\begin{array}{l}\textbf{viscosity subsolution } u \textbf{ of (5.1)}\\ \textbf{viscosity supersolution } v \textbf{ of (5.1)}\end{array}\right\} \Longrightarrow u \le v \textbf{ in } \overline{\Omega}}$$

Note that

the boundary condition is contained in the definition.

Using the above new definition, we shall formulate the boundary value problems in the viscosity sense. Given $F : \overline{\Omega} \times \mathbf{R} \times \mathbf{R}^n \times S^n \to \mathbf{R}$ and $B : \partial\Omega \times \mathbf{R} \times \mathbf{R}^n \times S^n \to \mathbf{R}$, we investigate general boundary value problems

$$\begin{cases} F(x, u, Du, D^2u) = 0 & \text{in } \Omega, \\ B(x, u, Du, D^2u) = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.2}$$

Setting G by

$$G(x, r, p, X) := \begin{cases} F(x, r, p, X) & \text{for } x \in \Omega, \\ B(x, r, p, X) & \text{for } x \in \partial\Omega, \end{cases}$$

we give the definition of boundary value problems (5.2) in the viscosity sense.

Definition. We call $u : \overline{\Omega} \to \mathbf{R}$ a viscosity subsolution (resp., supersolution) of (5.2) if it is a viscosity subsolution (resp., supersolution) of (5.1), where G is defined in the above.

We call $u : \overline{\Omega} \to \mathbf{R}$ a viscosity solution of (5.2) if it is both a viscosity sub- and supersolution of (5.2).

Remark. When F and B are continuous and G is given as above, G_* and G^* can be expressed in the following manner:

$$G_*(x, r, p, X) = \begin{cases} F(x, r, p, X) & \text{for } x \in \Omega, \\ \min\{F(x, r, p, X), B(x, r, p, X)\} & \text{for } x \in \partial\Omega, \end{cases}$$
$$G^*(x, r, p, X) = \begin{cases} F(x, r, p, X) & \text{for } x \in \Omega, \\ \max\{F(x, r, p, X), B(x, r, p, X)\} & \text{for } x \in \partial\Omega. \end{cases}$$

It is not hard to extend the existence and stability results corresponding to Theorem 4.3 and Proposition 4.8, respectively, to viscosity solutions in the above sense. However, it is not straightforward to show the comparison principle in this new setting. Thus, we shall concentrate our attention to the comparison principle, which implies the uniqueness (and continuity) of viscosity solutions.

The main difficulty to prove the comparison principle is that we have to "avoid" the boundary conditions for both of viscosity sub- and supersolutions.

To explain this, let us consider the case when G is given by (5.2). Let u and v be, respectively, a viscosity sub- and supersolution of (5.1). We shall observe that the standard argument in Theorem 3.7 does not work.

For $\varepsilon > 0$, suppose that $(x, y) \to u(x) - v(y) - (2\varepsilon)^{-1}|x - y|^2$ attains its maximum at $(x_\varepsilon, y_\varepsilon) \in \overline{\Omega} \times \overline{\Omega}$. Notice that there is **NO** reason to verify that $(x_\varepsilon, y_\varepsilon) \in \Omega \times \Omega$.

The worst case is that $(x_\varepsilon, y_\varepsilon) \in \partial\Omega \times \partial\Omega$. In fact, in view of Lemma 3.6, we find $X, Y \in S^n$ such that $((x_\varepsilon - y_\varepsilon)/\varepsilon, X) \in \overline{J}^{2,+}_{\overline{\Omega}} u(x_\varepsilon)$, $((x_\varepsilon - y_\varepsilon)/\varepsilon, Y) \in \overline{J}^{2,-}_{\overline{\Omega}} v(y_\varepsilon)$, the matrix inequalities in Lemma 3.6 hold for X, Y. Hence, we have

$$\min\left\{F\left(x_\varepsilon, u(x_\varepsilon), \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, X\right), B\left(x_\varepsilon, u(x_\varepsilon), \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, X\right)\right\} \leq 0$$

and

$$\max\left\{F\left(y_\varepsilon, v(y_\varepsilon), \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, Y\right), B\left(y_\varepsilon, v(y_\varepsilon), \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, Y\right)\right\} \geq 0.$$

However, even if we suppose that (3.21) holds for F and B "in $\overline{\Omega}$", we **cannot** get any contradiction when

$$F\left(x_\varepsilon, u(x_\varepsilon), \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, X\right) \leq 0 \leq B\left(y_\varepsilon, v(y_\varepsilon), \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, Y\right)$$

or

$$B\left(x_\varepsilon, u(x_\varepsilon), \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, X\right) \leq 0 \leq F\left(y_\varepsilon, v(y_\varepsilon), \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, Y\right).$$

It seems impossible to avoid this difficulty as long as we use $|x - y|^2/(2\varepsilon)$ as "test functions".

Our plan to go beyond this difficulty is to find new test functions $\phi_\varepsilon(x,y)$ (instead of $|x-y|^2/(2\varepsilon)$) so that the function $(x,y)\to u(x)-v(y)-\phi_\varepsilon(x,y)$ attains its maximum over $\overline{\Omega}\times\overline{\Omega}$ at an interior point $(x_\varepsilon,y_\varepsilon)\in\Omega\times\Omega$. To this end, since we will use several "perturbation" techniques, we suppose two hypotheses on F: First, we shall suppose the following continuity of F with respect to (p,X)-variables.

$$\left\{\begin{array}{c}\text{There is an } \omega_0\in\mathcal{M} \text{ such that}\\ |F(x,p,X)-F(x,q,Y)|\le\omega_0(|p-q|+\|X-Y\|)\\ \text{for } x\in\overline{\Omega}, p,q\in\mathbf{R}^n, X,Y\in S^n.\end{array}\right. \tag{5.3}$$

The next assumption is a bit stronger than the structure condition (3.21):

$$\left\{\begin{array}{c}\text{There is } \hat{\omega}_F\in\mathcal{M} \text{ such that}\\ \text{if } X,Y\in S^n \text{ and } \mu>1 \text{ satisfy}\\ -3\mu\begin{pmatrix} I & 0\\ 0 & I\end{pmatrix}\le\begin{pmatrix} X & 0\\ 0 & -Y\end{pmatrix}\le 3\mu\begin{pmatrix} I & -I\\ -I & I\end{pmatrix}, \text{ then}\\ F(y,p,Y)-F(x,p,X)\le\hat{\omega}_F(|x-y|(1+|p|+\mu|x-y|))\\ \text{for } x,y\in\overline{\Omega}, p\in\mathbf{R}^n, X,Y\in S^n.\end{array}\right. \tag{5.4}$$

5.1 Dirichlet problem

First, we consider Dirichlet boundary value problems (Dirichlet problems for short) in the above sense.

Assuming that viscosity sub- and supersolutions are continuous on $\partial\Omega$, we will obtain the comparison principle for them.

We now recall the classical Dirichlet problem

$$\left\{\begin{array}{rl}\nu u+F(x,Du,D^2u)=0 & \text{in } \Omega,\\ u-g=0 & \text{on } \partial\Omega.\end{array}\right. \tag{5.5}$$

Note that the Dirichlet problem of (5.5) in the viscosity sense is as follows:

$$\text{subsolution} \iff \left\{\begin{array}{cl}\nu u+F(x,Du,D^2u)\le 0 & \text{in } \Omega,\\ \min\{\nu u+F(x,Du,D^2u),u-g\}\le 0 & \text{on } \partial\Omega,\end{array}\right.$$

and

$$\text{supersolution} \iff \left\{\begin{array}{cl}\nu u+F(x,Du,D^2u)\ge 0 & \text{in } \Omega,\\ \max\{\nu u+F(x,Du,D^2u),u-g\}\ge 0 & \text{on } \partial\Omega.\end{array}\right.$$

We shall suppose the following property on the shape of Ω, which may be called an "interior cone condition" (see Fig 5.1):

$$\begin{cases} \text{For each } z \in \partial\Omega, \text{ there are } \hat{r}, \hat{s} \in (0,1) \text{ such that} \\ x - r\mathbf{n}(z) + r\xi \in \Omega \text{ for } x \in \overline{\Omega} \cap B_{\hat{r}}(z), r \in (0,\hat{r}) \text{ and } \xi \in B_{\hat{s}}(0). \end{cases} \tag{5.6}$$

Here and later, we denote by $\mathbf{n}(z)$ the unit outward normal vector at $z \in \partial\Omega$.

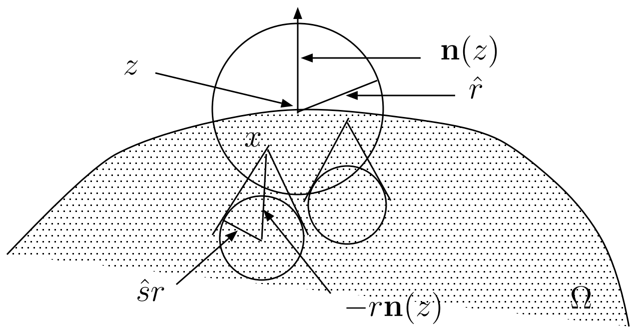

Fig 5.1

Theorem 5.1. *Assume that* $\nu > 0$, (5.3), (5.4) *and* (5.6) *hold. For* $g \in C(\partial\Omega)$, *we let* u *and* $v : \overline{\Omega} \to \mathbf{R}$ *be, respectively, a viscosity sub- and supersolution of* (5.5) *such that*

$$\liminf_{x \in \overline{\Omega} \to z} u^*(x) \geq u^*(z) \quad \textit{and} \quad \limsup_{x \in \overline{\Omega} \to z} v_*(x) \leq v_*(z) \quad \textit{for } z \in \partial\Omega. \tag{5.7}$$

Then, $u^* \leq v_*$ *in* $\overline{\Omega}$.

<u>Remark.</u> Notice that (5.7) implies the continuity of u^* and v_* on $\partial\Omega$.

<u>Proof.</u> Suppose that $\max_{\overline{\Omega}}(u^* - v_*) =: \theta > 0$. We simply write u and v for u^* and v_*, respectively.

<u>Case 1: $\max_{\partial\Omega}(u - v) = \theta$.</u> We choose $z \in \partial\Omega$ such that $(u - v)(z) = \theta$. We shall divide three cases:

<u>Case 1-1: $u(z) > g(z)$.</u> For $\varepsilon, \delta \in (0,1)$, where $\delta > 0$ will be fixed later, setting $\phi(x,y) := (2\varepsilon^2)^{-1}|x - y - \varepsilon\delta\mathbf{n}(z)|^2 - \delta|x - z|^2$, we let $(x_\varepsilon, y_\varepsilon) \in \overline{\Omega} \times \overline{\Omega}$ be the maximum point of $\Phi(x,y) := u(x) - v(y) - \phi(x,y)$ over $\overline{\Omega} \times \overline{\Omega}$.

Since $z-\varepsilon\delta\mathbf{n}(z) \in \overline{\Omega}$ for small $\varepsilon > 0$ by (5.6), $\Phi(x_\varepsilon, y_\varepsilon) \geq \Phi(z, z-\varepsilon\delta\mathbf{n}(z))$ implies that

$$\frac{|x_\varepsilon - y_\varepsilon - \varepsilon\delta\mathbf{n}(z)|^2}{2\varepsilon^2} \leq u(x_\varepsilon) - v(y_\varepsilon) - u(z) + v(z-\varepsilon\delta\mathbf{n}(z)) - \delta|x_\varepsilon - z|^2. \quad (5.8)$$

Since $|x_\varepsilon - y_\varepsilon| \leq M\varepsilon$, where $M := \sqrt{2}(\max_{\overline{\Omega}} u - \min_{\overline{\Omega}} v - u(z) + v(z) + 1)^{1/2}$, for small $\varepsilon > 0$, we may suppose that $(x_\varepsilon, y_\varepsilon) \to (\hat{x}, \hat{x})$ and $(x_\varepsilon - y_\varepsilon)/\varepsilon \to \hat{z}$ for some $\hat{x} \in \overline{\Omega}$ and $\hat{z} \in \mathbf{R}^n$ as $\varepsilon \to 0$ along a subsequence (denoted by ε again). Thus, from the continuity (5.7) of v at $z \in \partial\Omega$, (5.8) implies that

$$\theta \leq u(\hat{x}) - v(\hat{x}) - \delta|\hat{x} - z|^2,$$

which yields $\hat{x} = z$. Moreover, we have

$$\lim_{\varepsilon \to 0} \frac{|x_\varepsilon - y_\varepsilon - \varepsilon\delta\mathbf{n}(z)|^2}{\varepsilon^2} = 0,$$

which implies that

$$\lim_{\varepsilon \to 0} \frac{|x_\varepsilon - y_\varepsilon|}{\varepsilon} = \delta. \quad (5.9)$$

Furthermore, we note that $y_\varepsilon = x_\varepsilon - \varepsilon\delta\mathbf{n}(z) + o(\varepsilon) \in \Omega$ because of (5.6).

Applying Lemma 3.6 with Proposition 2.7 to $u(x) + \varepsilon^{-1}\delta\langle \mathbf{n}(z), x\rangle - \delta|x - z|^2 - 2^{-1}\delta^2$ and $v(y) + \varepsilon^{-1}\delta\langle \mathbf{n}(z), y\rangle$, we find $X, Y \in S^n$ such that

$$\left(\frac{x_\varepsilon - y_\varepsilon}{\varepsilon^2} - \frac{\delta}{\varepsilon}\mathbf{n}(z) + 2\delta(x_\varepsilon - z), X + 2\delta I\right) \in \overline{J}_{\overline{\Omega}}^{2,+} u(x_\varepsilon), \quad (5.10)$$

$$\left(\frac{x_\varepsilon - y_\varepsilon}{\varepsilon^2} - \frac{\delta}{\varepsilon}\mathbf{n}(z), Y\right) \in \overline{J}_{\overline{\Omega}}^{2,-} v(y_\varepsilon), \quad (5.11)$$

and

$$-\frac{3}{\varepsilon^2}\begin{pmatrix} I & O \\ O & I \end{pmatrix} \leq \begin{pmatrix} X & O \\ O & -Y \end{pmatrix} \leq \frac{3}{\varepsilon^2}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

Putting $p_\varepsilon := \varepsilon^{-2}(x_\varepsilon - y_\varepsilon) - \delta\varepsilon^{-1}\mathbf{n}(z)$, by (5.3), we have

$$F(x_\varepsilon, p_\varepsilon, X) - F(x_\varepsilon, p_\varepsilon + 2\delta(x_\varepsilon - z), X + 2\delta I) \leq \omega_0(2\delta|x_\varepsilon - z| + 2\delta). \quad (5.12)$$

Since $y_\varepsilon \in \Omega$ and $u(x_\varepsilon) > g(x_\varepsilon)$ for small $\varepsilon > 0$ provided $x_\varepsilon \in \partial\Omega$, in view of (5.10) and (5.11), we have

$$\nu(u(x_\varepsilon) - v(y_\varepsilon)) \leq F(y_\varepsilon, p_\varepsilon, Y) - F(x_\varepsilon, p_\varepsilon + 2\delta(x_\varepsilon - z), X + 2\delta I).$$

Combining this with (5.12) , by (5.4), we have

$$\nu(u(x_\varepsilon)-v(y_\varepsilon)) \le \omega_0(2\delta|x_\varepsilon-z|+2\delta)+\hat\omega_F\left(|x_\varepsilon-y_\varepsilon|\left(1+|p_\varepsilon|+\frac{|x_\varepsilon-y_\varepsilon|}{\varepsilon^2}\right)\right).$$

Sending $\varepsilon \to 0$ together with (5.9) in the above, we have

$$\nu\theta \le \omega_0(2\delta)+\hat\omega_F(2\delta^2),$$

which is a contradiction for small $\delta > 0$, which only depends on θ and ν.

Case 1-2: $v(z) < g(z)$. To get a contradiction, we argue as above replacing $\phi(x,y)$ by $\psi(x,y) := (2\varepsilon^2)^{-1}|x-y+\varepsilon\delta\mathbf{n}(z)|^2 - \delta|x-z|^2$ so that $x_\varepsilon = y_\varepsilon - \varepsilon\delta\mathbf{n}(z)+o(\varepsilon) \in \Omega$ for small $\varepsilon > 0$. Note that we need here the continuity of u on $\partial\Omega$ in (5.7) while the other one in (5.7) is needed in Case 1-1. (See also the proof of Theorem 5.3 below.)

Case 1-3: $u(z) \le g(z)$ and $v(z) \ge g(z)$. This does not occur because $0 < \theta = (u-v)(z) \le 0$.

Case 2: $\sup_{\partial\Omega}(u-v) < \theta$. In this case, using the standard test function $|x-y|^2/(2\varepsilon)$ (without $\delta|x-z|^2$ term), we can follow the same argument as in the proof of Theorem 3.7. $\square$

Remark. Unfortunately, without assuming the continuity of viscosity solutions on $\partial\Omega$, the comparison principle fails in general.

In fact, setting $F(x,r,p,X) \equiv r$ and $g(x) \equiv -1$, consider the function

$$u(x) := \begin{cases} 0 & \text{for } x \in \Omega, \\ -1 & \text{for } x \in \partial\Omega. \end{cases}$$

Note that $u^* \equiv 0$ and $u_* \equiv u$ in $\overline{\Omega}$, which are respectively a viscosity sub- and supersolution of $G(x,u,Du,D^2u) = 0$ in $\overline{\Omega}$. Therefore, this example shows that the comparison principle fails in general without assumption (5.7).

5.2 State constraint problem

The state constraint boundary condition arises in a typical optimal control problem. Thus, if the reader is more interested in the PDE theory, he/she may skip Proposition 5.2 below, which explains why we will adapt the "state constraint boundary condition" in Theorem 5.3.

To explain our motivation, we shall consider Bellman equations of first-order.

$$\sup_{a\in A}\{\nu u - \langle g(x,a), Du\rangle - f(x,a)\} = 0 \quad \text{in } \Omega.$$

Here, we use the notations in section 4.2.1.

We introduce the following set of controls: For $x \in \overline{\Omega}$,

$$\mathcal{A}(x) := \{\alpha(\cdot) \in \mathcal{A} \mid X(t;x,\alpha) \in \overline{\Omega} \quad \text{for } t \geq 0\}.$$

We shall suppose that

$$\mathcal{A}(x) \neq \emptyset \quad \text{for all } x \in \overline{\Omega}. \tag{5.13}$$

Also, we suppose that

$$\begin{cases} (1) & \sup_{a\in A}\left(\|f(\cdot,a)\|_{L^\infty(\Omega)} + \|g(\cdot,a)\|_{W^{1,\infty}(\Omega)}\right) < \infty, \\ (2) & \sup_{a\in A}|f(x,a) - f(y,a)| \leq \omega_f(|x-y|) \quad \text{for } x,y \in \overline{\Omega}, \end{cases} \tag{5.14}$$

where $\omega_f \in \mathcal{M}$.

We are now interested in the following the optimal cost functional:

$$u(x) := \inf_{\alpha\in\mathcal{A}(x)} \int_0^\infty e^{-\nu t} f(X(t;x,\alpha),\alpha(t))dt.$$

Proposition 5.2. *Assume that* $\nu > 0$, (5.13) *and* (5.14) *hold. Then, we have* (1) u *is a viscosity subsolution of*

$$\sup_{a\in A}\{\nu u - \langle g(x,a), Du\rangle - f(x,a)\} \leq 0 \quad \textit{in } \Omega,$$

(2) u *is a viscosity supersolution of*

$$\sup_{a\in A}\{\nu u - \langle g(x,a), Du\rangle - f(x,a)\} \geq 0 \quad \textit{in } \overline{\Omega}.$$

Remark. We often say that u satisfies the state constraint boundary condition when it is a viscosity supersolution of

$$\text{“}F(x,u,Du,D^2u) \geq 0 \quad \text{in } \partial\Omega\text{”}.$$

Proof. In fact, at $x \in \Omega$, it is easy to verify that the dynamic programming principle (Theorem 4.4) holds for small $T > 0$. Thus, we may show Theorem 4.5 replacing $\mathbf{R}^n$ by Ω.

Hence, it only remains to show (2) on $\partial\Omega$. Thus, suppose that there are $\hat{x} \in \partial\Omega$, $\theta > 0$ and $\phi \in C^1(\overline{\Omega})$ such that $(u_* - \phi)(\hat{x}) = 0 \leq (u_* - \phi)(x)$ for $x \in \overline{\Omega}$, and

$$\sup_{a \in A} \{\nu\phi(\hat{x}) - \langle g(\hat{x}, a), D\phi(\hat{x})\rangle - f(\hat{x}, a)\} \leq -2\theta.$$

Then, we will get a contradiction.

Choose $x_k \in \overline{\Omega} \cap B_{1/k}(\hat{x})$ such that $u_*(\hat{x}) + k^{-1} \geq u(x_k)$ and $|\phi(\hat{x}) - \phi(x_k)| < 1/k$. In view of (5.14), there is $t_0 > 0$ such that for any $\alpha \in \mathcal{A}(x_k)$ and large $k \geq 1$, we have

$$\nu\phi(X_k(t)) - \langle g(X_k(t), \alpha(t)), D\phi(X_k(t))\rangle - f(X_k(t), \alpha(t)) \leq -\theta \quad \text{for } t \in (0, t_0),$$

where $X_k(t) := X(t; x_k, \alpha)$. Thus, multiplying $e^{-\nu t}$ and then, integrating it over $(0, t_0)$, we have

$$\phi(x_k) \leq e^{-\nu t_0}\phi(X_k(t_0)) + \int_0^{t_0} e^{-\nu t} f(X_k(t), \alpha(t))dt - \frac{\theta}{\nu}(1 - e^{-\nu t_0}).$$

Since we have

$$u(x_k) \leq \frac{2}{k} + e^{-\nu t_0}u(X_k(t_0)) + \int_0^{t_0} e^{-\nu t} f(X_k(t), \alpha(t))dt - \frac{\theta}{\nu}(1 - e^{-\nu t_0}),$$

taking the infimum over $\mathcal{A}(x_k)$, we apply Theorem 4.4 to get

$$0 \leq \frac{2}{k} - \frac{\theta}{\nu}(1 - e^{-\nu t_0}),$$

which is a contradiction for large k. $\square$

Motivated by this proposition, we shall consider more general second-order elliptic PDEs.

Theorem 5.3. *Assume that* $\nu > 0$, (5.3), (5.4), (5.6) *and* (5.12) *hold. Let* $u : \overline{\Omega} \to \mathbf{R}$ *be, respectively, a viscosity sub- and supersolution of*

$$\nu u + F(x, Du, D^2u) \leq 0 \quad \text{in } \Omega,$$

and

$$\nu v + F(x, Dv, D^2v) \geq 0 \quad \text{in } \overline{\Omega}.$$

Assume also that

$$\liminf_{x \in \overline{\Omega} \to z} u^*(x) \geq u^*(z) \quad \text{for } z \in \partial\Omega. \tag{5.15}$$

Then, $u^* \le v_*$ *in* $\overline{\Omega}$.

<u>*Remark.*</u> In 1986, Soner first treated the state constraint problems for deterministic optimal control (*i.e.* first-order PDEs) by the viscosity solution approach.

We note that we do not need continuity of v on $\partial\Omega$ while we need it for Dirichlet problems. For further discussion on the state constraint problems, we refer to Ishii-Koike (1996).

We also note that the proof below is easier than that for Dirichlet problems in section 5.1 because we only need to avoid the boundary condition for viscosity subsolutions.

<u>*Proof.*</u> Suppose that $\max_{\overline{\Omega}}(u^* - v_*) =: \theta > 0$. We shall write u and v for u^* and v_*, respectively, again.

We may suppose that $\max_{\partial\Omega}(u - v) = \theta$ since otherwise, we can use the standard procedure to get a contradiction.

Now, we proceed the same argument in Case 1-2 in the proof of Theorem 5.1 (although it is not precisely written).

For $\varepsilon, \delta > 0$, setting $\phi(x,y) := (2\varepsilon^2)^{-1}|x - y + \varepsilon\delta\mathbf{n}(z)|^2 + \delta|x - z|^2$, where $\mathbf{n}$ is the unit outward normal vector at $z \in \partial\Omega$, we let $(x_\varepsilon, y_\varepsilon) \in \overline{\Omega}\times\overline{\Omega}$ the maximum point of $u(x) - v(y) - \phi(x,y)$ over $\overline{\Omega}\times\overline{\Omega}$. As in the proof of Theorem 3.4, we have

$$\lim_{\varepsilon\to 0}(x_\varepsilon, y_\varepsilon) = (z,z) \quad \text{and} \quad \lim_{\varepsilon\to 0}\frac{|x_\varepsilon - y_\varepsilon|}{\varepsilon} = \delta. \tag{5.16}$$

Since $x_\varepsilon = y_\varepsilon - \varepsilon\delta\mathbf{n}(z) + o(\varepsilon) \in \Omega$ for small $\varepsilon > 0$, in view of Lemma 3.6 with Proposition 2.7, we can find $X, Y \in S^n$ such that

$$\left(\frac{x_\varepsilon - y_\varepsilon}{\varepsilon^2} + \frac{\delta}{\varepsilon}\mathbf{n}(z) + 2\delta(x_\varepsilon - z), X + 2\delta I\right) \in \overline{J}^{2,+}_{\overline{\Omega}} u(x_\varepsilon),$$

$$\left(\frac{x_\varepsilon - y_\varepsilon}{\varepsilon^2} + \frac{\delta}{\varepsilon}\mathbf{n}(z), Y\right) \in \overline{J}^{2,-}_{\overline{\Omega}} v(y_\varepsilon),$$

and

$$-\frac{3}{\varepsilon^2}\begin{pmatrix} I & O \\ O & I \end{pmatrix} \le \begin{pmatrix} X & O \\ O & -Y \end{pmatrix} \le \frac{3}{\varepsilon^2}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

Setting $p_\varepsilon := \varepsilon^{-2}(x_\varepsilon - y_\varepsilon) + \delta\varepsilon^{-1}\mathbf{n}(z)$, we have

$$\begin{aligned}\nu(u(x_\varepsilon) - v(y_\varepsilon)) &\le F(y_\varepsilon, p_\varepsilon, Y) - F(x_\varepsilon, p_\varepsilon + 2\delta(x_\varepsilon - z), X + 2\delta I) \\ &\le \omega_0(2\delta|x_\varepsilon - z| + 2\delta) + \hat{\omega}_F\left(|x_\varepsilon - y_\varepsilon|\left(1 + |p_\varepsilon| + \frac{|x_\varepsilon - y_\varepsilon|}{\varepsilon}\right)\right).\end{aligned}$$

Hence, sending $\varepsilon \to 0$ with (5.16), we have

$$\nu\theta \le \omega_0(2\delta) + \hat{\omega}_F(2\delta^2),$$

which is a contradiction for small $\delta > 0$. $\Box$

5.3 Neumann problem

In the classical theory and modern theory for weak solutions in the distribution sense, the (inhomogeneous) Neumann condition is given by

$$\langle \mathbf{n}(x), Du(x) \rangle - g(x) = 0 \quad \text{on } \partial\Omega,$$

where $\mathbf{n}(x)$ denotes the unit outward normal vector at $x \in \partial\Omega$.

In Dirichlet and state constraint problems, we have used a test function which forces one of x_ε and y_ε to be in Ω. However, in the Neumann boundary value problem (Neumann problem for short), we have to avoid the boundary condition for viscosity sub- and supersolutions simultaneously. Thus, we need a new test function different from those in sections 5.1 and 5.2.

We first define the signed distance function from Ω by

$$\rho(x) := \begin{cases} \inf\{|x-y| \mid y \in \partial\Omega\} & \text{for } x \in \Omega^c, \\ -\inf\{|x-y| \mid y \in \partial\Omega\} & \text{for } x \in \Omega. \end{cases}$$

In order to obtain the comparison principle for the Neumann problem, we shall impose a hypothesis on Ω (see Fig 5.2):

$$\begin{cases} (1) & \text{There is } \hat{r} > 0 \text{ such that} \\ & \Omega \subset (B_{\hat{r}}(z + \hat{r}\mathbf{n}(z)))^c \text{ for } z \in \partial\Omega. \\ (2) & \text{There is a neighborhood } N \text{ of } \partial\Omega \text{ such that} \\ & \rho \in C^2(N), \text{ and } D\rho(x) = \mathbf{n}(x) \text{ for } x \in \partial\Omega. \end{cases} \tag{5.17}$$

Remark. This assumption (1) is called the "uniform exterior sphere condition". Since $|x - z - \hat{r}\mathbf{n}(z)| \ge \hat{r}$ for $z \in \partial\Omega$ and $x \in \overline{\Omega}$, we have

$$\langle \mathbf{n}(z), x - z \rangle \le \frac{|x-z|^2}{2\hat{r}} \quad \text{for } z \in \partial\Omega \text{ and } x \in \overline{\Omega}. \tag{5.18}$$

It is known that when $\partial\Omega$ is "smooth" enough, (2) of (5.17) holds true.

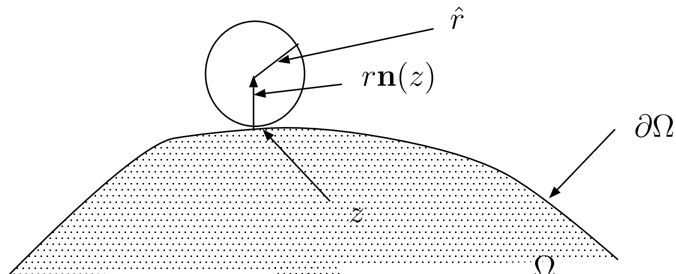

Fig 5.2

We shall consider the inhomogeneous Neumann problem:

$$\begin{cases} \nu u + F(x, Du, D^2u) = 0 & \text{in } \Omega, \\ \langle \mathbf{n}(x), Du \rangle - g(x) = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.19}$$

Remember that we adapt the definition of viscosity solutions of (5.19) for the corresponding G in (5.2).

Theorem 5.4. *Assume that* $\nu > 0$, (5.3), (5.4) *and* (5.17) *hold. For* $g \in C(\partial\Omega)$, *we let* u *and* $v : \overline{\Omega} \to \mathbf{R}$ *be a viscosity sub- and supersolution of* (5.19), *respectively.*

Then, $u^* \leq v_*$ *in* $\overline{\Omega}$.

Remark. We note that we do not need any continuity of u and v on $\partial\Omega$.

Proof. As before, we write u and v for u^* and v_*, respectively.

As in the proof of Theorem 3.7, we suppose that $\max_{\overline{\Omega}}(u - v) =: \theta > 0$. Also, we may suppose that $\max_{\partial\Omega}(u - v) = \theta$.

Let $z \in \partial\Omega$ be a point such that $(u - v)(z) = \theta$. For small $\delta > 0$, we see that the mapping $x \in \overline{\Omega} \to u(x) - v(y) - \delta|x - z|^2$ takes its strict maximum at z.

For small $\varepsilon, \delta > 0$, where $\delta > 0$ will be fixed later, setting $\phi(x, y) := (2\varepsilon)^{-1}|x - y|^2 - g(z)\langle \mathbf{n}(z), x - y \rangle + \delta(\rho(x) + \rho(y) + 2) + \delta|x - z|^2$, we let $(x_\varepsilon, y_\varepsilon) \in \overline{\Omega} \times \overline{\Omega}$ be the maximum point of $\Phi(x, y) := u(x) - v(y) - \phi(x, y)$ over $\overline{\Omega \cap N} \times \overline{\Omega \cap N}$, where N is in (5.17).

Since $\Phi(x_\varepsilon, y_\varepsilon) \geq \Phi(z, z)$, as before, we may extract a subsequence, which is denoted by $(x_\varepsilon, y_\varepsilon)$ again, such that $(x_\varepsilon, y_\varepsilon) \to (\hat{x}, \hat{x})$. We may suppose $\hat{x} \in \partial\Omega$. Since $\Phi(\hat{x}, \hat{x}) \geq \limsup_{\varepsilon \to 0} \Phi(x_\varepsilon, y_\varepsilon)$, we have

$$u(\hat{x}) - v(\hat{x}) - \delta|\hat{x} - z|^2 \geq \theta,$$

which yields $\hat{x} = z$. Moreover, we have

$$\lim_{\varepsilon \to 0} \frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon} = 0. \tag{5.20}$$

Applying Lemma 3.6 to $u(x) - \delta(\rho(x) + 1) - g(z)\langle \mathbf{n}(z), x\rangle - \delta|x - z|^2$ and $-v(y) - \delta(\rho(y) + 1) + g(z)\langle \mathbf{n}(z), y\rangle$, we find $X, Y \in S^n$ such that

$$\big(p_\varepsilon + \delta \mathbf{n}(x_\varepsilon) + 2\delta(x_\varepsilon - z), X + \delta D^2\rho(x_\varepsilon) + 2\delta I\big) \in \overline{J}^{2,+}_{\overline{\Omega}} u(x_\varepsilon), \tag{5.21}$$

$$\big(p_\varepsilon - \delta \mathbf{n}(y_\varepsilon), Y - \delta D^2\rho(y_\varepsilon)\big) \in \overline{J}^{2,-}_{\overline{\Omega}} v(y_\varepsilon), \tag{5.22}$$

where $p_\varepsilon := \varepsilon^{-1}(x_\varepsilon - y_\varepsilon) + g(z)\mathbf{n}(z)$, and

$$-\frac{3}{\varepsilon}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \le \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le \frac{3}{\varepsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

When $x_\varepsilon \in \partial\Omega$, by (5.18), we calculate in the following manner:

$$\begin{aligned} \langle \mathbf{n}(x_\varepsilon), D_x\phi(x_\varepsilon, y_\varepsilon)\rangle &= \langle \mathbf{n}(x_\varepsilon), p_\varepsilon + \delta\mathbf{n}(x_\varepsilon) + 2\delta(x_\varepsilon - z)\rangle \\ &\ge -\frac{|x_\varepsilon - y_\varepsilon|^2}{2\hat{r}\varepsilon} + g(z)\langle \mathbf{n}(x_\varepsilon), \mathbf{n}(z)\rangle + \delta - 2\delta|x_\varepsilon - z|. \end{aligned}$$

Hence, given $\delta > 0$, we see that

$$\langle \mathbf{n}(x_\varepsilon), D_x\phi(x_\varepsilon, y_\varepsilon)\rangle - g(x_\varepsilon) \ge \frac{\delta}{2} \quad \text{for small } \varepsilon > 0.$$

Thus, by (5.21), this yields

$$\nu u(x_\varepsilon) + F(x_\varepsilon, p_\varepsilon + \delta\mathbf{n}(x_\varepsilon) + 2\delta(x_\varepsilon - z), X + \delta D^2\rho(x_\varepsilon) + 2\delta I) \le 0. \tag{5.23}$$

Of course, if $x_\varepsilon \in \Omega$, then the above inequality holds from the definition.

On the other hand, similarly, if $y_\varepsilon \in \partial\Omega$, then

$$\langle \mathbf{n}(y_\varepsilon), -D_y\phi(x_\varepsilon, y_\varepsilon)\rangle - g(y_\varepsilon) \le -\frac{\delta}{2} \quad \text{for small } \varepsilon > 0.$$

Hence, by (5.22), we have

$$\nu v(y_\varepsilon) + F(y_\varepsilon, p_\varepsilon - \delta\mathbf{n}(y_\varepsilon), Y - \delta D^2\rho(y_\varepsilon)) \ge 0. \tag{5.24}$$

Using (5.3) and (5.4), by (5.23) and (5.24), we have

$$\begin{aligned}\nu(u(x_\varepsilon)-v(y_\varepsilon)) &\le F(y_\varepsilon,p_\varepsilon,Y)-F(x_\varepsilon,p_\varepsilon,X)+2\omega_0(\delta M)\\ &\le \hat{\omega}_F\left(|x_\varepsilon-y_\varepsilon|\left(1+|p_\varepsilon|+\frac{|x_\varepsilon-y_\varepsilon|}{\varepsilon}\right)\right)+2\omega_0(\delta M),\end{aligned}$$

where $M := 3+\sup_{x\in N\cap\overline{\Omega}}(2|x-z|+|D^2\rho(x)|)$. Sending $\varepsilon\to 0$ with (5.20) in the above, we have

$$\nu\theta \le 2\omega_0(\delta M),$$

which is a contradiction for small $\delta>0$. □

5.4 Growth condition at $|x|\to\infty$

In the standard PDE theory, we often consider PDEs in unbounded domains, typically, in $\mathbf{R}^n$. In this subsection, we present a technique to establish the comparison principle for viscosity solutions of

$$\nu u + F(x,Du,D^2u)=0 \quad \text{in } \mathbf{R}^n. \tag{5.25}$$

We remind the readers that in the proofs of comparison results we always suppose $\max_{\overline{\Omega}}(u-v)>0$, where u and v are, respectively, a viscosity sub- and supersolution. However, considering $\Omega := \mathbf{R}^n$, the maximum of $u-v$ might attain its maximum at "$|x|\to\infty$". Thus, we have to choose a test function $\phi(x,y)$, which forces $u(x)-v(y)-\phi(x,y)$ to takes its maximum at a point in a compact set.

For this purpose, we will suppose the linear growth condition (for simplicity) for viscosity solutions.

We rewrite the structure condition (3.21) for $\mathbf{R}^n$:

$$\left\{\begin{array}{c}\text{There is an } \omega_F\in\mathcal{M} \text{ such that if } X,Y\in S^n \text{ and } \mu>1 \text{ satisfy}\\ -3\mu\begin{pmatrix} I & 0\\ 0 & I\end{pmatrix}\le\begin{pmatrix} X & 0\\ 0 & -Y\end{pmatrix}\le 3\mu\begin{pmatrix} I & -I\\ -I & I\end{pmatrix},\\ \text{then } F(y,\mu(x-y),Y)-F(x,\mu(x-y),X)\\ \le \omega_F(|x-y|(1+\mu|x-y|)) \text{ for } x,y\in\mathbf{R}^n.\end{array}\right. \tag{5.26}$$

We will also need the Lipschitz continuity of $(p,X)\to F(x,p,X)$, which is stronger than (5.3).

$$\left\{\begin{array}{c}\text{There is } \mu_0>0 \text{ such that } |F(x,p,X)-F(x,q,Y)|\\ \le \mu_0(|p-q|+\|X-Y\|) \text{ for } x\in\mathbf{R}^n, p,q\in\mathbf{R}^n, X,Y\in S^n.\end{array}\right. \tag{5.27}$$

Proposition 5.5. *Assume that $\nu > 0$, (5.26) and (5.27) hold. Let u and $v : \mathbf{R}^n \to \mathbf{R}$ be, respectively, a viscosity sub- and supersolution of (5.25). Assume also that there is $C_0 > 0$ such that*

$$u^*(x) \le C_0(1+|x|) \quad \text{and} \quad v_*(x) \ge -C_0(1+|x|) \quad \text{for } x \in \mathbf{R}^n. \tag{5.28}$$

Then, $u^* \le v_*$ in $\mathbf{R}^n$.

<u>Proof.</u> We shall simply write u and v for u^* and v_*, respectively.

For $\delta > 0$, we set $\theta_\delta := \sup_{x\in\mathbf{R}^n}(u(x) - v(x) - 2\delta(1+|x|^2))$. We note that (5.28) implies that there is $z_\delta \in \mathbf{R}^n$ such that $\theta_\delta = u(z_\delta) - v(z_\delta) - 2\delta(1+|z_\delta|^2)$. Set $\theta := \limsup_{\delta\to 0}\theta_\delta \in \mathbf{R}\cup\{\infty\}$.

When $\theta \le 0$, since

$$(u-v)(x) \le 2\delta(1+|x|^2) + \theta_\delta \quad \text{for } \delta > 0 \text{ and } x \in \mathbf{R}^n,$$

we have $u \le v$ in $\mathbf{R}^n$.

Thus, we may suppose $\theta \in (0,\infty]$. Setting $\Phi_\delta(x,y) := u(x) - v(y) - (2\varepsilon)^{-1}|x-y|^2 - \delta(1+|x|^2) - \delta(1+|y|^2)$ for $\varepsilon, \delta > 0$, where $\delta > 0$ will be fixed later, in view of (5.28), we can choose $(x_\varepsilon, y_\varepsilon) \in \mathbf{R}^n \times \mathbf{R}^n$ such that $\Phi_\delta(x_\varepsilon, y_\varepsilon) = \max_{(x,y)\in\mathbf{R}^n\times\mathbf{R}^n}\Phi_\delta(x,y) \ge \theta_\delta$.

As before, extracting a subsequence if necessary, we may suppose that

$$\lim_{\varepsilon\to 0}\frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon} = 0. \tag{5.29}$$

By Lemma 3.6 with Proposition 2.7, putting $p_\varepsilon := (x_\varepsilon - y_\varepsilon)/\varepsilon$, we find $X, Y \in S^n$ such that

$$(p_\varepsilon + 2\delta x_\varepsilon, X + 2\delta I) \in \overline{J}^{2,+}u(x_\varepsilon),$$

$$(p_\varepsilon - 2\delta y_\varepsilon, Y - 2\delta I) \in \overline{J}^{2,-}v(y_\varepsilon),$$

and

$$-\frac{3}{\varepsilon}\begin{pmatrix} I & O \\ O & I \end{pmatrix} \le \begin{pmatrix} X & O \\ O & -Y \end{pmatrix} \le \frac{3}{\varepsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

Hence, we have

$$\begin{aligned}
&\nu(u(x_\varepsilon) - v(y_\varepsilon)) \\
\le\ & F(y_\varepsilon, p_\varepsilon - 2\delta y_\varepsilon, Y - 2\delta I) - F(x_\varepsilon, p_\varepsilon + 2\delta x_\varepsilon, X + 2\delta I) \\
\le\ & F(y_\varepsilon, p_\varepsilon, Y) - F(x_\varepsilon, p_\varepsilon, X) + 2\delta\mu_0(2 + |x_\varepsilon| + |y_\varepsilon|) \\
\le\ & \omega_F\left(|x_\varepsilon - y_\varepsilon|\left(1 + \frac{|x_\varepsilon - y_\varepsilon|}{\varepsilon}\right)\right) + \nu\delta(2 + |x_\varepsilon|^2 + |y_\varepsilon|^2) + C\delta,
\end{aligned}$$

where $C = C(\mu_0, \nu) > 0$ is independent of $\varepsilon, \delta > 0$. For the last inequality, we used "$2ab \leq \tau a^2 + \tau^{-1}b^2$ for $\tau > 0$".

Therefore, we have

$$\nu\theta \leq \omega_F \left(|x_\varepsilon - y_\varepsilon| \left(1 + \frac{|x_\varepsilon - y_\varepsilon|}{\varepsilon} \right) \right) + C\delta.$$

Sending $\varepsilon \to 0$ in the above together with (5.29), we get $\nu\theta \leq C\delta$, which is a contradiction for small $\delta > 0$. □