# 2 Definition

In this section, we derive the definition of viscosity solutions of (1.1) via the vanishing viscosity method.

We also give some basic properties of viscosity solutions and equivalent definitions using "semi-jets".

## 2.1 Vanishing viscosity method

When the notion of viscosity solutions was born, in order to explain the reason why we need it, many speakers started in their talks by giving the following typical example called the eikonal equation:

$$|Du|^2 = 1 \quad \text{in } \Omega. \tag{2.1}$$

We seek $C^1$ functions satisfying (2.1) under the Dirichlet condition:

$$u(x) = 0 \quad \text{for } x \in \partial\Omega. \tag{2.2}$$

However, since there is no classical solution of (2.1)-(2.2) (showing the non-existence of classical solutions is a good exercise), we intend to derive a reasonable definition of weak solutions of (2.1).

In fact, we expect that the following function (the distance from $\partial\Omega$) would be the unique solution of this problem (see Fig 2.1):

$$u(x) = \text{dist}(x, \partial\Omega) := \inf_{y\in\partial\Omega} |x - y|.$$

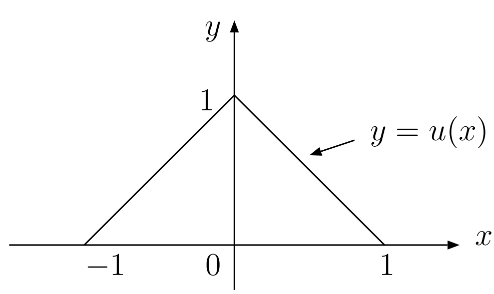


Fig 2.1

If we consider the case when $n = 1$ and $\Omega = (-1, 1)$, then the expected solution is given by

$$u(x) = 1 - |x| \quad \text{for } x \in [-1, 1]. \tag{2.3}$$

Since this function is $C^\infty$ except at $x = 0$, we could decide to call $u$ a weak solution of (2.1) if it satisfies (2.1) in $\Omega$ except at finite points.

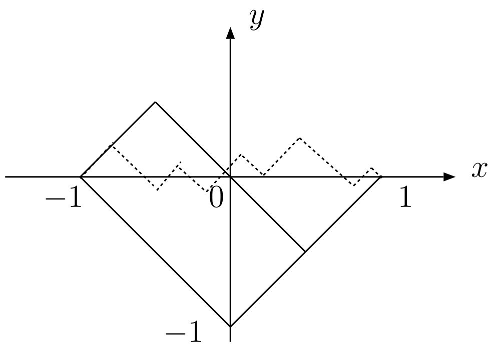


Fig 2.2

However, even in the above simple case of (2.1), we know that there are infinitely many such weak solutions of (2.1) (see Fig 2.2); for example, $-u$ is the weak solution and

$$u(x) = \begin{cases} x+1 & \text{for } x \in [-1, -\frac{1}{2}), \\ -x & \text{for } x \in [-\frac{1}{2}, \frac{1}{2}), \\ x-1 & \text{for } x \in [\frac{1}{2}, 1], \end{cases} \qquad \ldots etc.$$

Now, in order to look for an appropriate notion of weak solutions, we introduce the so-called vanishing viscosity method; for $\varepsilon > 0$, we consider the following PDE as an approximate equation of (2.1) when $n = 1$ and $\Omega = (-1, 1)$:

$$\begin{cases} -\varepsilon u_\varepsilon'' + (u_\varepsilon')^2 = 1 & \text{in } (-1, 1), \\ \qquad u_\varepsilon(\pm 1) = 0. \end{cases} \tag{2.4}$$

The first term, $-\varepsilon u_\varepsilon''$, in the left hand side of (2.4) is called the vanishing viscosity term (when $n = 1$) as $\varepsilon \to 0$.

By an elementary calculation, we can find a unique smooth function $u_\varepsilon$ in the following manner: We first note that if a classical solution of (2.4) exists, then it is unique. Thus, we may suppose that $u'_\varepsilon(0) = 0$ by symmetry. Setting $v_\varepsilon = u'_\varepsilon$, we first solve the ODE:

$$\begin{cases} -\varepsilon v'_\varepsilon + v_\varepsilon^2 = 1 \quad \text{in } (-1,1), \\ \qquad v_\varepsilon(0) = 0. \end{cases} \tag{2.5}$$

It is easy to see that the solution of (2.5) is given by

$$v_\varepsilon(x) = -\tanh\left(\frac{x}{\varepsilon}\right).$$

Hence, we can find $u_\varepsilon$ by

$$u_\varepsilon(x) = -\varepsilon \log\left(\frac{\cosh\left(\frac{x}{\varepsilon}\right)}{\cosh\left(\frac{1}{\varepsilon}\right)}\right) = -\varepsilon \log\left(\frac{e^{\frac{x}{\varepsilon}} + e^{-\frac{x}{\varepsilon}}}{e^{\frac{1}{\varepsilon}} + e^{-\frac{1}{\varepsilon}}}\right).$$

It is a good exercise to show that $u_\varepsilon$ converges to the function in (2.3) uniformly in $[-1,1]$.

Remark. Since $\hat{u}_\varepsilon(x) := -u_\varepsilon(x)$ is the solution of

$$\begin{cases} \varepsilon u'' + (u')^2 = 1 \quad \text{in } (-1,1), \\ \qquad u(\pm 1) = 0, \end{cases}$$

we have $\hat{u}(x) := \lim_{\varepsilon\to 0} \hat{u}_\varepsilon(x) = -u(x)$. Thus, if we replace $-\varepsilon u''$ by $+\varepsilon u''$, then the limit function would be different in general.

To define weak solutions, we adapt the properties which hold for the (uniform) limit of approximate solutions of PDEs with the "minus" vanishing viscosity term.

Let us come back to general second-order PDEs:

$$F(x, u, Du, D^2u) = 0 \quad \text{in } \Omega. \tag{2.6}$$

We shall use the following definition of classical solutions:

**Definition.** We call $u : \Omega \to \mathbf{R}$ a **classical subsolution** (resp., **supersolution, solution**) of (2.6) if $u \in C^2(\Omega)$ and

$$F(x, u(x), Du(x), D^2u(x)) \le 0 \quad (\text{resp., } \ge 0, \ = 0) \quad \text{in } \Omega.$$

Remark. If $F$ does not depend on $X$-variables (*i.e.* $F(x, u, Du) = 0$; first-order PDEs), we only suppose $u \in C^1(\Omega)$ in the above in place of $u \in C^2(\Omega)$.

Throughout this text, we also suppose the following monotonicity condition with respect to $X$-variables:

**Definition.** We say that $F$ is (degenerate) **elliptic** if

$$\begin{cases} F(x, r, p, X) \le F(x, r, p, Y) \\ \text{for all } x \in \Omega, r \in \mathbf{R}, p \in \mathbf{R}^n, X, Y \in S^n \text{ provided } X \ge Y. \end{cases} \tag{2.7}$$

We notice that if $F$ does not depend on $X$-variables (*i.e.* $F = 0$ is the first-order PDE), then $F$ is automatically elliptic.

We also note that the left hand side $F(x, r, p, X) = -\text{trace}(X)$ of the Laplace equation (1.2) is elliptic.

We will derive properties which hold true for the (uniform) limit (as $\varepsilon \to +0$) of solutions of

$$-\varepsilon \triangle u + F(x, u, Du, D^2u) = 0 \quad \text{in } \Omega \quad (\varepsilon > 0). \tag{2.8}$$

Note that since $-\varepsilon\text{trace}(X) + F(x, r, p, X)$ is "uniformly" elliptic (see in section 3 for the definition) provided that $F$ is elliptic and $F(x, r, p, X) \le C|X|$ for $(x, r, p) \in \Omega \times \mathbf{R} \times \mathbf{R}^n$, it is easier to solve (2.8) than (2.6) in practice. See [13] for instance.

**Proposition 2.1.** *Assume that $F$ is elliptic. Let $u_\varepsilon \in C^2(\Omega) \cap C(\overline{\Omega})$ be a classical subsolution (resp., supersolution) of* (2.8)*. If $u_\varepsilon$ converges to $u \in C(\Omega)$ (as $\varepsilon \to 0$) uniformly in any compact sets $K \subset \Omega$, then, for any $\phi \in C^2(\Omega)$, we have*

$$F(x, u(x), D\phi(x), D^2\phi(x)) \le 0 \quad (\textit{resp.}, \ \ge 0)$$

*provided that $u - \phi$ attains its maximum (resp., minimum) at $x \in \Omega$.*

Remark. When $F$ does not depend on $X$-variables, we only need to suppose $\phi$ and $u_\varepsilon$ to be in $C^1(\Omega)$ as before.

Proof. We only give a proof of the assertion for subsolutions since the other one can be shown in a symmetric way.

Suppose that $u-\phi$ attains its maximum at $\hat{x} \in \Omega$ for $\phi \in C^2(\Omega)$. Setting $\phi_\delta(y) := \phi(y) + \delta|y-\hat{x}|^4$ for small $\delta > 0$, we see that

$$(u-\phi_\delta)(\hat{x}) > (u-\phi_\delta)(y) \quad \text{for } y \in \Omega \setminus \{\hat{x}\}.$$

(This tiny technique to replace a maximum point by a "strict" one will appear in Proposition 2.2.)

Let $x_\varepsilon \in \overline{\Omega}$ be a point such that $(u_\varepsilon - \phi_\delta)(x_\varepsilon) = \max_{\overline{\Omega}}(u_\varepsilon - \phi_\delta)$. Note that $x_\varepsilon$ also depends on $\delta > 0$.

Since $u_\varepsilon$ converges to $u$ uniformly in $B_r(\hat{x})$ and $\hat{x}$ is the unique maximum point of $u - \phi_\delta$, we note that $\lim_{\varepsilon\to 0} x_\varepsilon = \hat{x}$. Thus, we see that $x_\varepsilon \in \Omega$ for small $\varepsilon > 0$. Notice that if we argue by $\phi$ instead of $\phi_\delta$, the limit of $x_\varepsilon$ might differ from $\hat{x}$.

Thus, at $x_\varepsilon \in \Omega$, we have

$$-\varepsilon \triangle u_\varepsilon(x_\varepsilon) + F(x_\varepsilon, u_\varepsilon(x_\varepsilon), Du_\varepsilon(x_\varepsilon), D^2u_\varepsilon(x_\varepsilon)) \le 0.$$

Since $D(u_\varepsilon - \phi_\delta)(x_\varepsilon) = 0$ and $D^2(u_\varepsilon - \phi_\delta)(x_\varepsilon) \le 0$, in view of ellipticity, we have

$$-\varepsilon \triangle \phi_\delta(x_\varepsilon) + F(x_\varepsilon, u_\varepsilon(x_\varepsilon), D\phi_\delta(x_\varepsilon), D^2\phi_\delta(x_\varepsilon)) \le 0.$$

Sending $\varepsilon \to 0$ in the above, we have

$$F(\hat{x}, u(\hat{x}), D\phi_\delta(\hat{x}), D^2\phi_\delta(\hat{x})) \le 0.$$

Since $D\phi_\delta(\hat{x}) = D\phi(\hat{x})$ and $D^2\phi_\delta(\hat{x}) = D^2\phi(\hat{x})$, we conclude the proof. □

**Definition.** We call $u : \Omega \to \mathbf{R}$ a **viscosity subsolution** (resp., **supersolution**) of (2.6) if, for any $\phi \in C^2(\Omega)$,

$$F(x, u(x), D\phi(x), D^2\phi(x)) \le 0 \quad (\text{resp., } \ge 0)$$

provided that $u - \phi$ attains its maximum (resp., minimum) at $x \in \Omega$.

We call $u : \Omega \to \mathbf{R}$ a **viscosity solution** of (2.6) if it is both a viscosity sub- and supersolution of (2.6).

*Remark.* Here, we have given the definition to "general" functions **but** we will often suppose that they are (semi-)continuous in Theorems etc.

In fact, in our propositions in sections 2.1, we will suppose that viscosity sub- and supersolutions are continuous.

However, all the proposition in section 2.1 can be proved by replacing upper and lower semi-continuity for viscosity subsolutions and supersolutions, respectively.

We will introduce general viscosity solutions in section 3.3.

**Notation.** In order to memorize the correct inequality, we will often say that $u$ is a viscosity subsolution (resp., supersolution) of

$$F(x, u, Du, D^2u) \leq 0 \quad (\text{resp., } \geq 0) \quad \text{in } \Omega$$

if it is a viscosity subsolution (resp., supersolution) of (2.6).

**Proposition 2.2.** *For $u : \Omega \to \mathbf{R}$, the following* (1) *and* (2) *are equivalent:*

$$\begin{cases} (1) & u \text{ is a viscosity subsolution (resp., supersolution) of (2.6)}, \\ (2) & \text{if } 0 = (u - \phi)(\hat{x}) > (u - \phi)(x) \ (\text{resp., } < (u - \phi)(x)) \\ & \text{for } \phi \in C^2(\Omega), \hat{x} \in \Omega \text{ and } x \in \Omega \setminus \{\hat{x}\}, \\ & \text{then } F(\hat{x}, \phi(\hat{x}), D\phi(\hat{x}), D^2\phi(\hat{x})) \leq 0 \ (\text{resp., } \geq 0). \end{cases}$$

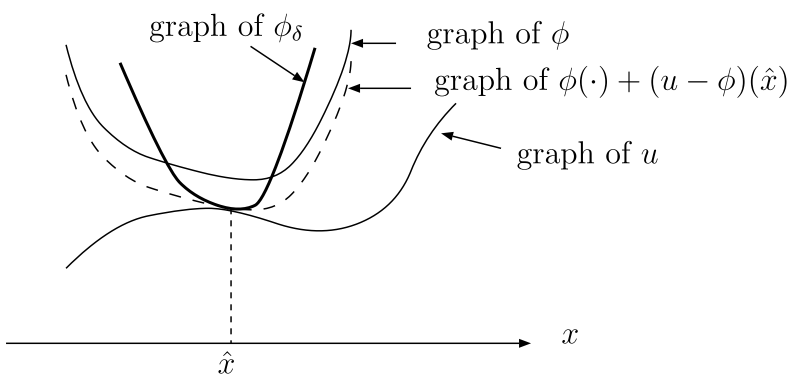


Fig 2.3

*Proof.* The implication (1) $\Rightarrow$ (2) is trivial.

For the opposite implication in the subsolution case, suppose that $u - \phi$ attains a maximum at $\hat{x} \in \Omega$. Set

$$\phi_\delta(x) = \phi(x) + \delta|x - \hat{x}|^4 + (u - \phi)(\hat{x}).$$

See Fig 2.3. Since $0 = (u - \phi_\delta)(\hat{x}) > (u - \phi_\delta)(x)$ for $x \in \Omega \setminus \{\hat{x}\}$, (2) gives

$$F(\hat{x}, \phi_\delta(\hat{x}), D\phi_\delta(\hat{x}), D^2\phi_\delta(\hat{x})) \le 0,$$

which implies the assertion. $\Box$

By the next proposition, we recognize that viscosity solutions are right candidates of weak solutions when $F$ is elliptic.

**Proposition 2.3.** *Assume that $F$ is elliptic. A function $u : \Omega \to \mathbf{R}$ is a classical subsolution* (*resp., supersolution*) *of* (2.6) *if and only if it is a viscosity subsolution* (*resp., supersolution*) *of* (2.6) *and $u \in C^2(\Omega)$.*

Proof. Suppose that $u$ is a viscosity subsolution of (2.6) and $u \in C^2(\Omega)$. Taking $\phi \equiv u$, we see that $u - \phi$ attains its maximum at any points $x \in \Omega$. Thus, the definition of viscosity subsolutions yields

$$F(x, u(x), Du(x), D^2u(x)) \le 0 \quad \text{for } x \in \Omega.$$

On the contrary, suppose that $u \in C^2(\Omega)$ is a classical subsolution of (2.6).

Fix any $\phi \in C^2(\Omega)$. Assuming that $u - \phi$ takes its maximum at $x \in \Omega$, we have

$$D(u - \phi)(x) = 0 \quad \text{and} \quad D^2(u - \phi)(x) \le 0.$$

Hence, in view of ellipticity, we have

$$0 \ge F(x, u(x), Du(x), D^2u(x)) \ge F(x, u(x), D\phi(x), D^2\phi(x)). \quad \Box$$

We introduce the sets of upper and lower semi-continuous functions: For $K \subset \mathbf{R}^n$,

$$USC(K) := \{u : K \to \mathbf{R} \mid u \text{ is upper semi-continuous in } K\},$$

and

$$LSC(K) := \{u : K \to \mathbf{R} \mid u \text{ is lower semi-continuous in } K\}.$$

Remark. Throughout this book, we use the following maximum principle for semi-continuous functions:

$\boxed{\text{An upper semi-continuous function in a compact set attains its maximum.}}$

We give the following lemma which will be used without mentioning it. Since the proof is a bit technical, the reader may skip it over first.

**Proposition 2.4.** *Assume that* $u \in USC(\overline{\Omega})$ (*resp.,* $u \in LSC(\overline{\Omega})$) *is a viscosity subsolution* (*resp., supersolution*) *of* (2.6) *in* $\Omega$.

*Then, for any open set* $\Omega' \subset \Omega$, $u$ *is a viscosity subsolution* (*resp., supersolution*) *of* (2.6) *in* $\Omega'$.

*Proof.* We only show the assertion for subsolutions since the other can be shown similarly.

For $\phi \in C^2(\Omega')$, by Proposition 2.2, we suppose that for some $\hat{x} \in \Omega'$,

$$0 = (u - \phi)(\hat{x}) > (u - \phi)(y) \quad \text{for all } y \in \Omega' \setminus \{\hat{x}\}.$$

For simplicity, we shall suppose $\hat{x} = 0$.

Choose $r > 0$ such that $B_{2r} \subset \Omega'$. We then choose $\xi_k \in C^\infty(\mathbf{R}^n)$ $(k = 1, 2)$ such that $0 \le \xi_k \le 1$ in $\mathbf{R}^n$, $\xi_1 + \xi_2 = 1$ in $\mathbf{R}^n$,

$$\xi_1 = 1 \quad \text{in } B_r, \quad \text{and} \quad \xi_2 = 1 \quad \text{in } \mathbf{R}^n \setminus B_{2r}.$$

We define $\psi = \xi_1 \phi + M\xi_2$, where $M = \sup_{\overline{\Omega}} u + 1$. Since it is easy to verify that $\psi \in C^2(\mathbf{R}^n)$, and $0 = (u - \psi)(0) > (u - \psi)(x)$ for $x \in \Omega \setminus \{0\}$, we leave the proof to the reader. This concludes the proof. $\square$

## 2.2 Equivalent definitions

We present equivalent definitions of viscosity solutions. However, since we will need those in the proof of uniqueness for second-order PDEs,

<u>the reader may postpone this subsection until section 3.3.</u>

First, we introduce "semi"-jets of functions $u : \Omega \to \mathbf{R}$ at $x \in \Omega$ by

$$J^{2,+}u(x) := \left\{ (p, X) \in \mathbf{R}^n \times S^n \;\middle|\; \begin{array}{ll} u(y) \le & u(x) + \langle p, y - x \rangle \\ & + \frac{1}{2}\langle X(y - x), y - x \rangle \\ & + o(|y - x|^2) \quad \text{as } y \in \Omega \to x \end{array} \right\}$$

and

$$J^{2,-}u(x) := \left\{ (p,X) \in \mathbf{R}^n \times S^n \;\middle|\; \begin{array}{ll} u(y) \geq & u(x) + \langle p, y-x \rangle \\ & +\frac{1}{2}\langle X(y-x), y-x \rangle \\ & +o(|y-x|^2) \quad \text{as } y \in \Omega \to x \end{array} \right\}.$$

Note that $J^{2,-}u(x) = -J^{2,+}(-u)(x)$.

<u>*Remark.*</u> We do not impose any continuity for $u$ in these definitions.

We recall the notion of "small order $o$" in the above: For $k \geq 1$,

$$\begin{array}{c} f(x) \leq o(|x|^k) \quad (\text{resp.,} \ \geq o(|x|^k)) \quad \text{as } x \to 0 \\ \Longleftrightarrow \left\{ \begin{array}{l} \text{there is } \omega \in C([0,\infty),[0,\infty)) \text{ such that } \omega(0)=0, \text{ and} \\ \displaystyle\sup_{x \in B_r \setminus \{0\}} \frac{f(x)}{|x|^k} \leq \omega(r) \quad \left(\text{resp.,} \inf_{x \in B_r \setminus \{0\}} \frac{f(x)}{|x|^k} \geq -\omega(|x|)\right) \end{array} \right. \end{array}$$

In the next proposition, we give some basic properties of semi-jets: (1) is a relation between semi-jets and classical derivatives, and (2) means that semi-jets are "defined" in dense sets of $\Omega$.

**Proposition 2.5.** *For $u : \Omega \to \mathbf{R}$, we have the following:*
(1) *If $J^{2,+}u(x) \cap J^{2,-}u(x) \neq \emptyset$, then $Du(x)$ and $D^2u(x)$ exist and,*

$$J^{2,+}u(x) \cap J^{2,-}u(x) = \{(Du(x), D^2u(x))\}.$$

(2) *If $u \in USC(\Omega)$ (resp., $u \in LSC(\Omega)$), then*

$$\Omega = \left\{ x \in \Omega \;\middle|\; \exists x_k \in \Omega \text{ such that } J^{2,+}u(x_k) \neq \emptyset, \lim_{k\to\infty} x_k = x \right\}$$

$$\left(\textit{resp.}, \ \Omega = \left\{ x \in \Omega \;\middle|\; \exists x_k \in \Omega \text{ such that } J^{2,-}u(x_k) \neq \emptyset, \lim_{k\to\infty} x_k = x \right\}\right).$$

<u>*Proof.*</u> The proof of (1) is a direct consequence from the definition.
We give a proof of the assertion (2) only for $J^{2,+}$.
Fix $x \in \Omega$ and choose $r > 0$ so that $\overline{B}_r(x) \subset \Omega$. For $\varepsilon > 0$, we can choose $x_\varepsilon \in \overline{B}_r(x)$ such that $u(x_\varepsilon) - \varepsilon^{-1}|x_\varepsilon - x|^2 = \max_{y \in \overline{B}_r(x)}(u(y) - \varepsilon^{-1}|y-x|^2)$. Since $|x_\varepsilon - x|^2 \leq \varepsilon(\max_{\overline{B}_r(x)} - u(x))$, we see that $x_\varepsilon$ converges to $x \in \overline{B}_r(x)$

as $\varepsilon \to 0$. Thus, we may suppose that $x_\varepsilon \in B_r(x)$ for small $\varepsilon > 0$. Hence, we have

$$u(y) \le u(x_\varepsilon) + \frac{1}{\varepsilon}(|y-x|^2 - |x_\varepsilon - x|^2) \quad \text{for all } y \in \overline{B}_r(x).$$

It is easy to check that $(2(x_\varepsilon - x)/\varepsilon, 2\varepsilon^{-1}I) \in J^{2,+}u(x_\varepsilon)$. $\square$

We next introduce a sort of closure of semi-jets:

$$\overline{J}^{2,\pm}u(x) := \left\{ (p,X) \in \mathbf{R}^n \times S^n \;\middle|\; \begin{array}{c} \exists x_k \in \Omega \text{ and } \exists (p_k, X_k) \in J^{2,\pm}u(x_k) \\ \text{such that } (x_k, u(x_k), p_k, X_k) \\ \to (x, u(x), p, X) \text{ as } k \to \infty \end{array} \right\}.$$

**Proposition 2.6.** *For $u : \Omega \to \mathbf{R}$, the following (1), (2), (3) are equivalent.*

$$\begin{cases} (1) & u \text{ is a viscosity subsolution (resp., supersolution) of (2.6).} \\ (2) & \text{For } x \in \Omega \text{ and } (p,X) \in J^{2,+}u(x) \text{ (resp., } J^{2,-}u(x)), \\ & \text{we have } F(x,u(x),p,X) \le 0 \text{ (resp., } \ge 0). \\ (3) & \text{For } x \in \Omega \text{ and } (p,X) \in \overline{J}^{2,+}u(x) \text{ (resp., } \overline{J}^{2,-}u(x)), \\ & \text{we have } F(x,u(x),p,X) \le 0 \text{ (resp., } \ge 0). \end{cases}$$

Proof. Again, we give a proof of the assertion only for subsolutions.

Step 1: (2) $\Longrightarrow$ (3). For $x \in \Omega$ and $(p,X) \in \overline{J}^{2,+}u(x)$, we can find $(p_k, X_k) \in J^{2,+}u(x_k)$ with $x_k \in \Omega$ such that $\lim_{k\to\infty}(x_k, u(x_k), p_k, X_k) = (x, u(x), p, X)$ and

$$F(x_k, u(x_k), p_k, X_k) \le 0,$$

which implies (3) by sending $k \to \infty$.

Step 2: (3) $\Longrightarrow$ (1). For $\phi \in C^2(\Omega)$, suppose also $(u-\phi)(x) = \max(u-\phi)$. Thus, the Taylor expansion of $\phi$ at $x$ gives

$$u(y) \le u(x) + \langle D\phi(x), y-x\rangle + \frac{1}{2}\langle D^2\phi(x)(y-x), y-x\rangle + o(|x-y|^2) \quad \text{as } y \to x.$$

Thus, we have $(D\phi(x), D^2\phi(x)) \in J^{2,+}u(x) \subset \overline{J}^{2,+}u(x)$.

Step 3: (1) $\Longrightarrow$ (2). For $(p,X) \in J^{2,+}u(x)$ $(x \in \Omega)$, we can find nondecreasing, continuous $\omega : [0,\infty) \to [0,\infty)$ such that $\omega(0) = 0$ and

$$u(y) \le u(x) + \langle p, y-x\rangle + \frac{1}{2}\langle X(y-x), y-x\rangle + |y-x|^2\omega(|y-x|) \quad (2.9)$$

as $y \to x$. In fact, by the definition of $o$, we find $\omega_0 \in C([0,\infty),[0,\infty))$ such that $\omega_0(0)=0$, and

$$\omega_0(r) \geq \sup_{y\in B_r(x)\setminus\{x\}} \frac{1}{|x-y|^2}\left\{u(y)-u(x)-\langle p,y-x\rangle-\frac{1}{2}\langle X(y-x),y-x\rangle\right\},$$

we verify that $\omega(r) := \sup_{0\leq t\leq r}\omega_0(t)$ satisfies (2.9).

Now, we define $\phi$ by

$$\phi(y) := \langle p,y-x\rangle+\frac{1}{2}\langle X(y-x),y-x\rangle+\psi(|x-y|),$$

where

$$\psi(t) := \int_t^{\sqrt{3}t}\left(\int_s^{2s}\omega(r)dr\right)ds \geq t^2\omega(t).$$

It is easy to check that

$$(D\phi(x),D^2\phi(x))=(p,X) \quad \text{and} \quad (u-\phi)(x)\geq(u-\phi)(y) \quad \text{for } y\in\Omega.$$

Therefore, we conclude the proof. $\square$

Remark. In view of the proof of Step 3, we verify that for $x \in \Omega$,

$$J^{2,+}u(x)=\left\{(D\phi(x),D^2\phi(x))\in\mathbf{R}^n\times S^n \;\middle|\; \begin{array}{l}\exists\phi\in C^2(\Omega) \text{ such that } u-\phi\\ \text{attains its maximum at } x\end{array}\right\},$$

$$J^{2,-}u(x)=\left\{(D\phi(x),D^2\phi(x))\in\mathbf{R}^n\times S^n \;\middle|\; \begin{array}{l}\exists\phi\in C^2(\Omega) \text{ such that } u-\phi\\ \text{attains its minimum at } x\end{array}\right\}.$$

Thus, we intuitively know $J^{2,\pm}u(x)$ from their graph.

Example. Consider the function $u\in C([-1,1])$ in (2.3). From the graph below, we may conclude that $J^{2,-}u(0)=\emptyset$, and $J^{2,+}u(0)=(\{1\}\times[0,\infty))\cup(\{-1\}\times[0,\infty))\cup((-1,1)\times\mathbf{R})$. See Fig 2.4.1 and 2.4.2.

We omit how to obtain $J^{2,\pm}u(0)$ of this and the next examples.

We shall examine $J^{2,\pm}$ for discontinuous functions. For instance, consider the Heaviside function:

$$u(x) := \begin{cases} 1 & \text{for } x\geq 0,\\ 0 & \text{for } x<0.\end{cases}$$

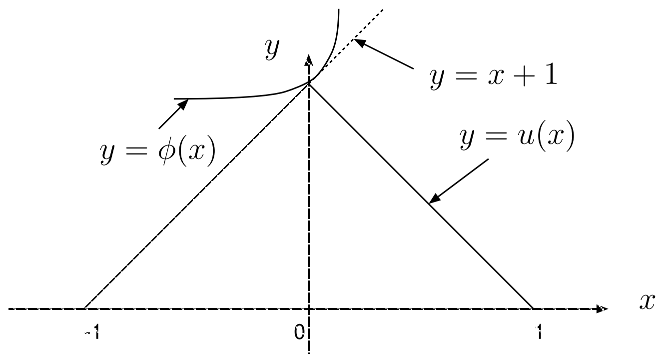


Fig 2.4.1

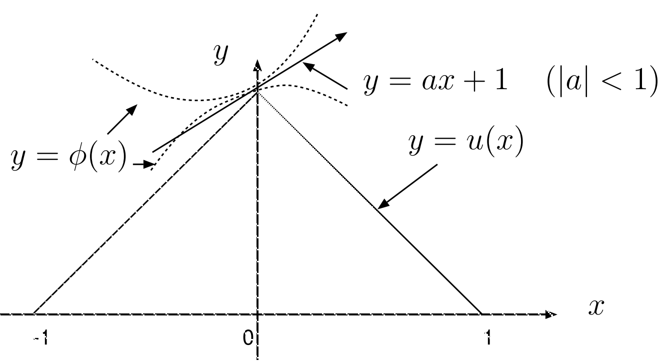


Fig 2.4.2

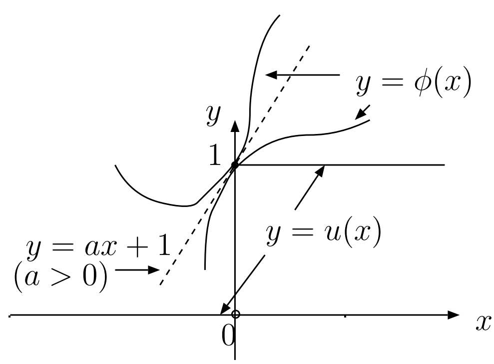


Fig 2.5

We see that $J^{2,-}u(0) = \emptyset$ and $J^{2,+}u(0) = (\{0\} \times [0,\infty)) \cup ((0,\infty) \times \mathbf{R})$. See Fig 2.5.

In order to deal with "boundary value problems" in section 5, we prepare some notations: For a set $K \subset \mathbf{R}^n$, which is not necessarily open, we define semi-jets of $u : K \to \mathbf{R}$ at $x \in K$ by

$$J_K^{2,+}u(x) := \left\{ (p,X) \in \mathbf{R}^n \times S^n \;\middle|\; \begin{array}{ll} u(y) \le & u(x) + \langle p, y-x \rangle \\ & +\frac{1}{2}\langle X(y-x), y-x \rangle \\ & +o(|y-x|^2) \quad \text{as } y \in K \to x \end{array} \right\},$$

$$J_K^{2,-}u(x) := \left\{ (p,X) \in \mathbf{R}^n \times S^n \;\middle|\; \begin{array}{ll} u(y) \ge & u(x) + \langle p, y-x \rangle \\ & +\frac{1}{2}\langle X(y-x), y-x \rangle \\ & +o(|y-x|^2) \quad \text{as } y \in K \to x \end{array} \right\},$$

and

$$\overline{J}_K^{2,\pm}u(x) := \left\{ (p,X) \in \mathbf{R}^n \times S^n \;\middle|\; \begin{array}{c} \exists x_k \in K \text{ and } \exists (p_k, X_k) \in J_K^{2,\pm}u(x_k) \\ \text{such that } (x_k, u(x_k), p_k, X_k) \\ \to (x, u(x), p, X) \text{ as } k \to \infty \end{array} \right\}.$$

Remark. It is obvious to verify that

$$x \in \Omega \quad \Longrightarrow \quad J_\Omega^{2,\pm}u(x) = J_{\overline{\Omega}}^{2,\pm}u(x) \quad \text{and} \quad \overline{J}_\Omega^{2,\pm}u(x) = \overline{J}_{\overline{\Omega}}^{2,\pm}u(x).$$

For $x \in \Omega$, we shall simply write $J^{2,\pm}u(x)$ (resp., $\overline{J}^{2,\pm}u(x)$) for $J_\Omega^{2,\pm}u(x) = J_{\overline{\Omega}}^{2,\pm}u(x)$ (resp, $\overline{J}_\Omega^{2,\pm}u(x) = \overline{J}_{\overline{\Omega}}^{2,\pm}u(x)$).

Example. Consider $u(x) \equiv 0$ in $K := [0,1]$. It is easy to observe that $J^{2,+}u(x) = J_K^{2,+}u(x) = \{0\} \times [0,\infty)$ provided $x \in (0,1)$. It is also easy to verify that

$$J_K^{2,+}u(0) = (\{0\} \times [0,\infty)) \cup ((0,\infty) \times \mathbf{R}),$$

and

$$J_K^{2,-}u(0) = (\{0\} \times (-\infty,0]) \cup ((-\infty,0) \times \mathbf{R}).$$

We finally give some properties of $J_{\overline{\Omega}}^{2,\pm}$ and $\overline{J}_{\overline{\Omega}}^{2,\pm}$. Since the proof is easy, we omit it.

**Proposition 2.7.** *For $u : \overline{\Omega} \to \mathbf{R}$, $\psi \in C^2(\overline{\Omega})$ and $x \in \overline{\Omega}$, we have*

$$J^{2,\pm}_{\overline{\Omega}}(u+\psi)(x) = (D\psi(x), D^2\psi(x)) + J^{2,\pm}_{\overline{\Omega}}u(x)$$

*and*

$$\overline{J}^{2,\pm}_{\overline{\Omega}}(u+\psi)(x) = (D\psi(x), D^2\psi(x)) + \overline{J}^{2,\pm}_{\overline{\Omega}}u(x).$$