APPENDIX B

An overview of the theory of p-adic uniformization.

By F. KATO

In this appendix we always denote by K, R, and π a complete discrete valuation field, its valuation ring, and a prime element of R, respectively. We assume that the residue field $k = R/\pi R$ is finite and consists of q elements. The Mumford-Kurihara-Mustafin uniformization is a procedure to construct nice analytic (and in many cases algebraic) varieties by taking discrete Schottky-type quotients of a certain p-adic analogue of symmetric space, so-called *Drinfeld symmetric space* Ω, or its variants.

1. Bruhat-Tits building.

First we survey the construction of a certain simplicial complex, called Bruhat-Tits building (attached to $\mathrm{PGL}(n+1, K)$), which will be closely related with the Drinfeld symmetric space. Let n be a positive integer and V an $n+1$ dimensional vector space over K. A *lattice* in V is a finitely generated R-submodule of V which spans V over K. Every lattice is therefore a free R-module of rank $n+1$. Let $\widetilde{\Delta}_0$ be the set of all lattices in V. We say that two lattices M_1 and M_2 are *similar* if there exists $\lambda \in K^\times$ such that $M_1 = \lambda M_2$. The similarity is obviously an equivalence relation. We denote by Δ_0 the set of all similarity classes of lattices in V.

Definition 1.1. The *Bruhat-Tits building* (attached to $\mathrm{PGL}(n+1, K)$) is the finite dimensional simplicial complex Δ with the vertex set Δ_0 defined as follows: A finite subset $\{\Lambda_0, \ldots, \Lambda_l\}$ of Δ_0 forms an l-simplex if and only if, after permuting indices if necessary, one can choose $M_i \in \Lambda_i$ for $0 \leq i \leq n+1$ such that

$$M_0 \supsetneqq M_1 \supsetneqq \cdots \supsetneqq M_l \supsetneqq \pi M_0.$$

The role of M_0 is by no means important; for example, one can shift the indices like $M_1 \supsetneqq \cdots \supsetneqq M_l \supsetneqq \pi M_0 \supsetneqq \pi M_1$.

To understand the structure of Δ, let us fix one vertex $\Lambda = [M]$. Suppose $\{\Lambda_0, \ldots, \Lambda_l\}$ is an l-simplex having Λ as a vertex; we may assume $\Lambda = \Lambda_0$, and can take $M_i \in \Lambda_i$ for $1 \leq i \leq l$ such that $M \supsetneqq M_1 \supsetneqq \cdots \supsetneqq M_l \supsetneqq \pi M$. Set $\overline{M}_i \colon = M_i/\pi M$. Then we get a flag

$$\overline{M} \supsetneqq \overline{M}_1 \supsetneqq \cdots \supsetneqq \overline{M}_l \supsetneqq 0$$

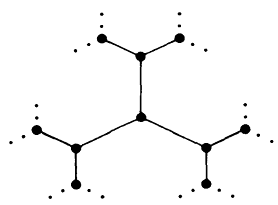
of length $l+1$ consisting of subspaces of the $n+1$-dimensional k-vector space $\overline{M} = M/\pi M$. Moreover, by the elementary algebra one finds that this yields a one-to-one correspondence between l-simplices containing Λ and length $l+1$ flags in $\overline{M} = M/\pi M$. By this, in particular, we know that simplices in Δ are at most n-dimensional, and there are indeed n-dimensional simplices, called chmabers, corresponding to flags of the full length $(= n+1)$. (Recall that by a chamber in an n-dimensional simplicial complex we mean a simplex having the maximal $(= n)$ dimension.) Moreover, we deduce

(i) Δ is an n-dimensional locally finite simplicial complex,
(ii) every simplex in Δ is a face of a chamber.

Furthermore, by the construction,

(iii) the group $\mathrm{Aut}_K(V)/K^\times (\cong \mathrm{PGL}(n+1, K))$ naturally acts on Δ simplicially.

Example 1.2 ($n = 1$ case)**.** In $n = 1$ case, the vertices adjacent to a fix vertex Λ are in one-to-one correspondence with the set of all k-rational points of the projective line $\mathbb{P}^1_k$; in particular, each vertex has exactly $q+1$ adjacent vertices. Moreover, it can be seen that the simplicial complex $\Delta = \Delta^1$ is a *tree*. The following picture shows how it looks like in case $q = 2$:



In general, the geometrical realization $|\Delta|$ of Δ is known to be a contractible topological space.

Example 1.3 ($n = 2$ case)**.** Already in case $n = 2$, the structure of $\Delta = \Delta^2$ is fairly complicated. For a fixed vertex Λ, the set of all vertices adjacent of Λ is divided in two; one for those corresponding to $q^2 + q + 1$ k-rational points in $\mathbb{P}^2_k$ and the other for those corresponding to $q^2 + q + 1$ k-rational lines in $\mathbb{P}^2_k$. So it is convenient to consider the surface B which is obtained by blowing-up of $\mathbb{P}^2_k$ at all its k-rational points. Two vertices adjacent to Λ forms with Λ a 2-simplex if and only if the corresponding lines in B intersect.

2. Drinfeld symmetric space.

To each lattice M in V we associate an R-scheme

$$\mathbb{P}(M) := \mathrm{Proj}(\mathrm{Sym}_R M).$$

Obviously the scheme $\mathbb{P}(M)$ depends only on the similarity class; so we should write as $\mathbb{P}(\Lambda)$, where $\Lambda = [M]$. The scheme $\mathbb{P}(M)$ is isomorphic to $\mathbb{P}^n_R$; but the isomorphism cannot be taken canonically, while there exists the *canonical* isomorphism $\mathbb{P}(V) \cong \mathbb{P}(M) \otimes_R K$ due to the canonical isomorphism $M \otimes_R K \cong V$. So we get a bijection:

$$\Delta_0 \xrightarrow{\sim} \left\{ (\mathbb{P}, \phi) \;\middle|\; \begin{array}{l} \mathbb{P} \xrightarrow{\sim} \mathbb{P}^n_R, \\ \phi \colon \mathbb{P} \otimes_R K \xrightarrow{\sim} \mathbb{P}(V) \end{array} \right\} / \sim,$$

where $(\mathbb{P}_1, \phi_1) \sim (\mathbb{P}_2, \phi_2)$ if and only if there exists an R-isomorphism $\Phi \colon \mathbb{P}_1 \xrightarrow{\sim} \mathbb{P}_2$ which makes the diagram

$$\begin{array}{ccc} \mathbb{P}_1 \otimes_R K & \xrightarrow{\Phi \otimes_R K} & \mathbb{P}_2 \otimes_R K \\ \phi_1 \downarrow & & \downarrow \phi_2 \\ \mathbb{P}(V) & = & \mathbb{P}(V) \end{array}$$

commutative.

Next we consider two vertices Λ_1 and Λ_2 which are adjacent with each other. The canonical isomorphism between the generic fibers induce a natural birational map between $\mathbb{P}(\Lambda_1)$ and $\mathbb{P}(\Lambda_2)$. More explicitly, taking representatives $M_i \in \Lambda_i$ $(i = 1, 2)$ so that $M_1 \supsetneq M_2 \supsetneq \pi M_1$, we have the birational map from $\mathbb{P}(\Lambda_1)$ to $\mathbb{P}(\Lambda_2)$ induced by $M_2 \hookrightarrow M_1$. We define the *join* $\mathbb{P}(\Lambda_1) \vee \mathbb{P}(\Lambda_2)$ to be the closure of the graph in $\mathbb{P}(\Lambda_1) \times_R \mathbb{P}(\Lambda_2)$ of this rational map. It is easily seen that the join $\mathbb{P}(\Lambda_1) \vee \mathbb{P}(\Lambda_2)$ is R-isomorphic to the blow-up of $\mathbb{P}(\Lambda_1)$ at the closed subscheme $\mathrm{Proj}(\mathrm{Sym}_k M_1/M_2)$. In particular, the closed fiber of $\mathbb{P}(\Lambda_1) \vee \mathbb{P}(\Lambda_2)$ consists of two rational varieties intersecting transversally.

Lemma 2.1. The operation $\vee$ is associative and commutative.

The proof is straightforward, and left to the reader. By the lemma, we can extend the construction to any finite subcomplex of Δ. More precisely, for a finite subcomplex S, we take the joins of all vertices in S; the resulting R-scheme is denoted by $\mathbb{P}(\mathsf{S})$. The scheme $\mathbb{P}(\mathsf{S})$ is a regular projective scheme over R having the following properties:

(i) The generic fiber is canonically isomorphic to $\mathbb{P}(V)$.
(ii) The closed fiber is a reduced normal crossing of non-singular rational varieties of which the dual graph coincides with S.

We want to do this for a general convex subcomplex of Δ; this cannot be done in terms of schemes because we have to consider a limit of blow-up's of $\mathbb{P}^n_R$ centered in the closed fiber. But we can do this in terms of formal schemes, or rigid analytic spaces. Let Δ_* be a convex subcomplex of Δ. For any finite subcomplex S of Δ_* we define $\widehat{\Omega}(\mathsf{S})$ to be the completion of $\mathbb{P}(\mathsf{S})$ along the closed fiber. Consider the maximal Zariski open subset $\widehat{\Omega}(\mathsf{S})'$ of $\widehat{\Omega}(\mathsf{S})$ such that, for any finite subcomplex T of Δ_* containing S, the induced morphism $\rho^{\mathsf{T}}_{\mathsf{S}} \colon \widehat{\Omega}(\mathsf{T}) \to \widehat{\Omega}(\mathsf{S})$ gives an isomorphism restricted to the pull-back of $\widehat{\Omega}(\mathsf{S})'$.

Definition 2.2. For a convex subcomplex Δ_* of Δ we define

$$\widehat{\Omega}(\Delta_*) \colon = \bigcup_{\mathsf{S}\colon \text{finite}} \widehat{\Omega}(\mathsf{S})',$$

where S runs through all finite subcomplex of Δ_*. The corresponding rigid analytic space (Raynaud generic fiber) is denoted by $\Omega(\Delta_*)$. In case $\Delta_* = \Delta$ we simply write $\widehat{\Omega} = \widehat{\Omega}(\Delta)$ and $\Omega = \Omega(\Delta)$. The rigid analytic space Ω is called the *Drinfeld symmetric space.*

The formal scheme $\widehat{\Omega}(\Delta_*)$ is formally locally of finite type. Its closed fiber is a normal crossing union of non-singular rational varieties, whose associated intersection graph (so called, dual graph) isomorphic to Δ_*. By definition, the group $\mathrm{PGL}(n+1, K)$ naturally acts on $\widehat{\Omega}$, and hence Ω.

Example 2.3. Let us consider $\widehat{\Omega}$ in $n = 2$. The closed fiber $\widehat{\Omega}_0$ consists of countably many components which are all isomorphic to the surface B considered in 1.3. In each component every k-rational line is a double curve in $\widehat{\Omega}_0$, and every k-rational point is a triple point.

Structure of Ω. First we note that, as a point set, we have an equality

$$\Omega = \mathbb{P}_K^{n,\mathrm{an}} \setminus \bigcup_{\mathrm{H} \in \mathcal{H}} \mathrm{H},$$

where $\mathcal{H}$ is the set of all K-rational hyperplanes.

Here we do not attempt to give a proof of this fact, but rather give a sketchy explanation in order to get a feeling. We limit ourselves to $n = 1$ case just for simplicity. The formal scheme $\widehat{\Omega}$ is, roughly speaking, the *limit* of successive blow-up's

$$\cdots \longrightarrow U_i \xrightarrow{\rho_i} U_{i-1} \longrightarrow \cdots \longrightarrow U_0$$

with $U_0 = \mathbb{P}_R^1$ and ρ_i the blow-up centered at all the k-rational points in the closed fiber of U_{i-1}. Let z be a point in the generic fiber $\mathbb{P}_K^1$ of each U_i. We consider the closure $\overline{z}$ of z in each U_i. Since U_i is a regular scheme, the section $\overline{z}$ intersects the closed fiber at a smooth point transversally. If z is *not* a K-rational point, then for sufficiently large N, $\overline{z}$ intersects the closed fiber of U_i at a non-k-rational smooth point for $i > N$. But, if z is K-rational, $\overline{z}$ intersects the closed fiber always at a k-rational point which is in the next blow-up center. So if we consider the limit, such formal sections determined by K-rational points does not exist any more, while those by non-K-rational points certainly exist. This explains the above equality in $n = 1$. For higher n we can apply the similar argument.

Next we discuss the analytic structure of Ω; viz. we show how to introduce an admissible covering to Ω; since we have constructed the rigid analytic space Ω from the formal model, we must have one such covering such that the associated formal model recovers $\widehat{\Omega}$. This can be actually described in an elegant way in terms of norms: A non-negative real valued function $\alpha\colon V \to \mathbb{R}_{\geq 0}$ is said to be a *norm* over K if it satisfies:

(i) $\alpha(x) > 0$ whenever $x \neq 0$.
(ii) For $a \in K$, $\alpha(ax) = |a|\alpha(x)$.
(iii) $\alpha(x+y) \leq \max\{\alpha(x), \alpha(y)\}$.

Two norms α and α' over K is said to be *similar* if $\alpha' = \lambda\alpha$ for some $\lambda \in \mathbb{R}_{>0}$. The set of all similarity classes of norms on V over K forms in an obvious way a topological space. The following fact is well-known ([**GI63**]):

Proposition 2.4. The topological space of similarlity classes of norms on V over K is $\mathrm{PGL}(n+1, K)$-equivaliantly homeomorphic to the topological realization $|\Delta|$ of the Bruhat-Tits building Δ.

The homeomorphism is given as follows: To any lattice M we need to associate a norm α_M. For a non-zero $x \in V$, the set $\{a \in K \mid ax \in M\}$ is a fractional ideal of R, and hence is of form (π^m). We then define $\alpha_M(x) = q^m$ for non-zero $x \in V$, and $\alpha_M(0) = 0$. It is easily verified that α_M is a norm. The similarity class of α_M depends only on the similarity class of M. Thus we get a mapping from Δ_0 to the set of similarity classes of norms. Let Λ_0 and Λ_1 be vertices adjacent with each other. We can choose a basis $e_0, \ldots, e_n$ of V such that $M_0 = \bigoplus_{i=0}^{n} Re_i$ and $M_1 = \bigoplus_{i=0}^{n-1} Re_i \oplus R\pi e_n$ give representatives of Λ_0 and Λ_1, respectively. Then the norms α_{M_0} and α_{M_1} are given by

$$\begin{aligned} \alpha_{M_0}(\sum_{i=0}^{n} a_i e_i) &= \max\{|a_0|, \ldots, |a_n|\}, \\ \alpha_{M_1}(\sum_{i=0}^{n} a_i e_i) &= \max\{|a_0|, \ldots, |a_{n-1}|, |\pi|^{-1}|a_n|\}, \end{aligned}$$

respectively. For $0 < t < 1$, the class of the norm

$$\alpha_t(\sum_{i=0}^{n} a_i e_i) = \max\{|a_0|, \ldots, |a_{n-1}|, |\pi|^{-t}|a_n|\}$$

corresponds to the point $t\Lambda_0 + (1-t)\Lambda_1$ on the edge connecting Λ_0 and Λ_1.

Now, to any point $z = (z_0 \colon \cdots \colon z_n) \in \mathbb{P}(V)$ (subject to some K-basis $\{e_0, \ldots, e_n\}$of V) we associate, up to invertible factor, a non-negative real valued function

$$\alpha_z(\sum_{i=0}^{n} a_i e_i) = \left| \sum_{i=0}^{n} a_i z_i \right|$$

on V. Note that *this determines a class of norm if and only if z is not lying in any K-rational hyperplane.* Thus we get a mapping

$$\rho \colon \Omega \longrightarrow |\Delta|.$$

The admissible covering in question is actually given by the collection of subsets of Ω consisting of pull-backs of each simplices of $|\Delta|$ by ρ. For example, notation being as above,

$$\rho^{-1}(\Lambda_0) = \{|z_0| = \cdots = |z_n| = 1\} \setminus \bigcup_H \{(z_0 \colon \cdots \colon z_n) \bmod \pi \in H\},$$

where H runs through all hyperplanes in $\mathbb{P}^n_k$, and

$$\rho^{-1}(t\Lambda_0 + (1-t)\Lambda_1) = \{|z_0| = \cdots = |z_{n-1}| = 1, |z_n| = |\pi|^{-t}\}$$

for $0 < t < 1$.

3. Uniformization.

Recall that, once we fix a K-basis of V, the group $\mathrm{PGL}(n+1, K)$ canonically acts on the Bruhat-Tits building Δ and the Drinfeld symmetric space Ω. We shall consider discrete groups in $\mathrm{PGL}(n+1, K)$ and the quotients of Δ and Ω by them. A subgroup $\Gamma \in \mathrm{PGL}(n+1, K)$ is said to be *hyperbolic* if it acts on Δ discretely and freely.

To a hyperbolic subgroup Γ we associate a convex subcomplex Δ_Γ of Δ by the following recipe: In general, a convex subcomplex generated by vertices of form

$$\left[\sum_{i=0}^{n} R \cdot \pi^{\sigma_i} \mathsf{Y}_i\right], \qquad \sigma_i \in \mathbb{Z}, \qquad 0 \leq i \leq n,$$

subject to a basis $\mathsf{Y}_0, \ldots, \mathsf{Y}_n$ of V, is said to be an *apartment*. The geometrical realization of each apartment is a triangulation of $\mathbb{R}^n$ of A_n-type, and Δ is the union of all apartments. There exists an obvious bijection between the set of all apartments and the set of all K-split maximal tori in $\mathrm{PGL}(n+1, K)$. For any K-split maximal tori T the untersection $\Gamma \cap T$ is a commutative discrete group of rank at most n. If this rank is n we call the apartment corresponding to T a Γ-*apartment*, and we define Δ_Γ to be the union of all Γ-apartments. Clearly the group Γ acts on Δ_Γ; if the quotient Δ_Γ/Γ is a finite simplicial complex, we say Γ is *normal hyperbolic*.

Now let us be given a normal hyperbolic subgroup Γ. For each $\gamma \in \Gamma$ and each finite subcomplex S of Δ_Γ, $\gamma\mathsf{S}$ is again a finite subcomplex of Δ_Γ, and then, γ induces an R-isomorphism $\mathbb{P}(\mathsf{S}) \cong \mathbb{P}(\gamma\mathsf{S})$ which induces on generic fibers $(= \mathbb{P}^n_K)$ exactly the linear automorphism γ. Considering all S, gluing, and taking Raynaud generic fiber give a rigid analytic automorphism of $\Omega(\Delta_\Gamma)$ which we denote again by γ. Thus we get an homomorphism

$$\Gamma \longrightarrow \mathrm{Aut}_{K\text{-rig.}}\Omega(\Delta_\Gamma).$$

Theorem 3.1 (Mustafin [**Mus78**], Kurihara [**Kur80**]). Let Γ be a normal hyperbolic subgroup of $\mathrm{PGL}(n+1, K)$ and Δ_Γ as above. Then there exists, unique up to isomorphisms, a rigid analytic space X_Γ, smooth, proper and of finite type over K, and a topological covering map $\mathsf{p}\colon \Omega(\Delta_\Gamma) \to X_\Gamma$ such that

(i) $\mathsf{p} \circ \gamma = \mathsf{p}$ for any $\gamma \in \Gamma$,
(ii) for $x, y \in \Omega(\Delta_\Gamma)$, $\mathsf{p}(x) = \mathsf{p}(y)$ if and only if $x = \gamma(y)$ for some $\gamma \in \Gamma$.

In the references, this theorem has been stated in terms of formal schemes; viz. they took quotients of $\widehat{\Omega}(\Delta_\Gamma)$ to get a formal scheme $\widehat{X}_\Gamma$ flat and formally of finite type over R. We have an admissible covering of X_Γ induced from that of $\Omega(\Delta_\Gamma)$; in terms of formal scheme, the closed fiber of $\widehat{X}_\Gamma$ is a divisor of normal crossings with rational components having the dual group isomorphic to Δ_Γ/Γ.

Algebraicity. In $n = 1$ case the rigid analytic curve X_Γ is algebraizable for any Γ ([**Mu72b**]). In higher dimensions, we know the following sufficient condition: Let Γ be a torsion-free *uniform lattice* in $\mathrm{PGL}(n+1, K)$, viz. a discrete co-compact subgroup of finite co-volume without elements of finite order. Then, since it is finite co-volume, we have $\Delta_\Gamma = \Delta$. Hence in this case, we are going to take a quotient of the Drinfeld symmetric space Ω.

Theorem 3.2 (loc. cit.). If Γ is a torsion-free uniform lattice, then the quotient rigid analytic variety X_Γ can be algebraizable to a non-singular projective variety having the ample canonical class.

In terms of formal scheme, this means that the relative canonical sheaf of $\widehat{X}_\Gamma/\mathrm{Spf}R$ is ample.

4. Examples.

Tate curve. Let T be the maxial torus consisting of all invertible diagonal matrices. We take n elements $q_i \in K^\times$ $(1 \leq i \leq n)$ with $|q_i| < 1$ and set $q_0 = 1$. Define $\Gamma \subset T$ to be the set of all diagonal matrices with (i,i)-entry an integer power of q_i for $0 \leq i \leq n$. Then Γ is a normal hyperbolic subgroup of $\mathrm{PGL}(n+1, K)$ with Δ_Γ the apartment corresponding to T. If $n = 1$, the curve $\Omega(\Delta_\Gamma)/\Gamma$ is known to be the so-called *Tate curve*; in this case $\Omega(\Delta_\Gamma)$ is simply $\mathbb{P}^1_K$ minus two points.

Mumford curve. Consider again the $n = 1$ case. The rigid analytic curve X_Γ with Γ generated by at least two elements is called a *Mumford curve.* A Mumford curve is, in fact, algebraized to a non-singular projective curve of genus greater than 1. Moreover, the formal model provided by the admissible covering inhereted from $\Omega(\Delta_\Gamma)$ gives a stable k-split multiplicative reduction of that curve. The converse is also known:

Theorem 4.1 (Mumford [**Mu72b**]). Let X be a smooth and proper rigid analytic curve over K, and suppose X admits a formal model with k-split multiplicative reduction. Then X is isomorphic to X_Γ for a unique Γ.

Surface case. In higher dimension it becomes extremely difficult to find normal hyperbolic groups; but there are several known examples of torsion-free uniform lattices. Let Γ is a torsion-free uniform lattice, and N the number of Γ-orbits in the vertex set Δ_0. Mumford [**Mu79**] calculated the several numerical invariants of the quotient surface X_Γ:

(i) $\chi(\mathcal{O}_{X_\Gamma}) = N(q-1)^2(q+1)/3$.
(ii) $(\mathrm{c}_{1,X_\Gamma})^2 = 3\mathrm{c}_{2,X_\Gamma} = 3N(q-1)^2(q+1)$.
(iii) $\mathfrak{q}(X_\Gamma) = 0$,

where $\mathfrak{q}$ denotes the irregularity. It is very interesting that the Mumford-Kurihara-Mustafin uniformization in two dimension always produces the surfaces which satisfies the equality in Miyaoka-Yau inequality. Mumford [**Mu79**] constructed one example of Γ in $q = 2$ such that $N = 1$; the resulting surface is one of the so-called "fake projective planes". Recently, Cartwright, Mantero, Steger, and Zappa constructed several examples of lattices in $q = 2$ and $q = 3$. Making use of this groups, in [**IK98**], Ishida and the author discussed other possible fake projective planes, and showed that there are at least three fake projective planes (incl. Mumford's one) which are not isomorphic to each other. We will see in more detail the construction of Mumford's fake projective plane in the next section.

Shimura variety case. Let D be a quaternion algebra over a totally real field F. We assume that D ramifies at every infinite place except one ∞_0. Let Γ_∞ be a subgroup of $D^\times/F^\times$ defined by some congruence condition; we shall assume that it is "sufficiently small". Then it is classically known that Γ_∞ acts discontinuously on the upper $\frac{1}{2}$ plane H such that the quotient H/Γ_∞ is a *projective* curve (Shimura curve of PEL-type). The canonical model Sh is defined over some ray-class field of F.

There must be a finite place ν at which B ramifies. Let p be the residue characteristic of ν. We assume that Γ_∞ is maximal at ν. Then the Shimura curve Sh has a bad reduction at ν, and due to Cherednik [**Čer76**] and Drinfeld [**Dri76**], the associated rigid analytic space over F_ν^{ur} has a p-adic uniformization, where F_ν^{ur} is the maximal unramified extension of F_ν. Let $\overline{D}$ be the quaternion algebra which has local invariants obtained precisely by switching those of D at ν and ∞_0 Then there exists a subgroup Γ_ν in $\overline{D}^\times/F^\times$ defined basically by the same congruence relation as that of Γ_∞ such that $\Omega\otimes F^{\mathrm{ur}}/\Gamma_\nu$ is isomorphic to the assocaied analytic space of Sh at ν. (This isomorphism actually decends to some finite unramified extension of F_ν, depending on the data of connected components of Sh.)

This story has been generalized to higher dimension by Rapoport-Zink [**RZ96**] and Boutot-Zink [**BZ95**]. The next section is devoted to explain one special example of the p-adic uniformization of a Shimura surface.

5. Mumford's fake projective plane.

Our example of p-adically uniformizable Shimura surface is constructed by means of some division algebra D. Th following construction is inspired by the Mumford's oroginal paper [**Mu79**].

Let $\zeta = \zeta_7$ be a primitive 7th-root of unity, and set $L\colon = \mathbb{Q}(\zeta)$; L is a cyclic extension of $\mathbb{Q}$ of degree 6 having an intermediate quadratic extension $K\colon = \mathbb{Q}(\lambda)$ ($\cong \mathbb{Q}(\sqrt{-7})$), where $\lambda = \zeta + \zeta^2 + \zeta^4$. The Galois group of L/K is generated by the Frobenius map $\sigma\colon \zeta \mapsto \zeta^2$, and that of $K/\mathbb{Q}$ is generated by the complex conjugation $z \mapsto \overline{z}$. Note that the prime 2 decomposes on K, more explicitly, $2 = \lambda\overline{\lambda}$ gives the prime factorization, and the prime 7 ramifies. We fix an infinite place $\varepsilon\colon K \hookrightarrow \mathbb{C}$ by $\lambda \mapsto \frac{-1+\sqrt{-7}}{2}$.

We set $\mu\colon = \lambda/\overline{\lambda}$, and define the central division algebra D over K of dimension 9 by

$$D\colon = \bigoplus_{i=0}^{2} L\Pi^i$$

with $\Pi^3 = \mu$ and $\Pi z = z^\sigma \Pi$ for $z \in L$. The involution $*$ on D is defined by $z^* = \overline{z}$ for $z \in L$ and $\Pi^* = \overline{\mu}\Pi^2$. It is easily verified that $*$ is positive.

We introduce a non-degenerate anti-symmetric $\mathbb{Q}$-bilinear form $\psi = \psi_b$ on $V = D$ by putting $b = (\lambda - \overline{\lambda}) - \overline{\lambda}\Pi + \overline{\lambda}\Pi^2$. Then define an algebraic group G over $\mathbb{Q}$ as follows: Given a $\mathbb{Q}$-algebra R, we set

$$G(R) \colon = \{g \in ((D \otimes R)^{\mathrm{opp}})^\times \mid \psi(xg, yg) = c(g)\psi(x,y),\ c(g) \in R^\times,\ x, y \in V \otimes R\}.$$

The algebraic group $G_{\mathbb{R}}$ is isomorphic to $\mathrm{GU}(2,1)$, which acts on the two dimensional complex unit-ball B. In the following we will construct an open compact subgroup C in $G(\mathbb{A}_{\mathrm{f}})$. By these data, we can define a Shimura variety

$$\mathcal{S}_C \colon = G(\mathbb{Q}) \setminus \mathsf{B} \times G(\mathbb{A}_{\mathrm{f}})/C.$$

The canonical model of $\mathcal{S}_C$ is defined over $E \colon = \varepsilon(K)$, which we denote by $\mathfrak{Sh}_C$.

The congruence condition. First we define an algebraic group I over $\mathbb{Q}$ by

$$I(R) \colon = \{g \in (\mathrm{M}_d(K) \otimes R)^\times \mid H^{-1}\, {}^t\overline{g} H \in R^\times\}$$

for any $\mathbb{Q}$-algebra R, where

$$H = \begin{bmatrix} 3 & \overline{\lambda} & \overline{\lambda} \\ \lambda & 3 & \overline{\lambda} \\ \lambda & \lambda & 3 \end{bmatrix}.$$

Then I is an inner form of G so that $G(\mathbb{A}_{\mathrm{f}}^2) \cong I(\mathbb{A}_{\mathrm{f}}^2)$.

To define the level structure, it suffices to define the prime-2-part C^2. We define C^2 in the form $C^2 = C^{2,7}C_7$ with $C_7 \in G(\mathbb{Q}_7) \cong I(\mathbb{Q}_7)$ and $C^{2,7}$ being the maximal one:

$$C^{2,7} \colon = \{g \in G(\mathbb{A}_{\mathrm{f}}^{2,7}) \mid \Gamma^{2,7} g = \Gamma^{2,7}\},$$

with $\Gamma^{2,7} = \Gamma \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^{2,7}$, where $\widehat{\mathbb{Z}}^{2,7} = \lim \mathbb{Z}/m\mathbb{Z}$ with the projective limit taken over m such that $(2, m) = (7, m) = 1$.

Now the component C_7 is defined as follows: we have an isomorphism

$$G(\mathbb{Q}_7) \xrightarrow{\sim} \{g \in \mathrm{GL}_3(\widetilde{\mathbb{Q}}_7) \mid g^* H g = c(g)H,\ c(g) \in \mathbb{Q}_7^\times\},$$

where $\widetilde{\mathbb{Q}}_7$ is the ramified quadratic extension of $\mathbb{Q}_7$, and $*$ is the matrix transposition followed by the Galois action. Let $\widetilde{C}_7$ be the maximal open compact subgroup of $G(\mathbb{Q}_7)$ consisiting of g which maps $\Gamma \otimes_{\mathbb{Z}} \mathbb{Z}_7$ to itself. We consider modulo $\sqrt{-7}$ reduction of $\widetilde{C}_7$, which is a subgroup G_0 in $\mathrm{GL}_3(\mathbb{F}_7)$. The matrix H mod $\sqrt{-7}$ is of rank 1, and has 2 dimensional null space N_0. Restricting elements in G_0 to N_0, we obtain a homomorphism

$$\pi \colon \widetilde{C}_7 \longrightarrow \mathrm{GL}_2(\mathbb{F}_7).$$

The last group has $2^6 3^2 7$ elements. Let S be a 2-Sylow subgroup of $\mathrm{GL}_2(\mathbb{F}_7) \cap \{g \mid \det g = \pm 1\}$. Now we set $C_7 \colon = \pi^{-1}(S)$.

Theorem 5.1. The canonical model Sh_C of the Shimura variety obtained by the data as above is a geometrically connected non-singular projective surface over $E = \mathbb{Q}(\sqrt{-7})$ such that the base change $\mathsf{Sh}_C \otimes_E E_\lambda$ is isomorphic over $\mathbb{Q}_2 \cong E_\lambda$ to the Mumford's fake projective plane [**Mu79**].

More precisely, the associated rigid analytic space to $\mathsf{Sh}_C \otimes_E E_\lambda$ is isomorphic to Ω/Γ with

$$\Gamma = I(\mathbb{Q}) \bigcap (J(\mathbb{Q}_2) \times C^2) \subset J(\mathbb{Q}_2) \cong \mathrm{PGL}(3, \mathbb{Q}_2),$$

where $I(\mathbb{Q})$ is considered as a subgroup of $I(\mathbb{A}_\mathrm{f}) \cong J(\mathbb{Q}_2) \times G(\mathbb{A}_\mathrm{f}^2)$. The proof of the theorem is in [**Kat01**], where two more fake projective planes are discussed as examples of Shimura varieties.