CHAPTER 8

Inductive complements

8.1. Examples

Roughly speaking the main idea of this chapter is to discuss the following inductive statement:

> if a two-dimensional pair $(X/Z, D)$ is lc but not klt and $-(K_X + D)$ is nef over Z, then $K_X + D$ is 1, 2, 3, 4 or 6-complementary.

It is known that this assertion is true when $-(K_X + D)$ is big over Z (see Proposition 5.3.1) and in the local case. Unfortunately examples 8.1.1 and 8.1.2 below shows that in general, this is false and some additional assumptions are needed. The main result is the Inductive Theorem 8.3.1 which is a generalization of 5.3.1.

EXAMPLE 8.1.1 ([**Sh3**]). Let $\mathcal{E}$ be a indecomposable vector bundle of rank two and degree 0 over an elliptic curve Z. Then $\mathcal{E}$ is a nontrivial extension

$$0 \longrightarrow \mathcal{O}_Z \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_Z \longrightarrow 0$$

(see e.g., [**Ha**]). Consider the ruled surface $X := \mathbb{P}_Z(\mathcal{E})$. Let $f\colon X \to Z$ be the projection and C a section corresponding to the above exact sequence. Then for the normal bundle of C in X we have $\mathcal{N}_{C/X} = \mathcal{O}_C$, hence $C|_C = 0$. In this situation we also have $-K_X \sim 2C$ (see [**Ha**]) and $(K_X + C)|_C = 0$. This yields $K_X|_C = 0$.

Since $\rho(X) = 2$, the Mori cone $\overline{NE}(X)$ is generated by two rays $R_1 = \mathbb{R}_+[F]$, where F is fiber of X and another ray, say R. Since $C^2 = 0$, C is nef and C generates R. In particular, both $-K_X$ and $-(K_X + C)$ are nef and numerically proportional to C.

We claim that $K_X + C$ is not n-complementary for any n. Indeed, otherwise we have $L \in |-m(K_X + C)|$ such that C is not a component of L. Then $L \cdot C = 0$ and $L \equiv mC$. The divisor $L - mC$ is trivial on fibers, hence $L - mC = f^*N$ for some $N \in \mathrm{Pic}(Z)$. Further, $C \cap L = \varnothing$. From this $(mC - L)|_C \sim 0$ (because $C|_C \sim 0$). Since $f|_C\colon C \to Z$ is an isomorphism, $f|_C^* N = (mC - L)|_C = 0$ gives $N \sim 0$, i.e., $L \sim mC$. Then the linear system $|L|$ determines on X a structure of an elliptic fibration $g\colon X \to \mathbb{P}^1$ with multiple fiber C. Hence $C|_C$ is an m-torsion element in $\mathrm{Pic}(C)$, a contradiction with $C|_C = 0$.

EXAMPLE 8.1.2. Let $X = \mathbb{P}^1 \times \mathbb{P}^1$. We fix a projection $f\colon X \to \mathbb{P}^1$. Let C, H_1, H_2 be different sections of f and F_1, F_2, F_3 different fibers. Consider the log divisor $K_X + D$, where

$$D := C + \frac{1}{7}H_1 + \frac{6}{7}H_2 + \frac{1}{2}F_1 + \frac{2}{3}F_2 + \frac{5}{6}F_3.$$

It is clear that $K_X + D$ is lc and numerically trivial. We claim that there are no regular complements of $K_X + D$. Indeed, assume that $K_X + D$ has a regular n-complement K_X+D^+. Then $K_F+\mathrm{Diff}_C(\frac{1}{2}F_1+\frac{2}{3}F_2+\frac{5}{6}F_3)$ is also n-complementary. Therefore $n = 6$ (see 4.1.12). On the other hand, by definition we have

$$D^+ \geq C + \frac{1}{6}H_1 + H_2 + \frac{1}{2}F_1 + \frac{2}{3}F_2 + \frac{5}{6}F_3,$$

a contradiction with $K_X + D^+ \equiv 0$.

The following example shows that under additional assumptions we can expect some inductive theorems even if $-(K + D)$ is not big.

EXAMPLE 8.1.3. Let $D = C + B$ be a boundary on $X := \mathbb{P}^2$ such that $K_X + D$ is lc and $C := \lfloor D \rfloor \neq 0$. Assume that $-(K_X + D)$ is nef. Then there exists a regular complement of $K_X + D$. Indeed, for some $n \in \mathcal{R}_2$ the log divisor $K_C + B|_C$ is n-complementary. Since $H^1(\mathbb{P}^2, \mathcal{L}) = 0$ for any invertible sheaf $\mathcal{L}$ on $\mathbb{P}^2$, n-complement (where $n \in \mathcal{R}_2$) on C can be extended to some $\mathbb{Q}$-divisor D^+ on X. We write $D^+ = C + B^+$. By Corollary 2.2.7, $K_X + D^+$ is lc near C. It is sufficient to show that $K_X + D^+$ is lc everywhere. But in the opposite case $K_X + C + \alpha B^+$ is not lc for some $\alpha < 1$. By Connectedness Lemma, $\mathrm{LCS}(X, C + \alpha B^+)$ is connected. This gives a contradiction.

8.2. Nonrational case

Now we consider the question 8.1 for the case when the surface X is nonrational.

THEOREM 8.2.1 ([**Sh3**]). *Let X be a normal projective nonrational surface and D a boundary on X such that $K_X + D$ is lc and $-(K_X + D)$ is nef. Assume that*

(i) *$K_X + D$ is not klt;*
(ii) *there is a boundary D' such that $D' \geq D$ and $K_X + D'$ is lc and numerically trivial.*

Then $K_X + D$ is n-complementary for $n \in \mathcal{R}_2$.

PROOF. By taking a log terminal modification we may assume that X is smooth, $K_X + D$ is dlt and $\lfloor D \rfloor \neq 0$ (see 3.1.1). Since $D \neq 0$, $\kappa(X) = -\infty$. So there is a morphism $f\colon X \to Z$ onto a curve Z of genus $g \geq 1$.

LEMMA 8.2.2. *Let $f\colon X \to Z$ be a contraction from a projective surface onto a curve of genus $g \geq 1$. Assume that $K_X + D$ is lc and $-(K_X + D)$ is nef. Furthermore, assume that the general fiber of f is a smooth rational curve. Then no components of* $\mathrm{Supp} D$ *are contained in fibers.*

PROOF. Let L be such a component. Replace (X, D) with a dlt modification. Then we may assume

(*) $\quad K_X + (1-\varepsilon)D$ is klt for $0 < \varepsilon \ll 1$. In particular, X is $\mathbb{Q}$-factorial.

If the fiber $f^{-1}(f(L))$ is reducible, there is its component $L_1 \neq L$ meeting L. Then $L_1^2 < 0$ and $(K_X + D - \varepsilon' L) \cdot L_1 < 0$ for $0 < \varepsilon' \ll 1$. Thus L_1 generates an extremal ray which is negative with respect to $K_X + (1-\varepsilon'')D - \varepsilon' L$ for $0 < \varepsilon'' \ll \varepsilon'$. By Contraction Theorem [**KMM**, 3-2-1] we can contract L_1 over Z. This contraction preserves all assumptions of the lemma as well as assumption (*) (however, we can lose the dlt property of (X, D)). Continuing the process, we get the situation when the fiber containing L is irreducible. Similarly, components of all reducible fibers can be contracted. We obtain a model $f' \colon X' \to Z$ such that all fibers are irreducible. Moreover f' is an extremal K-negative contraction. Hence $\rho(X'/Z) = 1$. By our construction, $K_{X'} + D'$ is lc, X' is $\mathbb{Q}$-factorial and $L' \subset \operatorname{Supp} D'$ is a fiber of f'. Let R be an extremal ray on X' other than that generated by fibers of f'. Then $R \cdot L' > 0$ and $(K_{X'} + D') \cdot R \leq 0$. Therefore $(K_{X'} + D' - \delta' L') \cdot R < 0$ for $\delta' > 0$. Hence there is a curve M on X' generating R (as above, if $K_{X'} + D'$ is not dlt, we can use Contraction Theorem for $K_{X'} + (1-\delta)D' - \delta' L'$, see also Appendix 11.2). In this situation, $M \simeq \mathbb{P}^1$ (see Proposition 11.2.5). But then the base curve Z also should be rational, a contradiction with $g \geq 1$. □

COROLLARY 8.2.3. *Notation as in Lemma 8.2.2. Then the pair (X, D) has at worst canonical singularities.*

SKETCH OF PROOF. Replace (X, D) with a suitable log terminal modification (see Proposition 3.1.2) and apply Lemma 8.2.2. □

Going back to the proof of Theorem 8.2.1, denote $C := \lfloor D \rfloor$ and $B := \{D\}$. Let L be a component of C. By Lemma 8.2.2, $f(L) = Z$. If $\rho(X/Z) > 1$, there is a curve E in a (reducible) fiber such that $E^2 < 0$ and $L \cdot E > 0$. Since $(K_X + D) \cdot E \leq 0$, E is a -1-curve and

$$-K_X \cdot E = L \cdot E = 1, \qquad E \cdot (D - L) = 0.$$

Hence we can contract E and by Lemma 4.3.2 and Remark 4.3.3 we can pull back complements under this contraction. Repeating the process, we reach the situation when $\rho(X/Z) = 1$. Thus we may assume that X is a (smooth) ruled surface over a nonrational curve Z, i.e., $X = \mathbb{P}_Z(\mathcal{E})$, where $\mathcal{E}$ is a rank two vector bundle on Z. Moreover, $K_X + D$ is dlt (see Proposition 1.1.6). Then

$$0 \geq (K_X + D) \cdot L \geq (K_X + L) \cdot L = 2p_a(L) - 2 \geq 0.$$

This implies

$$(K_X + D) \cdot L = (K_X + L) \cdot L = (D - L) \cdot L = 0. \tag{8.1}$$

Thus $p_a(L) = 1$ and L is a smooth elliptic curve. Hence $g = 1$.

CLAIM. *We may assume that $(K_X + L)^2 \geq 0$.*

PROOF. Let F be a general fiber of f. Since $\rho(X) = 2$, there is exactly one extremal ray $R \neq \mathbb{R}_+[F]$ on X. Assume that $(K_X + L)^2 < 0$. Then $-(K_X + L)$ is not nef, $D \neq L$ and $(K_X + L) \cdot R > 0$. If $L^2 \leq 0$, then L generates an extremal ray (see Proposition 11.2.1), so $R = \mathbb{R}_+[L]$. This contradicts (8.1). Therefore, $L^2 > 0$. In particular, $L \cdot R \geq 0$.

Further, if $L \cdot R > 0$, then R is negative with respect to $K_X + D - L$. Since $K_X + D - L$ is dlt, the ray R must be generated by a rational curve. This implies that Z is also rational, a contradiction. Finally, we have $L \cdot R = 0$. Then (8.1) implies $[D - L] \in R$ (recall that $D - L$ is effective). By the Hodge Index Theorem, $R^2 < 0$. Thus, $D - L = aL'$ and $R = \mathbb{R}_+[L']$, where $0 < a \leq 1$ and L' is an irreducible curve with $L'^2 < 0$. If $a < 1$, then R is negative with respect to $K_X + D + \varepsilon L'$ for $0 < \varepsilon \ll 1$. Again we have a contradiction. Hence $a = 1$ and $D = L + L'$. As in (8.1) we have

$$(K_X + D) \cdot L' = (K_X + L') \cdot L' = L \cdot L' = 0.$$

Since $\rho(X) = 2$, $-(K_X + L')$ is nef, so $(K_X + L')^2 \geq 0$. Replacing L with L' we get our assertion. □

Taking into account the equality $K_X^2 = 8(1 - g(Z)) = 0$, we obtain

$$0 \leq (K_X + L)^2 = L \cdot K_X, \qquad L^2 \leq 0.$$

Therefore L generates an extremal ray R on X (see Proposition 11.2.1). It cannot be K_X-negative (otherwise R is generated by a rational curve). Hence $L \cdot K_X = L^2 = 0$ and in relations above equalities hold. Thus inequality (8.1) gives that L and all the components $D_i \subset \operatorname{Supp}(D - L)$ are numerically proportional (because $\rho(X) = 2$). In particular,

$$(8.2) \qquad D_i^2 = 0, \quad \forall i \qquad \text{and} \qquad L \cap D_i = D_i \cap D_j = \varnothing, \quad \forall i, j.$$

Let F be a general fiber of f. Then $F \cdot (K_X + D) \leq 0$ and $K_X \cdot F = -2$. This yields $1 \leq C \cdot F \leq D \cdot F \leq 2$.

LEMMA 8.2.4. *Let $f\colon X \to Z$ be a ruled surface over an elliptic curve and let C be a reduced divisor such that $K_X + C \equiv 0$. Then $2(K_X + C) \sim 0$. Moreover, if C is reducible, then $K_X + C \sim 0$.*

PROOF. Since $K_X + C \equiv 0$, $K_X + C = f^*N$ for some integral divisor of degree 0 on Z. First we assume that C is irreducible. Then C is a smooth elliptic curve and we have $f_C^*N = (K_X + C)|_C = K_C = 0$. Hence $2N \sim 0$ because f_C is of degree two. Now we assume that $C = C_1 + C_2$, where C_1, C_2 are sections. Similarly, $f_{C_1}^*N = (K_X + C)|_C = K_C = 0$. Hence $N \sim 0$ because f_{C_1} is an isomorphism. □

Now we finish the proof of Theorem 8.2.1.

If $C \cdot F = 2$ (i.e., C is a 2-section of f), then $D = C$, $B = 0$ and $(K_X + C) \cdot F = 0$. Hence $K_X + C \equiv 0$. By Lemma 8.2.4, $2(K_X + C) \sim 0$, i.e., we have an 1 or 2-complement (with $C^+ = C$).

If $C \cdot F = 1$, then C is a section of f. Recall that $X = \mathbb{P}_Z(\mathcal{E})$, where $\mathcal{E}$ is a rank two vector bundle on Z. By assumption (ii) of the theorem, the surface X is not such as in Example 8.1.1. On the other hand, by (8.2) $C^2 = 0$ and the vector bundle $\mathcal{E}$ has even degree. From the classification of rank two vector bundles over elliptic curves (see e.g., [**Ha**]), we obtain that $\mathcal{E}$ is of splitting type. Hence there is a section C_1 such that $C \cap C_1 = \varnothing$. Write $B = \sum b_i B_i$. Then

$$\sum b_i B_i \cdot F \leq 1. \tag{8.3}$$

From 8.2.4 we have

$$-K_X \sim C + C_1, \tag{8.4}$$

where C, C_1 is a pair of disjoint sections.

By (8.4) and by the definition of complements, $D^+ = C + C_1$ is an 1-complement of $K_X + D$ if $b_i < 1/2$ whenever $B_i \neq C_1$. By (8.3) this does not hold only if $B = \frac{1}{2}B_1$, where B_1 is a 2-section. Consider this case. As in the proof of Lemma 8.2.4, $2(K_X + C + B) = f^*N$ and $f_C^*N = 2(K_X + C + B)|_C = 2K_C + B_1|_C = 0$. Therefore $N = 0$ and $2(K_X + C + B) \sim 0$. This proves our theorem. □

8.3. The Main Inductive Theorem

THEOREM 8.3.1 (Inductive Theorem, [**Sh3**]). *Let (X, D) be a projective log surface such that $D \in \Phi_{\mathbf{m}}$, $K_X + D$ is lc, but not klt and $-(K_X + D)$ is nef. Assume additionally that there exists a boundary $D' \leq D$ such that $K_X + D'$ is klt and $-(K_X + D')$ is nef and big. Then there exists a regular complement of $K_X + D$.*

Shokurov proved this theorem under weaker assumptions [**Sh3**]. In particular, he showed that if we remove condition $D \in \Phi_{\mathbf{m}}$ in the theorem, we obtain a weaker result: there exists an n-complement of $K_X + D$, where

$$n \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, \\ 23, 24, 25, 26, 27, 28, 29, 30, 31, 35, 36, 40, 41, 42, 43, 56, 57\}.$$

REMARK. In contrast with 5.3.1, in Theorem 8.3.1 we cannot say that any regular complement of $K_C + \mathrm{Diff}_C(B)$ can be extended to X. For example, let $X = \mathbb{P}^1 \times \mathbb{P}^1$, C is a fiber of $\mathrm{pr}_1 \colon X \to \mathbb{P}^1$ and $B := \frac{1}{2}(B' + B'') + \frac{2}{3}(B_1 + B_2 + B_3)$, where B', B'' are fibers of pr_1 and B_1, B_2, B_3 are fibers of pr_2. Then $K_X + C + B \equiv 0$, $K_C + \mathrm{Diff}_C(B) = K_C + \frac{2}{3}(P_1 + P_2 + P_3)$ is 3-complementary and $K_X + C + B$ has no 3-complements. Indeed, otherwise for a 3-complement, we have $B^+ \geq \frac{2}{3}(B' + B'' + B_1 + B_2 + B_3)$, a contradiction. However, $K_X + C + B$ has a 6-complement with $B^+ = B$.

COROLLARY 8.3.2. *Notation as in Theorem 8.3.1. We can take a regular complement $K_X + D^+$ so that $a(E, D) = -1$ implies $a(E, D^+) = -1$ for any E.*

PROOF. Since $D^+ \geq D$ (see Remark 4.2.8), we have $a(E, D^+) \leq a(E, D)$. □

PROOF OF THEOREM 8.3.1. If $-(K_X + D)$ is big, then by Proposition 5.3.1 there is a regular complement. Therefore we assume that $K_X + D \equiv 0$ or $\kappa(X, -(K_X + D)) = 1$.

8.3.3. Applying a minimal log terminal modification $f\colon \overline{X} \to X$ we may assume that $K_{\overline{X}} + \overline{D} = f^*(K_X + D)$ is dlt and $\lfloor \overline{D} \rfloor \neq 0$. Take the crepant pull back

$$f^*(K_X + D') = K_{\overline{X}} + \overline{D'}, \quad \text{with} \quad f_*\overline{D'} = D'.$$

By 1.1.6, $K_{\overline{X}} + \overline{D'}$ is klt, but it is not necessarily a boundary. Consider the new boundary

$$\overline{D''} := \overline{D'} + t(\overline{D} - \overline{D'}), \quad \text{where } 0 < 1 - t \ll 1.$$

Then $K_{\overline{X}} + \overline{D''}$ is klt and $-(K_{\overline{X}} + \overline{D''})$ is nef and big. Further, by 4.3.1 we can push-down complements. So we replace X, D, D' with $\overline{X}$, $\overline{D}$, $\overline{D''}$. Thus we assume now that $K_X + D$ is dlt, $\lfloor D \rfloor \neq 0$, $-(K_X + D)$ is nef, and there exists a boundary $D' \leq D$ such that $K_X + D'$ is klt and $-(K_X + D')$ is nef and big. By Lemma 5.4.1, X is rational. Set $C := \lfloor D \rfloor$ and $B := \{D\}$. The following lemma shows that the Mori cone $\overline{NE}(X)$ is polyhedral and generated by contractible extremal curves.

LEMMA 8.3.4. *Let $(X/Z, B)$ be a log variety such that $K_X + B$ is klt and $-(K_X + B)$ is nef and big over Z. Then there exists a new boundary $B' \geq B$ on X such that $K_X + B'$ is again klt and $-(K_X + B')$ is ample over Z.*

SKETCH OF PROOF. Let H be a very ample divisor on X (over Z). By Kodaira's Lemma, $|-n(K_X + B) - H| \neq \varnothing$ for some $n \gg 0$ (see e.g., [**KMM**, 0-3-4]). Take $L \in |-n(K_X + B) - H|$ and put $B' := B + \varepsilon L$. □

The lemma shows that we can contract all extremal rays on X. Moreover, if an extremal ray R on X is birational and generated by a curve L which is not contained in C, then the contraction preserves all assumptions (see 1.1.6). If additionally R is $(K_X + D)$-trivial, we can pull back regular complements by Proposition 4.3.2.

8.3.5. Division into cases. By Proposition 3.3.2, C has at most two connected components. As in proofs of Theorems 6.1.6 and 7.2.10 we distinguish the following cases: C is disconnected, C is a smooth elliptic curve or a wheel of smooth rational curves, $C \simeq \mathbb{P}^1$, C is a tree of rational curves. Also we should separate cases $K_X + C + B \equiv 0$ and $K_X + C + B \not\equiv 0$. If $K_X + C + B \not\equiv 0$, then by Base Point Free Theorem the linear system $|-m(K_X + C + B)|$ determines a contraction $\nu\colon X \to \mathbb{P}^1$. By Lemma 8.3.4, $-(K_X + D'')$ is ample for some boundary D'', hence a general fiber is rational. Then we consider cases when C is contained in a fiber of ν and C has a horizontal component.

8.3.6. Case: C is disconnected. By Proposition 3.3.2 there exists a contraction $f\colon X \to Z$ onto a curve such that $C = C_1 + C_2$ is a pair of two disjoint smooth sections (in particular, $K_X + D$ is plt). A general fiber F of f is $\mathbb{P}^1$ (see 8.3.5), so $-K_X \cdot F = C \cdot F = 2$, $B \cdot F = 0$ and B is contained in fibers of f. Since X is rational, $Z \simeq \mathbb{P}^1$. In our case $-(K_X + D)$ is numerically trivial on a general fiber of f, so it is numerically trivial on all fibers. Contracting curves in fibers we get the situation when $\rho(X/Z) = 1$. We can pull back all complements by Proposition 4.3.2. If $C_1^2 > 0$, then $-(K_X + (1-\varepsilon)C_1 + C_2 + B)$ is nef and big for any $\varepsilon > 0$. By Proposition 5.3.1 there exists a regular complement of $K_X + (1-\varepsilon)C_1 + C_2 + B$. By Corollary 4.1.8 $K_X + C_1 + C_2 + B$ also has a regular complement. Therefore we may assume that $C_1^2 \le 0$ and $C_2^2 \le 0$. Then both C_1 and C_2 generate extremal rays which must coincide because $\rho(X) = 2$. The ray cannot be birational, so $C_1^2 = C_2^2 = 0$. This shows that there exists a nonbirational contraction $g\colon X \to \mathbb{P}^1$ such that C_1 and C_2 are fibers of g. If $K_X + C_1 + C_2 + B \not\equiv 0$, then again $-(K_X + (1-\varepsilon)C_1 + C_2 + B)$ is ample for $\varepsilon > 0$. As above (by Proposition 5.3.1 and Corollary 4.1.8) there is a regular complement. Therefore we may assume that $K_X + C_1 + C_2 + B \equiv 0$. Now it is sufficient to verify $n(K_X + C + B) \sim 0$ for some $n \in \mathcal{R}_2$. By Lemma 5.1.3 the numerical equivalence in $\operatorname{Pic}(X)$ coincides with linear one. Therefore it is sufficient to show only that $n(K_X + C + B)$ is Cartier for some $n \in \mathcal{R}_2$. Take

$$n := \min\{r \in \mathbb{N} \mid rB \text{ is integral.}\}$$

Since $\lfloor B \rfloor = 0$, $n > 1$. By Theorem 7.2.11, K_X+C+B has a regular n_1-complement $K_X + C + B^+$ near C_1 (a fiber of $g\colon X \to \mathbb{P}^1$). Then $B^+ \ge B$ by Remark 4.2.8. This yields $B^+ = B$, $n_1 B$ is integral and $n_1(K_X + C + B) \sim 0$ near C_1 (because C_1 intersects all the components of B). Hence $n \mid n_1$. Similarly, we have a regular n_2-complement near C_2 and $n \mid n_2$. Let $n' := \operatorname{lcm}(n_1, n_2)$. Then $n'B$ is integral and $n'(K_X + C + B)$ is Cartier near C. Let F be a fiber of f and let $P_i := F \cap C_i$. By Adjunction,

$$0 \ge (K_X + C + F) \cdot F = \deg K_F + \deg \operatorname{Diff}_F(C)$$

and

$$\operatorname{Diff}_F(C) \ge P_1 + P_2.$$

Therefore $\operatorname{Diff}_F(C) = P_1 + P_2$ and X is smooth outside of C. Further, $n'(K_X + C + B)$ is Cartier everywhere on X and it is sufficient to show that $n' \in \mathcal{R}_2$. Assume the opposite. Then we have (up to permutations C_1 and C_2): $n_1 = 4$, $n_2 = 6$ and $n = 2$. Since $2B$ is integral, $B \in \Phi_{\mathbf{sm}}$. Corollary 2.2.9 gives $\operatorname{Diff}_{C_i}(B) \in \Phi_{\mathbf{sm}}$. By 4.1.12,

$$\operatorname{Diff}_{C_1}(B) = \frac{1}{2}Q_1 + \frac{3}{4}Q_2 + \frac{3}{4}Q_3, \quad \operatorname{Diff}_{C_2}(B) = \frac{1}{2}R_1 + \frac{2}{3}R_2 + \frac{5}{6}R_3,$$

where $Q_1, Q_2, Q_3 \in C_1$, $R_1, R_2, R_3 \in C_2$ are some points. On the other hand,

$$(C_1, \operatorname{Diff}_{C_1} B) \simeq (C_2, \operatorname{Diff}_{C_2} B)$$

(see [**Ut**, 12.3.4]) a contradiction. This proves our theorem in the case when C is disconnected.

8.3.7. Case: C is connected and $p_a(C) \geq 1$. By Lemma 8.3.8 below there is an 1-complement. Note that in this case the assumption $D \in \Phi_{\mathbf{m}}$ is not needed. First we claim that $K_X + C + B \equiv 0$. Indeed, by Adjunction we have

$$0 \geq (K_X + C + B) \cdot C \geq (K_X + C) \cdot C \geq \deg K_C \geq 0$$

If $K_X + C + B \not\equiv 0$, then C is contained in fibers of $\nu \colon X \to \mathbb{P}^1$. But ν has rational fibers (see 8.3.5), a contradiction.

LEMMA 8.3.8. *Let $(X, C + B)$ be a rational projective log surface, where C is the reduced and B is the fractional part of the boundary. Assume that $K_X + C + B$ is analytically dlt, $K_X + C + B \equiv 0$, C is connected and $p_a(C) \geq 1$. Then $B = 0$, $K_X + C \sim 0$, X is smooth along C and has only Du Val singularities outside.*

PROOF. By Lemma 6.1.7 and Remark 6.1.8, X is smooth and $B = 0$ near C. Replace X with a minimal resolution and $C + B$ with the crepant pull back. It is sufficient to show only that $B = 0$ and $K_X + C \sim 0$. Now we contract -1-curves on X. Since C is not a tree of rational curves, it cannot be contracted. This process preserves all the assumptions, so on each step $C \cap \operatorname{Supp} B = \varnothing$. Since every -1-curve E has positive intersection number with $C + B$, we have either $C \cdot E \geq 1$, $B \cdot E = 0$ or $C \cdot E = 0$, $B \cdot E > 0$. If $B \neq 0$, then whole $\operatorname{Supp} B$ also cannot be contracted. At the end we get $X = \mathbb{P}^2$ or $\mathbb{F}_n$ (a Hirzebruch surface). In the case $X = \mathbb{P}^2$, $C + B \equiv -K_X$ is ample. Hence $\operatorname{Supp}(C + B)$ is connected. If $B \neq 0$ we derive a contradiction. Consider the case $X = \mathbb{F}_n$. Then

$$0 = (K_X + C + B)|_C = (K_X + C)|_C + B|_C = K_C + B_C$$

and the last two terms are nonnegative. Therefore $(K_X + C)|_C = B|_C = 0$ and $p_a(C) = 1$. On the other hand, for a general fiber F of $\mathbb{F}_n \to \mathbb{P}^1$ one has

$$(K_X + C + B + F)|_F = K_F + (C + B)|_F.$$

In particular, $(C + B) \cdot F \leq 2$. Since $p_a(C) = 1$, C is not a section of $\mathbb{F}_n \to \mathbb{P}^1$ at a general point. Hence $C \cdot F = 2$ and $B \cdot F = 0$. Recall that $B \cdot C = 0$. Since $\rho(\mathbb{F}_n) = 2$, we have $B \equiv 0$ and $B = 0$. We proved that $p_a(C) = 1$, $B = 0$ and $K_X + C \sim 0$ in the case $X = \mathbb{F}_n$. Therefore on our original X one also has $B = 0$. By Proposition 4.3.2 we can pull back an 1-complement of $K_X + C^+ = K_X + C$ under contractions of -1-curves. □

8.3.9. Case: $C \simeq \mathbb{P}^1$ (the exceptional case) and $K_X + C + B \equiv 0$. In this case, $K_X + C + B$ is plt. We claim that after a number of birational contractions $B \in \Phi_{\mathbf{sm}}$. Indeed, otherwise there is a component B_i of B with coefficient $b_i \notin \Phi_{\mathbf{sm}}$ and $b_i > 6/7$. If $B_i^2 > 0$, then as in 8.3.6 we can reduce b_i a little so that $-(K_X + C + B)$ becomes big and obtain a regular complement by Proposition 5.3.1 (note that $b_i^+ = 1$). This complement is also a complement of our original $K_X + C + B$ by Proposition 4.1.7. If $B_i^2 < 0$, then we can contract B_i and pull back complements by Proposition 4.3.2 and Remark 4.3.3. Consider the case $B_i^2 = 0$. By Base Point Free Theorem the linear system $|mB_i|$ determines a contraction $f: X \to \mathbb{P}^1$ such that B_i is a fiber of f. Contracting curves $\neq C$ in reducible fibers we get the situation when $\rho(X) = 2$, i.e. $f: X \to \mathbb{P}^1$ is an extremal contraction. Let $g: X \to Z$ be another extremal contraction on X and F a nontrivial fiber of g. Then $F \simeq \mathbb{P}^1$ and $F \cap B_i \neq \varnothing$ (because F is not a fiber of f). Assume that g is nonbirational and F is sufficiently general. Then X is smooth along F, $\mathrm{Diff}_F(C+B) = (C+B)|_F$ and F intersects $\mathrm{Supp}(C+B)$ transversally. Hence $\mathrm{Diff}_F(C+B) \in \Phi_{\mathbf{m}}$ and we can write

$$\mathrm{Diff}_F(C+B) = \sum_j b_j P_{j,l} + \sum_{s=1}^{r} Q_s,$$

where $\{P_{j,1}, \dots, P_{j,r_j}\} = B_j \cap F$ and $\{Q_1, \dots, Q_r\} := C \cap F$. Moreover, the coefficient b_i of $\mathrm{Diff}_F(C+B)$ at points $B_i \cap F$ satisfies $6/7 < b_i < 1$. Further,

$$\deg \mathrm{Diff}_F(C+B) = \deg(-K_F) = 2.$$

Easy computations as in 4.1.12 show that this is impossible. Indeed,

$$2 = \deg \mathrm{Diff}_F(C+B) = r + \sum_j b_j r_j. \tag{8.5}$$

Clearly, $\sum_{j \neq i} b_j r_j = 2 - r - b_i r_i > 0$ (otherwise $b_i r_i = 2 - r \in \{0, 1, 2\}$ but $6/7 < b_i < 1$). Since $b_j \in \Phi_{\mathbf{m}}$, we have $\sum_{j \neq i} b_j r_j \geq 1/2$. Thus

$$2 - r = \sum_j b_j r_j \geq 1/2 + (6r_i)/7 > 1.$$

This gives us $r = 0$ and $r_i = 1$. Hence $\sum_{j \neq i} b_j r_j = 2 - b_i < 8/7$. It is easy to check that the last inequality has no solutions with $r_j \in \mathbb{N}$ and $b_j \in \Phi_{\mathbf{m}}$.

If g is birational and contract C (i.e., $C = F$), then $\mathrm{Diff}_C(B) \in \Phi_{\mathbf{m}}$ (by Corollary 2.2.9) and has a coefficient $> 6/7$. Moreover, $K_X + C + B$ is plt, so $K_C + \mathrm{Diff}_C(B)$ is klt. As above we derive a contradiction with $\deg \mathrm{Diff}_C(B) = 2$. Finally, if g is birational and does not contract C, we can replace X with Z. We get the situation when $\rho = 1$ and $B_i^2 > 0$, as above.

Thus we may assume now that $B \in \Phi_{\mathbf{sm}}$. By Base Point Free Theorem and assumptions of the theorem, there exists $n \in \mathbb{N}$ such that $n(K_X + C + B) \sim 0$. Let n be the index of $K_X + C + B$ (i.e., the minimal positive integer with this property) and $\varphi: X' \to X$ the log canonical n-cover (see 1.3). It is sufficient to show that $n \in \mathcal{R}_2$. Write

$$K_{X'} + C' = \varphi^*(K_X + C + B),$$

where $C' = \varphi^* C$. Then $K_{X'} + C'$ is linearly trivial and plt (see Proposition 1.2.1). By Adjunction every connected component of C' is a smooth elliptic curve.

First we assume that C' is connected. By construction, $K_{X'} + C' \sim 0$ and we can identify $\mathrm{Gal}(C'/C)$ with $\mathbb{Z}_n$. We claim that $\mathrm{Gal}(C'/C)$ contains no translations. Indeed, let $\xi \in \mathrm{Gal}(C'/C)$ a translation. Then we put $X'' := X'/\langle \xi \rangle$ and $C'' := C'/\langle \xi \rangle$. By Lemma 8.3.8, $K_{X''} + C''$ is linearly trivial and plt (because

$p_a(C'') = 1$). But then $n''(K_X + C + B) \sim 0$, where n'' is the degree of $X'' \to X$. By assumptions n is the smallest positive with this property. The contradiction shows that $\mathrm{Gal}(C'/C)$ contains no translations. Then $\mathrm{Gal}(C'/C)$ is a finite group of order 2, 3, 4, or 6 (see e.g., **[Ha]**).

If C' is disconnected, then $\mathrm{Gal}(X'/X)$ interchange connected components of C'. By Proposition 3.3.2 there is a contraction $X' \to Z'$ onto an elliptic curve with rational fibers such that components of $C' = C_1' + C_2'$ are sections. This contraction must be $\mathrm{Gal}(X'/X)$-equivariant because X' has a unique structure of a contraction with rational fibers. Set $G_0 := \mathrm{Ker}(\mathrm{Gal}(X'/X) \to \mathrm{Aut}(Z'))$. Since the ramification locus of $X' \to X$ does not contain components of C', $G_0 \simeq \mathbb{Z}_2$. As above we consider $X'' := X'/G_0$ and $C'' := C'/G_0$. Then C'' is a smooth elliptic curve, hence $K_{X''} + C'' \sim 0$ by Lemma 8.3.8. This contradicts our choice of n.

8.3.10. Case: C is a tree of rational curves and $K_X + C + B \equiv 0$. By Lemma 6.1.9, C is a chain and B has coefficients $= 1/2$ near C. As in case 8.3.9 we claim that after some birational contractions $B \in \Phi_{\mathbf{sm}}$. Let B_k be a component with nonstandard coefficient. If $B_k^2 \neq 0$, then we can argue as in case 8.3.9. The only nontrivial case is $B_k^2 = 0$ and $B_k \cap C = \varnothing$. Then again $|mB_k|$ determines a contraction $f\colon X \to \mathbb{P}^1$. Clearly, B_k is a fiber and C is contained in a fiber (because $C \cdot B_k = 0$). There is an extremal rational curve F which is not contained in fibers. Then $F \cap B_k \neq \varnothing$. If $F^2 < 0$, we contract F and replace X with a new birational model. If $F^2 = 0$ we derive a contradiction computing $\mathrm{Diff}_F(C + B)$ as in (8.5) of case 8.3.9.

Thus we may assume that $B \in \Phi_{\mathbf{sm}}$. As in case 8.3.9 take the log canonical n-cover $\varphi\colon X' \to X$. It is sufficient to show that $n \in \mathcal{R}_2$. Again we can write

$$\varphi^*(K_X + C + B) = K_{X'} + C' \sim 0,$$

where $C' := \varphi^* C$. Obviously, C' is reducible. We claim that $K_{X'} + C'$ is dlt. By Proposition 1.2.1 $K_{X'} + C'$ is plt outside of $\varphi^{-1}(\mathrm{Sing}C)$. Recall that the ramification divisor of φ is $\mathrm{Supp}B$. Hence none of irreducible components of the ramification divisor intersects $\mathrm{Sing}C$. At points $\mathrm{Sing}C$ the surface X is smooth, so φ is étale over $\mathrm{Sing}C$. Therefore $K_{X'} + C'$ is dlt and X' is smooth at points $\varphi^{-1}(\mathrm{Sing}C)$. Since $K_{X'} + C' \sim 0$, X' is smooth along C' (see 2.1.2) and $p_a(C') = 1$. By Lemma 6.1.7, C' is a wheel of smooth rational curves and by our construction $\mathrm{Gal}(X'/X) = \mathbb{Z}_n$ acts on C' faithfully. Let $C' = \sum_{i=1}^r C_i'$ be the irreducible decomposition and $P_1, \cdots, P_r$ singular points of C'. If $\mathrm{Gal}(X'/X)$ contains an element ξ such that $\xi \cdot C_i' = C_i'$ and $\xi \cdot P_i = P_i$ for all i, then $C'' := C'/\langle\xi\rangle$ is again a wheel of smooth rational curves. As in case 8.3.9 we derive a contradiction. Therefore $\mathrm{Gal}(X'/X)$ acts faithfully on the dual graph of C' and then it is a subgroup of the dihedral group $\mathfrak{D}_r$. The same arguments show that every $\xi \in \mathrm{Gal}(X'/X)$ has a fixed point on C'. This is possible only if $\mathrm{Gal}(X'/X) \simeq \mathbb{Z}_2$. Therefore $n = 2$ and $K_X + C + B$ is 2-complementary.

8.3.11. Case: $K_X + C + B \not\equiv 0$ and C is contained in a fiber of ν. We may contract all components of reducible fibers of $\nu\colon X \to \mathbb{P}^1$ which are different from components of C. Thus $C = \nu^{-1}(P)$, $P \in \mathbb{P}^1$ is a fiber and all other fibers of ν are irreducible. Again we may assume that ν-horizontal components of B have standard coefficients (otherwise some $b_i > 6/7$ and $-(K_X + C + B - \varepsilon B_i)$ is nef and big for $0 < \varepsilon \ll 1$, hence we can use Proposition 5.3.1 and Corollary 4.1.8). Note that the horizontal part B_{hor} of B is nontrivial (because a general fiber of ν is $\mathbb{P}^1$). Further, there is a regular n-semicomplement of $K_C + \operatorname{Diff}_C(B)$ (see Theorem 4.1.10). For sufficiently small $\varepsilon > 0$ the $\mathbb{Q}$-divisor $-(K_X+C+B-\varepsilon B_{\text{hor}})$ is nef and big. Thus we can extend n-semicomplements of $K_C + \operatorname{Diff}_C(B - \varepsilon B_{\text{hor}})$ from C by Proposition 5.3.1. If $n+1$ is not a denominator of coefficients of B_{hor}, then by Proposition 4.1.7 we obtain a regular complement of $K_X + C + B$. If C is reducible, then by Theorem 4.1.10 we can take $n = 2$. On the other hand, by Lemma 6.1.9 coefficients of B_{hor} are equal to $1/2$. Therefore $n + 1 = 3$ is not a denominator of coefficients of B_{hor} in this case and there is a 2-semicomplement of $K_X + C + B$. Now we assume that $C \simeq \mathbb{P}^1$. Then $\rho(X) = 2$. By 4.1.12 and by Corollary 2.2.9 we have the following possibilities:

$\operatorname{Diff}_S(B)$	n
$\frac{1}{2}P_1 + \cdots + \frac{1}{2}P_4$	2
$\frac{2}{3}P_1 + \frac{2}{3}P_2 + \frac{2}{3}P_3$	3
$\frac{1}{2}P_1 + \frac{3}{4}P_2 + \frac{3}{4}P_3$	4
$\frac{1}{2}P_1 + \frac{2}{3}P_2 + \frac{5}{6}P_3$	6

By Corollary 2.2.8, $n+1$ is not a denominator of coefficients of B_{hor} in all cases.

8.3.12. Case: $K_X + C + B \not\equiv 0$ and C is not contained in a fiber of ν. So, we assume that there is a horizontal component $C_i \subset C$ (i.e., such that $\nu(C_i) = \mathbb{P}^1$). It is clear that $-(K_X + C + B - \varepsilon C_i)$ is nef and big for a sufficiently small positive ε. If $C_i \subsetneq C$ (i.e., C is reducible), then the same trick as in 8.3.6 (using Corollary 4.1.8) gives the existence of regular complements. Thus we may assume that $C = C_i$ and $\nu(C) = \mathbb{P}^1$. In particular, K_X+C+B is plt. Contracting curves $\neq C$ in fibers we get the situation, when fibers are irreducible, i.e., $\rho(X) = 2$. We can pull back complements by 4.3.2. There are two subcases: (a) C is a section of ν, and (b) C is a 2-section of ν.

8.3.13. If C is a section of ν, then the horizontal part B_{hor} of B is nontrivial and either $B_{\text{hor}} = \frac{1}{2}B_1 + \frac{1}{2}B_2$ or $B_{\text{hor}} = \frac{1}{2}B_0$, where B_1, B_2 are sections and B_0 is a 2-section. As above we can take a regular n-complement for $K_X+C+B-\varepsilon B_{\text{hor}}$. If $n \neq 1$, then again this gives a regular complement of $K_X + C + B$ by Proposition 4.1.7. But on $C \simeq \mathbb{P}^1$ there exists a regular 2, 3, 4, or 6-complement for the boundary $\Delta := \operatorname{Diff}_C(B) \geq \operatorname{Diff}_C(B - \varepsilon B_{\text{hor}})$. Indeed, otherwise by Theorem 4.1.10 there is an 1-complement $K_C + \Delta^+$. By Corollary 2.2.9, $\Delta \in \Phi_{\mathbf{m}}$. Therefore $\Delta^+ \geq \Delta$ and Δ is supported in one or two points (because $\deg \Delta^+ = 2$).

Then $K_C + \Delta^+$ is also n-complement for any n. This shows that we have a regular complement in the case when C is a section.

8.3.14. Now let C be a 2-section of ν. Then B is contained in the fibers of X. Since $C \simeq \mathbb{P}^1$, the restriction $\nu\colon C \to \mathbb{P}^1$ has exactly two ramification points, say $P_1, P_2 \in C$. Put $Q_i := \nu(P_i)$ and $F_i := \nu^{-1}(Q_i)_{\mathrm{red}}$, $i = 1, 2$.

LEMMA 8.3.15. *Notation as in 8.3.14. Let F be a fiber of ν such that $F \neq F_1, F_2$ and $F \cap C = \{P', P''\}$ (where $P' \neq P''$). Then*

(i) $\mathrm{Sing} X \subset C \cup F_1 \cup F_2$;
(ii) *$K_X + C + F$ is lc and linearly trivial near F;*
(iii) *$K_X + F$ is plt;*
(iv) *$\mathrm{Diff}_C(B)$ is invariant under the natural Galois action of $\mathbb{Z}_2$ on $C \to \mathbb{P}^1$.*

PROOF. First we show that $K_X + C + F$ is lc. Assume the converse and regard X as an analytic germ near F. Let C_1, C_2 be analytic components of C. If $K_X + C_1 + C_2 + F$ is not lc near C_1, then $K_X + (1-\varepsilon)C_1 + C_2 + (1-\varepsilon)F$ is not, either. But in this case, $\mathrm{LCS}(X, (1-\varepsilon)C_1 + C_2 + (1-\varepsilon)F)$ is not connected. This contradicts Theorem 2.3.1.

Now, by Adjunction

$$-K_F \equiv \mathrm{Diff}_F(C) \geq P' + P''. \tag{8.6}$$

On the other hand,

$$\deg K_F + \deg \mathrm{Diff}_F(C) = (K_X + C) \cdot F \leq 0, \tag{8.7}$$
$$\deg \mathrm{Diff}_F(C_1 + C_2) \leq 2.$$

This gives that in (8.6) and (8.7) equalities hold. Hence $\mathrm{Diff}_F(C) = P' + P''$ and $\mathrm{Sing} X = \{P', P''\}$ near F and proves (i). By Theorem 2.1.3, $K_X + C + F$ is Cartier near F. Since $K_X + C + F \equiv 0$, we have $K_X + C + F \sim 0$. This proves (ii). (iii) easily follows by (ii). Further,

$$\mathrm{Diff}_F(C) = \left(1 - \frac{1}{m'}\right) P' + \left(1 - \frac{1}{m''}\right) P'', \qquad m', m'' \in \mathbb{N}.$$

To show (iv) we just note that $\nu\colon X \to \mathbb{P}^1$ is of type A^* of Theorem 7.1.12 near F (because $K_X + F$ is plt). In particular, we have $m' = m''$. □

As in Corollary 4.1.11 using that $\mathrm{Diff}_C(B) \in \Phi_{\mathbf{m}}$ and $\deg \mathrm{Diff}_C(B) \leq 2$, we have the following cases (up to permutations of P_1, P_2):

$$\mathrm{Diff}_C(B) = \begin{cases} \alpha P_1 + \beta P_2 \\ \alpha P' + \alpha P'' \\ \alpha P_1 + \frac{1}{2} P' + \frac{1}{2} P'' \\ \frac{1}{2} P_1 + \frac{2}{3} P' + \frac{2}{3} P'', \end{cases} \qquad \mathrm{Supp} B \subset \begin{cases} F_1 + F_2 \\ F \\ F_1 + F \\ F_1 + F, \end{cases} \tag{8.8}$$

where $F := \nu^{-1}(Q)_{\mathrm{red}}$ for some $Q \in \mathbb{P}^1$, $Q \neq Q_1, Q_2$, $\{P', P''\} = F \cap C$ and $\alpha, \beta \in \Phi_{\mathbf{m}}$.

Our strategy is very simple: we construct a boundary $B' \geq B$ such that B' is contained in fibers of ν, $K_X + C + B'$ is lc and numerically trivial. If $B' \in \Phi_{\mathbf{m}}$, then we can use proved cases with $K_X + C + B \equiv 0$. If $B' \notin \Phi_{\mathbf{m}}$, then we show that $n(K_X + C + B') \sim 0$ for some $n \in \mathcal{R}_2$. The numerical equivalence in $\mathrm{Pic}(X)$ coincides with linear one (see Lemma 5.1.3), it is sufficient to show only that $n(K_X + C + B')$ is Cartier. Note that $K_X + C + B'$ is trivial on fibers of ν (because B' is contained in fibers). On the other hand,

$$(K_X + C + B') \cdot C = \deg(K_C + \mathrm{Diff}_C(B')) = -2 + \deg \mathrm{Diff}_C(B').$$

Since $\rho(X) = 2$, we have

$$K_X + C + B' \equiv 0 \quad \Leftrightarrow \quad \deg \mathrm{Diff}_C(B') = 2. \tag{8.9}$$

LEMMA 8.3.16. *Let $\nu\colon X \to Z \ni o$ be a germ of a contraction from a surface to a curve, $F_1 := \nu^{-1}(o)_{\mathrm{red}}$, and $C \subset X$ a germ of a curve such that $C \cap F_1$ is one point. Assume that $\rho(X/Z) = 1$, $K_X + C$ is plt and numerically trivial. Then there is an 1 or 2-complement $K_X + C + \alpha' F_1$ (with $\alpha' \in \{\frac{1}{2}, 1\}$) such that $K_X + C + \alpha' F_1$ is not plt at $C \cap F_1$.*

PROOF. Note that a general fiber of ν is $\mathbb{P}^1$ and C is a 2-section of ν. Take α so that $K_X + C + \alpha F_1$ is maximally log canonical (i.e., α is maximal such that with the log canonical property of $K_X + C + \alpha F_1$, see 5.3.3). Then $0 < \alpha \leq 1$. We claim that $K_X + C + \alpha F_1$ is not plt at $C \cap F_1$. Indeed, otherwise $\mathrm{LCS}(X, C + \alpha F_1)$ is not connected near F_1. This is a contradiction with Proposition 3.3.1. In particular, $(X/Z \ni o, C + \alpha F_1)$ is not exceptional. By Theorem 7.2.11, $K_X + C + \alpha F_1$ has an 1, or 2-complement $K_X + C + R$ which is not plt at $C \cap F_1$. In particular, $R \neq 0$. Since C is a 2-section of ν, R has no horizontal components. Hence $R = F_1$ or $\frac{1}{2} F_1$. □

Now we consider possibilities of (8.8) step by step.

Subcase: $\mathrm{Diff}_C(B) = \alpha P_1 + \beta P_2$. Then X is smooth outside of $F_1 \cup F_2$. By Lemma 8.3.16 there are $\alpha', \beta' \in \{\frac{1}{2}, 1\}$ such that $\alpha' F_1 + \beta' F_2 \geq B'$, $K_X + C + \alpha' F_1 + \beta' F_2$ is lc and not plt at P_1, P_2 and $2(K_X + C + \alpha' F_1 + \beta' F_2) \sim 0$ near F_1, F_2. By Adjunction (see 2.2.6 and 2.2.7) $\mathrm{Diff}_C(\alpha' F_1 + \beta' F_2) = P_1 + P_2$. By (8.9), $K_X + C + \alpha' P_1 + \beta' P_2 \equiv 0$. Moreover, $2(K_X + C + \alpha' F_1 + \beta' F_2)$ is Cartier near F_1 and F_2. Therefore $2(K_X + C + \alpha' F_1 + \beta' F_2)$ is Cartier on X (because $\mathrm{Sing} X \subset F_1 \cup F_2$).

Subcase: $\mathrm{Diff}_C(B) = \alpha P' + \alpha P''$. Then $C \cap F = \{P', P''\}$ for some fiber F of ν. By Lemma 8.3.15 $K_X + C + F$ is lc. Since $\mathrm{Diff}_C(F) = P' + P''$, $K_X + C + F$ is numerically trivial. By the above cases with $K_X + C + B \equiv 0$ there is a regular complement of $K_X + C + F$ (actually, $p_a(C + F) = 1$ and we can use Case 8.3.7 to show the existence of an 1-complement).

Subcase: $\mathrm{Diff}_C(B) = \alpha P_1 + \frac{1}{2}P' + \frac{1}{2}P''$. By Lemma 8.3.16 there is $\alpha_1 > 0$ such that $2(K_X + C + \alpha_1 F_1 + B) \sim 0$ near F_1, $K_X + C + \alpha_1 F_1 + B$ is lc and not plt at P_1. Since $\mathrm{Diff}_C(\alpha_1 F_1 + B) = P_1 + \frac{1}{2}P' + \frac{1}{2}P''$, $K_X + C + \alpha_1 F_1 + B \equiv 0$. Again by the above cases with $K_X + C + B \equiv 0$ there is a regular complement of $K_X + C + B$ (more precisely, $K_X + C + \alpha_1 F_1 + B$ is not plt, so we can use Case 8.3.10).

Subcase: $\mathrm{Diff}_C(B) = \frac{1}{2}P_1 + \frac{2}{3}P' + \frac{2}{3}P''$. Then we take $B' = B + \alpha_1 F_1$ so that $\mathrm{Diff}_C(B') = \frac{2}{3}P_1 + \frac{2}{3}P' + \frac{2}{3}P''$. By (8.9), $K_X + C + B' \equiv 0$. We show that $6(K_X + C + B')$ is Cartier. First note that $2(K_X + C)$ (and $2(K_X + C + B')$) is Cartier along F_2. Indeed, $P_2 \in X$ is smooth and $C \cdot F_2 \in \mathbb{N}$. On the other hand, $\nu^{-1}(Q_2) \cdot C = 2$. Hence the multiplicity k of the fiber $\nu^{-1}(Q_2) = kF_2$ is at most 2 and $2F_2 \sim 0$ near F_2 over $\mathbb{P}^1$. By Lemma 8.3.16 there is a 2-complement $K_X + C + \alpha_1 F_2$ near F_2 which is not plt at P_2. If $\alpha_1 = 1$, then $2(K_X + C + F_2) \sim 2(K_X + C) \sim 0$ near F_2. Assume that $\alpha_1 = 1/2$. Then $P_2 = \mathrm{Diff}_C(\frac{1}{2}F_2) = \frac{1}{2}F_2|_C$. Hence $F_2 \cdot C = 2$ and the fiber $\nu^{-1}(Q_2) = F_2$ is not multiple. So F_2 is Cartier and X is smooth along F_2. This yields $2(K_X + C + \frac{1}{2}F_2) \sim 0$ near F_2.

Write

$$B = \gamma_1 F_1 + \gamma F, \qquad \gamma_1, \gamma \in \mathbf{\Phi_m}$$

and

$$\mathrm{Diff}_C(0) = \left(1 - \frac{1}{m_1}\right) P_1 + \left(1 - \frac{1}{m}\right) P' + \left(1 - \frac{1}{m}\right) P'', \quad m_1, m \in \mathbb{N}.$$

Then by Corollary 2.2.8,

$$1 - \frac{1}{m_1} + \frac{\gamma_1 n_1}{m_1} = \frac{1}{2}, \qquad 1 - \frac{1}{m} + \frac{\gamma n}{m} = \frac{2}{3},$$

where $n_1, n \in \mathbb{N}$. Since $\gamma_1, \gamma \geq 1/2$, we have $n_1 = n = 1$. This gives only the following possibilities for m:

$m = 1$ (i.e., X is smooth along F) and $\gamma = 2/3$;
$m = 3$ (i.e., case A^* of Theorem 7.1.12 with $m = 3$) and $\gamma = 0$.

It is easy to see that $3(K_X + C + B')$ is Cartier near F in both cases.

Now it is sufficient to show only that $6(K_X + C + B') \sim 0$ near F_1. Similarly, we obtain only the following possibilities for m_1:

$m_1 = 1$ (i.e., $P_1 \in X$ is smooth) and $\gamma_1 = 1/2$;
$m_1 = 2$ (i.e., $P_1 \in X$ is Du Val of type A_1) and $\gamma_1 = 0$,

Assume that $P_1 \in X$ is smooth. Since $K_X + C + \frac{1}{2}F_1$ is plt, F_1 intersects C transversally (see 4.4.4). Hence $B' = B + \frac{1}{6}F_1$ and the coefficient of F_1 in B' is $2/3$. We claim that $6(K_X + C + \frac{2}{3}F_1) \sim 0$ near F_1. Indeed, $C \cdot F_1 = 1$ and $C \cdot \nu^{-1}(Q_1) = 2$ implies that the multiplicity of the fiber $\nu^{-1}(Q_1)$ is at most 2 and $2F_1 \sim 0$ over $\mathbb{P}^1$. On the other hand, by Lemma 8.3.16 there is a 2-complement

$K_X + C + \alpha_1 F_1$ near F_1 which is not plt at P_1. By our assumptions, $K_X + C + \frac{1}{2}F_1$ is plt. Thus $\alpha_1 = 1$ and $2(K_X + C + F_1) \sim 2(K_X + C) \sim 0$ near F_1. This yields

$$6\left(K_X + C + \frac{2}{3}F_1\right) \sim 4F_1 \sim 0$$

near F_1.

Now we assume that $P_1 \in X$ is Du Val of type A_1. Then F_1 is not a component of B. As above, by Lemma 8.3.16 there is a 2-complement $K_X + C + \alpha_1 F_1$ near F_1 which is not plt at P_1. If $\alpha_1 = 1$, then $K_X + C + F_1$ is lc and by Theorem 2.1.3, $C \cdot F_1 = 1/2$. It is easy to see that in this case

$$\mathrm{Diff}_C\left(\frac{1}{3}F_1\right) = \left(\frac{1}{2} + \frac{1}{3}C \cdot F_1\right) P_1 = \frac{2}{3}P_1 \quad \text{near} \quad P_1.$$

Hence we can take $B' = B + \frac{1}{3}F_1$. Then near F_1 we have

$$6(K_X + C + B') = 6(K_X + C + F_1) - 4F_1 \sim -4F_1.$$

On the other hand, the multiplicity of the fiber $\nu^{-1}(Q_1)$ divides 4 (because $\nu^{-1}(Q_1) \cdot C = 2$ and $C \cdot F_1 = 1/2$). This gives as $4F_1 \sim 0$ and $6(K_X + C + B') \sim 0$ near F_1.

Finally, let $\alpha_1 = 1/2$. By Lemma 8.3.16 $K_X + C + \frac{1}{2}F_1$ is lc but not plt at P_1. Then

$$P_1 = \mathrm{Diff}_C\left(\frac{1}{2}F_1\right) = \left(\frac{1}{2} + \frac{1}{2}C \cdot F_1\right) P_1 \quad \text{near} \quad P_1.$$

Hence $C \cdot F_1 = 1$ and as above, $2F_1 \sim 0$. Similarly, we obtain $B' = B + \frac{1}{6}F_1$ and

$$6(K_X + C + B') = 6(K_X + C) + F_1 \sim -3F_1 + F_1 \sim 0.$$

near F_1.

This finishes the proof of Theorem 8.3.1. □

8.4. Corollaries

The following form of Theorem 8.3.1 is very important for applications.

COROLLARY 8.4.1 (see [**P1**]). *Let (X, D') be a log del Pezzo such that $D' \in \Phi_{\mathbf{m}}$. Assume also that there exists a boundary $D \geq D'$ such that $-(K_X + D)$ is nef and $K_X + D$ is not klt. Then $K_X + D'$ has a regular complement which is not klt.*

PROOF. If $K_X + D'$ is not klt, then there is a regular complement by Proposition 5.3.1. From now on we assume that $K_X + D'$ is klt. Replacing D with suitable $D' + \lambda(D - D')$ we also may assume that $K_X + D$ is lc (and not klt).

First we consider the case when $\lfloor D \rfloor \neq 0$. Let D_1 be a component of $\lfloor D \rfloor$. Replace δ_1 with 1:

$$D'' := D_1 + \sum_{i \neq 1} \delta_i D_i, \qquad D' \leq D'' \leq D.$$

If $-(K_X + D'')$ is nef, then we can apply 8.3.1 (because $\lfloor D'' \rfloor \neq 0$ and $D'' \in \Phi_{\mathbf{m}}$). Further, we assume that $-(K_X + D'')$ is not nef. Then there exists a $(K_X + D')$-nonpositive extremal ray R such that $(K_X + D'') \cdot R > 0$. If it is birational, then we contract it. Since $K_X + D$ is nonpositive on R, this preserves the lc property of $K_X + D$ and $K_X + D''$. We can pull back regular complements of $K_X + D''$ because $D'' \in \Phi_{\mathbf{m}}$ (now we are looking for regular complements of $K_X + D''$, see Proposition 4.3.2 and 4.3.3). Note also that $(D'' - D') \cdot R > 0$. Therefore D_1 is not contracted and on each step $K_X + D''$ is not klt. If on some step $-(K_X + D'')$ is nef, we are done. Otherwise continuing the process, we obtain a nonbirational extremal ray R on X such that $(K_X + D'') \cdot R > 0$. But on the other hand,

$$(K_X + D'') \cdot R \leq (K_X + D) \cdot R \leq 0,$$

a contradiction.

Consider now the case $\lfloor D \rfloor = 0$. Then $K_X + D$ is lc, but is not plt. Recall that $K_X + D'$ is klt. As in Proposition 3.1.4 we can construct a blowup $f\colon \widetilde{X} \to X$ with an irreducible exceptional divisor E such that $a(E, D) = -1$, the crepant pull back

$$K_{\widetilde{X}} + \widetilde{D} + E = f^*(K_X + D)$$

is lc and $K_{\widetilde{X}} + \widetilde{D}^\varepsilon + E$ is plt for any $\widetilde{D}^\varepsilon := \widetilde{D}' + \varepsilon(\widetilde{D} - \widetilde{D}')$, $\varepsilon < 1$. Here $\widetilde{D}$ and $\widetilde{D}'$ are proper transforms of D and D', respectively. Note also that $\rho(\widetilde{X}/X) = 1$. Write

$$K_{\widetilde{X}} + \widetilde{D}' + \alpha E = f^*(K_X + D'),$$

where $\alpha < 1$. Assume that there exists a curve C such that $(K_{\widetilde{X}} + \widetilde{D}' + E) \cdot C > 0$. Then $(\widetilde{D} - \widetilde{D}') \cdot C < 0$. Therefore C is a component of $(\widetilde{D} - \widetilde{D}')$ and $C^2 < 0$. Further, $C \cdot E > 0$. Hence $C \neq E$ and we can choose $\widetilde{D}^\varepsilon < \widetilde{D}$ so that $(K_{\widetilde{X}} + \widetilde{D}^\varepsilon) \cdot C < 0$. Therefore C is a $(K_{\widetilde{X}} + \widetilde{D}')$-negative extremal curve and its contraction preserves the lc property of $K_{\widetilde{X}} + \widetilde{D} + E$. Again we can pull back complements of $K_{\widetilde{X}} + \widetilde{D}' + E$ (see Proposition 4.3.2 and 4.3.3). Repeating the process, we get the situation when $-(K_{\widetilde{X}} + \widetilde{D}' + E)$ is nef. All the steps preserve the nef and big property of $-(K_{\widetilde{X}} + \widetilde{D}' + \alpha E)$. If $\alpha \geq 0$, then we apply Theorem 8.3.1 to $(\widetilde{X}, \widetilde{D}' + E, \widetilde{D}' + \alpha E)$. If $\alpha < 0$, then by the monotonicity, $-(K_{\widetilde{X}} + \widetilde{D}')$ is nef and big. Again apply Theorem 8.3.1 to $(\widetilde{X}, \widetilde{D}' + E, \widetilde{D}')$. This concludes the proof of the corollary. □

COROLLARY 8.4.2. *Let (X, B) be a log del Pezzo surface such that $B \in \Phi_{\mathbf{m}}$. Then (X, B) is nonexceptional if and only if there exists a regular complement $K_X + B^+$ which is not klt.*

COROLLARY 8.4.3 (cf. Corollary 5.3.4). *Let (X, B) be a log del Pezzo surface. Assume that $B \in \Phi_{\mathbf{m}}$ and $(K_X + B)^2 \geq 4$. Then there exists a regular complement of $K_X + B$. Moreover, there exists such a complement which is not klt (in particular, (X, B) is nonexceptional).*

SKETCH OF PROOF. If $K_X + B$ is not klt, this assertion follows by Proposition 5.3.1. Assume that $K_X + B$ is klt. Take $n \in \mathbb{N}$ so that $H := -n(K_X + B)$ is an integral (ample) Cartier divisor. By Riemann-Roch

$$\dim H^0(X, \mathcal{O}_X(H)) \geq \frac{H \cdot (H - K_X)}{2} + 1 = \frac{1}{2}n(n+1)(K_X + D)^2 - \frac{1}{2}n(K_X + D) \cdot D + 1 \geq 2n(n+1) + 1.$$

Pick a smooth point $P \in X$ and let $\mathfrak{m}_P$ be the ideal sheaf of P. Then

$$\dim H^0(X, \mathcal{O}_X/\mathfrak{m}_P^{2n}) = \dim \mathbb{C}[x,y]/(x,y)^{2n} = \frac{(2n+1)2n}{2} = (2n+1)n.$$

From the exact sequence

$$0 \longrightarrow \mathfrak{m}_P^{2n} \otimes \mathcal{O}_X(H) \longrightarrow \mathcal{O}_X(H) \longrightarrow \mathcal{O}_X/\mathfrak{m}_P^{2n} \otimes \mathcal{O}_X(H) \simeq \mathcal{O}_X/\mathfrak{m}_P^{2n} \longrightarrow 0$$

we see that

$$H^0(X, \mathfrak{m}_P^{2n} \otimes \mathcal{O}_X(H)) \neq 0.$$

Therefore there is $H' \in |H|$ such that $\operatorname{mult}_P(H') \geq 2n$. It is easy to see that $K_X + B + \frac{1}{n}H'$ is not klt. Then $K_X + B + \alpha H'$ is lc but not klt for some $\alpha \leq \frac{1}{n}$. Clearly, $-(K_X + B + \alpha H')$ is nef. Hence we can apply Corollary 8.4.1. □

Note that the above result can be improved: by taking $P \in \operatorname{Supp}(B)$ or $P \in \operatorname{Sing}(X)$ it is possible to find nonklt boundary $K_X + B + \alpha H'$ for smaller values of $(K_X + D)^2$. On the other hand, in the case $B = 0$ and X is smooth, it is well known that K_X is strongly 1-complementary.

COROLLARY 8.4.4 (see [**P1**]). *Let $X \ni P$ be a three-dimensional klt singularity, $f\colon (Y, S) \to X$ a plt blowup, and $K_X + D$ an n-complement which is not klt at P. Then one of the following holds*

(i) *$a(S, D) = -1$ and $K_Y + S + D_Y := f^*(K_X + D)$ is an n-complement of $K_Y + S$;*
(ii) *$a(S, D) > -1$ and then there exists a regular complement of $K_Y + S$ which is not plt.*

PROOF. (i) is obvious. Assume that $a(S, D) > -1$. Write

$$K_Y + aS + D_Y := f^*(K_X + D),$$

where $a = -a(S, D) < 1$ and D_Y is the proper transform of D. By assumptions $K_Y + aS + D_Y$ is lc and not klt (see 1.1.6). Therefore $K_Y + S + D_Y$ is not plt and we can take $0 < b \leq 1$ so that $K_Y + S + bD_Y$ is lc but not plt. It is easy to see that $-(K_Y + S + bD_Y)$ is f-ample. If $f(S) = P$, then $(S, \operatorname{Diff}_S(bD_Y))$ is a log Del Pezzo. By 8.4.1 (or by 5.3.1) there is a regular complement of $K_S + \operatorname{Diff}_S(0)$ and by 4.4.1 it can be extended to a complement of $K_Y + S$. Similarly, in the case when $f(S)$ is a curve, we can use Theorem 6.0.6. □

Similar to Theorem 8.3.1 one can prove the following

PROPOSITION 8.4.5 (cf. [**Bl**]). *Let (X, D) be a log Enriques surface (i.e., $K_X + D$ is lc and $K_X + D \equiv 0$). Assume that $K_X + D$ is not klt and $D \in \Phi_{\mathbf{m}}$. Then $n(K_X + D) \sim 0$ for some $n \in \mathcal{R}_2$. In particular, $D \in \Phi_{\mathbf{sm}}$.*

PROOF. By 4.2.8 it is sufficient to show the existence of a regular complement. If X is not rational, then the assertion follows by Theorem 8.2.1. Otherwise we can apply Theorem 8.3.1. The existence of D' in conditions of the theorem follows by Proposition 8.4.6 below. □

PROPOSITION 8.4.6. *Let (X, Λ) be a log Enriques surface. Assume that X is rational. Then there is a boundary $\Lambda' \leq \Lambda$ such that*

(i) *$K_X + \Lambda'$ is klt, and*
(ii) *$-(K_X + \Lambda')$ is nef and big.*

PROOF. Replace X with its minimal resolution and Λ with its crepant pull back. Then again (X, Λ) is a log Enriques surface. By Corollary 1.1.7 it is sufficient to construct Λ' on this new X. Further, there is a sequence of contractions of -1-curves $\varphi\colon X \to X^\bullet$, where $X^\bullet \simeq \mathbb{P}^2$ or $X^\bullet \simeq \mathbb{F}_n$, $n \geq 0$, $n \neq 1$. Put $\Lambda^\bullet := \varphi_*\Lambda$. Then $(X^\bullet, \Lambda^\bullet)$ is again a log Enriques surface. It is sufficient to construct $\Lambda'^\bullet \leq \Lambda^\bullet$ such that $K_{X^\bullet} + \Lambda'^\bullet$ is klt and $-(K_{X^\bullet} + \Lambda'^\bullet)$ is nef and big. Indeed, the crepant pull back Λ'' of $\Lambda'^\bullet$ satisfies (i) and (ii). However, Λ'' is not necessarily a boundary (i.e., effective). To avoid this one can take $\Lambda' = \Lambda'' + t(\Lambda - \Lambda'')$ for $0 < 1 - t \ll 1$.

Further, if $X^\bullet \simeq \mathbb{P}^2$ or $X^\bullet \simeq \mathbb{P}^1 \times \mathbb{P}^1$, then we take $\Lambda'^\bullet = 0$. In the case $X^\bullet \simeq \mathbb{F}_n$ with $n \geq 2$, we write $\Lambda^\bullet = \lambda\Sigma_0 + \Lambda^\circ$, where Σ_0 is the negative section of $X^\bullet = \mathbb{F}_n$, $0 \leq \lambda \leq 1$, $\Lambda^\circ \geq 0$, and Σ_0 is not a component of Λ°. It is easy to see

$$2 - n = -K_{X^\bullet} \cdot \Sigma_0 = \Lambda'^\bullet \cdot \Sigma_0 = -n\lambda + \Lambda^\circ \cdot \Sigma_0 \geq -n\lambda.$$

Hence, $\lambda \geq 1 - 2/n$. Thus we can take $\Lambda'^\bullet = (1 - 2/n)\Sigma_0$. □

8.5. Characterization of toric surfaces

Following Shokurov we prove Conjecture 2.2.18 in dimension two. Moreover, we prove a generalization of 2.2.18 for ρ_{num} instead of $\mathrm{rkWeil_{alg}}$ (recall that $\rho_{\mathrm{num}}(X)$ is the rank of the quotient of $\mathrm{Weil}(X)$ modulo numerical equivalence).

THEOREM 8.5.1 ([**Sh3**]). *Let $(X, D = \sum d_i D_i)$ be a projective log surface such that*

(i) *$K_X + D$ is lc, and*
(ii) *$-(K_X + D)$ is nef.*

Then

$$\sum d_i \leq \rho_{\mathrm{num}}(X) + 2. \tag{8.10}$$

If the equality holds, then $K_X + D \equiv 0$ and X has only rational singularities (in particular X is $\mathbb{Q}$-factorial).

PROOF. Assume that

$$\sum d_i - \rho_{\mathrm{num}}(X) - 2 \geq 0. \tag{8.11}$$

First we consider the case $K_X + D \equiv 0$.

Step 0. Apply a minimal log terminal modification as in 3.1.3. It is easy to see that this preserves the left hand side of (8.11). Thus we may assume that $K_X + D$ is dlt. In particular, $K_X + \{D\}$ is klt, X is $\mathbb{Q}$-factorial and $\rho_{\mathrm{num}}(X) = \rho(X)$.

Step 1. Write $D = C+B$, where $C := \lfloor D \rfloor$ and $B := \{D\}$. Then $-(K_X+B) \equiv C+$(nef divisor). Hence K_X+B cannot be nef. Run (K_X+B)-MMP, i.e., contract birational extremal rays R such that $R\cdot(K_X+B) < 0$. The left hand side of (8.11) does not decrease. Of course, we can lose the dlt property of K_X+D, but properties (i)–(ii) are preserved. Moreover, if $C \neq 0$, then on each step we contract a curve R with $R \cdot C > 0$. In particular, whole C is not contracted.

At the end we get a nonbirational contraction $\varphi\colon X \to Z$.

Step 2. Assume that after Step 1 we get a Fano contraction φ with $\dim(Z) = 1$. Write $D = D^{\mathrm{vert}} + D^{\mathrm{hor}}$, where $D^{\mathrm{hor}} = \sum_{\mathrm{hor}} d_i D_i$ is the sum of all components such that $\varphi(D_i) = Z$ and $D^{\mathrm{vert}} = \sum_{\mathrm{vert}} d_j D_j$ is the sum of components which are fibers of φ. Let F be a general fiber of φ. Then by Adjunction

$$\begin{aligned} 0 \geq (K_X + D)\cdot F &= (K_X + D^{\mathrm{hor}} + F)\cdot F \\ &= \deg K_F + D^{\mathrm{hor}}\cdot F \geq -2 + D^{\mathrm{hor}}\cdot F. \end{aligned}$$

This gives $D^{\mathrm{hor}} \cdot F \leq 2$. In particular,

$$\sum_{\mathrm{hor}} d_i \leq 2, \qquad \sum_{\mathrm{vert}} d_j \geq 2 \tag{8.12}$$

(because $\rho(X) = 2$). Now, let R be the extremal ray of $\overline{NE}(X)$ other than $\mathbb{R}_+[F]$. Then $R\cdot D^{\mathrm{vert}} > 0$. Hence $R\cdot(K_X + (1-\varepsilon_1)D - \varepsilon D^{\mathrm{vert}}) < 0$ for $0 < \varepsilon_1 \ll \varepsilon \ll 1$. By Contraction Theorem there is a contraction $\psi\colon X \to Z_1$ of R.

Assume that $\dim Z_1 = 1$. Then, as above, we have $\sum_{\mathrm{vert}} d_j \leq 2$ (because components of D^{vert} are horizontal with respect to ψ). This yields equalities in (8.12) and (8.11). If $\dim Z_1 = 2$, then ψ is birational. Let E be the ψ-exceptional divisor. If E is a component of $C := \lfloor D \rfloor$, then again by Adjunction we have

$$0 \geq (K_X + D)\cdot E = \deg K_E + \deg \mathrm{Diff}_E(D-E), \quad \deg \mathrm{Diff}_E(D-E) \leq 2.$$

Since any component of D^{vert} meets E, by Corollary 2.2.8 we obtain $\sum_{\mathrm{vert}} d_j \leq 2$. This yields equalities in (8.12) and (8.11). Finally, if E is not a component of $C := \lfloor D \rfloor$, then we replace X with Z_1. Note that in this case we get strict inequality $\sum d_i - \rho(X) - 2 > 0$ in (8.11) and $\rho(X) = 1$. By the next two steps this is a contradiction.

Step 3. Assume that Z is a point (and $\rho(X) = 1$). Then $\sum d_i \geq 3$. We claim that after perturbation of coefficients one can obtain the case when $K_X + D$ is not klt. Indeed, assume that $K_X + D$ is klt. Let H be the ample generator of $\mathrm{Pic}(X) \simeq \mathbb{Z}$ (see Lemma 5.1.3) and let $D_i \equiv a_i H$, $a_i > 0$. Without loss of generality we may assume that $a_1 \leq a_2 \leq \cdots$. Take $D^t = D + t(D_i - D_j)$, where $i < j$ and $0 < t \leq d_j$. Clearly, $-(K_X + D^t)$ is again nef and D^t is effective.

Moreover, for D^t the left hand side of (8.11) remains the same. If $K_X + D^{t_0}$ is lc but not klt for some for $0 \le t_0 \le d_j$, then D^{t_0} gives the required boundary. If $K_X + D^t$ is klt for $t = d_j$, then we replace D with D^{d_j} continue the process with another pair D_i, D_j. Since the last procedure reduces the number of components of D, this process terminates. At the end we get the situation when $K_X + D$ is not klt.

Step 4. Now we consider the case when $K_X + D$ is not klt. Apply steps 0–2 again. On Step 2 in (8.11) the equality holds. So we assume that $\rho(X) = 1$ and $C := \lfloor D \rfloor \neq 0$. For any component $C_i \subset C$ by Adjunction we have

$$2 \ge -\deg K_{C_i} = \deg \operatorname{Diff}_{C_i}(D - C_i) \tag{8.13}$$

On the other hand, all components of D intersect C_i and

$$\deg \operatorname{Diff}_{C_i}(D - C_i) \ge \sum d_j - 1 \ge 2 \tag{8.14}$$

(see Corollary 2.2.8). Therefore $\sum d_j = 3$ and $\deg K_{C_i} = 2$. This completes the proof in the case $K_X + D \equiv 0$.

Consider the case $K_X + D \not\equiv 0$. As in Step 0 we may assume that $K_X + D$ is dlt. Further, similar to Step 1, run $(K_X + D)$-MMP. This preserves assumption (8.11). Let $\varphi \colon X \to X'$ be a birational extremal contraction, $D' := \varphi_* D$, and E the exceptional curve. Clearly, $K_X + D \equiv \varphi^*(K_{X'} + D') + aE$, where $a \in \mathbb{Q}$. Then $0 > (K_X + D) \cdot E = aE^2$. Hence $a > 0$. Assume that $K_{X'} + D' \equiv 0$ and H a hyperplane section of X. Then $0 \ge (K_X + D) \cdot H = aE \cdot H > 0$, a contradiction. Therefore $K_{X'} + D' \not\equiv 0$. We can replace (X, D) with (X', D') and continue the process. At the end we get a log surface (X, D) with a nonbirational $(K_X + D)$-negative extremal contraction $\phi \colon X \to Z$. In particular $\rho(X) \le 2$. If Z is a point, then $\rho(X) = 1$ and $-(K_X + D)$ is ample. Take $n \ge 0$ so that the divisor $-n(K_X + D)$ is integral and very ample. Let $G \in |-n(K_X + D)|$ be a general member. Then $K_X + D + \frac{1}{n}G$ is dlt and numerically trivial. In this case, by the proved inequality (8.10), $3 \le \sum d_i < \sum d_i + 1/n \le 3$, a contradiction. If Z is a curve, then we can use the arguments of Step 2. Thus $4 \le \sum d_i = \sum_{\text{hor}} d_i + \sum_{\text{vert}} d_j < 4$ (because we have strict inequality in (8.12)). The last contradiction proves that $K_X + D \equiv 0$.

Assume that X has at least one nonrational singularity $P \in X$. Clearly, K_X is not klt at P and $P \notin \operatorname{Supp}(D)$. Then by Corollary 6.1.11 $P \in X$ is a simple elliptic or cusp singularity. As in Step 0, let $\varphi \colon (\tilde{X}, \tilde{D}) \to (X, D)$ be a minimal log terminal modification. If $\lfloor \tilde{D} \rfloor$ is connected, then $\lfloor \tilde{D} \rfloor = \varphi^{-1}(P)$ and $p_a\left(\lfloor \tilde{D} \rfloor\right) = 1$. By Lemma 8.3.8 $\tilde{D} = \lfloor \tilde{D} \rfloor$ and $D = 0$, a contradiction. Therefore $C := \varphi^{-1}(P)$ is a connected component of $\lfloor \tilde{D} \rfloor$. Denote $\tilde{B} := \tilde{D} - \lfloor \tilde{D} \rfloor$ and $C' := \lfloor \tilde{D} \rfloor - C$. Our assumption (8.11) implies that $\tilde{B} \neq 0$. Then $C \cap C' = C \cap \operatorname{Supp}(\tilde{B}) = \varnothing$. By Proposition 3.3.2 there is a contraction $\psi \colon \tilde{X} \to Z$ with rational fibers onto a curve Z such that C and C' are (smooth) disjoint sections. Then $\tilde{B}$ has no horizontal components. Let R be a $(K_{\tilde{X}} + \tilde{D} - \varepsilon\tilde{B})$-negative extremal rational curve. Since

$p_a(Z) = p_a(C) = 1$, R cannot be horizontal. On the other hand, $\rho(\tilde{X}) \geq 3$ (because $R \cdot \tilde{B} > 0$). Therefore the contraction of R is birational. This contraction reduces the left hand side of (8.11), a contradiction. This proves Theorem 8.5.1. □

THEOREM 8.5.2 ([**Sh3**]). *Let (X, C) be a projective log surface with a reduced boundary $C = \sum_{i=1}^{r} C_i$ such that $K_X + C$ is lc and $-(K_X + C)$ is nef. Assume also*

$$r \geq \rho_{\mathrm{num}}(X) + 2.$$

Then

(i) $r = \rho_{\mathrm{num}}(X) + 2 = \rho(X) + 2$;
(ii) $K_X + C \sim 0$ *(i.e., $K_X + C$ is 1-complementary);*
(iii) *C is connected and $p_a(C) = 1$;*
(iv) *the pair (X, C) is toric.*

PROOF. The assertion (i) follows by Theorem 8.5.1. This also shows that $K_X + C \equiv 0$. To prove (ii) we apply steps 0–4 of the proof of Theorem 8.5.1 to (X, C). At the end we obtain one of the following:

- $\rho(X) = 1$ and C has exactly three components. Clearly they intersect each other and does not pass through one point, so $p_a(C) \geq 1$ (cf. Proof of Corollary 10.1.2). By Lemma 8.3.8, $K_X + C$ is a 1-complement. According to 4.3.1 and 4.3.2, $K_X + C$ on our original X is 1-complementary.
- $\rho(X) = 2$ and C has exactly four components. Moreover, there is an extremal contraction $\varphi \colon X \to Z$ onto a curve. By discussions in Step 2 of the proof of Theorem 8.5.1 (especially, (8.12)), we have a decomposition $C = C^{\mathrm{hor}} + C^{\mathrm{vert}}$ such that both C^{hor} and C^{vert} have two irreducible components. Any component of C^{vert} meets all components of C^{hor}. As in (10.2) we have $p_a(C) \geq 1$. Finally, as above, $K_X + C$ is 1-complementary.

By Proposition 3.3.2, $C \subset \mathrm{LCS}(X, C)$ is connected. Since $K_X + C$ is Cartier, $\mathrm{Diff}_C(0) = 0$ (see 2.2.4). Thus $K_C = 0$ and $p_a(C) = 1$. This proves (iii). The assertion of (iv) follows by the lemma below. □

LEMMA 8.5.3. *Let (X, C) be a projective log surface such that C is reduced and connected. Assume that C has exactly $\rho_{\mathrm{num}}(X) + 2$ components, $K_X + C$ is lc and linearly trivial. Then (X, C) is a toric pair.*

PROOF. Let $\varphi \colon \overline{X} \to X$ be minimal lt modification of (X, C). Write $\varphi^*(K_X + C) = K_{\overline{X}} + \overline{C}$, where $\overline{C}$ is reduced and $\varphi_*(\overline{C}) = C$. The exceptional divisor of φ is contained in $\overline{C}$. Hence on $(\overline{X}, \overline{C})$ all our conditions hold. So it is sufficient to prove our assertion for $(\overline{X}, \overline{C})$. By Lemma 8.3.8, $p_a(\overline{C}) = 1$, $\overline{C}$ a wheel of smooth rational curves, $\overline{X}$ is smooth along $\overline{C}$ and has only Du Val singularities outside. Run $K_{\overline{X}}$-MMP. By (8.10), on each step we contracted a component of $\overline{C}$ (which is contained into the smooth locus of $\overline{X}$). Thus our MMP is a sequence of contractions of -1-curves. At the end we obtain a Fano contraction $\psi \colon (\widehat{X}, \widehat{C}) \to Z$, where $\widehat{X}$ has only Du Val singularities, $K_{\widehat{X}} + \widehat{C}$ is lc (in fact it is analytically dlt) and numerically trivial. The sequence of transformations $\overline{X} \to \widehat{X}$ is a sequence of blowups of divisors with discrepancies $a(\cdot, \widehat{C}) = -1$. They must preserve the action of a two-dimensional torus (if $(\widehat{X}, \widehat{C})$ is a toric pair). Thus it is sufficient to show that the pair $(\widehat{X}, \widehat{C})$ is toric.

If $\rho(\widehat{X}) = 1$, then C has exactly three components which are Cartier divisors. Therefore $\widehat{X}$ is a log del Pezzo of Fano index $r(\widehat{X}) \geq 3$ (see 10.2.3). By Lemma 10.2.4, $\widehat{X} \simeq \mathbb{P}^2$ and $\widehat{C} = \widehat{C}_1 + \widehat{C}_2 + \widehat{C}_3$, where the $\widehat{C}_i$ are lines. Obviously, $(\widehat{X}, \widehat{C})$ is toric in this case. Finally, assume that $\dim Z = 1$. Then $\widehat{C}$ has exactly four components and by Lemma 8.3.8 they form a wheel of smooth rational curves. It is an easy exercise to prove that $Z \simeq \mathbb{P}^1$ and the fibers of ψ are rational. Therefore $C = C_1^{\mathrm{hor}} + C_2^{\mathrm{hor}} + C_1^{\mathrm{vert}} + C_2^{\mathrm{vert}}$, where C_1^{hor}, C_2^{hor} are disjoint sections of ψ and C_1^{vert}, C_2^{vert} are fibers. We claim that $\widehat{X}$ is smooth. Indeed, by construction, $\widehat{X}$ is smooth along $\widehat{C}$. Let F be a fiber of ψ different from C_1^{vert}, C_2^{vert}. Take c so that $K_{\widehat{X}} + \widehat{C} + cF$ is maximally lc. If $c < 1$, then $\mathrm{LCS}(\widehat{X}, \widehat{C} + cF)$ has three connected components near F. This contradicts Proposition 3.3.1. Hence $K_{\widehat{X}} + \widehat{C} + F$ is lc. By Adjunction $\deg \mathrm{Diff}_F(\widehat{C}) = 2$. On the other hand, $\mathrm{Diff}_F(\widehat{C}) \geq P_1 + P_2$, where $P_i = F \cap C_i^{\mathrm{hor}}$. Hence $\mathrm{Diff}_F(\widehat{C}) = P_1 + P_2$ and $\widehat{X}$ is smooth along F.

Thus we have shown that $\widehat{X}$ is smooth. Then $\widehat{X} \simeq \mathbb{F}_n$ and ψ is the natural projection $\mathbb{F}_n \to \mathbb{P}^1$. Since C_1^{hor}, C_2^{hor} are disjoint sections one of them, say C_1^{hor}, must be the minimal section Σ_0. Now, it is easy to show that the pair $(\widehat{X}, \widehat{C})$ is toric. $\square$