CHAPTER 1

Preliminary results

1.1. Singularities of pairs

List of notations.

$\equiv$	numerical equivalence
$\sim$	linear equivalence
$\sim_{\mathbb{Q}}$	$\mathbb{Q}$-linear equivalence
$\mathcal{K}(X)$	function field of X
$D \approx D'$	D and D' gives the same valuation of $\mathcal{K}(X)$
$\rho(X)$	Picard number of X, rank of the Néron-Severi group
$Z_1(X/Z)$	group of 1-cycles on X over Z (see [**KMM**])
$N_1(X/Z)$	quotient of $Z_1(X/Z)$ modulo numerical equivalence (cf.)
$\overline{NE}(X/Z)$	Mori cone (see [**KMM**])
$\mathrm{Weil}(X)$	group of Weil divisors, i.e., the free abelian group generated by prime divisors on X
$\mathrm{Weil}_{\mathrm{lin}}(X)$, $\mathrm{Weil}_{\mathrm{alg}}(X)$	quotients of $\mathrm{Weil}(X)$ modulo linear and algebraic equivalence respectively.

All varieties are assumed to be algebraic varieties defined over the field $\mathbb{C}$. By a *contraction* we mean a projective morphism $f\colon X \to Z$ of normal varieties such that $f_*\mathcal{O}_X = \mathcal{O}_Z$ (i.e., having connected fibers). We call a birational contraction a *blowdown* or *blowup*, depending on our choice of initial variety.

A *boundary* on a variety X is a $\mathbb{Q}$-Weil divisor $D = \sum d_i D_i$ with coefficients $0 \le d_i \le 1$. If we have only $d_i \le 1$, we say that D is a *subboundary*. All varieties are usually considered supplied with boundary (or subboundary) as an additional structure. If D is a boundary, then we say that (X, D) is a *log variety* or *log pair*. Moreover, if we have a contraction $f\colon X \to Z$, then we say that (X, D) is a *log variety* over Z and denote it simply by $(X/Z, D)$. If $\dim Z > 0$, we often consider Z as a germ near some point $o \in Z$. To specify this we denote the corresponding log variety by $(X/Z \ni o, D)$.

Given a birational morphism $f\colon X \to Y$, the boundary D_Y on Y is usually considered as the image of the boundary D_X on X: $D_Y = f_* D_X$. The integral part of a $\mathbb{Q}$-divisor $D = \sum d_i D_i$ is defined in the usual way: $\lfloor D \rfloor := \sum \lfloor d_i \rfloor D_i$, where $\lfloor d_i \rfloor$ is the greatest integer such that $\lfloor d_i \rfloor \le d_i$. The (round up) upper integral part $\lceil D \rceil$ and the fractional part $\{D\}$ are similarly defined.

A log resolution is a resolution $f\colon \widetilde{X} \to X$ of singularities of X such that the union $\left(\bigcup \widetilde{D}_i\right) \cup \operatorname{Exc}(f)$ of proper transforms* of all the D_i and the exceptional locus the exceptional locus $\operatorname{Exc}(f)$ is a divisor with simple normal crossings.

Let X be a normal variety, D a $\mathbb{Q}$-divisor on X, and $f\colon \widetilde{X} \to X$ any projective birational morphism, where X and $\widetilde{X}$ are normal. Assume that $K_X + D$ is $\mathbb{Q}$-Cartier. Then we can write

$$K_{\widetilde{X}} + \widetilde{D} = f^*(K_X + D) + \sum_E a(E, D)E, \tag{1.1}$$

where $\widetilde{D}$ is the proper transform D and $a(E, D) \in \mathbb{Q}$. The numbers $a(E, D)$ depends only on X, D and the discrete valuation of the field $\mathcal{K}(X)$ corresponding to E (i.e., they do not depend on f). They are called *discrepancies* or *discrepancy coefficients.* Define

$$\operatorname{discrep}(X, D) := \inf_E\{a(E, D) \mid E \text{ is an exceptional divisor over } X\}.$$

We also put for nonexceptional divisors

$$a(E, D) := \begin{cases} -d_i & \text{if } E = D_i; \\ 0 & \text{otherwise.} \end{cases}$$

Let us say that the pair (X, D) has

terminal singularities, if $\operatorname{discrep}(X, D) > 0$;
canonical singularities, if $\operatorname{discrep}(X, D) \geq 0$;
Kawamata log terminal (klt) singularities, if $\operatorname{discrep}(X, D) > -1$ and $\lfloor D \rfloor \leq 0$;
purely log terminal (plt) singularities, if $\operatorname{discrep}(X, D) > -1$;
log canonical (lc) singularities, if $\operatorname{discrep}(X, D) \geq -1$;
ε-log terminal (ε-lt) singularities, † if $\operatorname{discrep}(X, D) > -1 + \varepsilon$;
ε-log canonical (ε-lc) singularities, if $\operatorname{discrep}(X, D) \geq -1 + \varepsilon$;
divisorial log terminal (dlt) singularities, if $a(E_i, D) > -1$ for all exceptional divisors E_i of some log resolution f whose exceptional locus consists of divisors (or it is empty).

In these cases we also say simply that $K_X + D$ is lc (resp. klt, etc.) We usually omit D if it is trivial.

1.1.1. For the klt and lc properties the inequalities $a(E, D) > -1$ (≥ -1) can be checked for exceptional divisors of some log resolution (see [**KMM**, 0-2-12]). The plt property of (X, D) is equivalent to the existence of a log resolution $f\colon \widetilde{X} \to X$ such that $a(E, D) > -1$ for all exceptional divisors of f and the proper transform $\widetilde{\lfloor D \rfloor}$ of $\lfloor D \rfloor$ on $\widetilde{X}$ is smooth. It is easy to see that if $K_X + D$ is lc, then D is a subboundary. In the two-dimensional case we can use Mumford's numerical

*The proper transform is sometimes also called the birational or strict transform.

†Note that our definition of ε-lt pairs is weaker than that given by Alexeev [**A**]: we do not claim that $-d_i > -1 + \varepsilon$.

pull back of any Weil divisor, so all the above definitions can be given in this situation numerically, without the $\mathbb{Q}$-Cartier assumption (see e.g., [**S1**] and 11.2).

EXAMPLE 1.1.2. Let X be a smooth surface and $D = D_1 + D_2$ a pair of smooth curves intersecting transversally at one point. The identity map is a log resolution, so (X, D) is dlt. However, the blowup of the point of intersection gives an exceptional divisor E with discrepancy $a(E, D) = -1$. Hence (X, D) is not plt. If D is an irreducible curve with a node on a nonsingular surface X, then the pair (X, D) is not dlt. This shows that the dlt condition is not local.

EXAMPLE 1.1.3. Let $Q \subset \mathbb{C}^4$ be a quadratic cone given by $xy = zt$ and D its hyperplane section $\{x = 0\}$. Then $D = D_1 + D_2$, where D_1 and D_2 are planes in $\mathbb{C}^4$. There is a small resolution $f\colon \widetilde{Q} \to Q$ with exceptional locus $\mathbb{P}^1$. The intersection of the proper transforms of the planes D_1 and D_2 is a line on $\widetilde{Q}$. So f is a log resolution. However, (Q, D) is not dlt because f is small.

In [**Ut**] the notion of weakly Kawamata log terminal singularity was introduced. Later it was proved that this is equivalent to the dlt property [**Sz**]. The very close (but wider) class of weakly log terminal pairs was considered in [**KMM**] and [**Sh2**].

Log varieties with dlt singularities form a convenient class of varieties in which the log Minimal Model Program (log MMP) works [**KMM**]. In particular, these singularities are rational [**KMM**, 1-3-1], [**KM**, 5.22] and the Cone Theorem and Contraction Theorem hold for these varieties [**KMM**, 4.2.1, 3-2-1]. Log canonical singularities are not necessarily rational. However, it was shown in [**Sh4**] that reasonable log MMP also works in this category.

More precisely, the property of a $\mathbb{Q}$-factorial log variety to have klt (resp. dlt) singularities is preserved under contractions of extremal rays and flips, i.e., they form classes of log varieties closed under the log MMP. We refer to [**KMM**] for technical details of this theory; see also [**Sh4**] and (for two dimensional case) Appendix 11.2, [**A**], [**KK**]. Note also that all distinctions between different notions of log terminal singularities arise only if D has components with coefficient 1.

The following property can be obtained directly from the definitions.

PROPOSITION 1.1.4 ([**Sh2**], [**Ut**, 2.17]). *Let X be a normal variety and $D = \sum d_i D_i$ a subboundary on X such that $K_X + D$ is a $\mathbb{Q}$-Cartier divisor.*

(i) *If $D' \le D$ and $K_X + D$ is lc (resp. ε-lt, dlt, plt or klt) and $K_X + D'$ is $\mathbb{Q}$-Cartier, then $K_X + D'$ also is lc (resp. ε-lt, dlt, plt or klt);*

(ii) *If $K_X + D$ is dlt, then there exists $\varepsilon > 0$ such that all $\mathbb{Q}$-Cartier divisors $K_X + D'$ also are dlt for all $D' = \sum d'_i D_i$ with $d'_i \le \min\{d_i + \varepsilon, 1\}$;*

(iii) *If $K_X + D$ is plt (resp. klt) and $K_X + D + D'$ is lc, then $K_X + D + tD'$ is plt (resp. klt) for all $t < 1$.*

REMARK 1.1.5. The formula (1.1) can be written as

$$K_{\tilde{X}} + D' = f^*(K_X + D), \tag{1.2}$$

where $D' := \widetilde{D} + \sum a_i E_i$. In particular, $D = f_* D'$. Then the lc property of $K_X + D$ is equivalent to that D' is a subboundary. In this case, $K_{\widetilde{X}} + D'$ is called the *crepant pull back* of $K_X + D$.

This trivial remark has the following useful generalization

PROPOSITION 1.1.6 ([**Ko2**]). *Let $f: Y \to X$ be a birational contraction and D a subboundary on X such that $K_X + D$ is $\mathbb{Q}$-Cartier. As in* (1.2) take the crepant pull back

$$K_Y + D_Y = f^*(K_X + D), \qquad \text{with} \qquad D = f_* D_Y. \tag{1.3}$$

Then

(i) $K_X + D$ is lc (resp. klt) $\iff$ $K_Y + D_Y$ is lc (resp. klt);
(ii) $K_X + D$ is plt (resp. dlt) $\iff$ $K_Y + D_Y$ is plt (resp. dlt) and f does not contract components of D_Y with coefficient 1;
(iii) $K_X + D$ is ε-lt $\iff$ $K_Y + D_Y$ is ε-lt and f does not contract components of D_Y with coefficient $\geq 1 - \varepsilon$.

COROLLARY 1.1.7 ([**Ut**, 2.18]). *Let $f: (Y, D_Y) \to (X, D)$ be a birational contraction, where $D = f_* D_Y$. Assume that $K_X + D$ is $\mathbb{Q}$-Cartier. If $K_Y + D_Y$ is lc (resp. klt) and f-(numerically) nonpositive, then $K_X + D$ is lc (resp. klt).*

EXAMPLE 1.1.8. Let X be a normal toric variety and D the reduced toric boundary on X. Then $K_X + D$ is lc. This follows by 1.1.6, from the fact that $K_X + D \sim 0$ and from the existence of toric resolutions.

1.2. Finite morphisms and singularities of pairs

Let $f: Y \to X$ be a finite surjective morphism of normal varieties and $D = \sum d_i D_i$ a subboundary on X. We assume that the ramification divisor is contained in Supp$D$ (we allow D to have coefficients $= 0$). Define a $\mathbb{Q}$-divisor B on Y by the condition

$$K_Y + B = f^*(K_X + D). \tag{1.4}$$

Write B as $B = \sum b_{i,j} B_{i,j}$, where $f(B_{i,j}) = D_i$, and $r_{i,j}$ the ramification index along $B_{i,j}$ (i.e., at the general point of $B_{i,j}$). By the Hurwitz formula we have

$$b_{i,j} = 1 - r_{i,j}(1 - d_i). \tag{1.5}$$

Hence B is also subboundary. Note however that B may not be a boundary even if D is.

PROPOSITION 1.2.1 ([**Sh2**, §2], [**Ut**, 20.3], [**Ko2**]). *Notation as above. Then $K_X + D$ is lc (resp. plt, klt) if and only if $K_Y + B$ is lc (resp. is plt, klt).*

Propositions 1.1.6 and 1.2.1 show that the classes of klt, plt and lc singularities of pairs are very natural and are closed under birational and finite morphisms. The implication $\Longrightarrow$ also holds for dlt pairs if f is étale in codimension one [**Sz**].

SKETCH OF PROOF. Let $g\colon X' \to X$ be a birational morphism. Consider the base change

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow{f} & X \end{array}$$

where Y' is a dominant component of the normalization of $Y \times_X X'$. As in (1.3), write

$$K_{X'} + D' = g^*(K_X + D) \qquad \text{and} \qquad K_{Y'} + B' = h^*(K_Y + B),$$

and similar to (1.4) we can write

$$K_{Y'} + B' = f'^*(K_{X'} + D'),$$

where by (1.5) the coefficients of B' are

$$b'_{i,j} = 1 - r'_{i,j}(1 - d'_i).$$

Let $E := B'_{i,j}$ be an h-exceptional divisor and $F := f'(E)$. Then this formula can be rewritten as

$$a(E, B) + 1 = r'_{i,j}(a(F, D) + 1), \qquad r'_{i,j} \le \deg f. \tag{1.6}$$

This yields $a(E, B) \ge a(F, D)$ and all the implications $\Longrightarrow$. The implications $\Longleftarrow$ follow by (1.6) and from the (nontrivial) fact that each exceptional divisor E over Y can be obtained in the way specified above (see [**Ko2**, 3.17]). □

Note that we have shown more:

$$1 + \operatorname{discrep}(X, D) \le 1 + \operatorname{discrep}(Y, B) \le (\deg f)(1 + \operatorname{discrep}(X, D)). \tag{1.7}$$

The following particular case of Proposition 1.2.1 is very interesting for applications.

COROLLARY 1.2.2. *If a morphism $f\colon Y \to X$ is étale in codimension one, then $K_X + D$ is lc (resp. plt, klt) if and only if $K_Y + f^*D$ is lc (resp. plt, klt).*

From Proposition 1.2.1 it is easy also to obtain the following

COROLLARY 1.2.3. *Let Y be a variety with at worst klt (resp. lc) singularities and $Y \to X$ a finite surjective morphism. Then X also has at worst klt (resp. lc) singularities.*

In particular, all quotient singularities are klt. However, the converse is true only in dimension two.

REMARK 1.2.4 ([**R2**, 3.1]). Let $G \subset \mathrm{GL}_n(\mathbb{C})$ be a finite subgroup without quasi-reflections. Then $\mathbb{C}^n/G$ has only canonical singularities if and only if for every element $g \in G$ of order r and for any primitive rth root of unity ε, the diagonal form of the action of g is

$$g\colon x_i \longrightarrow \varepsilon^{a_i} x_i \quad \text{with} \quad 0 \le a_i < r, \quad \text{and} \quad \sum a_i \ge r.$$

EXAMPLE 1.2.5 ([**Ca**]). Let $(Y \ni o)$ be a Du Val singularity and $f\colon (Y \ni o) \to (X \ni P)$ a quotient by an involution. Write $K_Y = f^*(K_X + \frac{1}{2}\Delta)$, where Δ is the ramification divisor. Then $(X, \frac{1}{2}\Delta)$ is (1/2)-lt. There is an explicit list of all such involutions and quotients [**Ca**].

EXAMPLE 1.2.6 ([**K**], [**Mo**, 7.2]). Let $X \ni P$ be a germ of a three-dimensional terminal singularity. By [**RY**], a general divisor $F \in |-K_X|$ has only Du Val singularities. Then according to Inversion of Adjunction 2.2.6 (see Example 2.2.12) $K_X + F$ is plt. From this, for general $S \in |-2K_X|$ the divisor $K_X + \frac{1}{2}S$ is also plt. Consider the double cover $f\colon Y \to X$ with ramification divisor S. Then Y has only klt singularities and $K_Y \sim 0$. Hence the singularities of Y are canonical of index one.

The existence of a good divisor $S \in |-2K_X|$ in the global case or for extremal contractions $X \to Z$ is a much more difficult problem. For example, it is sufficient for the existence of three-dimensional flips [**K**].

1.3. Log canonical covers

The following construction is well known (see e.g., [**Sh2**, 2.4], [**K3**], [**K**, 8.5], [**Ko1**]). Let X be a normal variety and $D = \sum d_i D_i$ a boundary such that $m(K_X + D) \sim 0$. We take m to be the least positive integer satisfying this condition. Such m is called the *index* of $K_X + D$. Assume that $K_X + D$ is lc and $d_i \in \{1 - 1/k \mid k \in \mathbb{N} \cup \{\infty\}\}$ for all i (i.e. all the d_i are standard, see 2.2.5). Then the natural map $\mathcal{O}_X(-m(K_X + D)) \to \mathcal{O}_X$ defines an $\mathcal{O}_X$-algebra structure on $\sum_{i=0}^{m-1} \mathcal{O}_X(\lfloor -iK_X - iD \rfloor)$. Put

$$Y := \operatorname{Spec}\left(\sum_{i=0}^{m-1} \mathcal{O}_X(\lfloor -iK_X - iD \rfloor)\right)$$

and $\varphi\colon Y \to X$ the projection. Then Y is irreducible, φ is a cyclic Galois $\mathbb{Z}_m$-cover. Put $B := \varphi^*(\lfloor D \rfloor)$. Then the ramification divisor (i.e. codimension one ramification locus) of φ is $\operatorname{Supp}(D - \lfloor D \rfloor)$. Further, the ramification index along D_i is r_i, where $d_i = 1 - 1/r_i$. Therefore,

$$\varphi^*(K_X + D) = K_Y + B \sim 0.$$

By 1.2.1, $K_Y + B$ is lc. Moreover, $K_X + D$ is plt (resp. klt) if and only if $K_Y + B$ is plt (resp. klt).

EXERCISE 1.3.1. Let $(X \ni P)$ be a Du Val singularity of type D_n (given by the equation $x^2 + y^2 z + z^{n-1} = 0$) and H be a general hyperplane section. Show that the double cover ramified along H is a lc singularity of index one. Write down the equation of this singularity.

Finally, we present some Bertini's type results.

PROPOSITION 1.3.2. *Let X be a normal variety and D a $\mathbb{Q}$-divisor on X. Let $\mathcal{H}$ be a base point free linear system and $H \in \mathcal{H}$ a general member.*

(i) *Then* $K_X + D$ *is lc if and only if* $K_X + D + H$ *is lc.*
(ii) *Assume additionally that* $\lfloor D \rfloor = 0$. *Then* $K_X + D$ *is klt if and only if* $K_X + D + H$ *is plt.*

PROOF. It is sufficient to show only the implications $\Longrightarrow$. Let $f\colon Y \to X$ be a log resolution of (X, D), $E_1, \ldots, E_r$ exceptional divisors and D_Y, H_Y proper transforms of D and H, respectively. By Bertini's theorem, $D_Y + H_Y$ is a simple normal crossing divisor, so f is also a log resolution of $(X, D + H)$. We can choose $H \in \mathcal{H}$ so that H does not contain $f(E_1), \ldots, f(E_r)$. Thus we have $a(E_i, D+H) = a(E_i, D) \geq -1$. This implies the first part of our proposition. In the second part we can use remark in 1.1.1. □

PROPOSITION 1.3.3 ([**R2**, 1.13], [**KM**, 5.17]). *Let* X *be a normal variety and* D *a* $\mathbb{Q}$*-divisor on* X. *Let* $\mathcal{H}$ *be a base point free linear system and* $H \in \mathcal{H}$ *a general member. Assume that* $K_X + D$ *is klt (resp. plt, lc, canonical, terminal). Then* $K_H + D|_H$ *is klt (resp. plt, lc, canonical, terminal).*