Chapter 5

Combinatorics

5.1 The Orlik-Solomon Algebra

Let $E^1 = \oplus_{H\in\mathcal{A}} \mathbb{C}e_H$ and let $E = E(\mathcal{A}) = \Lambda(E^1)$ be the exterior algebra of E^1. If $|\mathcal{A}| = n$, then $E = \oplus_{p=0}^{n} E^p$, where $E^0 = \mathbb{C}$, E^1 agrees with its earlier definition and E^p is spanned over $\mathbb{C}$ by all $e_{H_1} \cdots e_{H_p}$ with $H_k \in \mathcal{A}$. Define a $\mathbb{C}$–linear map $\partial_E = \partial : E \to E$ by $\partial 1 = 0$, $\partial e_H = 1$ and for $p \geq 2$

$$\partial(e_{H_1} \cdots e_{H_p}) = \sum_{k=1}^{p} (-1)^{k-1} e_{H_1} \cdots \widehat{e_{H_k}} \cdots e_{H_p}$$

for all $H_1, \ldots, H_p \in \mathcal{A}$. If $S = \{H_1, \ldots, H_p\}$, write $e_S = e_{H_1} \cdots e_{H_p}$, $\cap S = H_1 \cap \cdots \cap H_p$ and $|S| = p$. If $p = 0$, we agree that $S = \{\ \}$ is the empty tuple, $e_S = 1$, and $\cap S = V$. If $\cap S \neq \emptyset$, then we call S **dependent** when $r(\cap S) < |S|$ and **independent** when $r(\cap S) = |S|$. This agrees with linear dependence and independence of the hyperplanes in S.

Definition 5.1.1. *Let $\mathcal{A}$ be an affine arrangement. Let $I(\mathcal{A})$ be the ideal of $E(\mathcal{A})$ generated by*

$$\{e_S \mid \cap S = \emptyset\} \cup \{\partial e_S \mid S \text{ is dependent}\}.$$

The Orlik-Solomon algebra $\mathsf{A}(\mathcal{A})$ is defined by $\mathsf{A}(\mathcal{A}) = E(\mathcal{A})/I(\mathcal{A})$.

The grading of E induces a grading on A. The following basic properties of $\mathsf{A}^{\cdot}(\mathcal{A})$ will be needed in the sequel [OT1, 3.56, 3.72]:

Theorem 5.1.2. *(1) Let $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ be a deletion-restriction triple of a nonempty arrangement. Then there are exact sequences for $q \geq 0$:*

$$0 \to \mathsf{A}^q(\mathcal{A}') \to \mathsf{A}^q(\mathcal{A}) \to \mathsf{A}^{q-1}(\mathcal{A}'') \to 0.$$

(2) For $q \geq 0$

$$\mathsf{A}^q(\mathcal{A}) = \bigoplus_{\substack{X \in L \\ r(X)=q}} \mathsf{A}^q(\mathcal{A}_X).$$

Let a_H be the image of e_H under the natural projection. It is clear that these elements generate $\mathsf{A}^{\cdot}$ as an algebra. This algebra has a topological interpretation [OT1, 5.90]:

Theorem 5.1.3. *Let $\mathcal{A}$ be an arrangement. The map $a_H \to \omega_H$ induces an isomorphism of graded algebras:*

$$\iota : \mathsf{A}^{\cdot}(\mathcal{A}) \xrightarrow{\sim} \mathsf{B}^{\cdot}(\mathcal{A}). \quad \square$$

Definition 5.1.4. *Let $a_\lambda = \sum_{H \in \mathcal{A}} \lambda_H a_H$. Since $a_\lambda \wedge a_\lambda = 0$, exterior product with a_λ provides a complex $(\mathsf{A}^{\cdot}(\mathcal{A}), a_\lambda \wedge)$*

$$0 \to \mathsf{A}^0(\mathcal{A}) \xrightarrow{a_\lambda \wedge} \mathsf{A}^1(\mathcal{A}) \xrightarrow{a_\lambda \wedge} \ldots \xrightarrow{a_\lambda \wedge} \mathsf{A}^r(\mathcal{A}) \to 0. \tag{1}$$

It follows from Theorems 4.2.6 and 5.1.3 that under certain conditions on λ

$$H^p(M, \mathcal{L}) \simeq H^p(\mathsf{A}^{\cdot}(\mathcal{A}), a_\lambda \wedge).$$

This completes the transformation of the analytic problem into a problem in combinatorics. In order to solve this combinatorial problem, we need additional tools.

5.2 The NBC Complex

We introduce a simplicial complex, $\mathsf{NBC}(\mathcal{A})$, called the nbc (= no-broken-circuit) complex of $\mathcal{A}$. This complex depends on a **linear order** in $\mathcal{A}$. Different linear orders may result in different complexes, but it follows from our results that the number of simplexes in each dimension and the homotopy type of the complex is independent of the linear order in $\mathcal{A}$. Thus we may ignore dependence on the linear order.

Write $\mathcal{A} = \{H_1, \ldots, H_n\}$ and let $I = \{1, \ldots, n\}$ be the index set. The standard linear order is $H_i \prec H_j$ if $i < j$. A maximal independent set is called a **frame**. (Many authors call such sets bases but we wish to avoid this clash of terminology.) Every frame has cardinality $r = r(\mathcal{A})$. An inclusion-minimal dependent set is called a **circuit**. A **broken circuit** is a set S for which there exists $H \prec \min(S)$ such that $\{H\} \cup S$ is a circuit. The collection of nonempty subsets of $\mathcal{A}$ which have nonempty intersection and contain no broken circuits is called **nbc**. Since **nbc** is closed under taking subsets, it forms a simplicial complex called the **nbc complex** of $\mathcal{A}$, denoted by NBC. (Many authors call this complex BC.) We agree to include the empty set in **nbc** and the empty simplex of dimension -1 in $\mathsf{NBC}(\mathcal{A})$, see [B, p.27]. This results in reduced homology and cohomology. NBC is a pure $(r-1)$-dimensional complex consisting of independent sets. An $(r-1)$-dimensional simplex of NBC is called an **nbc frame**. A simplex of NBC is **ordered** if its vertices are linearly ordered. We agree to write every element of **nbc** in the standard linear order.

Definition 5.2.1. *Let $\hat{L} = L \setminus \{V\}$. Define a map $\nu : \hat{L} \to \mathcal{A}$ by*

$$\nu(X) = \min(\mathcal{A}_X).$$

Let $P = (X_1 > \cdots > X_q)$ be a flag of elements of $\hat{L}$. Define

$$\nu(P) = \{\nu(X_1), \ldots, \nu(X_q)\}.$$

Let $S = \{H_{i_1}, \ldots, H_{i_q}\}$ be an independent q-tuple with $H_{i_1} \prec \cdots \prec H_{i_q}$. Define a flag

$$\xi(S) = (X_1 > \cdots > X_q)$$

*of $\hat{L}$, where $X_p = \bigcap_{k=p}^{q} H_{i_k}$ for $1 \le p \le q$. A flag $P = (X_1 > \cdots > X_q)$ is called an **nbc flag** if $P = \xi(S)$ for some $S \in$ **nbc**. Let $\xi($**nbc**$)$ denote the set of **nbc** flags.*

Lemma 5.2.2. *The maps ξ and ν induce bijections*

$$\xi : \mathbf{nbc} \longrightarrow \xi(\mathbf{nbc}), \qquad \nu : \xi(\mathbf{nbc}) \longrightarrow \mathbf{nbc}$$

which are inverses of each other.

Proof. Let $S = \{H_{i_1}, \ldots, H_{i_q}\} \in \mathbf{nbc}$. We show first that $\nu \circ \xi(S) = S$. Suppose $\xi(S) = (X_1 > \cdots > X_q)$. Then $H_{i_p} \supseteq X_p$. If $\min \mathcal{A}_{X_p} \prec H_{i_p}$, then $\{\min \mathcal{A}_{X_p}, H_{i_p}, \ldots, H_{i_q}\}$ is dependent and must contain a circuit. It follows that $\{H_{i_p}, \ldots, H_{i_q}\}$ contains a broken circuit. This contradicts $S \in \mathbf{nbc}$. Therefore $\min \mathcal{A}_{X_p} = H_{i_p}$ for $1 \le p \le q$. This implies that $\nu(\xi(S)) = S$, so the map $\xi : \mathbf{nbc} \longrightarrow \xi(\mathbf{nbc})$ is bijective and $\nu \circ \xi : \mathbf{nbc} \longrightarrow \mathbf{nbc}$ is the identity map. Thus these maps are inverses of each other. □

Lemma 5.2.3. *We have*

$$\xi(\mathbf{nbc}) = \{(X_1 > \cdots > X_q) \mid \nu(X_1) \prec \nu(X_2) \prec \cdots \prec \nu(X_q), \quad r(X_p) = q - p + 1 \ (1 \le p \le q)\}.$$

Proof. By Lemma 5.2.2, the left hand side is contained in the right hand side. Conversely, let $P = (X_1 > \cdots > X_q)$ belong to the right hand side. We will show that $\nu(P) \in \mathbf{nbc}$. It is sufficient to derive a contradiction assuming that $\nu(P)$ itself is a broken circuit. There exists $H \in \mathcal{A}$ such that (1) $H \prec \nu(X_1)$ and (2) $\{H\} \cup \nu(P)$ is a circuit. Since

$$X_1 \subseteq \nu(X_1) \cap \cdots \cap \nu(X_q) \subseteq H,$$

we have $H \in \mathcal{A}_{X_1}$ and thus $\nu(X_1) = \min(\mathcal{A}_{X_1}) \preceq H$. This contradicts (1). Thus $\nu(P) \in \mathbf{nbc}$. Since $X_p \subseteq \nu(X_p) \cap \cdots \cap \nu(X_q)$ for $1 \le p \le q$, and $r(X_p) = q-p+1 = r(\nu(X_p) \cap \cdots \cap \nu(X_q))$, we have $P = \xi \circ \nu(P)$ so $P \in \xi(\mathbf{nbc})$. □

It follows from [OT1, Ch.3] that the number of p-simplexes, c_p, in NBC is independent of the linear order and we have

$$\sum_{p=-1}^{r-1} c_p t^{p+1} = (-t)^{\ell}\chi(\mathcal{A}, -t^{-1}). \tag{2}$$

If $\mathcal{A}$ is a real arrangement, then it follows from Theorem 3.3.9 that the cardinality of the set of chambers in the real complement equals $\sum_{p=-1}^{r-1} c_p$.

Example 5.2.4. *Recall the Selberg arrangement of Figure 3.1. The corresponding simplicial complex* $\mathsf{NBC}(\mathcal{A})$ *is shown in Figure 5.1, where the 1-simplexes are the ordered* **nbc** *frames:*

$$\{\{1,3\},\{2,3\},\{1,4\},\{2,4\},\{1,5\},\{2,5\}\}.$$

We see in Figure 3.1 that the complement of $\mathcal{A}$ *has 12 chambers. This agrees with formula (2) since* $c_{-1} = 1$, $c_0 = 5$, $c_1 = 6$.

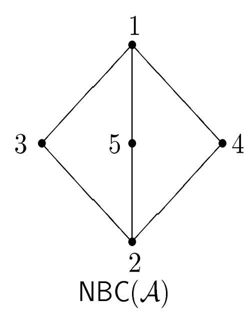

Figure 5.1: The Selberg arrangement, II

It follows from [OT1, 3.55] that the elements of **nbc** give a basis for $\mathsf{A}^{\cdot}$.

Theorem 5.2.5. *If* $S = \{H_{i_1}, \dots, H_{i_p}\}$, *write* $a_S = a_{H_{i_1}} \cdots a_{H_{i_p}} \in \mathsf{A}^{\cdot}$. *We agree that* $a_{\emptyset} = 1$. *Then the set* $\{a_S \mid S \in \mathbf{nbc}\}$ *is a* $\mathbb{C}$*-basis for* $\mathsf{A}^{\cdot}$.

Example 5.2.6. *Recall the Selberg arrangement of Figure 3.1. We write* a_j *in place of* a_{H_j} *and let* $a_{i,j} = a_i \wedge a_j$. *Theorem 5.2.5 and Example 5.2.4 provide the following basis for* A*:*

$$\{1, a_1, a_2, a_3, a_4, a_5, a_{1,3}, a_{2,3}, a_{1,4}, a_{2,4}, a_{1,5}, a_{2,5}\}.$$

We agree to delete the *last* hyperplane $H_n \in \mathcal{A}$ in the deletion-restriction triple $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ of Definition 3.1.3. Thus $\mathcal{A}' = \mathcal{A} - \{H_n\}$ and $\mathcal{A}'' = \mathcal{A}^{H_n}$. The linear order in $\mathcal{A}'$ is inherited from $\mathcal{A}$. It is the standard order. The linear order in $\mathcal{A}''$ is determined by labeling each hyperplane $K \in \mathcal{A}''$ by the smallest hyperplane $\nu(K) = \min(\mathcal{A}_K)$ of $\mathcal{A}$ containing it. Clearly $\nu(K) \prec H_n$ for all $K \in \mathcal{A}''$. Let $\mathbf{nbc}' = \mathbf{nbc}(\mathcal{A}')$, $\mathsf{NBC}' = \mathsf{NBC}(\mathcal{A}')$, $\mathbf{nbc}'' = \mathbf{nbc}(\mathcal{A}'')$, and $\mathsf{NBC}'' = \mathsf{NBC}(\mathcal{A}'')$.

Lemma 5.2.7. *Let* $\{X_1, \ldots, X_p\} \subseteq \mathcal{A}''$. *Then*

$$\{X_1, \ldots, X_p\} \in \mathbf{nbc}'' \iff \{\nu(X_1), \ldots, \nu(X_p), H_n\} \in \mathbf{nbc}.$$

Proof. ($\Rightarrow$): Suppose $\{X_1, \ldots, X_p\} \in \mathbf{nbc}''$. If $\{\nu(X_1), \ldots, \nu(X_p), H_n\}$ contains a broken circuit, then there exists an integer k with $1 \le k \le p$ and a hyperplane $H \in \mathcal{A}'$ with $H \prec \nu(X_k)$ such that $\{H, \nu(X_k), \ldots, \nu(X_p), H_n\}$ is linearly dependent. Thus $\{H \cap H_n, X_k, \ldots, X_p\}$ is also linearly dependent and $\nu(H \cap H_n) \preceq H \prec \nu(X_k)$. This implies that $\{X_k, \ldots, X_p\}$ contains a broken circuit, which is a contradiction.

($\Leftarrow$): Suppose $\{\nu(X_1), \ldots, \nu(X_p), H_n\} \in \mathbf{nbc}$. If $\{X_1, \ldots, X_p\}$ contains a broken circuit, then there exists an integer k with $1 \le k \le p$ and a hyperplane $X \in \mathcal{A}''$ with $X \prec X_k$ such that $\{X, X_k, \ldots, X_p\}$ is linearly dependent. Thus $\{\nu(X), \nu(X_k), \ldots, \nu(X_p), H_n\}$ is also linearly dependent and $\nu(X) \prec \nu(X_k)$. This implies that $\{\nu(X_k), \ldots, \nu(X_p), H_n\}$ contains a broken circuit, which is a contradiction. □

Lemma 5.2.8. *If* $\{H_{i_1}, \ldots, H_{i_p}, H_n\} \in \mathbf{nbc}$, *then* $\nu(H_{i_k} \cap H_n) = H_{i_k}$ *for* $1 \le k \le p$.

Proof. We may assume $p = 1$ without loss of generality. In general $\nu(H_{i_1} \cap H_n) \preceq H_{i_1}$. If $\nu(H_{i_1} \cap H_n) \prec H_{i_1}$, then $\{\nu(H_{i_1} \cap H_n), H_{i_1}, H_n\}$ is linearly dependent. Thus $\{H_{i_1}, H_n\}$ contains a broken circuit, which is a contradiction. □

Theorem 5.2.9. *Write* $\overline{\mathbf{nbc}}'' = \{\{\nu S'', H_n\} \mid S'' \in \mathbf{nbc}''\}$. *There is a disjoint union*

$$\mathbf{nbc} = \mathbf{nbc}' \cup \overline{\mathbf{nbc}}''.$$

The next result is well known to experts. We provide an elementary topological argument based on [OT1, 4.109].

Theorem 5.2.10. *Let* $\mathcal{A}$ *be an* ℓ*-arrangement of rank* $r = r(\mathcal{A}) \ge 1$. *Then* NBC = NBC($\mathcal{A}$) *has the homotopy type of a wedge of spheres,* $\vee_{\beta(\mathcal{A})} S^{r-1}$. *If* $\beta(\mathcal{A}) = 0$, *then* NBC *is contractible.*

Proof. The assertion holds for $r = 1$ since $\beta(\mathcal{A}) = |\mathcal{A}| - 1$ and NBC consists of $|\mathcal{A}|$ points.

If $\mathcal{A}$ is an arrangement with $r \ge 2$, then NBC is path connected. It is enough to show that vertices corresponding to distinct hyperplanes $H_i, H_j \in \mathcal{A}$, $i < j$, are connected. If $X = H_i \cap H_j \neq \emptyset$, then $r(X) = 2 \le r$. If $H_i = \nu(X)$, then $\{H_i, H_j\}$ is a 1-simplex in NBC. If $\nu(X) \prec H_i$, then $\{\nu(X), H_i\}$ and $\{\nu(X), H_j\}$ are both 1-simplexes in NBC. Thus the vertices H_i and H_j are connected. If $H_i \cap H_j = \emptyset$ then there exists H_k with $H_i \cap H_k \neq \emptyset$ and $H_j \cap H_k \neq \emptyset$ so H_i and H_j are connected via H_k.

If v is a vertex of NBC then its **star**, $\mathrm{st}(v)$, consists of all open simplexes whose closure contains v. The closure, $\overline{\mathrm{st}(v)}$, is a cone with cone point v. Let $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ be a triple with respect to the last hyperplane H_n. Then $\overline{\mathrm{st}(H_n)}$ consists of all simplexes belonging to the set $\{S \in \mathbf{nbc} \mid S \cup \{H_n\} \in \mathbf{nbc}\}$. Also NBC' consists of all simplexes S of NBC with $H_n \notin S$. Thus we have

$$\mathsf{NBC} = \overline{\mathrm{st}(H_n)} \cup \mathsf{NBC}'. \tag{3}$$

By Lemma 5.2.7, $\nu\{X_1, \ldots, X_p\} \in \overline{\mathrm{st}(H_n)} \cap \mathsf{NBC}'$. So the map ν induces a simplicial map

$$\nu : \mathsf{NBC}'' \longrightarrow \overline{\mathrm{st}(H_n)} \cap \mathsf{NBC}'.$$

This map is obviously injective. It is also surjective by Lemmas 5.2.8 and 5.2.7. Thus the two simplicial complexes are isomorphic:

$$\mathsf{NBC}'' \simeq \overline{\mathrm{st}(H_n)} \cap \mathsf{NBC}'. \tag{4}$$

If $\mathcal{A}$ is an arrangement with $r \geq 3$, then NBC is simply connected. We use induction on $|\mathcal{A}|$. Since $|\mathcal{A}| \geq r$, the induction starts with $|\mathcal{A}| = r$. In this case, $\mathcal{A}$ is isomorphic to the Boolean arrangement. Since any subset of $\mathcal{A}$ is a simplex of NBC, NBC is contractible. Now $\overline{\mathrm{st}(H_n)}$ is a cone with cone point H_n. In particular, it is simply connected. Since $|\mathcal{A}'| < |\mathcal{A}|$, the induction hypothesis implies that NBC' is simply connected. Finally, $r(\mathcal{A}'') = r - 1 \geq 2$, so it follows that NBC'' is path connected. Thus, by (3) and (4), van Kampen's theorem implies that NBC is simply connected.

Next we want to compute the homology groups of NBC. Integer coefficients are understood. Consider the Mayer–Vietoris sequence for the excisive couple $\{\overline{\mathrm{st}(H_n)}, \mathsf{NBC}'\}$. Using (3) and (4) we get the long exact sequence

$$\begin{aligned} \ldots \to H_p(\overline{\mathrm{st}(H_n)}) \oplus H_p(\mathsf{NBC}') &\xrightarrow{(i_1, i_2)_*} H_p(\mathsf{NBC}) \\ \xrightarrow{\partial_*} H_{p-1}(\overline{\mathrm{st}(H_n)} \cap \mathsf{NBC}') &\xrightarrow{(j_1, -j_2)_*} H_{p-1}(\overline{\mathrm{st}(H_n)}) \oplus H_{p-1}(\mathsf{NBC}') \to \ldots \end{aligned}$$

The fact that $\overline{\mathrm{st}(H_n)}$ is contractible, together with (4), gives

$$\ldots \to H_p(\mathsf{NBC}') \xrightarrow{i_{1*}} H_p(\mathsf{NBC}) \xrightarrow{\partial_*} H_{p-1}(\mathsf{NBC}'') \xrightarrow{j_{1*}} H_{p-1}(\mathsf{NBC}') \to \ldots \tag{5}$$

Next we show that

$$H_p(\mathsf{NBC}) = \begin{cases} 0 & \text{if } p \neq r-1, \\ \text{free of rank } \beta(\mathcal{A}) & \text{if } p = r-1. \end{cases}$$

We use induction on r, and for fixed r on $|\mathcal{A}|$. We have established the assertion for $r = 1$ and arbitrary $|\mathcal{A}|$. The assertion is also correct for arbitrary r when $|\mathcal{A}| = r$, since in that case $\mathcal{A}$ is the Boolean arrangement, NBC is contractible and it follows from [OT1, 2.51] that $\beta(\mathcal{A}) = 0$. For the induction step we assume that the result holds for all arrangements $\mathcal{B}$ with $r(\mathcal{B}) < r$ and for all arrangements $\mathcal{B}$ with $r(\mathcal{B}) = r$

and $|\mathcal{B}| < |\mathcal{A}|$. Consider the exact sequence (5). Here we need a case distinction. If H_n is a separator (Definition 3.1.3), then $r(\mathcal{A}') < r$. In this case $\mathcal{A}' = \mathcal{A}'' \times \Phi_1$. Thus $\chi(\mathcal{A}', t) = t\chi(\mathcal{A}'', t)$, so Proposition 3.1.4 implies that $\chi(\mathcal{A}, t) = (t-1)\chi(\mathcal{A}'', t)$ and hence $\beta(\mathcal{A}) = 0$. On the other hand, $X \cap H_n \neq \emptyset$ for all $X \in L(\mathcal{A}') \setminus \{V\}$ so $\mathsf{NBC} = \overline{\mathrm{st}(H_n)}$, which is contractible. If H_n is not a separator, then for $p \neq r-1$ the induction hypothesis implies that $H_p(\mathsf{NBC}') = H_{p-1}(\mathsf{NBC}'') = 0$ and hence $H_p(\mathsf{NBC}) = 0$. For $p = r-1$, the induction hypothesis implies that $H_{p-1}(\mathsf{NBC}'')$ is free of rank $\beta(\mathcal{A}'')$ and $H_p(\mathsf{NBC}')$ is free of rank $\beta(\mathcal{A}')$. The conclusion follows from Proposition 3.3.3.

This allows completion of the proof. For $r = 2$, the complex NBC is 1–dimensional and hence it has the homotopy type of a wedge of circles whose number equals the rank of $H_1(\mathsf{NBC})$. We showed above that this rank is $\beta(\mathcal{A})$. For $r \geq 3$, the complex NBC is simply connected. It follows from the homology calculation and the Hurewicz isomorphism theorem that $\pi_i(\mathsf{NBC}) = 0$ for $1 \leq i < r-1$ and $\pi_{r-1}(\mathsf{NBC}) \simeq H_{r-1}(\mathsf{NBC}; \mathbb{Z})$. The last group is free of rank $\beta(\mathcal{A})$. □

5.3 The βnbc Set

Ziegler [Zi] defined a subset $\beta\mathbf{nbc}(\mathcal{A})$ of $\mathbf{nbc}(\mathcal{A})$ of cardinality $|\beta\mathbf{nbc}(\mathcal{A})| = \beta(\mathcal{A})$. It has the property that if the simplexes corresponding to $\beta\mathbf{nbc}$ are removed from the complex NBC, the remaining simplicial complex is contractible. The set $\beta\mathbf{nbc}$ is used to construct a basis for the only nontrivial cohomology group $H^{r-1}(\mathsf{NBC})$.

Definition 5.3.1. *A frame B is called a β***nbc frame** *if B is an* **nbc** *frame and for every $H \in B$ there exists $H' \prec H$ in $\mathcal{A}$ such that $(B \setminus \{H\}) \cup \{H'\}$ is a frame. Let $\beta\mathbf{nbc}(\mathcal{A})$ be the set of all βnbc frames. When $\mathcal{A}$ is empty, we agree that $\beta\mathbf{nbc}(\mathcal{A}) = \emptyset$.*

We need to determine what happens to these frames under deletion and restriction. Recall that we have agreed to delete the *last* hyperplane $H_n \in \mathcal{A}$. Let $\beta\mathbf{nbc} = \beta\mathbf{nbc}(\mathcal{A})$, $\beta\mathbf{nbc}' = \beta\mathbf{nbc}(\mathcal{A}')$, and $\beta\mathbf{nbc}'' = \beta\mathbf{nbc}(\mathcal{A}'')$. Ziegler [Zi, Theorem 1.5] proved the following important $\beta\mathbf{nbc}$ recursion:

Theorem 5.3.2. *Write $\overline{\beta\mathbf{nbc}}'' = \{\{\nu B'', H_n\} \mid B'' \in \beta\mathbf{nbc}''\}$. If H_n is a separator, then $\beta\mathbf{nbc} = \emptyset$. Otherwise, there is a disjoint union*

$$\beta\mathbf{nbc} = \beta\mathbf{nbc}' \cup \overline{\beta\mathbf{nbc}}''.$$

When $\ell = 1$, we agree that $\beta\mathbf{nbc}''$ is empty, so $\overline{\beta\mathbf{nbc}}'' = \{H_n\}$.

For an nbc frame $B \in \mathbf{nbc}$ let $B^* \in C^{r-1}(\mathsf{NBC})$ denote the $(r-1)$-cochain dual to B. Thus for an nbc frame $B' \in \mathbf{nbc}$, B^* is determined by the formula

$$\langle B^*, B' \rangle = \begin{cases} 1 & \text{if } B' = B \\ 0 & \text{otherwise.} \end{cases}$$

The next result follows from Ziegler's recursion theorem 5.3.2:

Theorem 5.3.3 ([FT]). *The set* $\{[B^*] \mid B \in \beta\mathbf{nbc}\}$ *is a basis for* $H^{r-1}(\mathsf{NBC})$.

Proof. If H_n is a separator, then $\beta\mathbf{nbc} = \emptyset$. In this case NBC is contractible as we saw in the proof of Theorem 5.2.10. Suppose that H_n is not a separator. Recall (3) and (4). The Mayer-Vietoris cohomology exact sequence reduces to

$$0 \to H^{r-2}(\overline{\mathrm{st}(H_n)} \cap \mathsf{NBC}') \xrightarrow{\partial^*} H^{r-1}(\mathsf{NBC}) \xrightarrow{(i_1,i_2)^*} H^{r-1}(\overline{\mathrm{st}(H_n)}) \oplus H^{r-1}(\mathsf{NBC}') \to 0$$

because $H^p(\mathsf{NBC}) = 0$, $H^p(\mathsf{NBC}') = 0$ and $H^{p-1}(\mathsf{NBC}'') = 0$ when $p \neq r-1$ by Theorem 5.2.10. We describe the connecting morphism ∂^* explicitly. If B'' is an $(r-2)$-simplex of NBC'', then $\nu B''$ is an $(r-2)$-simplex of $\overline{\mathrm{st}(H_n)} \cap \mathsf{NBC}'$ and

$$\{\nu B''\}^* \in C^{r-2}(\overline{\mathrm{st}(H_n)} \cap \mathsf{NBC}').$$

The natural map

$$C^{r-2}(\overline{\mathrm{st}(H_n)}) \oplus C^{r-2}(\mathsf{NBC}') \xrightarrow{(j_1,-j_2)^\#} C^{r-2}(\overline{\mathrm{st}(H_n)} \cap \mathsf{NBC}')$$

sends the element $(\{\nu B''\}^*, 0)$ to $\{\nu B''\}^*$. Note that every $(r-1)$–simplex in $\overline{\mathrm{st}(H_n)}$ includes H_n as a vertex and $\{\nu B'', H_n\}$ is the unique $(r-1)$–simplex in $\overline{\mathrm{st}(H_n)}$ which contains $\nu B''$. Thus the coboundary map

$$\delta : C^{r-2}(\overline{\mathrm{st}(H_n)}) \to C^{r-1}(\overline{\mathrm{st}(H_n)})$$

sends $\{\nu B''\}^*$ to $\{\nu B'', H_n\}^*$. This diagram chase shows that

$$\partial^* ([\{\nu B''\}^*]) = [\{\nu B'', H_n\}^*] \in H^{r-1}(\mathsf{NBC}).$$

Now we get the desired result by induction on $|\mathcal{A}|$ using Ziegler's recursion theorem 5.3.2. ☐

Example 5.3.4. *Recall the Selberg arrangement of Examples 3.1.1, 5.2.4. Here* $\beta\mathbf{nbc} = \{\{2,4\},\{2,5\}\}$. *We see in Figure 3.1 that the complement of* $\mathcal{A}$ *has 2 bounded chambers. This agrees with the cardinality of* $\beta\mathbf{nbc}$*, see Theorem 3.3.9. The cohomology classes* $[\{2,4\}^*]$ *and* $[\{2,5\}^*]$ *form a basis for* $H^1(\mathsf{NBC})$.