Chapter 4

Non-Resonance Theorems

4.1 Logarithmic Complex

We review classical Hodge theory for an ℓ-dimensional compact complex projective manifold X. Let $\mathcal{O} = \mathcal{O}_X$ denote the sheaf of germs of holomorphic functions on X and let $\Omega^{\cdot} = \Omega^{\cdot}_X$ be the de Rham complex of germs of holomorphic differential forms on X with the exterior differentials, where $\Omega^0 = \mathcal{O}$. Let $\mathbb{C} = \mathbb{C}_X$ denote the constant sheaf on X. It follows from the Poincaré Lemma that the sequence $0 \to \mathbb{C} \to \Omega^{\cdot}$ is exact. Let σ be the stupid filtration. The spectral sequence associated with the filtered complex $(\Omega^{\cdot}, \sigma)$, is:

$$E_1^{p,q} = H^q(X, \Omega^p) \Rightarrow E_\infty^{p+q} = H^{p+q}(X, \mathbb{C}).$$

Theorem 4.1.1 (Hodge). *This spectral sequence degenerates at the E_1 term. As a consequence, there is a decomposition*

$$H^n(X, \mathbb{C}) = \bigoplus_{p+q=n} H^q(X, \Omega^p)$$

which is called the **Hodge decompostion**.

Next recall "the non-compact version" or "the mixed version" of the Hodge decomposition. Let X be a complex quasiprojective manifold, a Zariski open set of a compact projective manifold. As a corollary of Hironaka's resolution theorem, we know that there exists a (smooth) projective manifold $\bar{X}$ such that $Y = \bar{X} \setminus X$ is a normal crossing divisor. Each $x \in \bar{X}$ has a coordinate neighborhood V_x with coordinate system $(z_1, z_2, \ldots, z_\ell)$ and an integer k $(0 \le k \le \ell)$ such that $z_1(x) = z_2(x) = \cdots = z_\ell(x) = 0$ and Y is defined locally by the equation $z_1 z_2 \ldots z_k = 0$.

Definition 4.1.2. *For each $x \in \bar{X}$ and $p \ge 0$, define the $\mathcal{O}_x$–module*

$$\Omega^p(\log Y)_x = \{\omega \mid \ \omega \text{ is the germ of a meromorphic } p\text{–form such that } (z_1 z_2 \ldots z_k)\omega \in \Omega^p \text{ and } (z_1 z_2 \ldots z_k) d\omega \in \Omega^{p+1}\}.$$

Then the set

$$\Omega^p(\log Y) = \bigcup_{x\in\bar{X}} \Omega^p(\log Y)_x$$

has the structure of a sheaf of $\mathcal{O}_{\bar{X}}$–modules on $\bar{X}$. Note that $\Omega^p(\log Y)_x = \Omega^p_x$ if $x \notin Y$. Let $\Omega^\cdot(\log Y)$ denote this complex with the ordinary exterior differentials.

Let $\mathbf{H}$ denote hypercohomology. The next theorem shows that the cohomology of X with constant coefficients is computed using this logarithmic complex.

Theorem 4.1.3 (Deligne [D2]). *Let $j : X \hookrightarrow \bar{X}$ be the inclusion map.*
(1) The inclusion $\Omega^\cdot(\log Y) \hookrightarrow j_*\Omega^\cdot_X$ is a quasiisomorphism.
(2) $H^m(X,\mathbb{C}) = \mathbf{H}^m(\bar{X}, \Omega^\cdot(\log Y))$.
(3) The spectral sequence

$$E_1^{p,q} = H^q(\bar{X}, \Omega^p(\log Y)) \Rightarrow E_\infty^{p+q} = H^{p+q}(X,\mathbb{C}).$$

degenerates at the E_1 term. As a consequence, there is a decomposition

$$H^n(X,\mathbb{C}) = \bigoplus_{p+q=n} H^q(\bar{X}, \Omega^p(\log Y)).$$

The problem of computing the cohomology of X with coefficients in a nontrivial local system was also solved by Deligne. Fix $x \in \bar{X}$. It has a coordinate neighborhood V_x with coordinate system $(z_1, z_2, \dots, z_\ell)$ and an integer k $(0 \le k \le \ell)$ such that $z_1(x) = z_2(x) = \cdots = z_\ell(x) = 0$ and Y is locally defined by the equation $z_1 z_2 \dots z_k = 0$. In what follows, we consider only this neighborhood of x. Thus we may assume that $\bar{X} = D^\ell$ is the unit polydisk in $\mathbb{C}^\ell$ and that $X = (D^*)^k \times D^{\ell-k}$. Let γ_j be a loop around the hyperplane $z_j = 0$. Define a locally constant sheaf of rank one, $\mathcal{L}$, on X corresponding to the representation $\rho : \pi_1(X) \to \mathbb{C}^*$ satisfying

$$\rho(\gamma_j) = c_j \in \mathbb{C}^* \quad (j = 1, \dots, k).$$

Define

$$\mathcal{O}_X(\mathcal{L}) = \mathcal{O}_X \otimes_{\mathbb{C}_X} \mathcal{L},$$
$$\Omega^p_X(\mathcal{L}) = \Omega^p_X \otimes_{\mathbb{C}_X} \mathcal{L} \quad (p = 1, \dots, \ell).$$

Next we study how we can extend $\mathcal{O}_X(\mathcal{L})$ to a locally free sheaf $\mathcal{O}_{\bar{X}}(\bar{\mathcal{L}})$ with some rank-one local system $\bar{\mathcal{L}}$ on $\bar{X}$. For simplicity, let us consider the special (but important) case of $n = k = 1$. In this case $\bar{X} = D$, $X = D^*$, and $Y = \{0\}$. Write $c = c_1$, $z = z_1$. For $x \in D^*$, choose a basis v_x for $\mathcal{L}_x$ so $\mathcal{L}_x = \mathbb{C}v_x$. Choose $\lambda \in \mathbb{C}$ such that $c = \exp(-2\pi i\lambda)$. (Note that λ is determined by c only modulo $\mathbb{Z}$.) Then the collection $\{z^\lambda \otimes v_x\}_{x\in D^*}$ determines a section on D^* invariant under monodromy. Thus it determines a rank-one local system $\bar{\mathcal{L}}$ on D with a global section v_0:

$$v_0 = z^\lambda \otimes v_x \quad (\text{for each } x \in D^*).$$

Then $\mathcal{O}_D(\bar{\mathcal{L}})$ is an extension of $\mathcal{O}_{D^*}(\mathcal{L})$ from D^* to D:

$$\mathcal{O}(\bar{\mathcal{L}})|_{D^*} = \mathcal{O}_{D^*}(\mathcal{L}).$$

Let $j : D^* \hookrightarrow D$ be the inclusion map. We have an exact sequence

$$0 \to \mathcal{L} \to \mathcal{O}(\mathcal{L}) \xrightarrow{\nabla} \Omega^1(\mathcal{L})$$

on D^* in which $\nabla(f \otimes v_x) = df \otimes v_x \quad (f \in \mathcal{O})$. If we use the global section v_0 in order to describe the map ∇, then we have

$$\begin{aligned}\nabla(f \otimes v_0) &= \nabla(f z^\lambda \otimes v_x) = d(f z^\lambda) \otimes v_x \\ &= f\lambda z^{\lambda-1} dz \otimes v_x + z^\lambda df \otimes v_x = (df + \lambda f \frac{dz}{z}) \otimes v_0.\end{aligned}$$

Therefore we can extend the map ∇ to $\bar{\nabla} : \mathcal{O}(\bar{\mathcal{L}}) \to \Omega^1(\log Y)(\bar{\mathcal{L}})$ by

$$\bar{\nabla}(f \otimes v_0) = (df + \lambda f \frac{dz}{z}) \otimes v_0.$$

It is easy to see that

$$\ker \bar{\nabla} = \begin{cases} z^{-\lambda} \otimes v_0 & \text{if } \lambda \in \mathbb{Z}_{\leq 0} \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, consider the kernel of

$$j_*\nabla : j_*\mathcal{O}(\mathcal{L}) \to j_*\Omega^1(\mathcal{L}).$$

It is again easy to see that

$$\ker(j_*\nabla) = \begin{cases} z^{-\lambda} \otimes v_0 & \text{if } \lambda \in \mathbb{Z} \\ 0 & \text{otherwise.} \end{cases}$$

Therefore we have

$$\ker \bar{\nabla} = \ker(j_*\nabla) \text{ if } \lambda \notin \mathbb{Z}_{>0}.$$

This observation is the essential part of the following

Theorem 4.1.4 (Deligne[D1]). *Suppose that Y is a normal crossing divisor in a complex projective manifold $\bar{X}$. Let $X = \bar{X} \setminus Y$. Let $\mathcal{L}$ denote a locally constant sheaf on X. Suppose that $\bar{\mathcal{L}}$ is a rank-one locally constant sheaf on $\bar{X}$ such that*

(1) $\mathcal{O}_{\bar{X}}(\bar{\mathcal{L}})|_X = \mathcal{O}_X(\mathcal{L})$, and

(2) the integrable connection $\nabla : \mathcal{O}_X(\mathcal{L}) \to \Omega^1(\mathcal{L})$, whose kernel is equal to $\mathcal{L}$, can be extended to a connection $\bar{\nabla} : \mathcal{O}_{\bar{X}}(\bar{\mathcal{L}}) \to \Omega^1_{\bar{X}}(\log Y)(\bar{\mathcal{L}})$ which can be expressed as

$$\bar{\nabla}(f \otimes v_0) = (df + \sum_{i=1}^{k} \lambda_i \frac{dz_i}{z_i}) \otimes v_0$$

locally at $y \in Y$, where $z_1 \dots z_k = 0$ is a defining equation for Y, v_0 is a basis for $\bar{\mathcal{L}}$ near y, and

$$\lambda_i \in \mathbb{C} \setminus \mathbb{Z}_{>0} \quad (i = 1, \cdots, k).$$

Then the inclusion

$$(\Omega^{\cdot}_{\bar{X}}(\log Y)(\bar{\mathcal{L}}), \bar{\nabla}) \hookrightarrow (j_* \Omega^{\cdot}_X(\mathcal{L}), j_* \nabla)$$

is a quasiisomorphism.

4.2 The Arrangement Case

Now we return to arrangements of hyperplanes and study their Hodge theory. Arnold [Ar] conjectured and Brieskorn [Bri] proved:

Theorem 4.2.1. *Let $\mathsf{B}^{\cdot}(\mathcal{A})$ denote the graded $\mathbb{C}$-algebra generated by 1 and the forms ω_H, $H \in \mathcal{A}$ and call $\mathsf{B}(\mathcal{A})$ the* **Brieskorn algebra** *of $\mathcal{A}$. The inclusion $\mathsf{B}^{\cdot} \subset \Omega^{\cdot}_M$ induces isomorphism of graded algebras*

$$\mathsf{B}^{\cdot}(\mathcal{A}) \simeq H^{\cdot}(M(\mathcal{A}), \mathbb{C}).$$

Choose projective coordinates $u_0, \dots, u_\ell$ so that $H_\infty = \ker(u_0)$ is the hyperplane at infinity. Then the defining equation of $N(\mathcal{A}_\infty)$ is $u_0 \tilde{Q} = 0$ where $\tilde{Q}$ is the homogenization of Q.

Proposition 4.2.2. *Let $\tilde{\alpha}_H$ be the homogenized polynomial of α_H. For each $H \in \mathcal{A}$, define a global meromorphic form $\tilde{\omega}_H$ on $\mathbb{CP}^\ell$ by*

$$\tilde{\omega}_H = \frac{d\tilde{\alpha}_H}{\tilde{\alpha}_H} - \frac{du_0}{u_0} \quad (H \in \mathcal{A}).$$

Then

$$H^{\cdot}(M, \mathbb{C}) \simeq \bigwedge (\tilde{\omega}_H \mid H \in \mathcal{A}).$$

Proof. Note that the global meromorphic 1-form $\tilde{\omega}_H$ is the unique extension of the form ω_H from $\mathbb{C}^\ell$ to $\mathbb{CP}^\ell$. Therefore the restriction map

$$\tilde{\omega}_H \mapsto \tilde{\omega}_H|_{\mathbb{C}^\ell} = \omega_H$$

gives an isomorphism

$$\bigwedge(\tilde{\omega}_H \mid H \in \mathcal{A}) \simeq \bigwedge(\omega_H \mid H \in \mathcal{A})$$

as $\mathbb{C}$–algebras. The second algebra is isomorphic to $H^\cdot(M,\mathbb{C})$ by Theorem 4.2.1. □

Theorem 4.2.3 (Esnault-Schechtman-Viehweg [ESV]). *Given the arrangement $\mathcal{A}$ in $\mathbb{C}^\ell$ with projective closure $\mathcal{A}_\infty$ in $\mathbb{CP}^\ell$, let $N_\infty = N(\mathcal{A}_\infty)$ be its divisor. Suppose that $\bar{X}$ is a (nonsingular) complex projective manifold which is the result of successively blowing up $\mathbb{CP}^\ell$, $\tau : \bar{X} \to \mathbb{CP}^\ell$. Assume that $Y = \tau^{-1}(N_\infty)$ has normal crossings. Let $X = \bar{X} \setminus Y$ and $M = \mathbb{C}^\ell \setminus \bigcup_{H\in\mathcal{A}} H$. Then*

(1) $\Gamma(\bar{X}, \Omega^n(\log Y)) \simeq H^n(M,\mathbb{C}) \simeq \bigwedge^n_{\mathbb{C}}(\tilde{\omega}_H \mid H \in \mathcal{A})$ $(n = 0, \ldots, \ell)$, and

(2) $H^q(\bar{X}, \Omega^p(\log Y)) = 0 \quad (q \geq 1)$.

Proof. By Proposition 4.2.2 we have

$$H^n(M,\mathbb{C}) \simeq \bigwedge^n(\tilde{\omega}_H \mid H \in \mathcal{A}).$$

Note that the pullback of each $\tilde{\omega}_H$ by τ belongs to $\Gamma(\bar{X}, \Omega^1(\log Y))$. On the other hand, by Theorem 4.1.3 (3), there is a decomposition

$$H^n(X,\mathbb{C}) = \bigoplus_{p+q=n} H^q(\bar{X}, \Omega^p(\log Y)).$$

Thus we have

$$H^n(M,\mathbb{C}) \simeq \bigwedge^n(\tilde{\omega}_H \mid H \in \mathcal{A}) \hookrightarrow \Gamma(\bar{X}, \Omega^n(\log Y)) \hookrightarrow H^n(X,\mathbb{C}).$$

Since $X \simeq M$, these two monomorphisms are isomorphisms. This proves (1). We obtain (2) by applying Theorem 4.1.3 again. □

Now we want to apply Deligne's results on local system cohomology to arrangements. In general $N(\mathcal{A}_\infty) \subset \mathbb{CP}^\ell$ is not a normal crossing divisor. Our first task is to use resolution of singularities to blow up linear subspaces of $N(\mathcal{A}_\infty)$ to make it a normal crossing divisor. The blow up of $\mathbb{C}^\ell$ at the origin is the space $\tilde{\mathbb{C}}^\ell = U_1 \cup \cdots \cup U_\ell$ where each $U_j = \mathbb{C}^\ell$ and identifications are given by the map $\tau : \tilde{\mathbb{C}}^\ell \to \mathbb{C}^\ell$ as follows. Let U_j have coordinates $\{y_k^{(j)}\}$ and let the domain $\mathbb{C}^\ell$ have coordinates $\{u_i\}$. Then

$$\tau^*(u_i) = \begin{cases} y_i^{(j)} y_j^{(j)} & \text{on } U_j \text{ if } j \neq i \\ y_i^{(i)} & \text{on } U_i. \end{cases}$$

The exceptional divisor is $E = \tau^{-1}(0)$. It is a copy of $\mathbb{CP}^{\ell-1}$.

There is an embedding of $\tilde{\mathbb{C}}^\ell$ in $\mathbb{C}^\ell \times \mathbb{CP}^{\ell-1}$. Let $\mathbb{CP}^{\ell-1}$ have coordinates $(w_1 : \ldots : w_\ell)$. Then $\tilde{\mathbb{C}}^\ell = \{u_i w_j - u_j w_i = 0 \mid i \neq j\}$. Here $y_i^{(j)} = w_i/w_j$ if $i \neq j$ and $y_i^{(i)} = u_i$.

Now suppose $\mathcal{A}$ is a central arrangement and $T(\mathcal{A})$ is the origin. Let $\mathbb{P}\mathcal{A}$ be the projective quotient of $\mathcal{A}$ and let $\mathbb{P}N$ be its divisor. Note that N is the complex cone over $\mathbb{P}N$. We have a commutative diagram

$$
\begin{array}{ccccc}
\mathbb{P}N & \subset & \mathbb{CP}^{\ell-1} & & \\
 & & \pi \uparrow & \nwarrow pr_2 & \\
 & & \tilde{\mathbb{C}}^\ell & \subset & \mathbb{C}^\ell \times \mathbb{CP}^{\ell-1} \\
 & & \tau \downarrow & \swarrow pr_1 & \\
N & \subset & \mathbb{C}^\ell & &
\end{array}
$$

and $\tau^{-1}(N) = \pi^{-1}(\mathbb{P}N) \cup E$.

Theorem 4.2.4 (Varchenko (10. 8 in [V2])). *Let $\mathcal{A}_\infty$ be a projective arrangement in $X_0 = \mathbb{CP}^\ell$ and let $N_\infty = N(\mathcal{A}_\infty)$ be its divisor. Let $\tau_1 : X_1 \to X_0$ be the result of blowing up points in $\mathsf{D}_0(\mathcal{A}_\infty)$. For $2 \leq s \leq \ell - 1$, let $\tau_s : X_s \to X_{s-1}$ be the result of blowing up the proper transforms of the $(s-1)$-dimensional dense spaces in $\mathsf{D}_{s-1}(\mathcal{A}_\infty)$ by $\tau_{s-1} \circ \tau_{s-2} \circ \cdots \circ \tau_1$. Let $\bar{X} = X_{\ell-1}$ and $\tau = \tau_{\ell-1} \circ \cdots \circ \tau_1$. Then $\bar{X}$ is nonsingular and $Y = \tau^{-1}(N_\infty)$ is a normal crossing divisor.*

Proof. Let $p \in \mathbb{CP}^\ell$. It suffices to prove that p has a neighborhood U such that $\tau^{-1}(U \cap N_\infty)$ is a normal crossing divisor. Thus we may assume that p is the origin in $\mathbb{C}^\ell$ and $\mathcal{A}$ is a central arrangement with $p \in T(\mathcal{A})$. We argue by induction on ℓ. There is nothing to prove when $\ell = 1$. For $\ell > 1$ there are three cases to consider. Write $T = T(\mathcal{A})$ and $N = N(\mathcal{A})$.

(1) If $p \neq T$, then $\dim T = d > 0$. Consider the projection $q : \mathbb{C}^\ell \to \mathbb{C}^\ell/T = \mathbb{C}^{\ell-d}$. Let $\mathcal{A}^{es} = q(\mathcal{A})$ be the corresponding essential arrangement. Then $q^{-1}(\mathsf{D}_j(\mathcal{A}^{es})) = \mathsf{D}_{j+d}(\mathcal{A})$. Let $N^{es} = N(\mathcal{A}^{es})$ and let τ^{es} be its resolution. Then $\tau^{-1}(N) = \tau^{-1}q^{-1}(N^{es}) = q^{-1}(\tau^{es})^{-1}(N^{es})$. The conclusion follows from the fact that $(\tau^{es})^{-1}(N^{es})$ is a normal crossing divisor.

$$
\begin{array}{ccc}
X_{\ell-1} & \xrightarrow{\pi} & Y_{\ell-2} \\
\hat{\tau}\downarrow & & \downarrow\sigma \\
X_1 & \xrightarrow{\pi} & Y_0 \\
\tau_1\downarrow & & \\
X_0 & &
\end{array}
$$

(2) If $p = T$ is a dense edge, then p is the center of τ_1. We argue by induction on ℓ. There is a bijection between $\mathsf{D}_j(\mathcal{A})$ and $\mathsf{D}_{j-1}(\mathbb{P}\mathcal{A})$ for $j \geq 1$. Thus we

may assume by induction that we have a resolution $\sigma : Y_{\ell-2} \to Y_0 = \mathbb{CP}^{\ell-1}$ of $\mathbb{P}\mathcal{A}$ so that $Y_{\ell-2}$ is nonsingular and $\sigma^{-1}(\mathbb{P}N)$ is a normal crossing divisor. In the commutative diagram above $\hat{\tau} = \tau_{\ell-1} \circ \cdots \circ \tau_2$. Since $\tau_1^{-1}(N) = \pi^{-1}(\mathbb{P}N) \cup E$, we have $\tau^{-1}(N) = \pi^{-1}\sigma^{-1}(\mathbb{P}N) \cup \hat{\tau}^{-1}E$. The first divisor on the right has normal crossings by the induction hypothesis. The preimage of the nonsingular divisor E intersects it transversely, hence $\tau^{-1}(N)$ is a normal crossing divisor.

(3) If $p = T$ is not a dense edge, then $\mathcal{A}$ has a decomposition into indecomposable subarrangements $\mathcal{A} = \mathcal{A}_1 \uplus \cdots \uplus \mathcal{A}_m$ with $m > 1$. Here $\dim T(\mathcal{A}_i) > 0$ for all i. Thus we may use the first case to blow up $\mathbb{CP}^\ell$ for each $\mathcal{A}_i$ by the construction provided above so the preimage of each $N(\mathcal{A}_i)$ has normal crossings. It follows from Lemma 3.2.7 that $\mathsf{D}_j(\mathcal{A}) = \bigcup_{i=1}^m \mathsf{D}_j(\mathcal{A}_i)$. Since $\mathcal{A}_1, \ldots, \mathcal{A}_m$ may be defined in distinct sets of variables, the resolution τ is the composition of the m separate resolutions of the $\mathcal{A}_i$. □

Define a global 1-form on $\mathbb{CP}^\ell$ by

$$\tilde{\omega}_\lambda = \sum_{H \in \mathcal{A}} \lambda_H \tilde{\omega}_H.$$

It is the unique extension of the 1-form ω_λ. Thus the residue of $\tilde{\omega}_\lambda$ along $\bar{H} \in \mathcal{A}_\infty$ is λ_H and the residue of $\tilde{\omega}_\lambda$ along H_∞ is $-\sum_{H\in\mathcal{A}} \lambda_H$. Therefore let $\lambda_\infty = -\sum_{H\in\mathcal{A}} \lambda_H$ be the weight of H_∞. For $Z \in L(\mathcal{A}_\infty)$, define $\lambda_Z \in \mathbb{C}$ by

$$\lambda_Z = \sum_{H \in (\mathcal{A}_\infty)_Z} \lambda_H.$$

Lemma 4.2.5. *Let E be a component of the normal crossing divisor Y in the resolution of Theorem 4.2.4 and let $g = 0$ be a local defining equation of E. If E first appeared when we blew up a proper transform of the dense edge $Z \in \mathsf{D}(\mathcal{A}_\infty)$, then the coefficient of dg/g in $\tau^*\tilde{\omega}_\lambda$ is λ_Z.*

Theorem 4.2.6 ([ESV][STV]). *If $\lambda_Z \notin \mathbb{Z}_{>0}$ for every $Z \in \mathsf{D}(\mathcal{A}_\infty)$, then*

$$H^p(M, \mathcal{L}_\lambda) \simeq H^p(\mathsf{B}^\cdot(\mathcal{A}), \omega_\lambda \wedge).$$

Proof. Let $\tau : \bar{X} \to \mathbb{CP}^\ell$ be given as in Theorem 4.2.4 by successively blowing up the proper transforms of the dense edges of $\mathcal{A}_\infty$. Then $Y = \tau^{-1}(N(\mathcal{A}_\infty))$ has normal crossings. Let $X = \bar{X} \setminus Y$. Note that τ gives a biholomorphic map between M and X. Recall $\Phi_\lambda = \prod_{H\in\mathcal{A}} \alpha_H^{\lambda_H}$ as in Proposition 2.1.2 where we showed that $\mathcal{L}_\lambda$ is a locally constant sheaf whose local sections are isomorphic to constant multiples of Φ_λ^{-1}. Recall that the integrable connection

$$\nabla_\lambda : \mathcal{O}_M(\mathcal{L}_\lambda) \to \Omega^1_M(\mathcal{L}_\lambda)$$

satisfies $\nabla_\lambda(f \otimes \Phi_\lambda^{-1}) = (df) \otimes \Phi_\lambda^{-1}$ $(f \in \mathcal{O}_M)$. Pull back ∇_λ by τ and we have

$$\tau^*\nabla_\lambda : \mathcal{O}_X(\tau^*\mathcal{L}_\lambda) \to \Omega^1_X(\tau^*\mathcal{L}_\lambda)$$

satisfying $(\tau^*\nabla_\lambda)(f \otimes \tau^*\Phi_\lambda^{-1}) = (df) \otimes \tau^*\Phi_\lambda^{-1} \quad (f \in \mathcal{O}_X)$. Define

$$\bar{\nabla} : \mathcal{O}_{\bar{X}}(\bar{\mathcal{L}}) \to \Omega^1_{\bar{X}}(\bar{\mathcal{L}})$$

by

$$\bar{\nabla}(f \otimes 1) = (df + f(\tau^*\omega_\lambda)) \otimes 1 \quad (f \in \mathcal{O}_{\bar{X}}),$$

where $\bar{\mathcal{L}}$ is equal to the (global) constant sheaf $\mathbb{C}_{\bar{X}}$. Write $\mathcal{L} = \tau^*\mathcal{L}_\lambda$ and $\nabla = \tau^*\nabla_\lambda$. Then $X, \bar{X}, Y, \mathcal{L}, \bar{\mathcal{L}}, \nabla, \bar{\nabla}$ satisfy the assumptions of Theorem 4.1.4. Thus we have

$$\begin{aligned}
H^p(\mathsf{B}^{\cdot}(\mathcal{A}), \omega_\lambda\wedge) &= H^p(\mathsf{B}^{\cdot}(\mathcal{A}), d + \omega_\lambda\wedge) \\
&= H^p(\Gamma(\bar{X}, \Omega^{\cdot}(\log Y)), d + \tau^*\tilde{\omega}_\lambda\wedge) && \text{(Theorem 4.2.3 (1))} \\
&\simeq \mathbf{H}^p(\bar{X}, (\Omega^{\cdot}(\log Y), d + \tau^*\tilde{\omega}_\lambda\wedge)) && \text{(Theorem 4.2.3 (2))} \\
&\simeq \mathbf{H}^p(\bar{X}, (\Omega^{\cdot}(\log Y)(\bar{\mathcal{L}}), \bar{\nabla})) \\
&\simeq \mathbf{H}^p(\bar{X}, (j_*\Omega^{\cdot}(\mathcal{L}), j_*\nabla)) && \text{(Theorem 4.1.4)} \\
&\simeq \mathbf{H}^p(X, (\Omega^{\cdot}(\mathcal{L}), \nabla)) && (j \text{ is a Stein map}) \\
&\simeq H^p(X, \mathcal{L}) \\
&\simeq H^p(X, \tau^*\mathcal{L}_\lambda) \\
&\simeq H^p(M, \mathcal{L}_\lambda).
\end{aligned}$$

□

In Chapter 5 we introduce some combinatorial tools. These are used in Chapter 6 to compute the groups $H^p(\mathsf{B}^{\cdot}(\mathcal{A}), \omega_\lambda\wedge)$.