12 Applications to semilinear wave and Klein - Gordon equations

12.1 Application to semilinear wave equation

In this Chapter we consider semilinear wave equation

$$\Box u = F_\lambda(u) \tag{12.1.1}$$

in $\mathbf{R}^{n+1}$. Here the nonlinearity $F_\lambda(u)$ is a C^1 function of u for any real $\lambda > 1$ so that the following estimate

$$\left| \left(\frac{\partial}{\partial u}\right)^j F_\lambda(u) \right| \leq C|u|^{\lambda - j} \quad , \quad j = 0, 1 \tag{12.1.2}$$

is fulfilled for u close to zero. Here the constant C may depend on j, λ, but C is independent of u. A typical model is $F_\lambda(u) = |u|^\lambda$. If $f, g \in C_0^\infty(\mathbf{R}^n)$ are fixed, we shall consider the corresponding Cauchy problem for (12.1.1) with initial data

$$u(0, x) = \varepsilon f(x), \ \partial_t u(0, x) = \varepsilon g(x). \tag{12.1.3}$$

For $\varepsilon > 0$ small enough solutions of the Cauchy problem (12.1.1) and (12.1.3) are called small amplitude solutions. Instead of fixing the functions f, g and taking $\varepsilon > 0$ small enough we could take initial data of the form

$$u(0, x) = f(x), \ \partial_t u(0, x) = g(x) \tag{12.1.4}$$

and assume that suitable Sobolev norms of the initial data are sufficiently small. All results of this section could be formulated for these small data solutions, but for sake of simplicity we shall consider only the case of small amplitude solutions.

Our main goal then is to find, for a given n, the sharp range of powers for which one always has a global weak solution of (12.1.1), (12.1.3), if $\varepsilon > 0$ is small enough.

Note that, even in the linear case, where one solves an inhomogeneous equation with a Lipschitz forcing term, in general one can only obtain weak solutions.

Let us now give some historical background. In 1979, John [26] showed that when $n = 3$ global solutions always exist if $\lambda > 1 + \sqrt{2}$ and $\varepsilon > 0$ is small. He also showed that the power $1 + \sqrt{2}$ is critical in the sense that no such result can hold if $\lambda < 1 + \sqrt{2}$ and $F_\lambda(u) = |u|^\lambda$. It was shown sometime later by Schaeffer [47] that there can also be blowup for arbitrarily small data in $(1 + 3)$-dimensions when $\lambda = 1 + \sqrt{2}$.

The number $1 + \sqrt{2}$ appears to have first arisen in Strauss' work [53] on scattering for small-amplitude semilinear Schrödinger equations. Based on this, he made the conjecture in [54] that when $n \geq 2$ global solutions of (12.1.1),(12.1.3) should always exist if ε is small and λ is greater than a critical power which is the solution of the quadratic equation

$$(n - 1)\lambda_c^2 - (n + 1)\lambda_c - 2 = 0, \ \lambda_c > 1. \tag{12.1.5}$$

This conjecture was shortly verified when $n = 2$ by Glassey [18]. John's blowup results were then extended by Sideris [49], showing that, for all n, there can be blowup for arbitrarily small data if $\lambda < \lambda_c$. On the other hand, Zhou [64] showed that when $n = 4$, in which case $\lambda_c = 2$, there is always global existence for small data if $\lambda > \lambda_c$. This result has recently been extended to dimensions $n \leq 8$ in Lindblad and Sogge [33]. Here it was also shown that, under the assumption of spherical symmetry, for arbitrary $n \geq 3$ global solutions of (12.1.1), (12.1.3) exist if $\lambda > \lambda_c$ and ε is small enough. For odd spatial dimensions, the last result was obtained independently by Kubo [30].

In [14] it is shown that the assumption of spherical symmetry can be removed. Specifically, we have the following.

Theorem 12.1.1 *Let $n \geq 3$ and assume that F_λ satisfying (12.1.2) is fixed with $\lambda_c < \lambda \leq (n+3)/(n-1)$. Then if $\varepsilon > 0$ is sufficiently small (12.1.2) has a unique (weak) global solution u verifying*

$$(1 + |t^2 - |x|^2|\,)^\gamma u \in L^{\lambda+1}(\mathbf{R}^{n+1}), \tag{12.1.6}$$

for some γ satisfying

$$1/\lambda(\lambda+1) < \gamma < (\,(n-1)\lambda - (n+1)\,)/2(\lambda+1). \tag{12.1.7}$$

Note that our condition on γ only makes sense if $\lambda > \lambda_c$.

In Theorem 12.1.1 we have only considered powers smaller than the conformally invariant power $\lambda_{\text{conf}} = (n+3)/(n-1)$ since it was already known that there is global existence for powers larger than λ_{conf}. (see [33]).

We shall prove Theorem 12.1.1 using the estimate of Theorem 11.1.2.

To do so let us first notice that by shifting the time variable by $R > 0$ Theorem 11.1.2 yield

$$\begin{aligned} &\|((t+R)^2 - |x|^2)^{\alpha/2} u\|_{L^q(\mathbf{R}^{n+1})} \leq \\ &\leq C\|((t+R)^2 - |x|^2)^{\beta/2} F\|_{L^p(\mathbf{R}^{n+1})} \end{aligned} \tag{12.1.8}$$

for

$$\frac{n-1}{2(n+1)} \leq \frac{1}{q} \leq \frac{1}{2}\,, \ \frac{1}{p} = 1 - \frac{1}{q}$$

$$\alpha < n - 1 - \frac{2n}{q} \ , \beta > \frac{2}{q}.$$

Here u is a solution of $\Box u = F$ with zero initial data and F is satisfying

$$\text{supp } F(s,y) \subset \{(s,y); |y| \leq s + R - 1\}.$$

It is more convenient to use this equivalent version of the estimate from Theorem 11.1.2. The key step will be to use it to establish the following

Lemma 12.1.1 *Let $u_{-1} \equiv 0$, and for $m = 0, 1, 2, 3, \ldots$ let u_m be defined recursively by requiring*

$$\Box u_m = F_\lambda(u_{m-1})$$

$$u_m(0,x) = \varepsilon f(x), \ \partial_t u_m(0,x) = \varepsilon g(x),$$

where $f, g \in C_0^\infty(\mathbf{R}^n)$ vanishing outside the ball of radius $R-1$ centered at the origin are fixed. Then if $\lambda_c < \lambda \le (n+3)/(n-1)$, fix γ satisfying

$$\frac{2}{\lambda(\lambda+1)} < \gamma < \frac{(n-1)\lambda - (n+1)}{\lambda+1}$$

and set

$$A_m = \|((t+R)^2 - |x|^2)^{\gamma/2} u_m\|_{L^{\lambda+1}(\mathbf{R}^{n+1})},$$

$$B_m = \|((t+R)^2 - |x|^2)^{\gamma}(u_m - u_{m-1})\|_{L^{\lambda+1}(\mathbf{R}^{n+1})}.$$

Then there is an $\varepsilon_0 > 0$, depending on λ F_λ, γ and the data (f, g) so that for $m = 0, 1, 2, \ldots$

(12.1.9) $$A_m \le 2A_0 \text{ and } 2B_{m+1} \le B_m, \text{ if } \varepsilon < \varepsilon_0.$$

Proof. Because of the support assumptions on the data, domain of dependence considerations imply that u_m, and hence $F_\lambda(u_m)$, must vanish if $|x| > t + R - 1$. It is also standard that the solution u_0 of the free wave equation $\Box u_0 = 0$ with the above data satisfies

$$u_0 = O(\epsilon(1+t)^{-(n-1)/2}(1 + |t - |x||)^{-(n-1)/2}).$$

For example this estimate is established in [8] by the aid of Penrose conformal transform.

Using this estimate one finds that

$$A_0 \le C_0 \varepsilon,$$

for some uniform constant C_0.

To complete the induction argument let us first notice that for $j, m \ge 0$, $u_{m+1} - u_{j+1}$ has zero Cauchy data at $t = 0$ and

$$\Box(u_{m+1} - u_{j+1}) = V_\lambda(u_m, u_j)(u_m - u_j),$$

where by (12.1.2),

$$V_\lambda(u_m, u_j) = O((|u_m| + |u_j|)^{\lambda-1}).$$

Since we are assuming that

$$\gamma < 2n(1/2 - 1/q) - 1, \text{ and } \lambda\gamma > 1/q, \ q = \lambda + 1,$$

if we apply (12.1.8) with $\alpha = \gamma$, $\beta = \gamma\lambda$ and Hölder's inequality we therefore obtain with $q = \lambda + 1, p = (\lambda + 1)/\lambda$

$$\begin{aligned} &\|((t+R)^2 - |x|^2)^{\gamma/2}(u_{m+1} - u_{j+1})\|_{L^q} \\ &\le C_1 \|((t+R)^2 - |x|^2)^{\lambda\gamma/2} V_p(u_m, u_j)(u_m - u_j)\|_{L^p} \\ &\le C_1 \left(C_2(\|((t+R)^2 - |x|^2)^{\gamma/2} u_m\|_{L^q} + \|((t+R)^2 - |x|^2)^{\gamma/2} u_j\|_{L^q})\right)^{\lambda-1} \times \\ &\times \|((t+R)^2 - |x|^2)^{\gamma/2}(u_m - u_j)\|_{L^q}, \end{aligned}$$

for certain constants C_j which are uniform if above λ, γ and F_λ are fixed. Based on this we conclude that

$$\begin{aligned} &\|((t+R)^2 - |x|^2)^{\gamma/2}(u_{m+1} - u_{j+1})\|_{L^q} \\ &\le C_1 \left(C_2(A_m + A_j)\right)^{\lambda-1} \|((t+R)^2 - |x|^2)^{\gamma/2}(u_m - u_j)\|_{L^q}. \end{aligned} \tag{12.1.10}$$

If $j = -1$, then $A_j = 0$ and hence we conclude that

$$A_{m+1} \le A_0 + A_m/2 \quad \text{if } C_1(C_2A_m)^{\lambda-1} \le 1/2.$$

By the earlier bound for A_0, this yields the first part of (12.1.9) if

$$C_1(2C_2C_0\varepsilon_0)^{\lambda-1} < 1/2.$$

If we take $j = m - 1$ in (12.1.10), we also obtain the other half of (12.1.9) if this condition is satisfied, which completes the proof.

Using the lemma we easily get the existence part of Theorem 12.1.1. If $\varepsilon > 0$ in (12.1.3) is small and if u_m are as above we notice from the second half of (12.1.9) that u_m converges to a limit u in L^q and hence in the sense of distributions. Since (12.1.1) and the bounds for B_{m+1} yield

$$\|F_\lambda(u_{m+1}) - F_\lambda(u_m)\|_{L^p} = O(2^{-m}),$$

and hence

$$F_\lambda(u_m) \to F_\lambda(u)$$

in L^p, we conclude that u must converge to a weak solution of (12.1.1) which must satisfy (12.1.6) by the bounds for A_m. Since the proof of the bound for B_{m+1} yields the uniqueness part, this completes our argument showing that the weighted Strichartz estimates imply Theorem 12.1.1.

12.2 Application to semilinear Klein - Gordon equation

In this section we shall consider the Klein-Gordon equation

$$\Box u - u = \pm u|u|^{\lambda-1} \tag{12.2.1}$$

in $\mathbf{R}^{n+1}$. If $g_0, g_1 \in C_0^\infty(\mathbf{R}^n)$ are fixed, we shall consider the corresponding Cauchy problem for (12.2.1) with initial data

$$u(0,x) = \varepsilon g_0(x), \ \partial_t u(0,x) = \varepsilon g_1(x). \tag{12.2.2}$$

It is easy to see that a combination between the energy identity

$$\frac{1}{2}\int |\nabla_{t,x} u(t,x)|^2 dx + \frac{1}{2}\int |u(t,x)|^2 dx \pm \int \frac{|u(t,x)|^{\lambda+1}}{\lambda+1} dx = const \tag{12.2.3}$$

and the Sobolev embedding

$$\|u(t,.)\|_{L^q} \leq C\|u(t,.)\|_{H^1} \tag{12.2.4}$$

with $1/q = 1/(\lambda+1) \geq (n-2)/2n$ leads to uniform bound of $\|u(t,.)\|_{H^1}$ for $\varepsilon > 0$ small enough.

Hence, when $n = 1,2$ we have no upper restrictions on $\lambda > 1$ and when $n \geq 3$ we need

$$\lambda \leq \frac{n+2}{n-2}$$

so that for $\varepsilon > 0$ small enough we shall have a global (weak) solution

$$u \in L^\infty((0,\infty) : H^1(\mathbf{R}^n)).$$

The key problem we shall discuss is the decay of the solution. We shall concentrate our analysis to the case of space dimension $n = 1,2,3$.

More precisely, our goal is to establish the following decay of the solution

$$|u(t,x)| \leq C(1+t+|x|)^{-n/2} \tag{12.2.5}$$

for the solution of the semilinear Klein – Gordon equation.

To establish this type of decay we shall assume that the power λ of the nonlinearity is strictly bigger than the critical value

$$\lambda_{\text{cr}} = 1 + \frac{2}{n}.$$

The main difficulty is the fact that singularity of the nonlinear term is forcing us to consider only weak solutions and therefore we have no right to differentiate many times the equation (12.2.1).

Our main idea is to combine the Sobolev inequality on the hyperboloid with curvature -1, see (9.2.10), together with the energy inequality on this hyperboloid, see (4.3.10).

To apply this plan we shall derive higher order derivative estimate of type (4.3.10) for the solution of the linear Klein – Gordon equation

$$\Box u - u = -F. \tag{12.2.6}$$

Take any real number $\rho_0 > 1$. The term

$$v = (1 - \Delta_X - \Delta_{\mathbb{S}^{n-1}})^{s/2} u \tag{12.2.7}$$

is a solution of the Klein-Gordon equation

$$\Box v - v = -G, \quad G = -(1 - \Delta_X - \Delta_{\mathbb{S}^{n-1}})^{s/2} F \tag{12.2.8}$$

in the domain $\{t^2 - |x|^2 \geq \rho_0^2, t > 0\}$. Indeed writing the D'Alembert operator in the form (8.2.9), $\Box = -\partial_\rho^2 - \frac{n}{\rho}\partial_\rho + \frac{1}{\rho^2}\Delta_X$, we see that $(1 - \Delta_X - \Delta_{\mathbb{S}^{n-1}})^{s/2}$ and $\Box$ commute.

For any fixed $\rho > \rho_0$ we consider the hyperboloid

$$X_\rho = \{t^2 - |x|^2 = \rho^2, t > 0\}$$

and our goal is to control the energy of v over X_ρ. Multiplying (12.2.8) by $-\partial_t v$, we obtain the identity

$$\sum_{\mu=0}^{n} \partial_\mu P^\mu = 2G\partial_t v, \tag{12.2.9}$$

where

$$\partial_0 = \partial_t, \quad \partial_j = \partial_{x_j}, \quad j = 1, \dots, n; \tag{12.2.10}$$

$$P^0 = |\nabla_{t,x} v|^2 + |v|^2, \quad P^j = -2\partial_t v \partial_j v, \quad j = 1, \dots, n, \tag{12.2.11}$$

P^0, P^j are the components of the energy-momentum tensor. We integrate (12.2.9) into domain

$$D_{\rho,\rho_0} = \{(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n; t > 0, \rho_0^2 < t^2 - |x|^2 < \rho^2\}.$$

Thus we get

$$\int_{\partial D_{\rho,\rho_0}} \sum_{\mu=0}^{n} \nu_\mu P^\mu(t,x) d\Sigma_{t,x} = \int_{D_{\rho,\rho_0}} 2G\partial_t v(t,x) dx dt, \tag{12.2.12}$$

where $\partial D_{\rho,\rho_0}$ is the boundary of the domain D_{ρ,ρ_0}, ν is the outward normal and $\Sigma_{t,x}$ is the surface element with respect to the Riemannian metric. The boundary $\partial D_{\rho,\rho_0}$ for $\rho > \rho_0$ consists of the hyperboloids X_ρ and X_{ρ_0}. For $(t,x) \in X_\rho$ we have

$$\nu = \nu(t,x) = (\rho^2 + 2|x|^2)^{-1/2}(\sqrt{|x|^2 + \rho^2}, -x) \tag{12.2.13}$$

while

$$d\Sigma_{t,x} = \frac{\sqrt{\rho^2 + 2|x|^2}}{\sqrt{|x|^2 + \rho^2}} dx.$$

Applying Lemma 4.3.1, with $\rho > t_0$ and $(t,x) \in X_\rho$ we get

$$\int_{X_\rho} \sum_{\mu=0}^{n} \nu_\mu(t,x) P^\mu(t,x) d\Sigma_{t,x}$$

$$\geq \int_{\mathbf{R}^n} |v(\sqrt{\rho^2+|x|^2},x)|^2 dx + \int_{\mathbf{R}^n} \rho^2 |\partial_\rho v(\sqrt{\rho^2+|x|^2},x)|^2 \frac{dx}{\rho^2+|x|^2}$$

$$\geq \rho^n \|u(\rho\cdot)\|^2_{H^{s,1/2}(X)} + \rho^n \|\partial_\rho v(\rho\cdot)\|^2_{L^2(X)}. \tag{12.2.14}$$

Our next step is to estimate from above the term

$$\int_{X_{\rho_0}} \sum_{\mu=0}^{n} \nu_\mu P^\mu(t,x) d\Sigma_{t,x}.$$

From the representation formula (12.2.11) we see that

$$\sum_{\mu=0}^{n} \nu_\mu P^\mu(t,x) \leq C(|v(t,x)|^2 + |\nabla_{t,x} v(t,x)|^2).$$

Using the relation

$$(t^2 - |x|^2)\partial_t = t\rho_0 \partial_\rho - \sum_{j=1}^{n} x_j Y_{0j},$$

we see that for $(t,x) \in X_{\rho_0}$ we have

$$|\partial_t v(t,x)) \leq C\nu_0(|\partial_\rho v(t,x)| + \sum_{j=1}^{n} |Y_{0j} v(t,x)|).$$

In a similar way we get a more general inequality, namely we have

$$|\nabla_{t,x} v(t,x)) \leq C\nu_0(|\partial_\rho v(t,x)| + \sum_{j=1}^{n} |Y_{0j} v(t,x)| + \sum_{j,k=1}^{n} |Y_{jk} v(t,x)|).$$

So we arrive at

$$|\int_{X_{\rho_0}} \sum_{\mu=0}^{n} \nu_\mu P^\mu(t,x) d\Sigma_{t,x}| \leq C\|u(\rho_0\cdot)\|^2_{H^{s+1,1/2}(X)} +$$

$$C\|\partial_\rho u(\rho_0\cdot)\|^2_{H^{s,3/2}(X)}.$$

An application of the Gronwall inequality for the energy identity (12.2.12) (in the same way as it was done in the proof of energy estimate (4.3.10) for Klein-Gordon equation) gives

$$\rho^{n/2}\|u(\rho\cdot)\|_{H^{s,1/2}(X)} \leq C\|u(\rho_0\cdot)\|_{H^{s+1,1/2}(X)}$$

$$+C\|\partial_\rho u(\rho_0\cdot)\|_{H^{s,3/2}(X)} + C\int_{\rho_0}^{\rho} \|F(\sigma\cdot)\|_{H^{s,1/2}(X)} \sigma^{n/2} d\sigma. \tag{12.2.15}$$

Note that the constant C in the above inequality might depend on ρ_0, but when this parameter varies in a bounded interval, the constant C is uniform and independent of ρ.

After small translation in time, namely $t \to t + t_0$, where $t_0 = 1 + R$ and R is the radius of the ball containing the support of the initial data, we can assume that the Cauchy problem for Klein–Gordon equation is given by (12.2.6) with the initial data

$$u_{|t=t_0} = g_0, \quad \partial_t u_{|t=t_0} = g_1 \tag{12.2.16}$$

are such that

$$\text{supp} g_0 \cap \text{supp} g_1 \subset \{|x| < R\}$$

and

$$\|g_0\|_{H^{s+1}(\mathbb{R}^n)} + \|g_1\|_{H^s(\mathbb{R}^n)} < \varepsilon.$$

For F we assume that

$$\text{supp} F \subset \{|x| \le t-1\}. \tag{12.2.17}$$

Then a finite dependence domain argument for Klein-Gordon equation assures that

$$\text{supp} u \subset \{|x| \le t-1\}.$$

To evaluate the weighted Sobolev norm

$$\|u(\rho_0 \cdot)\|_{H^{s+1,1/2}(X)} + \|\partial_\rho u(\rho_0 \cdot)\|_{H^{s,3/2}(X)}$$

we take $\rho_0 > t_0$ and use the inclusion

$$\{t^2 - |x|^2 = \rho_0^2\ ,\ |x| \le t-1\} \subset \{|x| \le t-1, t_0 \le t \le t_1\},$$

where t_1 is chosen so that the intersection of $X_{\rho_0} = \{t^2 - |x|^2 = \rho_0^2\}$ with the cone $\{|x| = t-1\}$ is contained in the plane $\{t = t_1\}$, i.e.

$$t_1 = \frac{1+\rho_0^2}{2}.$$

This argument shows that the restriction of the solution u on the upper branch of the hyperboloid X_{ρ_0} has a compact support.

Using the property (9.2.5) in combination with Theorem 7.5.1, we get

$$\|u(\rho_0 \cdot)\|_{H^{s,a}(X)} \le C\|u(\rho_0 \cdot)\|_{H^{s,0}(X)} \tag{12.2.18}$$

and

$$\|\partial_\rho u(\rho_0 \cdot)\|_{H^{s,a}(X)} \le C\|\partial_\rho u(\rho_0 \cdot)\|_{H^{s,0}(X)} \tag{12.2.19}$$

for any positive number a. Further, we shall establish the following.

Lemma 12.2.1 *For any integer $s \geq 0$ we have*

$$\sum_{|\alpha|\leq s+1} \|Y^\alpha u(\rho_0\cdot)\|_{L^2(X)} + \sum_{|\alpha|\leq s} \|Y^\alpha \partial_\rho u(\rho_0\cdot)\|_{L^2(X)}$$
$$\leq C(\rho_0)\|g_0\|_{H^{s+1}} + C(\rho_0)\|g_1\|_{H^s}$$
$$+C(\rho_0)\|F\|_{H^s((t_0,t_1)\times\mathbf{R}^n)}.$$

Proof. Let $|\alpha| = s$. Then $v = Y^\alpha u$ is a solution of the Cauchy problem

$$\Box v - v = -G \ , \ G = Y^\alpha u,$$
$$v(t_0, x) = g^0_\alpha \ , \ \partial_t v(t_0, x) = g^1_\alpha. \tag{12.2.20}$$

Taking the limit $\rho \to \rho_0$ in the energy estimate (4.3.10), we find

$$\int_{\mathbf{R}^n} |v(\sqrt{\rho_0^2+|x|^2}, x)|^2 dx + \int_{\mathbf{R}^n} \frac{\rho_0^2}{\rho_0^2+|x|^2} |\nabla_{t,x} v(\sqrt{\rho_0^2+|x|^2}, x)|^2 dx \leq$$
$$\leq C(\|g^0_\alpha\|^2_{H^1} + \|g^1_\alpha\|^2_{L^2}) +$$
$$+C\left(\int_{t_0}^{t_1}\left(\int_{\mathbf{R}^n}|G(\tau,x)|^2 dx\right)^{1/2} d\tau\right)^2. \tag{12.2.21}$$

For $(\sqrt{\rho_0^2+|x|^2}, x) \in \mathrm{supp} u$ and ρ_0 ($\rho_0 \geq t_0 \geq 1$) varying in a compact set we see that $|x|$ is bounded and this observation leads to the estimate

$$\int_{\mathbf{R}^n} |v(\sqrt{\rho_0^2+|x|^2}, x)|^2 dx + \int_{\mathbf{R}^n} \frac{\rho_0^2}{\rho_0^2+|x|^2} |\nabla_{t,x} v(\sqrt{\rho_0^2+|x|^2}, x)|^2 dx$$
$$\geq C\|u(\rho_0\cdot)\|_{H^{s+1}(X)} + C\|\partial_\rho u(\rho_0\cdot)\|_{H^s(X)}.$$

To evaluate the Sobolev norms of the initial data, we take into account the relation

$$v(t_0, x) = (Y^\alpha u)(t_0, x) = \sum_{|\beta|\leq|\alpha|} c_\beta(t_0, x)\partial^\beta_{t,x} u(t_0, x),$$

where $c_\beta(t, x)$ is a polynomial in t, x of order $|\beta|$. Using the relation $\partial_t^2 u = \Delta u + F$, we can compute the time derivatives of u of order ≥ 2 and in this way we obtain the estimate

$$\|v(t_0,\cdot)\|_{H^1(\mathbf{R}^n)} \leq C\|g_0\|_{H^{s+1}} + C\|g_1\|_{H^s}$$
$$+C\sum_{k=0}^{s-1} \|\partial_t^k F(t_0,\cdot)\|_{H^{s-k-1}(\mathbf{R}^n)}.$$

From the Sobolev inequality (or the trace theorem) we have

$$\|\partial_t^k F(t_0,\cdot)\|_{H^{s-k-1}(\mathbf{R}^n)} \leq C\|F\|_{H^s((t_0,t_1)\times\mathbf{R}^n)}$$

so we obtain the estimate

$$\|v(t_0,\cdot)\|_{H^1(\mathbf{R}^n)} = \|g^0_\alpha\|_{H^1} \leq \\ \leq C\|g_0\|_{H^{s+1}} + C\|g_1\|_{H^s} + \\ +C\|F\|_{H^s((t_0,t_1)\times\mathbf{R}^n)}.$$

In a similar way we have

$$\|\partial_t v(t_0,\cdot)\|_{L^2(\mathbf{R}^n)} = \|g^1_\alpha\|_{L^2} \leq \\ \leq C\|g_0\|_{H^{s+1}} + C\|g_1\|_{H^s} + \\ +C\|F\|_{H^s((t_0,t_1)\times\mathbf{R}^n)}.$$

From these estimates and (12.2.21) we get the desired estimate.

This completes the proof.

The above Lemma implies that we have the estimate

$$\|u(\rho_0\cdot)\|_{H^{s+1}(X)} + \|\partial_\rho u(\rho_0\cdot)\|_{H^s(X)} \\ \leq C(\rho_0)\|g_0\|_{H^{s+1}} + C(\rho_0)\|g_1\|_{H^s} \\ +C(\rho_0)\|F\|_{H^s((t_0,t_1)\times\mathbf{R}^n)} \tag{12.2.22}$$

for any integer $s \geq 0$. Using an interpolation argument, we see it is true for any non-negative real number s.

Combining the inequality (12.2.22) together with (12.2.15), we find

$$\rho^{n/2}\|u(\rho\cdot)\|_{H^{s,1/2}(X)} \leq C\|g_0\|_{H^{s+1}} + C\|g_1\|_{H^s} \\ +C\|F\|_{H^s((t_0,t_1)\times\mathbf{R}^n)} + C\int_{\rho_0}^{\rho} \|F(\sigma\cdot)\|_{H^{s,1/2}(X)}\sigma^{n/2}d\sigma, \tag{12.2.23}$$

where $\rho_0 \geq t_0 > 1$ and $t_1 = (1+\rho_0^2)/2$.

On the other hand, the classical energy estimate (4.3.3) for the Klein - Gordon equation gives

$$\|u\|_{H^{s+1}((t_0,t_1)\times\mathbf{R}^n)} \leq C\|g_0\|_{H^{s+1}} + C\|g_1\|_{H^s} \\ +C\|F\|_{H^s((t_0,t_1)\times\mathbf{R}^n)}. \tag{12.2.24}$$

Now we can examine the asymptotic behavior of the weak solution $u(t,x)$ of the semilinear Klein-Gordon equation (12.2.1).

For the purpose we shall study only the case $t \to +\infty$, since the case when t tends to $-\infty$ is similar.

Proof of decay estimate for Klein - Gordon equation. After small translation in time, namely $t \to t + t_0$, where $t_0 = 1 + R$ and R is the radius of the ball containing the support of the initial data, we can assume that the Cauchy problem for Klein–Gordon equation is given by (4.3.1) with u, F supported in the light cone

$\{|x| < t-1\}$ and the initial data (4.3.2), where $g_0, g_1 \in H^{s+1}(\mathbb{R}^n) \times H^s(\mathbb{R}^n)$ are fixed and

$$\mathrm{supp} g_0 \cap \mathrm{supp} g_1 \subset \{|x| < R\},$$

moreover

$$\|g_0\|_{H^{s+1}(\mathbb{R}^n)} + \|g_1\|_{H^s(\mathbb{R}^n)} < \varepsilon.$$

Given any $s > n/2$, $\rho_0 \geq t_0$ and $\rho \geq \rho_0$, we define the norm

$$X(u) = X_{s,\rho}(u) =$$

$$= \sup_{\rho_0 \leq \sigma \leq \rho} \sigma^{n/2} \|u(\sigma \cdot)\|_{H^{s,1/2}(X)} + \|u\|_{H^{s+1}((t_0,t_1)\times\mathbb{R}^n)}. \tag{12.2.25}$$

We recall that $t_1 = (1+\rho_0^2)/2$. Our goal is to evaluate the operator

$$u \to N(u),$$

defined by

$$\Box N(u) - N(u) = \pm |u|^\lambda \tag{12.2.26}$$

with initial data

$$N(u)(t_0, x) = g_0(x), \ \partial_t N(u)(t_0, x) = g_1(x) \tag{12.2.27}$$

that satisfy $\|g_0\|_{H^{s+1}} + \|g_1\|_{H^s} < \varepsilon$. We shall establish the estimates:

$$X(N(u)) \leq C\varepsilon + C(X(u))^\lambda \tag{12.2.28}$$

$$X(N(u) - N(\omega)) \leq CX(u-\omega)\,(X(u) + X(\omega))^{\lambda-1} \tag{12.2.29}$$

It is important to notice that the constant C in these estimates must be independent of the parameter $\rho \geq t_0$ used in the definition of the norm $X(u)$. In this way these estimates lead to the existence of a unique solution of

$$u = N(u). \tag{12.2.30}$$

In fact, taking the recurrent sequence

$$u_{-1} = 0, \ u_j = N(u_{j-1}), \quad j = 0, 1, 2, \ldots$$

we apply (12.2.28) and we see inductively that

$$X(u_j) \leq C_1 \varepsilon \tag{12.2.31}$$

where $C_1 = 2C$ and $0 \leq \varepsilon \leq \varepsilon_0$ with $\varepsilon_0 = \varepsilon_0(C_1)$ sufficiently small.

Once the uniform estimate (12.2.31) is checked, we use (12.2.29) once more and find

$$X(u_{j+1} - u_j) \leq C\varepsilon^{\lambda-1} X(u_j - u_{j-1}) < \frac{1}{2} X(u_j - u_{j-1}) \tag{12.2.32}$$

for $\varepsilon \le \varepsilon_0$ sufficiently small. From (12.2.32) we get inductively

$$X(u_j - u_{j-1}) \le \frac{C}{2^j}$$

and we see that $\{u_j\}$ is a Cauchy sequence in the Banach space $\{u; X(u) < \infty\}$ converging to the unique solution of (12.2.30).

Finally we see that

$$(1 + t + |x|)^{n/2}|u(t,x)| \le CX_{s,\rho}(u) \tag{12.2.33}$$

provided $\rho \ge \sqrt{t^2 - |x|^2}$. Indeed, for $t + |x|$ bounded this follows from classical Sobolev inequality. For t large enough (say $t \ge t_0^2$) we can choose $\rho = \sqrt{t^2 - |x|^2} \ge t_0$ and $\Omega = (t,x)/\rho$. Then applying Sobolev embedding from (9.2.10), we get

$$\rho^{n/2}\Omega_0^{n/2}|u(\rho\Omega)| \le C\rho^{n/2}\|u(\rho\,\cdot)\|_{H^{s,1/2}(X)} \le CX_{s,\rho}(u). \tag{12.2.34}$$

This shows we have (12.2.33).

On the other hand, in view of (12.2.31), the norm $X_{s,\rho}(u)$ is uniformly bounded with respect to ρ; we conclude

$$|u(t,x)| \le C(1 + t + |x|)^{-n/2}$$

in $\{|x| < t\}$.

Therefore, to obtain the decay estimate (12.2.5) it remains only to establish (12.2.28), (12.2.29). For simplicity we shall obtain only (12.2.28), since the proof of (12.2.29) is similar.

First, we apply the classical energy estimate (12.2.24) for the Klein - Gordon equation and find

$$\|N(u)\|_{H^{s+1}((t_0,t_1)\times\mathbf{R}^n)} \le C\|g_0\|_{H^{s+1}} + C\|g_1\|_{H^s} + C\||u|^\lambda\|_{H^s((t_0,t_1)\times\mathbf{R}^n)}.$$

From Lemma 6.6.1 and Sobolev inequality in $\mathbf{R}^n$, we get

$$\|N(u)\|_{H^s((t_0,t_1)\times\mathbf{R}^n)} \le C\varepsilon + CX(u)^\lambda. \tag{12.2.35}$$

The next step is to evaluate the first term of our norm $X(u)$. From (12.2.23) we have

$$\begin{aligned}\rho^{n/2}\|N(u)(\rho\cdot)\|_{H^{s,1/2}(X)} &\le C\|g_0\|_{H^{s+1}} + C\|g_1\|_{H^s} \\ &+C\||u|^\lambda\|_{H^s((t_0,t_1)\times\mathbf{R}^n)} + C\int_{\rho_0}^{\rho}\||u(\sigma\cdot)|^\lambda\|_{H^{s,1/2}(X)}\sigma^{n/2}d\sigma,\end{aligned} \tag{12.2.36}$$

Our assumption on the initial data guarantees that

$$\|g_0\|_{H^{s+1}} + \|g_1\|_{H^s} \le \varepsilon.$$

As above we have

$$\||u|^\lambda\|_{H^s((t_0,t_1)\times\mathbf{R}^n)} \le CX(u)^\lambda.$$

Applying Theorem 9.2.2, we obtain

$$\||u(\sigma\cdot)|^\lambda\|_{H^{s,1/2}(X)} \le C\|u(\sigma\cdot)\|_{H^{s,1/2}(X)}\|u(\sigma\cdot)\|^{\lambda-1}_{L^\infty(X)}.$$

A comparison with (12.2.34) shows that

$$\Omega_0^{n/2}|u(\sigma\Omega)| \le \frac{C}{\sigma^{n/2}}X_{s,\rho}(u),$$

$$\sigma^{n/2}\|u(\sigma\cdot)\|_{H^{s,1/2}(X)} \le CX_{s,\rho}(u).$$

Hence, we arrive at

$$\int_{\rho_0}^{\rho} \||u(\sigma\cdot)|^\lambda\|_{H^{s,1/2}(X)}\sigma^{n/2}d\sigma \le CX(u)^\lambda \int_{\rho_0}^{\rho} \sigma^{-(\lambda-1)n/2}d\sigma.$$

Our assumption that $\lambda > 1 + 2/n$ implies that

$$\int_{\rho_0}^{\rho} \sigma^{-(\lambda-1)n/2}d\sigma < \infty$$

so we have

$$\int_{\rho_0}^{\rho} \||u(\sigma\cdot)|^\lambda\|_{H^{s,1/2}(X)}\sigma^{n/2}d\sigma \le CX(u)^\lambda.$$

This observation leads to the estimate

$$\rho^{n/2}\|N(u)(\rho\cdot)\|_{H^{s,1/2}(X)} \le C\varepsilon + X(u)^\lambda.$$

This estimate and (12.2.35) give (12.2.28) and completes the proof of (12.2.5).

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