5 Stationary phase method and pseudodifferential operators

5.1 Stationary phase method

In this section we shall give a brief review of the methods to study the asymptotic behavior of oscillatory integrals of type

$$I(R) = \int_{\mathbf{R}^n} e^{iR\phi(x)} f(x)dx \tag{5.1.1}$$

as $R > 0$ tends to infinity.

Here $f(x), \phi(x)$ are smooth functions defined on $\mathbb{R}^n$ with $\phi(x)$ being real-valued.

First, we consider the case, when the phase function $\varphi(x)$ has no critical points. More precisely, we consider the case, when there exist $\delta > 0, \delta \leq 1$ and $C > 0$ so that

$$|\nabla\phi(x)| \geq C^{-1} < x >^{\delta}, \quad < x >^2 = 1 + |x|^2, \tag{5.1.2}$$

$$|\partial_x^\alpha \nabla\phi(x)| \leq C < x >^{\delta - |\alpha|} \tag{5.1.3}$$

for any $x \in \operatorname{supp} f$.

Lemma 5.1.1 *Suppose the assumptions (5.1.2), (5.1.3) are fulfilled and $f(x)$ is a smooth function with compact support. Then for any integer $N \geq 0$ and for any $\varepsilon > 0$ we have*

$$|I(R)| \leq \frac{C}{R^N} \sum_{|\alpha| \leq N} \| < x >^{-N\delta - N + |\alpha| + n/2 + \varepsilon} \partial^\alpha f \|_{L^2(\mathbb{R}^n)}.$$

Proof. Given any first order differential operator

$$L(x, \partial_x) = (\sum_{j=1}^n a_j(x)\partial_{x_j}) + b(x),$$

we denote by L^* its adjoint operator with respect to the inner product in $L^2(\mathbb{R}^n)$, i.e.

$$L^*(x, \partial_x) = -(\sum_{j=1}^n \overline{a_j(x)}\partial_{x_j}) + \overline{b(x)} + \sum_{j=1}^n \partial_{x_j} \overline{a_j(x)}.$$

Therefore, for any couple f, g of smooth compactly supported functions on $\mathbb{R}^n$ we have

$$(Lf, g)_{L^2(\mathbb{R}^n)} = (f, L^* g)_{L^2(\mathbb{R}^n)}. \tag{5.1.4}$$

Let $L(x, \partial_x)$ be the differential operator, such that its adjoint is

$$L^* = i^{-1} \sum_{k=1}^n \frac{\partial_{x_k}\phi}{|\nabla\phi|^2} \partial_{x_k},$$

where $\nabla = (\partial_{x_1}, ..., \partial_{x_n})$.

It is clear that

$$L^*(e^{iR\phi}) = Re^{iR\phi}.$$

Then (5.1.4) implies that

$$I(R) = \frac{1}{R^N} \int_{\mathbb{R}^n} e^{iR\phi} L^N(f) dx.$$

In order to evaluate $L^N(f)$, we shall establish inductively with respect to N that L^N can be represented as

$$L^N = \sum_{|\alpha| \leq N} a_\alpha^N(x) \partial_x^\alpha, \tag{5.1.5}$$

where the coefficients satisfy suitable decay estimates. To formulate precisely this statement, given any real number m we denote by S^m the class of all smooth functions $a(x)$ such that for any multiindex β there exists $C = C(\beta)$ so that

$$|(< x > \partial_x)^\beta a(x)| \leq C < x >^m .$$

Our goal is to show that the coefficients in (5.1.5) satisfy

$$a_\alpha^N(x) \in S^{-\delta N - N + |\alpha|}. \tag{5.1.6}$$

For $N = 1$ we have

$$L = i^{-1} \sum_{k=1}^n \frac{\partial_{x_k} \phi}{|\nabla \phi|^2} \partial_{x_k} + b(x),$$

where $b(x)$ is constant times

$$\sum_{k=1}^n \partial_{x_k} (\partial_{x_k} \phi / |\nabla \phi|^2).$$

Therefore, we have to show that

$$\frac{\nabla \phi(x)}{|\nabla \phi(x)|^2} \in S^{-\delta}. \tag{5.1.7}$$

Indeed, consider the function

$$v \in \mathbb{R}^n \setminus 0 \to \chi(v) = v/|v|^2.$$

Then the function in (5.1.7) can be represented as $\chi(\nabla\phi)$. Moreover, for any multiindex α we can represent $\partial_x^\alpha \chi(\nabla\phi)$ as a linear combination of terms of type

$$(\partial_v^\beta \chi)(\nabla\phi)(\partial_x^{\gamma_1} \nabla\phi)...(\partial_x^{\gamma_{|\beta|}} \nabla\phi)$$

with $|\beta| \leq |\alpha|$ and

$$\gamma_1 + \ldots + \gamma_{|\beta|} = \alpha.$$

Since

$$|\partial_v^\beta \chi(v)| \leq C|v|^{-1-|\beta|}$$

and $|v| = |\nabla\phi| \geq C^{-1} < x >^{\delta}$, we have

$$|(\partial_v^\beta \chi)(\nabla\phi)| \leq C_1 < x >^{-\delta(1+|\beta|)} .$$

Applying (5.1.3), we find

$$|\partial_x^\alpha \nabla\phi| \leq C < x >^{\delta-|\alpha|}$$

so (5.1.7) is established. Using the trivial property

$$a \in S^m \Longrightarrow \partial_x^\alpha a \in S^{m-\alpha}, \tag{5.1.8}$$

we obtain (5.1.6) and this implies

$$|L^N(f)(x)| \leq C < x >^{-\delta N - N + |\alpha|} \sum_{|\alpha|\leq N} |\partial_x^\alpha f(x)|.$$

Applying the Cauchy inequality, we complete the proof of the Lemma.

As an application we shall consider the oscillatory integral

$$A(x,\xi) = \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \mathrm{e}^{-i(z,\eta)} a(x+z, \xi+\eta) b(x,z,\xi,\eta) dz d\eta, \tag{5.1.9}$$

where $a(x,\xi)$ is a smooth function on $\mathbf{R}^n \times \mathbf{R}^n$ belonging to the class of symbols $S^{m,k}$, defined as follows.

Definition 5.1.1 *A smooth function $a(x,\xi)$ belongs to $S^{m,k}$, (m, k are real numbers) if for any integer $N \geq 0$ one can find a constant $C = C(N)$ so that*

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C < \xi >^{m-|\beta|} < x >^{k-|\alpha|}$$

for $|\alpha| + |\beta| \leq N$.

Further, we set

$$S^{-\infty,-\infty} = \cap_{m,k} S^{m,k}.$$

Moreover,

$$b(x,z,\xi,\eta) = (1 - \varphi(\eta/(1+|\xi|)))(1 - \varphi(z/(1+|x|)))$$

where the cut-off function $\varphi(x)$ in (5.1.9) is such that $\varphi(x) = 1$ for $|x| \leq 1/4$ and $\varphi(x) = 0$ for $|x| \geq 1/2$.

We shall establish the following.

Lemma 5.1.2 *If $a \in S^{m,k}$, then the oscillatory integral A in (5.1.9) belongs to* $S^{-\infty,-\infty}$.

Proof. Taking $\phi(y,z) = (y,z), R = 1$, we see that the oscillatory integral A has the form (5.1.1). Taking $\delta = 1$, we see that the assumptions (5.1.2) and (5.1.3) are fulfilled. Thus, for any integer $N \geq 1$ we have

$$|A(x,\xi)| \leq C \sum_{|\alpha|+|\beta|\leq N} \int\int B(x,z,\xi,\eta)dzd\eta,$$

where

$$B(x,z,\xi,\eta) = (1+|z|)^{-2N-2\delta+n+2\varepsilon} \times \times(1+|\eta|)^{-2N-2\delta+n+2\varepsilon}|\partial_z^\alpha \partial_\eta^\beta a(x+z,\xi+\eta)|^2. \tag{5.1.10}$$

The integration above is over $|z| \geq (1+|x|)/4$ and $|\eta| \geq (1+|\xi|)/4$. This observation implies that for any integer $N_1 \geq 1$ we have the estimate

$$|A(x,\xi)| \leq C(1+|x|)^{-N_1}(1+|\xi|)^{-N_1}.$$

In a similar way we estimate the derivatives of A and get

$$|\partial_x^\alpha \partial_\xi^\beta A(x,\xi)| \leq C(1+|x|)^{-N_1}(1+|\xi|)^{-N_1}.$$

This completes the proof of the lemma.

In a similar way, we can consider the oscillatory integral

$$A(x,\xi) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i(z,\eta)} a_1(x,z,\xi,\eta)dzd\eta, \tag{5.1.11}$$

where

$$a_1(x,z,\xi,\eta) = a(x+z,\xi+\eta)(1-\varphi(\eta/(1+|\xi|)))\varphi(z/(1+|x|)).$$

Now we can use the argument of the proof of Lemma 5.1.1 and use the operator

$$L = (\frac{\eta}{i|\eta|^2}, \nabla_z).$$

Then integrating by parts as it was done in Lemma 5.1.1, we get

$$|A(x,\xi)| \leq C \sum_{|\alpha|=N} \int\int (1+|\eta|)^{-N} |\partial_z^\alpha(\varphi(z/(1+|x|))a(x+z,\xi+\eta))|dzd\eta.$$

Here the integration is over $|z| \leq (1+|x|)/2$ so on the integration domain the weights $1+|x+z|$ and $1+|x|$ are equivalent. Then the definition 5.1.1 shows that we have the estimate

$$|\partial_z^\alpha(\varphi(z/(1+|x|))a(x+z,\xi+\eta))| \leq C < x >^{k-|\alpha|} .$$

Choosing $N \geq 1$ sufficiently large, we get

Lemma 5.1.3 *If $a \in S^{m,k}$, then the oscillatory integral A in (5.1.11) belongs to $S^{-\infty,-\infty}$.*

Our next step is to consider the case when

$$I(R) = \int_{\mathbf{R}^n} \mathrm{e}^{iR(Qx,x)} f(x)dx, \tag{5.1.12}$$

where Q is a constant symmetric invertible matrix. Then the assumption (5.1.2) is not satisfied. For this case stationary phase method gives the following.

Lemma 5.1.4 *For any real number $s > n/2$ we have the estimate*

$$|I(R)| \leq CR^{-n/2}\|f\|_{H^s}.$$

Proof. We have seen in (3.4.1) that the Fourier transform of the distribution $\mathrm{e}^{iR(Qx,x)}$ is constant times

$$R^{-n/2}\mathrm{e}^{-i(Q^{-1}\xi,\xi)/4R}.$$

Therefore, applying Plancherel identity, we get

$$|I(R)| \leq CR^{-n/2}\int_{\mathbf{R}^n} |\hat{f}(\xi)|d\xi.$$

Applying the Cauchy inequality, we complete the proof.

We can obtain asymptotic expansion for $I(R)$. In fact, we have the expansion

$$\mathrm{e}^{-i(Q^{-1}\xi,\xi)/4R} = \sum_{k=0}^{N-1}(-i/4R)^k(Q^{-1}\xi,\xi)^k/k! + r_N,$$

where the remainder $r_N(\xi)$ satisfies the estimate

$$|r_N(\xi)| \leq C_N|(Q^{-1}\xi,\xi)|^N R^{-N}.$$

Therefore, we have the asymptotic expansion

$$I(R) = \sum_{k=0}^{N-1} I_k(R) + \sigma_N(R), \tag{5.1.13}$$

where

$$I_k(R) = \frac{Ci^{-k}}{k!(4R)^{k+n/2}}(Q^{-1}D_x, D_x)f(0)$$

and the remainder $\sigma_N(R)$ satisfies the estimate

$$|\sigma_N(R)| \leq \frac{C_N}{R^{N+n/2}}\|f\|_{H^{2N+s}}$$

with $s > n/2$.

5.2 Pseudodifferential operators

Consider pseudodifferential operators of type

$$P_m(x, D_x)(f)(x) = \int e^{ix\xi} p(x,\xi)\hat{f}(\xi)d\xi,$$

where $p(x,\xi) \in S^m$. Recall that (see [22]) the class S^m of symbols is formed by smooth functions $p(x,\xi)$ defined in $\mathbb{R}^n \times \mathbb{R}^n$ such that for any multiindices α and β there exists a constant $C_{\alpha,\beta}$ so that

(5.2.1) $$|\partial_x^\alpha \partial_\xi^\beta p(x,\xi)| \leq C(1+|\xi|)^{m-|\beta|}$$

for any $x \in \mathbf{R}^n$ and $\xi \in \mathbb{R}^n$. The real number m is called order of the operator.

The class $S^{-\infty}$ is the intersection of S^m for all real values of m.

Here below we shall list some simple properties of the class S^m.

Problem 5.2.1 *If $p(x,\xi) \in S^m$, then for any α, β we have*

$$\partial_x^\alpha \partial_\xi^\beta p(x,\xi) \in S^{m-|\beta|}.$$

Problem 5.2.2 *If $p \in S^m$ and $q \in S^k$, then*

$$pq \in S^{m+k}.$$

Problem 5.2.3 *If $p, q \in S^m$, then $p \pm q \in S^m$.*

Next we shall define asymptotic expansions in the classes of symbols. Let $P \in S^m$. Given any decreasing sequence $m_j, j = 0,1,2,..$ of real numbers with $m_0 = m$ and

$$m_j \to -\infty$$

and any sequence $P_{m_j} \in S^{m_j}$ of symbols, the asymptotic expansion

(5.2.2) $$P \sim \sum_{j=0}^{\infty} P_{m_j}$$

means that there exists (eventually another) decreasing sequence μ_j of real numbers with

$$\mu_j \to -\infty$$

such that for any integer $k \geq 1$ we have

$$P - \sum_{j=0}^{k-1} P_{m_j} \in S^{\mu_k}. \tag{5.2.3}$$

Problem 5.2.4 *If*

$$p \sim \sum_{j=0}^{\infty} p_{m_j}$$

and

$$q \sim \sum_{j=0}^{\infty} p_{m_j}$$

show that $p - q \in S^{-\infty}$.

One of the basic points in the theory of pseudodifferential operators is the following.

Lemma 5.2.1 *(see [22]) Suppose* $p_{m_j} \in S^{m_j}$*, where* $m_j, j = 0, 1, 2, \ldots$ *is a decreasing sequence or real numbers, tending to* $-\infty$*. Then there exists a symbol* $p \in S^{m_0}$*, so that*

$$p \sim \sum_{j=0}^{\infty} p_{m_j}.$$

For simplicity we shall consider here only the case $p_{m_j} = p_{m_j}(\xi)$.

Lemma 5.2.2 *Suppose* $p_{m_j}(\xi) \in S^{m_j}$*, where* $m_j, j = 0, 1, 2, \ldots$ *is a decreasing sequence or real numbers, tending to* $-\infty$*. Then there exists a symbol* $p(\xi) \in S^{m_0}$*, so that*

$$p \sim \sum_{j=0}^{\infty} p_{m_j}.$$

Proof. Let $\varphi(\xi)$ be a smooth function, such that $\varphi(\xi) = 1$ for $|\xi| \leq 1/2$ and $\varphi(\xi) = 0$ for $|\xi| \geq 1$. The key point in the proof is to find a decreasing sequence

$$\varepsilon_j \to 0,$$

so that for any multiindex α with $|\alpha| \leq j$ we have

$$|\partial_\xi^\alpha(\varphi(\varepsilon_j \xi) p_{m_j}(\xi))| \leq 2^{-j}(1 + |\xi|)^{m_{j-1} - |\alpha|}. \tag{5.2.4}$$

In fact, we have the estimate

$$|\partial_\xi^\alpha \varphi(\varepsilon\xi)| \le C(1+|\xi|)^{-|\alpha|}.$$

for any $\varepsilon \in (0,1)$. Here the constant $C > 0$ is independent of ε. This estimate and the assumption $p_{m_j} \in S^{m_j}$ imply that we have

$$|\partial_\xi^\alpha(\varphi(\varepsilon_j\xi)p_{m_j}(\xi))| \le C(1+|\xi|)^{m_j-|\alpha|}.$$

Since $m_j > m_{j-1}$ and $\varepsilon|\xi| \ge 1/2$, we get

$$|\partial_\xi^\alpha(\varphi(\varepsilon_j\xi)p_{m_j}(\xi))| \le C\varepsilon^{m_j-m_{j-1}}(1+|\xi|)^{m_{j-1}-|\alpha|}.$$

Thus, choosing $C\varepsilon^{m_j-m_{j-1}} \le 2^{-j}$, we obtain (5.2.4).

Taking

$$p(\xi) = \sum_{j=0}^{\infty}(1-\varphi(\varepsilon_j\xi))p_{m_j}(\xi),$$

we see that $p(\xi)$ is a well-defined smooth function.

Further, we have

$$p - \sum_{j=0}^{r-1} p_j = -\sum_{j=0}^{r-1}(1-\varphi(\varepsilon_j\xi))p_{m_j} + R_r(\xi),$$

where the remainder

$$R_r(\xi) = \sum_{j=r}^{\infty}\varphi(\varepsilon_j\xi)p_{m_j}(\xi)$$

satisfies

$$|\partial_\xi^\alpha R_r(\xi)| \le 2^{-r}(1+|\xi|)^{m_{r-1}-|\alpha|}.$$

Hence,

$$p - \sum_{j=0}^{r-1} p_{m_j} \in S^{m_{j-1}}.$$

This completes the proof of the Lemma.

Further, if P_m and Q_s are two pseudodifferential operators of orders m and s with symbols $p(x,\xi)$ and $q(x,\xi)$ respectively, then their product P_mQ_s is a pseudodifferential operator of order $m+s$ with symbol (modulo symbols in $S^{-\infty}$)

$$\sum_{\alpha}(-i)^{|\alpha|}\partial_\xi^\alpha p(x,\xi)\partial_x^\alpha q(x,\xi)/\alpha! \tag{5.2.5}$$

Given any pseudodifferential operator of order m, we can study its kernel $k(x,y)$ defined by

$$P(f)(x) = \int k(x,y)f(y)dy.$$

Formally, the kernel is the following oscillatory integral

$$k(x,y) = \int_{\mathbf{R}^n} e^{i(x-y)\xi} p(x,\xi)d\xi.$$

Lemma 5.2.3 *If $p \in S^m, m > -n$, then for any integer $M \geq 1$ we have*

$$|k(x,y)| \leq \frac{C}{|x-y|^{n+m}}(1+|x-y|)^{-M}$$

for $x \neq y$.

Proof. For simplicity we shall consider only the case of symbols $p = p(\xi)$, since in the general case $p(x,\xi)$ we can consider x as a parameter. Setting $z = x - y$, we shall evaluate the kernel

$$k(z) = \int_{\mathbf{R}^n} e^{iz\xi} p(\xi)d\xi. \tag{5.2.6}$$

Using integration by parts by means of the operators

$$(1+|z|^2)^{-1}(1-\Delta_\xi),$$

we see that the decay factor $(1+|z|^2)^{-M}$ can be obtained, so it is sufficient to show that

$$|k(z)| \leq \frac{C(1+|z|)^l}{|z|^{n+m}} \tag{5.2.7}$$

for some non-negative number $l \geq 0$ and for $z \neq 0$.

Let $\varphi(x)$ be a smooth compactly supported function, such that $\varphi(x) = 1$ near $x = 0$. Then

$$k(z) = k_1(z) + k_2(z),$$

where

$$k_1(z) = \int_{\mathbf{R}^n} e^{iz\xi} \varphi(|z|\xi)p(\xi)d\xi$$

and

$$k_2(z) = \int_{\mathbf{R}^n} e^{iz\xi}(1-\varphi(|z|\xi))p(\xi)d\xi.$$

For $k_1(z), m \geq 0$, we have

$$|k_1(z)| \leq \frac{C}{|z|^n} + \frac{C}{|z|^{n+m}} \leq \frac{C(1+|z|^m)}{|z|^{n+m}}.$$

For $k_1(z), m < 0$, we have

$$|k_1(z)| \leq \frac{C}{|z|^{n+m}}.$$

For $k_2(z)$ we integrate by parts by means of the operator $(z/|z|^2, \nabla_\xi)$ and get

$$|k_2(z)| \le \frac{C}{|z|^N} \int_{|\xi||z| \ge C} |q(z,\xi)| d\xi,$$

where

$$q(z,\xi) = \sum_{|\alpha|=N} \partial_\xi^\alpha ((1-\varphi(|z||\xi|))p(\xi)).$$

Since

$$|q(z,\xi)| \le \frac{C(1+|\xi|)^m}{|\xi|^N},$$

we get

$$|k_2(z)| \le \frac{C(1+|z|^m)}{|z|^{n+m}}$$

for $m \ge 0$ and

$$|k_2(z)| \le \frac{C}{|z|^{n+m}}$$

for $0 > m > -n$. Thus the estimate (5.2.7) is established and this completes the proof of the Lemma.

For the case $m < -n$ we can get (following the proof of the previous Lemma)

Lemma 5.2.4 *If $p \in S^m, m < -n$, then for any integer $M \ge 1$ one can find a constant $C = C(M)$ so that*

$$|k(x,y)| \le C(1+|x-y|)^{-M}$$

for $x \ne y$.

Applying the Young inequality (2.4.15), combined with the estimate from the previous Lemma we get

$$\|P(f)\|_{L^p(\mathbf{R}^n)} \le C\|f\|_{L^p(\mathbf{R}^n)} \tag{5.2.8}$$

for any pseudodifferential operator P of order $< -n$.

In general, the above estimate is true for any pseudodifferential operator of order 0.(see Theorem 18.1.11 in [22] for the case $p = 2$ and Chapter XI in [60] for example). More precisely, it is possible to give a more precise expression of the constant C in the estimate (5.2.8). Namely, we have (see Chapter XI in [60])

$$C = C_0 \max_{|\alpha| \le n+1, |\beta| \le n+1} \sup_{x \in \mathbf{R}^n, \xi \in \mathbf{R}^n - \{0\}} <\xi>^{-|\beta|} |\partial_x^\alpha \partial_\xi^\beta p(x,\xi)|, \tag{5.2.9}$$

where C_0 is universal constant , depending on the space dimension n and on p. This constant is independent of the symbol p.

From this estimate and the definition of the Sobolev space H^s_p it follows

$$\|P_m(f)\|_{L^p(\mathbf{R})^n} \leq C\|f\|_{H^m_p(\mathbf{R}^n)} \tag{5.2.10}$$

for any pseudodifferential operator P_m of order m.

It is easy to obtain the above estimate for the case of convolution type operator

$$P(f)(x) = \int_{\mathbf{R}^n} e^{-ix\xi}p(\xi)\hat{f}(\xi)d\xi.$$

In fact, applying Lemma 5.2.3, we easily see the assumptions of Stein's theorem 3.3.1 are satisfied, so P is a bounded operator in L^p.

Here below we shall use a slightly different class of symbols $S^{m,k}$. This class is used in the work of Cordes [9] and it consists of smooth functions defined in $\mathbf{R}^n \times \mathbf{R}^n$, so that

$$|\partial_x^\alpha \partial_\xi^\beta p(x,\xi)| \leq C(1+|\xi|)^{m-|\beta|}(1+|x|)^{k-|\alpha|}.$$

We have the following simple properties of these classes. We have the inclusion

$$S^{m,k} \subset S^m,$$

if $k \leq 0$. Moreover, if $p(x,\xi) \in S^{m,k}$, then for any α, β we have

$$\partial_x^\alpha \partial_\xi^\beta p(x,\xi) \in S^{m-|\beta|,k-|\alpha|}.$$

Further if $p_j \in S^{m_j,k_j}, j = 1,2$, then

$$p_1 p_2 \in S^{m_1+m_2,k_1+k_2}.$$

Set

$$S^{-\infty,-\infty} = \cap_{m,k} S^{m,k}.$$

The asymptotic expansions in the classes of symbols $S^{m,k}$ can be defined as follows. Let $P \in S^{m,k}$. Given any decreasing sequences $m_j, k_j, j = 0,1,2,..$ of real numbers with $m_0 = m, k_0 = k$ and

$$m_j \to -\infty \quad , \quad k_j \to -\infty$$

and any sequence $P_j \in S^{m_j,k_j}$ of symbols, the asymptotic expansion

$$P \sim \sum_{j=0}^{\infty} P_j \tag{5.2.11}$$

means that there exists (eventually another) decreasing sequences μ_j, ν_j of real numbers with

$$\mu_j \to -\infty \quad , \quad \nu_j \to -\infty.$$

such that for any integer $k \geq 1$ we have

$$P - \sum_{j=0}^{k-1} P_j \in S^{\mu_k, \nu_k}. \tag{5.2.12}$$

As above if we have two asymptotic expansions

$$p \sim \sum_{j=0}^{\infty} p_j$$

and

$$q \sim \sum_{j=0}^{\infty} p_j,$$

then $p - q \in S^{-\infty, -\infty}$.

Moreover, we have the following.

Lemma 5.2.5 *Suppose $p_j \in S^{m_j, k_j}$, where $m_j, k_j, j = 0, 1, 2, \ldots$ are a decreasing sequences or real numbers, tending to $-\infty$. Then there exists a symbol $p \in S^{m_0, k_0}$, so that*

$$p \sim \sum_{j=0}^{\infty} p_j.$$

Proof. It is sufficient to take

$$p(x, \xi) = \sum_{j=0}^{\infty} (1 - \varphi(\delta_j x))(1 - \varphi(\varepsilon_j \xi)) p_j(x, \xi),$$

where δ_j, ε_j are suitable decreasing sequences tending to 0. To choose them, we can use the argument of the proof of Lemma 5.2.2.

This completes the proof.

Given any pseudodifferential operator P, we shall find the symbol of the adjoint operator P^* defined by

$$(Pf, g)_{L^2(\mathbf{R}^n)} = (f, P^* g)_{L^2(\mathbf{R}^n)}.$$

Taking two smooth compactly supported functions f, g, we get

$$(Pf, g)_{L^2} = \int f(y) \overline{P^*(g)(y)} dy,$$

where

$$P^*(g)(y) = \int \int e^{i(y-x)\xi} \overline{p(x, \xi)} g(x) dx d\xi.$$

Hence, the adjoint operator is defined by

$$P^*(g)(x) = \int\int e^{i(x-y)\xi}\overline{p(y,\xi)}g(y)dyd\xi.$$

To show that P^* is also a pseudodifferential operator, it is necessary to show that modulo operators with symbols in $S^{-\infty,-\infty}$ we have

$$P^*(g)(x) = \int\int e^{ix\eta}q(x,\xi)g(y)dyd\xi.$$

In fact, using the representation

$$g(y) = c\int e^{iy\eta}\hat{g}(\eta)d\eta,$$

we see that $q(x,\xi)$ is given (formally!) by the oscillatory integral

$$q(x,\eta) = \int\int e^{i(x-y)(\xi-\eta)}\overline{p(y,\xi)}dyd\xi.$$

Setting $x-y=-z$, $\xi-\eta=\zeta$, we get

$$q(x,\eta) = \int\int e^{-iz\zeta}\overline{p(x+z,\eta+\zeta)}dzd\zeta.$$

Applying Lemma 5.1.2 and Lemma 5.1.3, we see it is sufficient to study the oscillatory integral

$$\tilde{q}(x,\eta) = \int\int e^{-iz\zeta}\,\overline{p(x+z,\eta+\zeta)}\varphi(z/(1+|x|))\varphi(\zeta/(1+|\eta|))dzd\zeta.$$

Here $\varphi(x)$ is a smooth compactly supported function such that $\varphi(x)=1$ near $x=0$.

Indeed, $\tilde{q}$ is a well – defined smooth function. Setting

$$R_1 = 1+|x| \quad , \quad R_2 = 1+|\eta|,$$

we have

$$R_1^{-n}R_2^{-n}\tilde{q}(x,\eta) = \int\int e^{-iR_1R_2z\zeta}\overline{p(x+R_1z,\eta+R_2\zeta)}\varphi(z)\varphi(\zeta))dzd\zeta.$$

Applying stationary phase method (see (5.1.13)), we get

$$q(x,\eta) \sim \sum_{k=0}^{\infty}\frac{(\partial_x,D_\eta)^k}{k!}\overline{p(x,\eta)}.$$

Once the asymptotic expansion of the adjoint operator is obtained, one can use the approach from [22] and obtain the results for the product of two pseudodifferential operators, L^p- boundedness as well as to construct parametrix for uniformly elliptic operators. We shall avoid a repetition of these standard steps and shall state only the needed results.

If p_1, p_2 are the symbols of the operators P_1, P_2 such that

$$p_j \in S^{m_j,k_j} \quad , \quad j = 1, 2,$$

then the product $P_1 P_2$ of these pseudodiffrerential operators has a symbol in $S^{m_1+m_2,k_1+k_2}$ and this symbol has an asymptotic expansion

$$\sum_{k=0}^{\infty} \frac{(\partial_y, D_\eta)^k}{k!}(p_1(x,\eta)p_2(y,\xi))|_{y=x,\eta=\xi}.$$

If P is a pseudodifferential operator with symbol in $S^{0,0}$, then for $1 < p < \infty$ we have

$$\|Pf\|_{L^p(\mathbf{R}^n)} \leq C\|f\|_{L^p(\mathbf{R}^n)}.$$

This estimate follows from the inclusion $S^{0,0} \subset S^0$ and the corresponding standard L^p estimates for pseudodifferential operators with symbols in S^0.

Given asymptotic expansion of the symbol $p \in S^{m,k}$

$$p \sim \sum_{j=0}^{\infty} p_j,$$

with $p_j \in S^{m-j,k-j}$, we shall call p_0 a principal symbol of p.

Assuming the principle symbol is uniformly elliptic, i.e. there exists a constant $\mu > 0$, so that

$$|p_0(x,\xi)| \geq \mu(1+|\xi|)^m(1+|x|)^k,$$

we can construct a parametrix for the corresponding pseudodifferential operator P. In this way we find a pseudodifferential operator $Q \in S^{-m,-k}$ so that $PQ - I$ and $QP - I$ are pseudodifferential operators with symbols in $S^{-\infty,-\infty}$.

Our next step is one variant of Gärding's inequality for pseudodifferential operators.

Theorem 5.2.1 *If P is a pseudodifferential operator with symbol $p \in S^{0,0}$ and if the principle symbol p_0 satisfies $p_0(x,\xi) \geq \mu > 0$ for any $x, \xi \in \mathbf{R}^n$, then*

$$(Pu, u)_{L^2(\mathbf{R}^n)} \geq -C\| < x >^{-N} (1-\Delta)^{-N} u\|^2_{L^2}$$

for any integer $N \geq 1$.

Proof. The symbol $q(x,\xi) = \sqrt{p(x,\xi)}$ belongs to $S^{0,0}$. This fact shows that one can find a pseudodifferential operator Q_0, so that $Q_0^*Q_0 - P$ is a pseudodifferential operator with symbol in $S^{-1,-1}$. Using the fact that Q_0 is uniformly elliptic we can find a pseudodifferential operator Q_1 with symbol in $S^{-1,-1}$, so that

$$(Q_0^* + Q_1^*)(Q_0 + Q_1) - P$$

has a symbol in $S^{-2,-2}$. Continuing this procedure, we can find Q so that

$$Q^*Q - P = R$$

is a pseudodifferential operator with symbol in $S^{-\infty,-\infty}$. For any such R we have the estimate

$$\|Ru\|_{L^2} \leq C\| < x >^{-N} (1-\Delta)^{-N}u\|_{L^2}.$$

This completes the proof.