3 Fourier transform and Sobolev spaces on flat space

3.1 Overview

In this Chapter we shall introduce two basic tools for the study of hyperbolic equations. Namely, we start with the Fourier transform and the main facts about Sobolev spaces in $\mathbf{R}^n$. We shall avoid a complete and detailed representation of the theory of Fourier transform and Sobolev spaces. Nevertheless, we shall underline only the points, which are important for a further generalization for the case of manifolds of curvature -1.

The reader can use [44], [21] , [43] [6] for more detailed information about the space of distributions, Fourier transform and the convolution.

3.2 Preliminary facts about holomorphic functions

Let $\mathbf{C}$ be the complex plane and let $U \subseteq \mathbf{C}$ be an open domain in this plane. Any point $z \in U$ can be represented as

$$z = x + iy,$$

where x, y are real numbers. A function

$$f : U \to \mathbf{C}$$

is $C^1(U)$ if the partial derivatives

$$\partial_x f(x+iy), \partial_y f(x+iy)$$

exist and are continuous functions. Of special interest are the vector fields

$$\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$$

and

$$\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y).$$

If $f \in C^1(U)$, then f is called holomorphic in U, if satisfies the equation

$$\partial_{\bar{z}} f(z) = 0, \quad z \in U.$$

One can see that a function $f : U \to \mathbf{C}$ is holomorphic in U if and only if

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists for any $z \in U$.

The most important formula in the elementary theory of holomorphic functions is the Cauchy theorem and the Cauchy formula.

Let Γ be a closed path in U and let $z \in \mathbf{C}$ be a point such that Γ does not pass through z. Then the index of z with respect to Γ is

$$\mathrm{Ind}_\Gamma(z) = \frac{1}{2\pi i}\int_\Gamma \frac{d\zeta}{\zeta - z}.$$

The Cauchy theorem states that if Γ is closed path in U such that $\mathrm{Ind}_\Gamma(w) = 0$ for any w outside U, then

$$\int_\Gamma f(\zeta)d\zeta = 0 \tag{3.2.1}$$

for any function holomorphic in $\mathbf{C}$. The corresponding Cauchy formula is

$$f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - z}d\zeta. \tag{3.2.2}$$

The condition $\mathrm{Ind}_\Gamma(w) = 0$ for w outside U is fulfilled for the case U is simply connected.

Also in the case of a simply connected domain U with smooth boundary ∂U for any function holomorphic in U and continuos in the closure of U we have the corresponding Cauchy formula

$$f(z) = \frac{1}{2\pi i}\int_{\partial U} \frac{f(\zeta)}{\zeta - z}d\zeta. \tag{3.2.3}$$

Applying for example the above formula for $\{z, |z - z_0| < \delta\} \subset U$, we obtain the estimate

$$|\partial_z^k f(z_0)| \le \frac{Mk!}{\delta^k}, \tag{3.2.4}$$

where

$$M = \sup_{|z-z_0|=\delta} |f(z)|.$$

This estimate guarantees that the formal Taylor series

$$\sum_{k=0}^{\infty} \partial_z^k f(z_0)(z - z_0)^k/k!$$

converges absolutely and uniformly for $|z - z_0|$ sufficiently small and moreover the series coincides with $f(z)$ for z sufficiently close to z_0.

Our next step is the study of holomorphic functions in the strip

$$S = \{z; 0 < \mathrm{Re} z < 1\}.$$

More precisely, given any real number γ we consider the class $F(\gamma)$ of all functions $f \in C^1(\bar{S})$ holomorphic in S and satisfying the estimate

$$|f(z)| \le Ce^{\gamma|\mathrm{Im} z|} \tag{3.2.5}$$

Lemma 3.2.1 *(Three lines Lemma.) If $f \in F(\gamma)$, then for any $\theta \in (0,1)$ we have*

$$|f(\theta)| \leq \|e^{\delta(i\ \cdot)^2} f(i\ \cdot)\|^{1-\theta}_{L^\infty(\mathbf{R})} \|e^{\delta(1+i\ \cdot)^2} f(1+i\ \cdot)\|^{\theta}_{L^\infty(\mathbf{R})}.$$

Proof. If $f \in F(\gamma)$, then we can consider the function

$$g(z) = e^{\delta z^2} f(z) a_0^{z-1} a_1^{-z},$$

where

$$a_j = \|e^{\delta(j+i\ \cdot)^2} f(j+i\cdot)\|_{L^\infty(\mathbf{R})},\ j = 0, 1.$$

It is clear that we can assume that a_j are positive numbers. Otherwise, if $a_1 = 0$, then we can replace a_1 by $a_1 + \varepsilon$. Then it is easy to see that $g \in F(-\delta_1)$ with $0 < \delta_1 < \delta$ so we have the estimate

$$|g(z)| \leq C e^{-\delta_1 |\mathrm{Im} z|}$$

for $\mathrm{Re} z \in [0, 1]$. Then this estimate enables us to extend the Cauchy formula (3.2.3) for the strip S.

From the Cauchy formula it follows the maximum principle, i.e.

$$\sup_{z \in S} |g(z)| \leq \max(\sup_{t \in \mathbf{R}} |g(it)|, \sup_{t \in \mathbf{R}} |g(1+it)|).$$

Since, $|g(it)| \leq 1$ and $|g(1+it)| \leq 1$, we get

$$|g(\theta + iy)| \leq 1.$$

Taking $y = 0$, we complete the proof of the lemma.

Remark. From the proof it is clear that we have the estimate

$$|f(\theta+iy)| \leq$$
$$e^{-\delta|\theta+iy|^2} \|e^{\delta(i\ \cdot)^2} f(i\ \cdot)\|^{1-\theta}_{L^\infty(\mathbf{R})} \|e^{\delta(1+i\ \cdot)^2} f(1+i\ \cdot)\|^{\theta}_{L^\infty(\mathbf{R})}.$$

For the case of a function

$$f : U \to V,$$

where $U \subseteq \mathbf{C}$ is an open domain and V is a topological vector space, we shall say that f is weakly holomorphic if Λf is a holomorphic function for any $\Lambda \in V'$. Then f is also strongly holomorphic in the sense that

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists in V for any $z \in U$.(see [44], Chapter 3)

We close this section by another important complex interpolation theorem. To formulate this theorem we shall denote by $\mathrm{L}(A, B)$ the Banach space of bounded operators from a Banach space A into the Banach space B.

Given any positive real numbers p_0, p_1 with $1 \leq p_0 < p_1 \leq \infty$, we recall the notation $L^{p_0}(\mathbf{R}^n) + L^{p_1}(\mathbf{R}^n)$ for the linear space

$$\{f : f = f_0 + f_1, f_0 \in L^{p_0}(\mathbf{R}^n), f_1 \in L^{p_1}(\mathbf{R}^n)\}.$$

The norm in this space we define as follows

$$\|f\|_{L^{p_0}+L^{p_1}} = \inf_{f=f_0+f_1} \|f_0\|_{L^{p_0}} + \|f_1\|_{L^{p_1}}.$$

Theorem 3.2.1 *(Stein interpolation theorem, see [43])*

Suppose $1 \leq p_0, p_1, q_0, q_1 \leq \infty, T(z)$ *is a continuous function from the strip* $0 \leq \mathbf{Re} z \leq 1$ *into* $\mathrm{L}(L^{p_0} + L^{p_1}; L^{q_0} + L^{q_1})$ *, holomorphic for* $0 < \mathbf{Re} z < 1$ *and satisfying the properties*

$$\|T(z)\|_{\mathrm{L}(L^{p_0};L^{q_0})} \leq C \exp(C|\mathbf{Im} z|) \quad \text{for} \quad \mathbf{Re} z = 0, \tag{3.2.6}$$

$$\|T(z)\|_{\mathrm{L}(L^{p_1};L^{q_1})} \leq C \exp(C|\mathbf{Im} z|) \quad \text{for} \quad \mathbf{Re} z = 1. \tag{3.2.7}$$

Then for any $\theta \in (0,1)$ *we have*

$$\|T(\theta)\|_{\mathrm{L}(L^p;L^q)} \leq C,$$

where

$$\frac{1}{p} = (1-\theta)\frac{1}{p_0} + \theta\frac{1}{p_1} \quad , \quad \frac{1}{q} = (1-\theta)\frac{1}{q_0} + \theta\frac{1}{q_1}. \tag{3.2.8}$$

Note that Riesz-Thorin interpolation theorem is a trivial corollary of this complex interpolation theorem.

Moreover, we can relax the condition

(Assumption H). $T(z)$ is a continuous function from the strip $0 \leq \mathbf{Re} z \leq 1$ into $\mathrm{L}(L^{p_0} + L^{p_1}; L^{q_0} + L^{q_1})$ and holomorphic for $0 < \mathbf{Re} z < 1$.

Indeed, let

$$q^* = \max(q_0, q_1), \quad p^* = \min(p_0, p_1).$$

Then for any compact subset K in $\mathbf{R}^n$ the embeddings

$$L^{q^*}(K) \to L^{q_0}(K) + L^{q_1}(K)$$

and

$$L^{p_0}(K) + L^{p_1}(K) \to L^{p^*}(K).$$

are continuos operators.

Here and below $L^p(K)$ denotes the space of all L^p functions having support in K.

Then we shall replace the **(Assumption H)** with the following one. For any two compact sets K, K' in $\mathbf{R}^n$ we have the property

(Assumption $H_{K,K'}$**)** $T(z)$ is a continuous function from the strip $0 \le \mathbf{Re} z \le 1$ into $\mathrm{L}(L^{p^*}(K); L^{q^*}(K'))$ and holomorphic for $0 < \mathbf{Re} z < 1$.

Then we have the following variant of Stein interpolation theorem using the weaker assumption **Assumption** $H_{K,K'}$

Theorem 3.2.2 *Suppose $1 \le p_0, p_1, q_0, q_1 \le \infty$ and the family $T(z)$ satisfies the* **Assumption** $H_{K,K'}$ *for any couple of compact subsets K, K' in $\mathbf{R}^n$. If the estimates*

$$\|T(z)\|_{\mathrm{L}(L^{p_0}(K);L^{q_0}(K'))} \le C \exp(C|\mathbf{Im} z|) \quad \text{for} \quad \mathbf{Re} z = 0, \tag{3.2.9}$$

$$\|T(z)\|_{\mathrm{L}(L^{p_1}(K);L^{q_1}(K'))} \le C \exp(C|\mathbf{Im} z|) \quad \text{for} \quad \mathbf{Re} z = 1 \tag{3.2.10}$$

are satisfied with some constant C independent of the choice of the couple K, K', then for any $\theta \in (0,1)$, the operator $T(\theta)$ is a bounded operator from L^p into L^q and we have

$$\|T(\theta)\|_{\mathrm{L}(L^p;L^q)} \le C.$$

Here p, q are chosen as in (3.2.8).

Proof: To see that the above theorem is valid, we replace the family $T(z)$ by

$$\chi_{K'} T(z) \chi_K,$$

where χ_K denotes the characteristic function of the set K. For the new family we apply the classical Stein interpolation Theorem 3.2.1 and see that

$$\|\chi_{K'} T(\theta) \chi_K f\|_{L^q} \le C \|f\|_{L^p},$$

where C is independent of K, K'. Further we replace K' by a sequence $K_m = \{x; |x| \le m\}$ and apply the fact that $f \in L^q$ if and only if for any integer $m > 1$ we have

$$\|\chi_{K_m} f\|_{L^q} \le C$$

with some constant C independent of m. Thus taking the limit as m tends to ∞, we get

$$\|T(\theta)\chi_K f\|_{L^q} \le C \|f\|_{L^p}$$

and see that $T(\theta)$ is a bounded operator from $L^p(K)$ into L^q. Note that the relation $\chi_K^2 = \chi_K$ leads to

$$\|T(\theta)\chi_K f\|_{L^q} \le C \|\chi_K f\|_{L^p}. \tag{3.2.11}$$

Further, we can see that $T(\theta)\chi_{K_m} f$ is a Cauchy sequence in L^q by application of (3.2.11) with $K = K_m - K_j, m > j$. This observation shows that the definition

$$T(\theta) f = \lim_{m \to \infty} T(\theta) \chi_{K_m} f$$

is quite consistent and in this way we see the $T(\theta)$ is a well – defined operator from L^p into L^q that satisfies

$$\|T(\theta)f\|_{L^q} \leq C\|f\|_{L^p}. \tag{3.2.12}$$

This completes the proof of the Theorem.

3.3 Distributions and Fourier transform in Euclidean space

The purpose of this section is to recall some basic notions and properties of the spaces , where the solutions of the hyperbolic equations are defined.

The first important space is $C_0^\infty(\mathbf{R}^n)$. This space consists of all smooth functions with compact support.

The space $C_0^\infty(\mathbf{R}^n)$ is nonempty linear vector space. Indeed, we can first construct a smooth function $f(x)$, such that $f(x) = 0$ for $x \leq 0$ and $f(x) > 0$ for $x > 0$. For the purpose take

$$f(x) = e^{-1/x}$$

for $x > 0$. Then $f(x)f(1-x)$ is a smooth function with support in the interval $[0,1]$.

Recall that the support of function $f(x)$ defined for $x \in \mathbf{R}^n$ is the closure of the set

$$\{x : f(x) \neq 0\}.$$

Sometimes this space is called space of test functions and is denoted by $D(\mathbf{R}^n)$. This space can be equipped with infinite number of semi norms. In fact, given any integer $N \geq 1$ we define

$$\|f\|_N = \max\{|\partial^\alpha f(x)|; |\alpha| \leq N\}, \tag{3.3.1}$$

where here and below $\partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$, $\alpha = (\alpha_1, \cdots, \alpha_n)$ is a multi index and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. It is well known (see [44]) that any vector space V with countable number of semi norms is metrizable, i.e. we can introduce a metric. In fact, if $\|.\|_1, ..., \|.\|_N ...$ are the corresponding semi norms, then

$$d(v,w) = \sum_{N=1}^{\infty} \frac{2^{-N}\|v-w\|_N}{1+\|v-w\|_N} \tag{3.3.2}$$

The space $C_0^\infty(R)$ with metric generated by the semi norms (3.3.1) is not complete.

In order to work with a complete space, i.e. space where any Cauchy sequence converges to an element of the space, we have to define the topology on $C_0^\infty(\mathbf{R}^n)$ using all the collection of semi norms $\|.\|_N$. For example a local basis of open neighborhoods of the origin can be defined by

$$V_N = \{f; \|f\|_N < 1/(N+1)\}, N = 0, 1, \cdots \tag{3.3.3}$$

Even the topology defined by this local basis is not complete. (see [44]). To define a complete topology the simplest way is to define the convergence of a sequence of functions $\{f_k\}_{k=1}^\infty$ to zero. Recall that this sequence converges to zero if there exists a compact set K such that $\text{supp} f_k \subseteq K$ for any integer $k \geq 1$ and $\|f_k\|_N$ tends to 0 as $k \to \infty$ for any integer $N \geq 0$. Applying the Arzela-Ascoli compactness theorem one can check that the topology determined by this convergence is complete. Moreover, this is locally convex topological vector space. Again we refer to [44] for a complete discussion of the topology on this space.

In a similar way one can consider the space $C^\infty(\mathbf{R}^n)$ consisting of all smooth functions. Now we have the following family of **semi norms**.

$$\|f\|_N = \max\{|\partial^\alpha f(x)|; |x| \leq N, |\alpha| \leq N\},$$

The above family of semi norms enables one directly to introduce a complete topology (even one can introduce a complete metric) by using (3.3.2).

The weakest space, where we shall look for solutions of the nonlinear partial differential equations, is the space of distributions $D'(\mathbf{R}^n)$ consisting of all linear continuous functionals on $C_0^\infty(\mathbf{R}^n)$. Given any distribution Λ we shall denote by

$$< \Lambda, f >$$

the action of the distribution (the linear functional) Λ on the test function $f \in C_0^\infty(\mathbf{R}^n)$. It is clear that

$$C_0^\infty(\mathbf{R}^n) \subset D'(\mathbf{R}^n)$$

and

$$< \Lambda, f >= \int \Lambda(x) f(x) dx$$

for $\Lambda \in C_0^\infty(\mathbf{R}^n)$.

A typical example of a distribution, which is not a test function, is the Dirac delta function δ defined by

$$< \delta, f >= f(0).$$

Since the space of distributions is the dual space to the space of test functions, we choose the topology on the space of distributions to be the weak topology on this dual space. This means that a sequence of distributions $\{\Lambda_k\}_{k=1}^\infty$ tends to zero if for any test function f we have $< \Lambda_k, f >$ tends to zero.

The space $D'(\mathbf{R}^n)$ equipped with this weak topology is a complete space.

Another useful characterization of the distributions is the following one. A linear functional Λ on $C_0^\infty(\mathbf{R}^n)$ is bounded if for any compact $K \subseteq \mathbf{R}^n$ there exist integer $k \geq 0$ and a positive real number C so that for any smooth function $\varphi(x)$ with compact support in K we have

$$| < \Lambda, \varphi > | \leq C \sum_{|\alpha| \leq k} \sup |\partial^\alpha \varphi(x)|.$$

Example. Let $\varphi(x)$ be smooth non-negative function such that $\varphi(0) > 0$. Given any $\varepsilon > 0$, we can define the function

$$\varphi_\varepsilon(x) = \varepsilon^{-n} \varphi(x/\varepsilon). \tag{3.3.4}$$

Then it is easy to see that

$$\lim_{\varepsilon \to 0} \varphi_\varepsilon = c\delta,$$

where

$$c = \int \varphi(x)dx > 0.$$

Another fact for distributions is the meaning of the identity

$$\Lambda = 0$$

in the sense of distributions. This means

$$< \Lambda, \varphi >= 0$$

for any test function φ.

A natural operation in the space of distribution is the differentiation defined by

$$< \partial^\alpha \Lambda, f >= (-1)^{|\alpha|} < \Lambda, \partial^\alpha f > .$$

Then $\partial^\alpha \Lambda$ is a bounded linear functional on $C_0^\infty(\mathbf{R}^n)$ provided $\Lambda \in D'(\mathbf{R}^n)$.

Given a distribution Λ on $\mathbf{R}$ with

$$\Lambda' = 0$$

in the sense of distributions, one can find a constant c such that

$$\Lambda = c$$

also in the sense of distributions.

Another important tool in the study of partial differential equations is the Fourier transform defined formally for function $f(x)$ by

$$\hat{f}(\xi) = \int f(x) e^{-ix\xi} dx \tag{3.3.5}$$

In order to be sure that the integral in the above definition is meaningful we recall the definition of the space $S(\mathbf{R}^n)$ of rapidly decreasing functions, where the Fourier transform is well defined. Namely, a smooth function $f(x)$ is called rapidly decreasing if the norm

$$\|f\|_{M,N} = \max\{(1 + |x|)^M |\partial^\alpha f(x)|; |\alpha| \le N\}$$

is bounded for any integers $M, N \ge 0$. It is well-known that the Fourier transform maps the space of rapidly decreasing functions into itself. The formula for the inverse Fourier transform has the form

$$f(x) = (2\pi)^{-n} \int \hat{f}(\xi) e^{ix\xi} d\xi$$

With $S'(\mathbf{R}^n)$ we shall denote the space of tempered distributions which is the dual to $S(\mathbf{R}^n)$. We have the inclusions

$$D(\mathbf{R}^n) \subset S(\mathbf{R}^n) \subset S'(\mathbf{R}^n) \subset D'(\mathbf{R}^n).$$

We refer again to [44] for detailed analysis of the topology on S'.

From (3.3.5) it follows that

$$\widehat{\partial^\alpha f}(\xi) = i^{|\alpha|}\xi^{|\alpha|}\hat{f}(\xi).$$

One can extend the Fourier transform for distributions in $S'(\mathbf{R}^n)$ by the aid of the relation

$$< \hat{u}, f >=< u, \hat{f} >$$

As an application we shall consider the following Cauchy problem for the linear wave equation

$$(-\partial_t^2 + \Delta)u = 0,$$
$$u(0,x) = 0 \ , \ \partial_t u(0,x) = f(x).$$

Applying the Fourier transform we get

$$(\partial_t^2 + |\xi|^2)\hat{u} = 0,$$
$$\hat{u}(0,x) = 0 \ , \ \partial_t \hat{u}(0,x) = \hat{f}(x), \tag{3.3.6}$$

where

$$\hat{u}(t,\xi) = \int u(t,x)e^{-ix\xi}dx$$

is the partial Fourier transform. The Cauchy problem for the second order ordinary differential equation in (3.3.6) can be solved directly. Applying the inverse Fourier transform we get

$$u(t,x) = (2\pi)^{-n}\int e^{ix\xi}\frac{\sin(|\xi|t)}{|\xi|}\hat{f}(\xi)d\xi \tag{3.3.7}$$

The Fourier transform can be extended in the Hilbert space L^2 as a unitary operator. In fact, we can denote by $(,)_{L^2}$ the scalar product in this Hilbert space. More precisely, for $f, g \in L^2$ we have

$$(f,g)_{L^2} = \int f(x)\bar{g}(x)dx$$

Using the inverse formula for the Fourier transform we get

$$(\hat{f},\hat{g})_{L^2} = (2\pi)^n(f,g)_{L^2}$$

and we arrive at the Plancherel formula

$$\|\hat{f}\|_{L^2} = (2\pi)^{n/2}\|f\|_{L^2}, \tag{3.3.8}$$

where $\|.\|_{L^2}$ is the norm in the Hilbert space L^2.

As another application we shall prove Hausdorff-Young inequality

$$\|\hat{f}\|_{L^q} \leq C\|f\|_{L^p} \tag{3.3.9}$$

for $1 \leq p \leq 2, 1/p + 1/q = 1$. To establish this inequality we shall check the inequality for $q = \infty$ and $q = 2$. The inequality for $q = \infty$ follows from the definition of the Fourier transform, while the estimate for $q = 2$ follow from the Plancherel identity . The application of Riesz-Thorin interpolation theorem implies (3.3.9).

Another classical operation in the space of rapidly decreasing functions is the convolution (denoted by $f * g$) of two functions $f, g \in S(\mathbf{R}^n)$. The convolution is defined by

$$(f * g)(x) = \int f(x-y)g(y)dy \tag{3.3.10}$$

Some important properties of the convolution are listed below.

$$f * g = g * f \tag{3.3.11}$$

$$(f * g) * h = f * (g * h) \tag{3.3.12}$$

A direct computation shows that we have the relation

$$\widehat{f * g} = \hat{f}\hat{g} \tag{3.3.13}$$

For $f, g \in S(\mathbf{R}^n)$ we have

$$\partial_x^\alpha (f * g) = (\partial_x^\alpha f) * g = f * \partial_x^\alpha g.$$

For a fixed $g \in S(\mathbf{R}^n)$ we can extend the map

$$f \in S(\mathbf{R}^n) \to f * g \in S(\mathbf{R}^n) \tag{3.3.14}$$

to a linear continuos map from the dual space $S'(\mathbf{R}^n)$ into itself. Then we have the relation

$$< f * g, \varphi > = < f, \breve{g} * \varphi >, \tag{3.3.15}$$

where $\breve{g}(x) = g(-x)$.

For the case, when $g \in S'(\mathbf{R}^n)$ we can define the map (3.3.14). A possibility to extend this map as a linear bounded operator in L^p for $1 < p < \infty$ is based on the following result of E.Stein (see Theorem 1 , Chapter 2 in [52])

Theorem 3.3.1 *Suppose the Fourier transform of g satisfies*

$$|\hat{g}(\xi)| \leq C, \tag{3.3.16}$$

while $g \in C^1(\mathbf{R}^n - 0)$ satisfies

$$|\nabla(g)(x)| \leq C|x|^{-n-1}, \tag{3.3.17}$$

for any $x \neq 0$. Then the map (3.3.14) can be extended as a linear bounded operator in L^p for $1 < p < \infty$.

Let $j(x)$ be a smooth compactly supported function such that

$$j(x) = ae^{1/(|x|^2-1)} \tag{3.3.18}$$

for $|x| \leq 1$ and $j(x) = 0$ for $|x| \geq 1$. The constant $a > 0$ is chosen so that

$$\int_{\mathbf{R}^n} j(x)dx = 1. \tag{3.3.19}$$

Another example is a smooth compactly supported function $\varphi(x)$ satisfying the property

$$\varphi(x) = \begin{cases} 1, & \text{if } |x| \leq 1; \\ 0, & \text{if } |x| \geq 2. \end{cases} \tag{3.3.20}$$

For the purpose let $\chi(x)$ be the characteristic function of the ball $\{|x| \leq 3/2\}$. Then for any $\varepsilon > 0$ consider the function

$$\varphi_\varepsilon(x) = j_\varepsilon * \chi,$$

where $j_\varepsilon(x) = \varepsilon^{-n} j(x/\varepsilon)$. Then for $\varepsilon >$ sufficiently small the function $\varphi_\varepsilon(x)$ i 1 for $|x| \leq 1$ and is 0 for $|x| \geq 2$.

One of the classical inequalities for this operation is the Young inequality having the form

$$\|f * g\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q} \tag{3.3.21}$$

for $1 \leq p, q, r \leq \infty$, and $1/p + 1/q = 1 + 1/r$.

Next we introduce Friedrich's molifiers. Let $j(x)$ be the function from (3.3.18). For each $u \in L^p$ and

$$u_\varepsilon(x) = j_\varepsilon * u$$

we have

$$\|u_\varepsilon\|_{L^p} \leq \|u\|_{L^p}$$

$(1 \leq p < \infty)$ and moreover

$$\|u - u_\varepsilon\|_{L^p} \to 0$$

as $\varepsilon \to 0$.

One can show further that for any $f \in S(\mathbf{R}^n)$ we have

$$j_\varepsilon * f \to f$$

in $S(\mathbf{R}^n)$ as $\varepsilon \to 0$.

This property enables us to show that $S(\mathbf{R}^n)$ is dense in $S'(\mathbf{R}^n)$.

A possible variation of the interpolation Theorem 3.2.2 is the case, when the assumption **Assumption** $H_{K,K'}$ is further relaxed. Namely, we assume that

$$T(z) : L^{p^*}(K) \to D',$$

and one can find a suitable approximation family $T_\varepsilon(z), \varepsilon > 0$ so that for any $f \in L^{p^*}(K)$ we have

(3.3.22) $$\lim_{\varepsilon\to 0} T_\varepsilon(z)f = T(z)f$$

with limit taken in the sense of distributions. For the approximation family we shall assume

(Assumption $H_{K,K',\varepsilon}$**)** $T_\varepsilon(z)$ is a continuous function from the strip $0 \le$ **Re**$z \le 1$ into $\mathrm{L}(L^{p^*}(K); L^{q^*}(K'))$ and holomorphic for $0 <$ **Re**$z < 1$.

As before we take

$$q^* = \max(q_0, q_1), \quad p^* = \min(p_0, p_1)$$

in the above assumption.

Then we turn to the following.

Theorem 3.3.2 *Suppose* $1 \le p_0, p_1, q_0, q_1 \le \infty$ *and the family* $T_\varepsilon(z)$ *satisfies the* **Assumption** $H_{K,K',\varepsilon}$ *for any couple of compact subsets* K, K' *in* $\mathbf{R}^n$ *and for any* $\varepsilon > 0$. *If the estimates*

(3.3.23) $$\|T_\varepsilon(z)\|_{\mathrm{L}(L^{p_0}(K);L^{q_0}(K'))} \le C\exp(C|\mathbf{Im}z|) \quad \text{for} \quad \mathbf{Re}z = 0,$$

(3.3.24) $$\|T_\varepsilon(z)\|_{\mathrm{L}(L^{p_1}(K);L^{q_1}(K'))} \le C\exp(C|\mathbf{Im}z|) \quad \text{for} \quad \mathbf{Re}z = 1$$

are satisfied with some constant C *independent of the choice of the couple* K, K' *and* $\varepsilon > 0$, *then for any* $\theta \in (0,1)$, *the operator* $T(\theta)$ *is a bounded operator from* L^p *into* L^q *and we have*

$$\|T(\theta)\|_{\mathrm{L}(L^p;L^q)} \le C.$$

Here p, q *are chosen as in (3.2.8).*

Proof: Let the compact set K be fixed. Take $f \in L^p(K)$ and extend f as zero outside K. For any $g \in C_0^\infty$ we have

$$\lim_{\varepsilon\to\infty} < T_\varepsilon(\theta)(f), g > = < T(\theta)(f), g > .$$

Now we are in position to apply the classical interpolation Theorem 3.2.2 for the family $T_\varepsilon(z)$ and for $\varepsilon > 0$ fixed. Thus we have the estimate

$$| < T_\varepsilon(\theta)(f), g > | \le C\|f\|_{L^p(K)}\|g\|_{L^{q'}},$$

where the index q' is conjugate to q i.e. $1/q + 1/q' = 1$ and the constant C is independent of $\varepsilon > 0, K$. Now we can use the following simple property. If $u_m, m = 1, 2, ...$ is a sequence of distributions, tending in the sense of distributions to u and if

$$| < u_m, g > | \le C\|g\|_{L^{q'}}$$

with C independent of $m > 1$, then the distribution u is a classical measurable function in L^q. This observation shows that $T(\theta)(f) \in L^q$ and

$$\|T(\theta)(f)\|_{L^q} \leq C\|f\|_{L^p(K)}.$$

In this way we arrived at the estimate (3.2.11). Then the argument following after (3.2.11), completes the proof of the Theorem .

3.4 Homogeneous distributions and Fourier transform of concrete functions

Fourier transform of a distribution $u \in S'(\mathbf{R}^n)$ can be defined on the basis of the relation

$$< \hat{u}, f >=< u, \hat{f} > .$$

Here $f \in S(\mathbf{R}^n)$ is a rapidly decreasing function and $< u, f >$ denotes the action of the functional $u \in S'(\mathbf{R}^n)$ on $f \in S(\mathbf{R}^n)$. The Fourier transform is a continuous linear operator in the space $S'(\mathbf{R}^n)$. In particular, if f_k is a sequence of elements in $S(\mathbf{R}^n)$ tending to f in $S'(\mathbf{R}^n)$, then $\hat{f}_k$ tend to $\hat{f}$.

This remark is of special interest when we want to compute the Fourier transform of a measurable function $f(x)$, homogeneous of degree $s \in \mathbf{R}$, i.e. for any $t > 0$ we have

$$f(tx) = t^s f(x)$$

for almost every $x \in \mathbf{R}^n$.

Any measurable homogeneous of order s function $f(x)$ satisfies the relation

$$< f, \varphi_t >= t^s < f, \varphi >, \tag{3.4.1}$$

where φ is arbitrary test function and $\varphi_t(x) = t^{-n}\varphi(x/t)$

On the basis of this fact we introduce the following.

Definition 3.4.1 *A distribution $f \in S'(\mathbf{R}^n)$ is homogeneous of order $s \in \mathbf{R}$, if*

$$< f, \varphi_t >= t^s < f, \varphi >, \tag{3.4.2}$$

where $\varphi \in S(\mathbf{R}^n)$ and $\varphi_t(x) = t^{-n}\varphi(x/t)$.

If $f \in S'(\mathbf{R}^n)$ is homogeneous of order s, then one can see that $\hat{f}$ is homogeneous of order $-n - s$.

Recall that $f \in L^p_{\mathrm{loc}}$ if $\varphi f \in L^p$ for any smooth compactly supported function φ.

Example. Let $x_+^s = x^s$ if $x > 0$ and $x_+^s = 0$ for $x \leq 0$. Then $x_+^s \in L^1_{\mathrm{loc}}(\mathbf{R})$ for $s > -1$.

Then for $1 \leq p < \infty$ we have

$$L^p_{\mathrm{loc}}(\mathbf{R}^n) \subset S'(\mathbf{R}^n)$$

The function x_+^s is a homogeneous distribution of order s for $s > -1$.

Example.

Let $x_-^s = x^s$ if $x < 0$ and $x_-^s = 0$ for $x \geq 0$. Then $x_-^s \in L^1_{\mathrm{loc}}(\mathbf{R})$ for $s > -1$.

As before x_-^s is a homogeneous distribution of order s for $s > -1$.

It can be shown that x_+^s can be extended as a distribution in $\mathbf{R}$ for $s \neq \{-1, -2, \ldots\}$. In a similar way if $f \in C^\infty(\mathbf{R}^n \backslash \{0\})$ is homogeneous function of order s, then it can be extended as a distribution in $\mathbf{R}^n$ for $s \neq \{-n, -n-1, \ldots\}$. (see Section 3.2 in [21]) Further, the results of L.Hörmander (see Theorem 7.1.18 in [21]) guarantee that f is a tempered distribution, i.e. $f \in S'(\mathbf{R}^n)$ so its Fourier transform is well defined.

Next we shall consider Fourier transform of some special functions (for more information we refer to some classical books as [11])

We start with the Fourier transform of the function

$$\mathrm{e}^{-x^2} \in S(\mathbf{R}^n),$$

i.e. our goal is to compute

$$\hat{f}(\xi) = \int_{\mathbf{R}^n} \mathrm{e}^{-ix\xi - x^2} dx.$$

We shall consider only the case $n = 1$, since the case $n \geq 2$ can be treated in a similar manner.

Using the identity

$$-ix\xi - x^2 = -(x + i\xi/2)^2 - \xi^2/4,$$

we get

$$\hat{f}(\xi) = \mathrm{e}^{-\xi^2/4} \int \mathrm{e}^{-(x+i\xi/2)^2} dx.$$

Let $\xi > 0$ for determinacy. Since the function

$$z \in \mathbf{C} \to \mathrm{e}^{-z^2}$$

is an entire function and for z in the strip $\{z; \mathrm{Im} z \in [0, \xi]\}$ we have the estimate

$$|\mathrm{e}^{-z^2}| \leq C(\xi)\mathrm{e}^{-x^2},$$

we can change the path of integration $\mathrm{Im} z = \xi$ in

$$\hat{f}(\xi) = \mathrm{e}^{-\xi^2/4} \int_{\mathrm{Im} z = \xi} \mathrm{e}^{-z^2} dz.$$

into $\mathrm{Im} z = 0$ so

$$\hat{f}(\xi) = \mathrm{e}^{-\xi^2/4} \int_{\mathrm{Im} z = 0} \mathrm{e}^{-z^2} dz = \mathrm{e}^{-\xi^2/4} \int_{\mathbf{R}} \mathrm{e}^{-x^2} dx.$$

To determine the constant

$$c = \int_{\mathbf{R}} e^{-x^2} dx,$$

we use the fact that

$$c^2 = \int_{\mathbf{R}} e^{-x_1^2} dx_1 \int_{\mathbf{R}} e^{-x_2^2} dx_2.$$

Introducing polar coordinates in $\mathbf{R}^2$

$$x = (x_1, x_2) = \rho(\cos\varphi, \sin\varphi),$$

where $\rho > 0, \varphi \in [0, 2\pi)$, we get

$$c^2 = 2\pi \int_0^\infty e^{-\rho^2} \rho d\rho = \pi.$$

So $c = \sqrt{\pi}$ and

$$\widehat{e^{-x^2}}(\xi) = \sqrt{\pi} e^{-\xi^2/4}$$

provided the space dimension is $n = 1$.

For $n-$dimensional case the Fourier transform $F(f)(\xi)$ of

$$f(x) = e^{-R^2 x^2}, x \in \mathbf{R}^n$$

is

$$F(e^{-R^2x^2})(\xi) = R^{-n/2} \pi^{n/2} e^{-\xi^2/(4R)}$$

for any space dimension $n \geq 1$ and any $R > 0$.

It is easy to extend the above result to the case, when $R \in \mathbf{C}$ and $\mathrm{Re}R > 0$.

In fact, we can use the fact that $z^{1/2}$ is a well - defined analytic function for $\mathrm{Re}z > 0$.

Using the formula

$$F(e^{-R^2x^2})(\xi) = R^{-n/2} \pi^{n/2} e^{-\xi^2/(4R)}$$

with $\mathrm{Re}R > 0$, we set

$$R = i + \varepsilon$$

and taking the limit as $\varepsilon \to 0$, we can establish the following property: the Fourier transform of the distribution

$$e^{ix^2} \in S'(\mathbf{R}^n)$$

is

$$e^{-i\pi n/4} \pi^{n/2} e^{-i\xi^2/4}.$$

Using a diagonalization for any symmetric matrix Q we obtain on the basis of the last argument the following.

Lemma 3.4.1 *If Q is a symmetric $n \times n$ matrix with determinant*

$$\det Q \neq 0,$$

then the Fourier transform of the distribution

$$e^{i(Qx,x)} \in S'(\mathbf{R}^n),$$

where $(.,.)$ is the scalar product in $\mathbf{R}^n$, is

$$e^{-i\pi \operatorname{sgn} Q/4}\pi^{n/2}|\det Q|^{-1/2}e^{-i(Q^{-1}\xi,\xi)/4}.$$

Here $\operatorname{sgn} Q$ *is the signature of the symmetric matrix Q.*

3.5 Sobolev spaces of integer order

The space of test functions $C_0^\infty(\mathbf{R}^n)$ is too strong, while the space of distribution is too weak to describe the space of solutions of partial differential equations. As intermediate spaces one can consider the Sobolev spaces $W_p^l(\mathbf{R}^n)$ defined for integers $l \geq 0$ and real numbers $1 \leq p < \infty$ as follows. A distribution $f \in S'(\mathbf{R}^n)$ if the norm

$$\|f\|_{W_p^l} = \sum_{|\alpha| \leq l} \|\partial^\alpha f\|_{L^p} \tag{3.5.1}$$

are finite.

Using Friedrich's molifiers, one can see that for any

$$u \in W_p^l$$

there exists a sequence of smooth compactly supported functions tending to u with respect to the norm (3.5.1).

This fact shows that the Sobolev space can be defined also as the closure of the space $C_0^\infty(\mathbf{R}^n)$ of test functions with respect to the norm (3.5.1).

The fundamental property of these Sobolev spaces is the inclusion

$$W_p^l \subseteq W_q^r \tag{3.5.2}$$

when $1 < p < q < \infty, 1/p - 1/q \leq (l-r)/n > 0$. The inclusion follows directly from the Sobolev inequality

$$\|f\|_{W_q^r} \leq C\|f\|_{W_p^l} \tag{3.5.3}$$

The proof of the Sobolev inequality (3.5.3) , we shall present, is based on suitable Sobolev identity. For the purpose we follow the approach from the book of Maz'ya [36]. Namely, we shall use the following Sobolev identity.

Lemma 3.5.1 *For any integer $l, 0 < l \le n$ and any real numbers $D > \delta > 0$ one can find smooth functions φ_β (β is multi index with $|\beta| < l$) supported in the unit ball and smooth functions ψ_α such that for any smooth function $u(x)$ and for $|x| < D$ we have the representation*

$$u(x) = \delta^{-n} \sum_{|\beta|<l} \left(\frac{x}{\delta}\right)^\beta \int_{|y|\le\delta} \varphi_\beta\left(\frac{y}{\delta}\right) u(y)dy + \tag{3.5.4}$$
$$+ \sum_{|\alpha|=l} \int_{|y|\le 2D} \psi_\alpha(x,r,\theta)\partial^\alpha u(y)\frac{dy}{r^{n-l}}$$

where $r = |x-y|, \theta = (x-y)/r$ and the functions $\psi_\alpha(x,r,\theta)$ satisfy the estimate

$$|\psi_\alpha(x,r,\theta)| \le C\left(\frac{D}{\delta}\right)^{n-1} \tag{3.5.5}$$

Proof. A scaling argument shows it is sufficient to consider the case $\delta = 1$. Let us take a smooth nonnegative function $\omega(y)$ supported in the unit ball so that

$$\int_{|y|\le 1} \omega(y)dy = 1. \tag{3.5.6}$$

Introduce functions

$$f(x;r,\theta) = -\frac{r^{l-1}}{(l-1)!}\int_r^\infty \omega(x+t\theta)t^{n-1}dt$$
$$F(x;r,\theta) = \sum_{k=0}^{l-1}(-1)^k \frac{\partial^k}{\partial r^k}u(x+r\theta)\frac{\partial^{l-1-k}}{\partial r^{l-1-k}}f(x;r,\theta) \tag{3.5.7}$$

We lose no generality assuming u has a compact support in $|x| \le 2D$. Since all the derivatives with respect to r of the function f at $r = 0$ up to order $l-2$ are zero we see that

$$F(x;0,\theta) = -u(x)\int_0^\infty \omega(x+t\theta)t^{n-1}dt$$

Now we apply the formula

$$0 = F(x;0,\theta) + \int_0^\infty \frac{\partial F}{\partial r}(x;r,\theta)dr$$

valid in view of the fact that u is compactly supported.

On the other hand, from (3.5.7) we get

$$\frac{\partial F}{\partial r}(x;r,\theta) = u\frac{\partial^l f}{\partial r^l} + (-1)^{l-1}\frac{\partial^l u}{\partial r^l}f.$$

Hence we get the identity

$$u(x)\int_0^\infty \omega(x+t\theta)t^{n-1}dt = \int_0^\infty u(x+r\theta)\frac{\partial^l f}{\partial r^l}(x;r,\theta)dr + \\ +(-1)^{l-1}\int_0^\infty \frac{\partial^l u}{\partial r^l}(x+r\theta)f(x;r,\theta)dr$$

Integrating this identity over $\theta \in \mathbf{S}^{n-1}$, we obtain

$$u(x) = \int u(y)\frac{\partial^l f}{\partial r^l}(x;r,\theta)\frac{dy}{r^{n-1}} + \\ +(-1)^{l-1}\int \frac{\partial^l u}{\partial r^l}(y)f(x;r,\theta)\frac{dy}{r^{n-1}}$$

Having in mind that we have the representation

$$\frac{\partial^l f}{\partial r^l}(x;r,\theta)\frac{1}{r^{n-1}} = \sum_{|\beta|<l} x^\beta \varphi_\beta(y),$$

we arrive at Sobolev identity (3.5.4) with

$$\psi_\alpha = c_\alpha \int_r^\infty \omega(x+t\theta)t^{n-1}dt. \tag{3.5.8}$$

The estimate (3.5.5) follows directly from (3.5.8).

This completes the proof of the Sobolev identity.

To prove the inequality (3.5.3) it is sufficient to combine the Sobolev identity with the Sobolev inequality of Lemma 2.4.1.

In fact, Lemma 2.4.1 allows us to estimate the Riesz potential operator

$$I_\lambda f(x) = \int |x-y|^{-\lambda} f(y)dy.$$

In fact, for $1/p - 1/q + \lambda/n = 1$ and $1 < p, q < \infty$ we have the following Adam's estimate

$$\|I_\lambda(f)\|_{L^q} \le C\|f\|_{L^p}. \tag{3.5.9}$$

The Sobolev identity shows that

$$u(x) = u^I(x) + u^{II}(x),$$

where

$$u^I(x) = \sum_{|\beta|<l} \delta^{-n-|\beta|} x^\beta \int_{|y|\le\delta} \varphi_\beta(y/\delta)u(y)dy.$$

Let us assume u is a smooth function with a compact support inside the ball of radius R and center at 0. Taking $\delta = R$ and $D = 2R$ in the Sobolev identity we get

$$\|u^I\|_{L^q(|y|\leq R)} \leq C \sum_{|\beta|<l} R^{-n-|\beta|} \left(\int_{|y|\leq R} |y|^{\beta q} dy \right)^{1/q} \int |u(y)| dy.$$

Applying the Hölder inequality, we get

$$\|u^I\|_{L^q(|y|\leq R)} \leq C R^{-n/p+n/q} \|u\|_{L^p},$$

where $C > 0$ is a constant independent of R. To estimate u^{II} we start with the estimate

$$|u^{II}(x)| \leq C \sum_{|\alpha|=l} I_{n-l}(|\partial^\alpha f|)(x).$$

Now we are in situation to apply Adams estimate (3.5.9) and get

$$\|u^{II}\|_{L^q} \leq C \sum_{|\alpha|=l} \|\partial^\alpha u\|_{L^p}$$

provided $1/q + 1 = 1/p + (n-l)/n$, i.e. for $1/p - 1/q = l/n$. Hence, we get

$$\|u\|_{L^q} \leq C R^{-n/p+n/q} \|u\|_{L^p} + C \sum_{|\alpha|=l} \|\partial^\alpha u\|_{L^p},$$

where $C > 0$ is independent of $R > 0$. Taking $R \to \infty$, we get

$$\|u\|_{L^q(\mathbb{R}^n)} \leq C \sum_{|\alpha|=l} \|\partial^\alpha u\|_{L^p(\mathbb{R}^n)} \tag{3.5.10}$$

provided u is a smooth compactly supported function and $l/n = 1/p - 1/q$.

Taking the closure of the space of smooth compactly supported functions with respect to the norm in W_p^l, we complete the proof of the Sobolev inequality (3.5.3).

We shall close this section with the following Sobolev inequality

$$\|f\|_{L^\infty} \leq C \|f\|_{W_p^s} \tag{3.5.11}$$

for $1 < p < \infty, 1/p < s/n$. An easy proof can be found when $p = 2$ by the aid of the Fourier transform and Plancherel identity. For the general case, one can use the Sobolev identity and the Hölder inequality.

Lemma 3.5.2 *(Hardy , Littlewood) Let $m(\xi) = c|\xi|^{-t}$ and consider the operator*

$$I(f)(x) = \int_{\mathbf{R}^n} e^{ix\xi} m(\xi) \hat{f}(\xi) d\xi.$$

Prove that for $1 < p \leq 2 \leq q < \infty$ and $1/p - 1/q = t/n$ this operator can be extended as a bounded operator from L^p to L^q.

On the other hand, for any real s one can define the Sobolev space $H^s(\mathbf{R}^n)$ as completition of the space of smooth functions with compact support with respect to the norm

$$\|f\|_{H^s} = \|(1-\Delta)^{s/2} f\|_{L^2} \tag{3.5.12}$$

Applying the Plancherel identity, we see that this norm is equivalent to the following one

$$\|(1+|\xi|^2)^{s/2}\hat{f}\|_{L^2}.$$

It is clear that this is a Hilbert space with scalar product

$$(f,g)_{H^s} = \int (1+|\xi|^2)^s \hat{f}(\xi)\bar{\hat{g}}(\xi)d\xi.$$

We have the following property

$$H^k = W_2^k.$$

Moreover, the dual space of H^s is H^{-s}.

Now we can formulate a result for existence and uniqueness of higher regularity solutions of the Cauchy problem

$$\begin{gathered}(-\partial_t^2 + \Delta - M^2)u = 0,\\ u(0,x) = f_0(x)\ ,\ \partial_t u(0,x) = f_1(x)\end{gathered} \tag{3.5.13}$$

for the linear Klein-Gordon equation. Namely, if the initial data $f = f_0 \times f_1$ belongs to the Hilbert space

$$H^s \times H^{s-1}$$

with $s \geq 1$, then the Cauchy problem (3.5.13) has a unique solution

$$u(t,.) \in \cap_{m=0}^{[s]} C^m([0,\infty); H^{s-m}),$$

where $[s]$ denotes the integer part of the real number s.

3.6 Gagliardo - Nirenberg inequality

Another useful interpolation inequality is the following variant of Gagliardo - Nirenberg inequality.

Lemma 3.6.1 *Suppose $f \in W^k_{q/k}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Then for any integer $k \geq 1$ and any $q \geq k$ we have*

$$\|f\|_{W^{k-1}_{q/(k-1)}(\mathbb{R}^n)} \leq C\|f\|^{1-1/k}_{W^k_{q/k}(\mathbb{R}^n)} \|f\|^{1/k}_{L^\infty(\mathbb{R}^n)}.$$

Proof. Let $\varphi(x)$ be a smooth compactly supported function such that $\varphi(x) = 1$ near $x = 0$. Then for any $q > 1$ and any integer $k \geq 0$ we know that

$$\varphi(x/R)f(x)$$

tends to $f \in W_q^k$ as R tends to infinity. Moreover, for $f \in L^\infty$ we have the uniform estimate

$$|\varphi(x/R)f(x)| \leq C|f(x)|.$$

This argument shows that without loss of generality we can assume f is compactly supported. We can reduce the proof to the case f is smooth compactly supported function, using Friedrich's molifiers.

Moreover, the simple estimate

$$\|f\|_{L^{q/(k-1)}} \leq C\|f\|_{L^{q/k}}^{1-1/k}\|f\|_{L^\infty}^{1/k}$$

enables us to reduce the proof to the case $n = 1$. Then we shall proceed by means of induction with respect to k.

We shall consider in details only the case

$$q \geq 2k$$

and shall give only the idea for the case $k \leq q < 2k$.

The identity

$$|f^{(k)}(x)|^{q/k} = f^{(k)}(x)|f^{(k)}(x)|^{-2+q/k}f^{(k)}(x)$$

and integration by parts give

$$\left|\int_{-\infty}^{\infty} |f^{(k)}(x)|^{q/k}dx\right| \leq$$

$$C\int_{-\infty}^{\infty} |f^{(k+1)}(x)||f^{(k)}(x)|^{-2+q/k}|f^{(k-1)}(x)|dx.$$

Applying the Hölder inequality, we get

$$\|f^{(k)}\|_{L^{q/k}}^{q/k} \leq C\|f^{(k+1)}\|_{L^{q/(k+1)}}\|f^{(k)}\|_{L^{q/k}}^{-2+q/k}\|f^{(k-1)}\|_{L^{q/(k-1)}}.$$

Applying the inductive assumption for $k - 1$, we get the desired estimate for k.

For the case $k \leq q < 2k$ we use the relation

$$|f^{(k)}(x)|^{q/k} = |f(x)|^A|f^{(k)}(x)|^{q/k}|f(x)|^{-A}$$

and applying Hölder inequality, we get an estimate of type

$$\int |f^{(k)}(x)|^{q/k}dx \leq C\left(\int |f(x)|^{Ar}dx\right)^{1/r}\left(\int |f(x)|^{-As}|f^{(k)}(x)|^2dx\right)^{1/s},$$

where $s = 2k/q$ and $r = 2k/(2k - q)$. Further, we make an integration by parts in

$$\int |f(x)|^{-sA} |f^{(k)}(x)|^2 dx$$

and use the inductive argument as it was done above in the case $q \geq 2k$.

This completes the proof of the Lemma.

The above estimate has few interesting corollaries.

Corollary 3.6.1 *Suppose $f \in W_p^k(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ with $p \geq 1$. Then for any integers $k > l \geq 1$ we have*

$$\|f\|_{W^l_{pk/l}(\mathbb{R}^n)} \leq C \|f\|^{l/k}_{W^k_p(\mathbb{R}^n)} \|f\|^{(k-l)/k}_{L^\infty(\mathbb{R}^n)}.$$

Corollary 3.6.2 *Suppose $f, g \in W_p^k(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ with $p \geq 1$. Then we have*

$$\|fg\|_{W^k_p(\mathbb{R}^n)} \leq C(\|f\|_{W^k_p(\mathbb{R}^n)} \|g\|_{L^\infty(\mathbb{R}^n)} + \|g\|_{W^k_p(\mathbb{R}^n)} \|f\|_{L^\infty(\mathbb{R}^n)})$$

Proof. It is sufficient to apply Leibniz rule in combination with the estimate of the previous Corollary.

Corollary 3.6.3 *Suppose $f \in W_p^k(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ with $p \geq 1$. Then for any $\lambda > k$ we have*

$$\||f|^\lambda\|_{W^k_p(\mathbb{R}^n)} \leq C \|f\|_{W^k_p(\mathbb{R}^n)} \|f\|^{\lambda-1}_{L^\infty(\mathbb{R}^n)}.$$

Proof. It is sufficient to apply induction with respect to k in combination with Gagliardo - Nirenberg inequality.

In fact, due to Leibniz rule we have

$$\partial_x^\alpha F(f) = \sum c_{\alpha_1,\dots,\alpha_m} F^m(f) \partial_x^{\alpha_1} f, \dots, \partial_x^{\alpha_m} f,$$

where the sum is over $m = 0, 1, \dots, |\alpha|$ and $|\alpha_1| + \dots + |\alpha_m| = |\alpha|$. Then with $|\alpha| = k$ we get

$$\||f|^\lambda\|_{W^k_p(\mathbb{R}^n)} \leq \sum \||f|^{\lambda-m}\|_{L^\infty} \|f\|_{W^{|\alpha_1|}_{pk/|\alpha_1|}} \dots \|f\|_{W^{|\alpha_m|}_{pk/|\alpha_m|}}$$

provided

$$|\alpha_1|/pk + \dots + |\alpha_m|/pk = 1/p.$$

Applying Corollary 3.6.1, we get the desired estimate.

This completes the proof.