Chapter 2

Preliminaries

§2.1. Two lemmas on Riccati's differential equations

First of all, we give two lemmas on ordinary differential equations of Riccati's type. These two lemmas are due to L. Hörmander [Ho1].

Lemma 2.1. Let $z = z(t)$ be a solution in [0,T] of the Riccati's differential equation:

$$\frac{dz}{dt} = a_0(t)z^2 + a_1(t)z + a_2(t), \tag{2.1.1}$$

where $a_j(t)$ $(j = 0, 1, 2)$ are continuous, $a_0(t) \geq 0$, and $T > 0$ is a given real number. Let

$$K = \int_0^T |a_2(t)|\, dt \cdot \exp\left(\int_0^T |a_1(t)|\, dt\right). \tag{2.1.2}$$

If

$$z(0) > K, \tag{2.1.3}$$

then it follows that

$$\int_0^T a_0(t)dt \cdot \exp\left(-\int_0^T |a_1(t)|\, dt\right) < (z(0) - K)^{-1}. \tag{2.1.4}$$

□

Proof. Let us first assume that $a_1(t) \equiv 0$ $(0 \le t \le T)$, and introduce

$$z_2(t) = \int_0^t |a_2(s)|ds.$$

Obviously,

$$z_2(0) = 0, \quad z_2(T) = K \quad \text{and} \quad 0 \le z_2(t) \le K, \quad \forall\, t \in [0,T].$$

Let z_1 be the solution of the Cauchy problem

$$\begin{cases} \frac{dz_1}{dt} = a_0(t)(z_1 - K)^2, \\ t = 0 : z_1 = z(0). \end{cases}$$

Then

$$(z(0) - K)^{-1} - (z_1(t) - K)^{-1} = \int_0^t a_0(s)ds$$

on the existence domain of $z_1 = z_1(t)$. Moreover, $z_1(t)$ is an increasing function of t. If $z_1 = z_1(t)$ exists in $[0,T]$, then

$$\int_0^t a_0(s)ds < (z(0) - K)^{-1}. \tag{2.1.4a}$$

Thus, in order to get (2.1.4a), it suffices to prove that $z_1 = z_1(t)$ exists in the whole interval $[0,T]$. Since on the existence domain of $z_1 = z_1(t)$

$$\frac{d(z_1(t) - z_2(t))}{dt} = a_0(t)(z_1(t) - K)^2 - |a_2(t)| \le a_0(t)(z_1(t) - z_2(t))^2 + a_2(t),$$

and $z_1(t) - z_2(t) = z(0)$ when $t = 0$, we obtain

$$z_1(t) - z_2(t) \le z(t) \quad \text{in} \quad [0,T]$$

as long as $z_1(t)$ exists. Therefore $z_1(t)$ can not become infinite in $[0,T]$, namely, $z_1 = z_1(t)$ exists in $[0,T]$. This proves (2.1.4a).

For the general case $a_1(t) \not\equiv 0$, we just make the following transformation

$$z(t) = Z(t) \exp\left(\int_0^t a_1(s)ds\right).$$

This reduces (2.1.1) to

$$\frac{dZ(t)}{dt} = a_0(t) \exp\left(\int_0^t a_1(s)ds\right) Z(t)^2 + a_2(t) \exp\left(-\int_0^t a_1(s)ds\right).$$

We apply the special case of the lemma already proved, and then get immediately the desired (2.1.4). Q.E.D.

For a fixed positive number T, consider equation (2.1.1). We have

Lemma 2.2. Suppose that $a_j(t)\,(j = 0,1,2)$ are continuous functions in [0,T]. Set

$$a_0^+(t) = \max\{a_0(t), 0\}, \tag{2.1.5}$$

and define K by (2.1.2). If

$$z_0 \geq 0, \tag{2.1.6}$$

$$\int_0^T a_0^+(t)dt \cdot \exp\left(\int_0^T |a_1(t)|\, dt\right) < (z_0 + K)^{-1} \tag{2.1.7}$$

and

$$\int_0^T |a_0(t)|\, dt \cdot \exp\left(\int_0^T |a_1(t)|\, dt\right) < K^{-1}, \tag{2.1.8}$$

where z_0 is a given real number. Then (2.1.1) has a unique solution $z = z(t)$ in [0,T] with $z(0) = z_0$, and the following estimates hold

$$(z(T))^{-1} \geq (z_0 + K)^{-1} - \int_0^T a_0^+(t)dt \cdot \exp\left(\int_0^T |a_1(t)|\, dt\right), \quad \text{if } z(T) > 0, \tag{2.1.9}$$

$$|z(T)|^{-1} \geq K^{-1} - \int_0^T |a_0(t)|\, dt \cdot \exp\left(\int_0^T |a_1(t)|\, dt\right), \quad \text{if } z(T) < 0. \tag{2.1.10}$$

□

Proof. We first prove this lemma in the special case $a_1(t) \equiv 0$ $(0 \leq t \leq T)$.

Let $z_2 = z_2(t)$ be still the integral of $|a_2|$ with $z_2(0) = 0$ and $z_2(T) = K$ and $z_1 = z_1(t)$ be the solution of the following initial value problem

$$\begin{cases} \frac{dz_1}{dt} = a_0^+(t)(z_1 + K)^2, \\ t = 0 : z_1 = z_0. \end{cases}$$

Then

$$(z_1(t) + K)^{-1} = (z_0 + K)^{-1} - \int_0^t a_0^+(s)ds.$$

By (2.1.7), $z_1 = z_1(t)$ exists in $[0,T]$ and $z_1(t)$ is an increasing function of t in $[0,T]$.

We now assume that $z(t)$ exists in $[0,T]$, and prove that (2.1.9)-(2.1.10) hold in this case.

Since

$$\frac{d(z_1(t)+z_2(t))}{dt} = a_0^+(t)(w_1(t)+K)^2 + |a_2(t)| \geq a_0^+(t)(z_1(t)+K)^2 + a_2(t),$$

and $z_1 + z_2 = z_0$ at $t = 0$, we get

$$z(t) \leq z_1(t) + z_2(t) \leq z_1(t) + K \quad \text{in} \quad [0,T].$$

Hence

$$z(T)^{-1} \geq (z_1(T)+K)^{-1} = (z_0+K)^{-1} - \int_0^T a_0^+(t)dt$$

if $z(T) > 0$, which proves (2.1.9).

On the other hand, if z has a zero in $[0,T]$, then we can apply (2.1.9) to $-z$, with z_0 replaced by 0 and to an interval starting at the zero of z. This gives (2.1.10).

If we do not assume a priori that $z(t)$ exists in $[0,T]$, it follows that (2.1.9)-(2.1.10) hold with T replaced by any smaller t such that a solution exists in $[0,t]$. Hence we have a fixed upper bound in any such interval. It follows at once that a solution does exist in $[0,T]$, for the considered set of t values is both open and closed.

Finally, when $a_1(t) \not\equiv 0$, we can reduce to the case already studied just as in the proof of Lemma 2.1. The proof is completed. Q.E.D.

§2.2. John's formula on decomposition of waves and generalized Hörmander's lemma

Suppose that on the domain under consideration, system (1.1) is hyperbolic and (1.4)-(1.5) hold.

Let

$$v_i = l_i(u)u \quad (i = 1,\cdots,n), \tag{2.2.1}$$

$$w_i = l_i(u)u_x \quad (i = 1, \cdots, n) \tag{2.2.2}$$

and

$$b_i(u) = l_i(u)B(u) \quad (i = 1, \cdots, n) \tag{2.2.3}$$

where

$$l_i(u) = (l_{i1}(u), \cdots, l_{in}(u)) \tag{2.2.4}$$

denotes the i-th left eigenvector.

By (1.4), it follows from (2.2.1)-(2.2.3) that

$$u = \sum_{k=1}^{n} v_k r_k(u), \tag{2.2.5}$$

$$u_x = \sum_{k=1}^{n} w_k r_k(u) \tag{2.2.6}$$

and

$$B(u) = \sum_{k=1}^{n} b_k(u) r_k(u). \tag{2.2.7}$$

Let

$$\frac{\mathrm{d}}{\mathrm{d}_i t} = \frac{\partial}{\partial t} + \lambda_i(u)\frac{\partial}{\partial x} \tag{2.2.8}$$

be the directional derivative along the i-th characteristic. Similar to [LZK1], we have (see [K3])

$$\frac{dv_i}{d_i t} = \sum_{j,k=1}^{n} \beta_{ijk}(u) v_j w_k + \sum_{j,k=1}^{n} \nu_{ijk}(u) v_j b_k(u) + b_i(u) \quad (i = 1, \cdots, n), \tag{2.2.9}$$

where

$$\beta_{ijk}(u) = (\lambda_k(u) - \lambda_i(u))\, l_i(u) \nabla r_j(u) r_k(u) \tag{2.2.10}$$

and

$$\nu_{ijk}(u) = -l_i(u) \nabla r_j(u) r_k(u). \tag{2.2.11}$$

Hence, we have

$$\beta_{iji}(u) \equiv 0, \quad \forall\, j. \tag{2.2.12}$$

It follows from (2.2.9) that

$$v_i(t,x) = v_i(0,\xi_i(0;t,x)) + \int_0^t \left[\sum_{j,k=1}^n \beta_{ijk}(u) v_j w_k + \sum_{j,k=1}^n \nu_{ijk}(u) v_j b_k(u) + b_i(u) \right] (\tau, \xi_i(\tau;t,x)) d\tau \quad (i=1,\cdots,n), \tag{2.2.13}$$

where v_i, w_i, $b_i(u)$, $\beta_{ijk}(u)$ and $\nu_{ijk}(u)$ are defined by (2.2.1)-(2.2.3) and (2.2.10)-(2.2.11) respectively, $\xi = \xi_i(\tau;t,x)$ stands for the i-th characteristic passing through (t,x) and satisfies

$$\begin{cases} \frac{d\xi}{d\tau} = \lambda_i(u(\tau,\xi(\tau;t,x))), \\ \tau = t: \quad \xi = x. \end{cases} \tag{2.2.14}$$

Noting (2.2.9) and (2.2.6), we have

$$\begin{aligned} \mathrm{d}\left[v_i\left(\mathrm{d}x - \lambda_i\left(u\right)\mathrm{d}t\right)\right] &= \left[\frac{\partial v_i}{\partial t} + \frac{\partial(\lambda_i(u)v_i)}{\partial x}\right] \mathrm{d}t \wedge \mathrm{d}x \\ &= \left[\frac{dv_i}{d_i t} + \left(\nabla\lambda_i\left(u\right)u_x\right)v_i\right] \mathrm{d}t \wedge \mathrm{d}x \\ &= \left[\sum_{j,k=1}^n \beta_{ijk}\left(u\right)v_j w_k + \sum_{j,k=1}^n \nu_{ijk}\left(u\right)v_j b_k(u) \right. \\ &\quad \left. + b_i(u) + \sum_{k=1}^n \left(\nabla\lambda_i\left(u\right)r_k\left(u\right)\right)v_i w_k\right] \mathrm{d}t \wedge \mathrm{d}x \qquad (2.2.15) \\ &= \left[\sum_{j,k=1}^n \tilde{\beta}_{ijk}\left(u\right)v_j w_k \right. \\ &\quad \left. + \sum_{j,k=1}^n \nu_{ijk}\left(u\right)v_j b_k(u) + b_i(u)\right] \mathrm{d}t \wedge \mathrm{d}x, \end{aligned}$$

where

$$\tilde{\beta}_{ijk}\left(u\right) = \beta_{ijk}\left(u\right) + \nabla\lambda_i\left(u\right)r_k\left(u\right)\delta_{ij}. \tag{2.2.16}$$

It follows from (2.2.12) that

$$\tilde{\beta}_{iji}\left(u\right) \equiv 0, \quad \forall\, j \neq i; \tag{2.2.17}$$

while

$$\tilde{\beta}_{iii}\left(u\right) = \nabla\lambda_i\left(u\right)r_i\left(u\right) \tag{2.2.18}$$

which identically vanishes only in the case that $\lambda_i(u)$ is linearly degenerate in the sense of P.D.Lax.

On the other hand, similar to [Jo] or [LZK1], we have (see [K3])

$$\frac{\mathrm{d}w_i}{\mathrm{d}_i t} = \sum_{j,k=1}^{n} \gamma_{ijk}(u)\, w_j w_k + (b_i(u))_x \quad (i = 1, \cdots, n), \tag{2.2.19}$$

where

$$\gamma_{ijk}(u) = \frac{1}{2}\{(\lambda_j(u) - \lambda_k(u))\, l_i(u)\, \nabla r_k(u)\, r_j(u) - \nabla\lambda_k(u)\, r_j(u)\, \delta_{ik} + (j|k)\}, \tag{2.2.20}$$

in which $(j|k)$ stands for all terms obtained by changing j and k in the previous terms.

It follows from (2.2.20) that

$$\gamma_{ijj}(u) \equiv 0, \quad \forall\, j \neq i \quad (i, j = 1, \cdots, n) \tag{2.2.21}$$

and

$$\gamma_{iii}(u) \equiv -\nabla\lambda_i(u)\, r_i(u) \quad (i = 1, \cdots, n). \tag{2.2.22}$$

When the i-th characteristic $\lambda_i(u)$ is linearly degenerate in the sense of P.D.Lax, we have

$$\gamma_{iii}(u) \equiv 0. \tag{2.2.23}$$

Similar to (2.2.13), by (2.2.19) we obtain

$$\begin{aligned} w_i(t,x) \;=\; & w_i(0, \xi_i(0; t, x)) + \\ & \int_0^t \left[\sum_{j,k=1}^{n} \gamma_{ijk}(u) w_j w_k + (b_i(u))_x\right] (\tau, \xi_i(\tau; t, x)) d\tau \quad (i = 1, \cdots, n), \end{aligned} \tag{2.2.24}$$

where $w_i, \gamma_{ijk}, b_i(u)$ and $\xi = \xi_i(\tau; t, x)$ are defined by (2.2.2), (2.2.20), (2.2.3) and (2.2.14) respectively.

Similar to (2.2.15), noting (2.2.19) and (2.2.6), we have

$$\mathrm{d}\left[w_i\left(\mathrm{d}x - \lambda_i(u)\, \mathrm{d}t\right)\right] = \left[\sum_{j,k=1}^{n} \tilde{\gamma}_{ijk}(u)\, w_j w_k + (b_i(u))_x\right] \mathrm{d}t \wedge \mathrm{d}x, \tag{2.2.25}$$

where

$$\begin{aligned}\tilde{\gamma}_{ijk}(u) &= \gamma_{ijk}(u) + \tfrac{1}{2}\left[\nabla\lambda_j(u)\, r_k(u)\,\delta_{ij} + (j|k)\right] \\ &= \tfrac{1}{2}\left(\lambda_j(u) - \lambda_k(u)\right) l_i(u)\left[\nabla r_k(u)\, r_j(u) - \nabla r_j(u)\, r_k(u)\right],\end{aligned} \tag{2.2.26}$$

then we get

$$\tilde{\gamma}_{ijj}(u) \equiv 0, \quad \forall\, i, j. \tag{2.2.27}$$

Moreover, it follows from (2.2.15) and (2.2.25) that

$$\frac{\partial v_i}{\partial t} + \frac{\partial(\lambda_i(u)\, v_i)}{\partial x} = \sum_{j,k=1}^{n} \tilde{\beta}_{ijk}(u)\, v_j w_k + \sum_{j,k=1}^{n} \nu_{ijk}(u)\, v_j b_k(u) + b_i(u) \quad (i = 1, \cdots, n) \tag{2.2.28}$$

and

$$\frac{\partial w_i}{\partial t} + \frac{\partial(\lambda_i(u)\, w_i)}{\partial x} = \sum_{j,k=1}^{n} \tilde{\gamma}_{ijk}(u)\, w_j w_k + (b_i(u))_x \quad (i = 1, \cdots, n). \tag{2.2.29}$$

Following L.Hörmander [Ho1], we get

Lemma 2.3. Suppose that $u = u(t,x)$ is a C^1 solution to system (1.1), τ_1 and τ_2 are two C^1 arcs which are never tangent to the i-th characteristic direction, and D is the domain bounded by τ_1, τ_2 and two i-th characteristic curves L_i^- and L_i^+, see Figure 1. Then we have

$$\begin{aligned}\int_{\tau_1} |v_i(\mathrm{d}x - \lambda_i(u)\,\mathrm{d}t)| \le \int_{\tau_2} &|v_i(\mathrm{d}x - \lambda_i(u)\,\mathrm{d}t)| + \\ &\iint_D \left|\sum_{j,k=1}^{n} \tilde{\beta}_{ijk}(u)\, v_j w_k\right| \mathrm{d}t\mathrm{d}x + \\ &\iint_D \left|\sum_{j,k=1}^{n} \nu_{ijk}(u) v_j b_k(u) + b_i(u)\right| \mathrm{d}t\mathrm{d}x\end{aligned} \tag{2.2.30}$$

and

$$\begin{aligned}\int_{\tau_1} |w_i(\mathrm{d}x - \lambda_i(u)\,\mathrm{d}t)| \le \int_{\tau_2} &|w_i(\mathrm{d}x - \lambda_i(u)\,\mathrm{d}t)| + \\ &\iint_D \left|\sum_{j,k=1}^{n} \tilde{\gamma}_{ijk}(u)\, w_j w_k + (b_i(u))_x\right| \mathrm{d}t\mathrm{d}x,\end{aligned} \tag{2.2.31}$$

where v_i, $\tilde{\beta}_{ijk}(u)$, $\nu_{ijk}(u)$, $b_i(u)$, w_i and $\tilde{\gamma}_{ijk}(u)$ are defined by (2.2.1), (2.2.16), (2.2.11), (2.2.3), (2.2.2) and (2.2.26) respectively. □

Proof. By Stokes' formula and noting that $(\mathrm{d}x - \lambda_i(u)\mathrm{d}t)$ has a fixed sign on τ_1 and τ_2, (2.2.30) easily follows from (2.2.15). The proof of (2.2.31) is similar (see [Hol]). Q.E.D.

Remark 2.1. Suppose that $A(u)$ and $B(u)$ are Lipschitz continuous, system (1.1) is hyperbolic on the domain under consideration, and (1.4)-(1.5) hold. Suppose furthermore that $u = u(t,x)$ be a Lipschitz solution to (1.1). Employing the difference technique, we can easily show that (2.2.13), (2.2.30)-(2.2.31) are still valid, and (2.2.24) holds a.e. in $\boldsymbol{R}^+ \times \boldsymbol{R}$, since the Rademacher theorem implies that any locally Lipschitz continuous function f: $\boldsymbol{R}^n \to \boldsymbol{R}^m$ is differentiable almost everywhere (see [Br]). □

§2.3. Equivalent definition of classical solutions

By means of the argument mentioned above, now we can give an equivalent definition of classical solutions to system (1.1) by the following

Proposition 2.1. Let $u = u(t,x)$ be a C^1 function with small L^∞ norm. Suppose that $A(u), B(u) \in C^1$ and system (1.1) is hyperbolic in a neighbourhood of $u = 0$. Then $u = u(t,x)$ satisfies (1.1) if and only if $v_i = v_i(t,x)$ $(i = 1, \cdots, n)$ satisfy (2.2.9), where $v_i = v_i(t,x)$ are defined by (2.2.1). □

Proof. The necessity is easily obtained from the preceding argument (see [K3]). Moreover, we do not require the smallness of L^∞ norm of $u = u(t,x)$.

It remains to prove the sufficiency.

Noting (2.2.1) and (1.2), we have

$$\begin{aligned} \frac{dv_i}{d_i t} &= l_i(u)\frac{du}{d_i t} + u^T\left(\nabla l_i^T(u)\frac{du}{d_i t}\right) \\ &= l_i(u)\left(u_t + A(u)u_x\right) + u^T\nabla l_i^T(u)\left(u_t + \lambda_i(u)u_x\right) \quad (i = 1, \cdots, n). \end{aligned} \tag{2.3.1}$$

On the other hand, by (1.4) we have

$$l_i(u)\nabla r_j(u) = -r_j^T(u)\nabla l_i^T(u). \tag{2.3.2}$$

Thus (2.2.9) becomes

$$\begin{aligned}\frac{dv_i}{d_it} = &\sum_{j,k=1}^{n} (\lambda_i(u) - \lambda_k(u))r_j^T(u)\nabla l_i^T(u)r_k(u)v_jw_k+ \\ &\sum_{j,k=1}^{n} r_j^T(u)\nabla l_i^T(u)r_k(u)v_jb_k(u) + b_i(u) \quad (i = 1,\cdots,n).\end{aligned} \tag{2.3.3}$$

By (1.2), (2.2.5)-(2.2.7) and (2.2.3), it follows from (2.3.3) that

$$\frac{dv_i}{d_it} = u^T\nabla l_i^T(u)\,(\lambda_i(u)u_x - A(u)u_x + B(u)) + l_i(u)B(u) \quad (i = 1,\cdots,n). \tag{2.3.4}$$

The combination of (2.3.1) and (2.3.4) leads to

$$\left(l_i(u) + u^T\nabla l_i^T(u)\right)(u_t + A(u)u_x - B(u)) = 0 \quad (i = 1,\cdots,n). \tag{2.3.5}$$

By (1.3) and the smallness of L^∞ norm of $u = u(t,x)$, from (2.3.5) we get (1.1) immediately. Thus the proof is finished. Q.E.D.

Proposition 2.2. Suppose that $A(u), B(u) \in C^1$, system (1.1) is hyperbolic, $u = u(t,x)$ is a C^1 function and satisfies (1.1), then $w = (w_1(t,x),\cdots,w_n(t,x))^T$ is a broad solution[1] to system (2.2.19), where $w_i = w_i(t,x)$ are defined by (2.2.2). □

Proof. Similar to the deriving process of (2.2.19) (see [K3]), we use the difference technique and then obtain (2.2.24) easily. (2.2.24) implies that $w = w(t,x)$ is a broad solution to system (2.2.19). The proof is completed. Q.E.D.

Remark 2.2. Throughout this paper, we only consider the classical solution to system (1.1), namely, C^1 solution to (1.1). In general, (2.2.19) no longer holds. Fortunately, by means of the difference technique, we can derive the integral equation (2.2.24) satisfied by w_i. In fact, we only use the integral equation (2.2.24) instead of the differential equation (2.2.19) in our proofs. We bear in our mind that (2.2.19) is satisfied formally by w_i and (2.2.24) holds actually when we mention equation (2.2.19) in the sequel. □

[1]See [Br] for the definition of the broad solution.