

Chapter 5

Deformations of Hyperbolic Structures

5.1 Introduction

Let Q be a compact 3-dimensional orbifold with link singularities. In most cases the manifold $Q - \Sigma$ obtained by removing the singular locus has a finite volume hyperbolic structure. If Q has a hyperbolic structure, it can be viewed as a cone-manifold structure with cone angles of the form $2\pi/m$. In the proof of the Orbifold Theorem we will attempt to connect the complete structure on $Q - \Sigma$, viewed as a cone-manifold with angles 0, with the desired orbifold structure via a family of cone-manifolds.

To study hyperbolic cone-manifold structures on Q with singularities along Σ , we first remove a neighbourhood of Σ from Q to obtain a compact manifold M with boundary consisting of tori. First we investigate when deformations of a hyperbolic structure on M exist. We will show that hyperbolic (or general (G, X)) structures on a compact manifold M are locally in 1–1 correspondence with nearby holonomy representations $\pi_1(M) \rightarrow G$ up to conjugacy. (If M has boundary, we may have to restrict to the complement of a small neighbourhood of the boundary ∂M .)

We then study how the deformed hyperbolic structures behave near the boundary of M . We will see that to find a nearby cone-manifold structure, it suffices to find a nearby holonomy representation for which the holonomy of each meridian is elliptic.

Next we discuss Thurston's analysis of representation spaces for 3-manifold groups into $PSL(2, \mathbb{C})$ and his theory of hyperbolic Dehn surgery. In particular, this implies that hyperbolic cone-manifold structures on Q with cone

angles, α_i , along the components of Σ exist for all sufficiently small values of α_i .

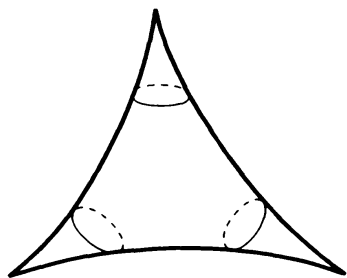
We finish the chapter with some examples, and some general conjectures on the global structure of hyperbolic Dehn surgery spaces.

5.2 Deformations and degenerations of surfaces

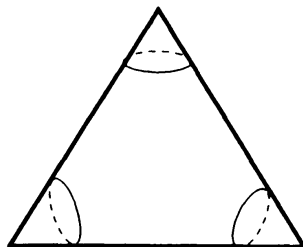
The proof of the orbifold theorem involves deforming a hyperbolic cone metric in an attempt to produce a hyperbolic orbifold. In the process the hyperbolic cone structure can degenerate. By analyzing how this happens one produces some other kind of geometric structure. In this section we will describe some two-dimensional analogues of the degenerations that occur in the proof of the orbifold theorem. The first example shows how Euclidean and spherical structures can arise and the second one suggests how an orbifold Seifert fibre structure can arise.

Example 5.1. Euclidean/Spherical transition

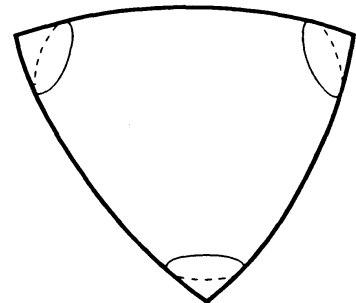
Given $\theta \in [0, 2\pi]$ there is a turnover $M(\theta) = S^2(\theta, \theta, \theta)$ with 3 cone angles θ . This is the double of a triangle of constant curvature with all angles $\theta/2$. It is hyperbolic, Euclidean or spherical depending on whether $\theta/2$ is less than, equal to, or greater than $\pi/3$. When the cone angle $\theta = 0$ this is interpreted as the double of an ideal hyperbolic triangle which gives a hyperbolic three-punctured sphere with three cusps. As $\theta \rightarrow 2\pi/3$ the diameter of $M(\theta)$ approaches 0. We may rescale the metric on $M(\theta)$ by multiplying the metric by a constant $\lambda = \text{diam}(M(\theta))^{-1}$ to obtain $M'(\theta) = \lambda \cdot M(\theta)$ of diameter 1. The curvature of $M'(\theta)$ is $\pm 1/\lambda^2$ and this goes to zero as $\theta \rightarrow 2\pi/3$. In this way one obtains a continuous family of cone metrics $M'(\theta)$ of varying constant curvature for $0 \leq \theta \leq 2\pi$.



$\theta < \pi/3$



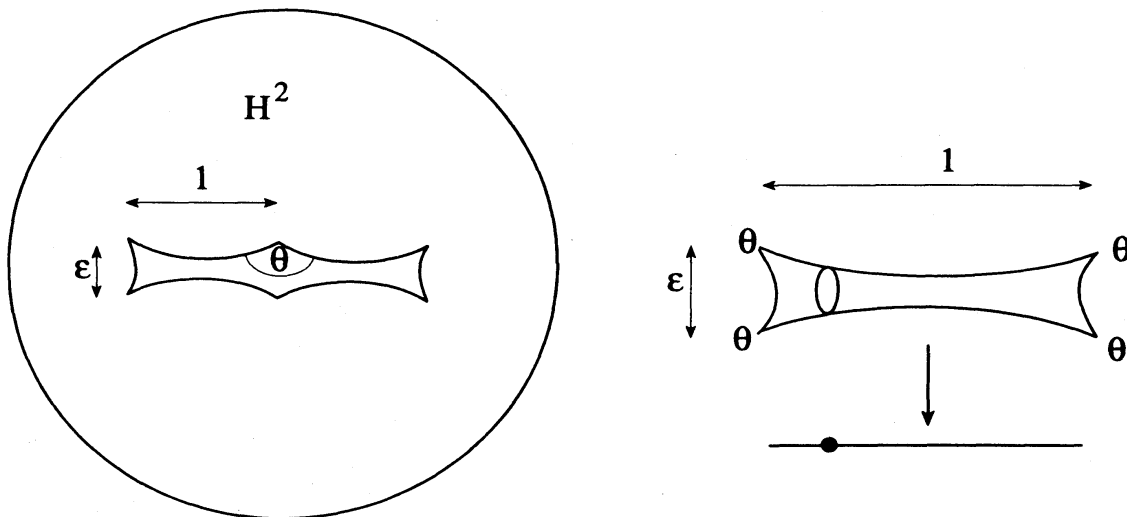
$\theta = \pi/3$



$\theta > \pi/3$

Example 5.2. *Degeneration to an Orbifold fibration*

The Gauss-Bonnet theorem implies there is no hyperbolic metric on a torus. It follows that a pillowcase is not a hyperbolic orbifold. In other words there is no hyperbolic cone metric on the sphere with 4 cone points each with cone angle π . Otherwise the 2-fold cover branched over these points would give a hyperbolic metric on the torus. However for each $\theta \in [0, \pi)$ there is a hyperbolic cone-manifold $M(\theta)$ with underlying space the sphere and four cone angles θ obtained by doubling a hyperbolic quadrilateral with all corner angles $\theta/2$.



This quadrilateral may be chosen to have diameter 1. Then the area of $M(\theta)$ approaches 0 as $\theta \rightarrow \pi$. The limit of $M(\theta)$ is an interval of length 1. For θ close to π there is an orbifold foliation of $M(\theta)$ by short circles plus two intervals joining the cone points. There is a map of $M(\theta)$ to the interval which collapses each circle to a point.

The orbifold fundamental group of a pillowcase has an infinite cyclic normal subgroup. A hyperbolic orbifold cannot have such a subgroup. As the cone angles increase to the orbifold angle of π , loops representing this subgroup shrink to points. In some sense this is forced by the holonomy representation which more and more nearly is a representation of the orbifold fundamental group. This produces an orbifold fibration. It is a two-dimensional version of the kind of collapsing that can happen with a three-dimensional orbifold.

5.3 General deformation theory

Of basic importance in the deformation theory of geometric structures on manifolds is the following observation. Given the holonomy representation $\rho : \pi_1(M) \rightarrow G$ for a (G, X) -structure on M , all nearby representations also correspond to geometric structures on M . (Compare [89], [80, chap. 5], [57].)

Let $\mathcal{R} = \text{Hom}(\pi_1(M), G)$ denote the space of representations $\pi_1(M) \rightarrow G$; the group G acts on \mathcal{R} by conjugation. If G is an algebraic (or analytic) group then \mathcal{R} has a natural structure as an algebraic (or analytic) variety; this gives a natural topology on \mathcal{R} . (This will be described in more detail in section 5.5 below).

Theorem 5.3 (Deformations exist).

Suppose that M is the interior of a compact manifold with boundary. Let $\rho : \pi_1(M) \rightarrow G$ be the holonomy representation of a (possibly incomplete) (G, X) -structure on M . Then there is a neighbourhood, U , of ρ in the representation space $\mathcal{R} = \text{Hom}(\pi_1(M), G)$ such that for each $\rho' \in U$ there is a (nearby) (G, X) -structure on M with holonomy ρ' .

Sketch of Proof. (See [31], [43] and [14] for more details.) Suppose that M is a connected (G, X) -manifold without boundary. Let \tilde{M} be the universal cover of M . Then there is a developing map $\text{dev} : \tilde{M} \rightarrow X$ which is a local diffeomorphism. This map is equivariant with respect to the holonomy representation $h : \pi_1(M) \rightarrow G$ in the sense that for all covering transformations g of \tilde{M}

$$\text{dev} \circ g = h(g) \circ \text{dev}.$$

Conversely if M is a smooth manifold (without a geometric structure) then given a homomorphism $h : \pi_1 M \rightarrow G$ and a map $\text{dev} : \tilde{M} \rightarrow X$ which is equivariant in the above sense and which is a local diffeomorphism we may use dev to pull-back the (G, X) -structure to \tilde{M} . The equivariance condition ensures this structure is preserved by covering transformations and therefore covers a (G, X) -structure on M . Hence a (G, X) -structure on M is determined by the pair (dev, h) satisfying the equivariance condition. There is an equivalence relation generated by *isotopy* and *thickening*. Two (G, X) -structures on M are *isotopic* if there is a (G, X) -diffeomorphism between them which is isotopic to the identity. Suppose that N is M minus a collar. We call M a *thickening* of N .

We now outline another approach to describing a geometric structure on a manifold (see Thurston [82, chapter 3]). Given only a holonomy representation $\rho : \pi_1(M) \rightarrow G$ we can construct a bundle $E = E_\rho \rightarrow M$ with fibre X . The developing map gives a section $s : M \rightarrow E$. This will now be described. There is a diagonal action of $\pi_1 M$ on $\tilde{M} \times X$ where the action on the first factor is by covering transformations and the action on the second factor is by the holonomy. The quotient $E = (\tilde{M} \times X)/\pi_1 M$ is a bundle as stated. The graph of the developing map:

$$\text{Graph}(\text{dev}) = \{ (\tilde{x}, \text{dev}(\tilde{x})) : \tilde{x} \in \tilde{M} \} \subset \tilde{M} \times X$$

is preserved by the action of $\pi_1 M$ and therefore defines a section $s : M \rightarrow E$. This section is defined by $s(x) = [(\tilde{x}, \text{dev}(\tilde{x}))]$. The notation $[(\tilde{x}, y)]$ denotes the projection of a point in $\tilde{M} \times X$ to E .

The product structure on $\tilde{M} \times X$ is preserved by the action of $\pi_1 M$ and so there is a *horizontal foliation* of E by leaves which are the projections of $\tilde{M} \times y$ for $y \in X$. This foliation determines a *flat connection* on E . A bundle over M with fibre X has a universal cover which is a bundle over \tilde{M} and the covering transformations act diagonally if and only if there is a foliation of E transverse to the fibres. Furthermore, the action on X is by elements of G if and only if the holonomy of the bundle is in G .

The statement that the developing map is a local diffeomorphism is equivalent to the statement that the graph of dev is transverse to the horizontal foliation. This is equivalent to the statement that the section s is transverse to the horizontal foliation of E .

It follows that a (G, X) -structure on M is determined by a flat X -bundle over M with holonomy in G together with a section s of this bundle which is transverse to the horizontal foliation.

Given a hyperbolic metric on M with holonomy ρ and a representation ρ' close to ρ then there is a bundle $E_{\rho'}$. The section $s : M \rightarrow E_\rho$ given by the (G, X) -structure on M is transverse to the horizontal foliation of E_ρ . The bundle $E_{\rho'}$ is close to E_ρ . One may construct a section $s' : M \rightarrow E_{\rho'}$ close to s . Since transversality is an open condition, if one restricts to a compact subset N of M which contains M minus a collar, then $s'|_N$ will still be transverse to the horizontal foliation, provided ρ' is close enough to ρ . Then the pair (ρ, s') determines a (G, X) -structure on $N \cong M$ with holonomy ρ' . \square

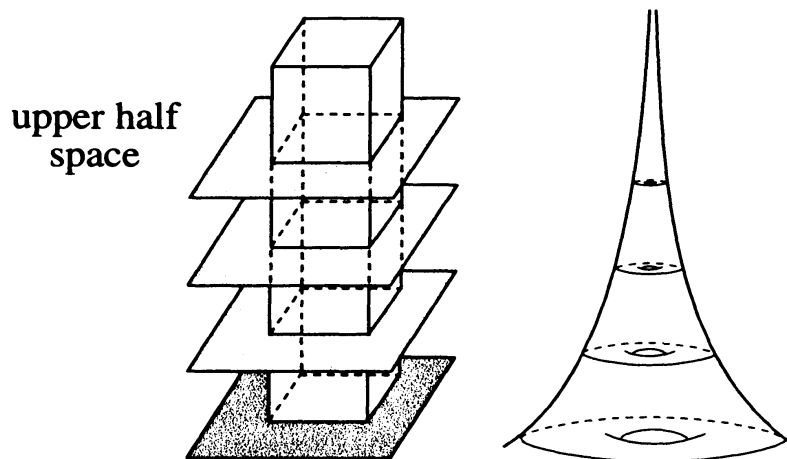
The previous result can be refined as follows. Let $\text{Def}(M)$ denote the *deformation space* consisting of (G, X) -structures on M modulo the equivalence relations of isotopy and thickening described above. This has a natural topology, where two points are close if they correspond to structures with developing maps $\tilde{M} \rightarrow X$ which are close in the C^∞ topology on compact subsets of \tilde{M} . The deformation space $\text{Def}(M)$ is locally parametrized by nearby holonomy representations in \mathcal{R} modulo the action of G by conjugation. (See Goldman [31] for a precise statement.)

5.4 Deforming hyperbolic cone-manifolds

Suppose we have a hyperbolic cone-manifold structure on Q with a link as the singular locus Σ . Theorem 5.3 applies to a (possibly incomplete) hyperbolic structure with no singularities on a manifold without boundary. To apply it to a cone-manifold structure, cut out a singular solid torus neighbourhood of each component of the cone locus. We then deform what remains, and glue in suitably deformed singular solid tori to get a new cone-manifold structure. This will be described in more detail below.

We will first investigate (incomplete) hyperbolic structures on $M = Q - \Sigma$ near the complete structure.

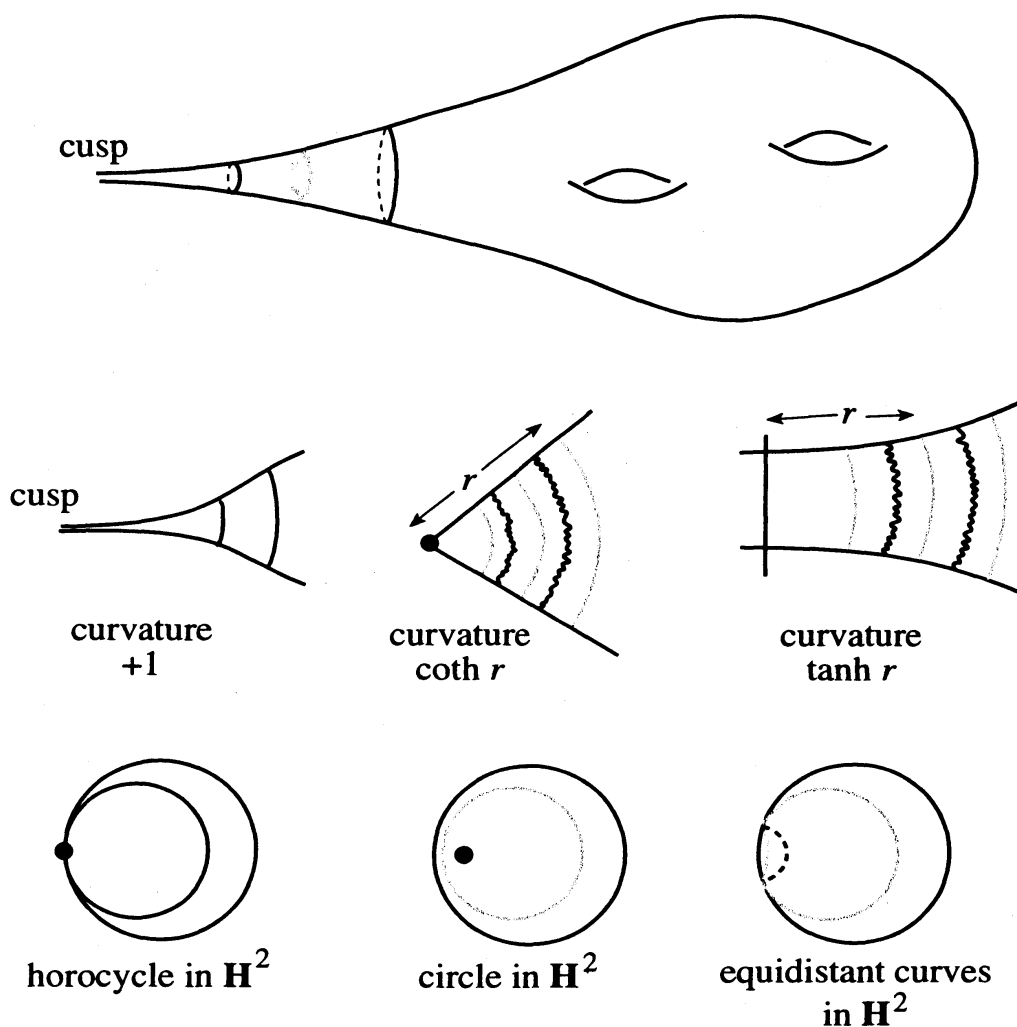
The *ends* of a complete, orientable hyperbolic 3-manifold M with finite volume are *cusps* which are topologically $T^2 \times [0, \infty)$ and are foliated by horospherical tori. They are obtained by dividing out the foliation of \mathbb{H}^3 by horospheres by a $\mathbb{Z} \oplus \mathbb{Z}$ lattice. Each torus has an induced flat metric which decreases exponentially as one moves out in the cusp; all geodesics in this flat metric have geodesic curvature $+1$ in M .



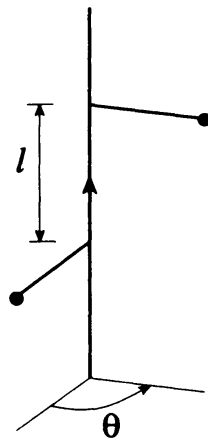
Near a parabolic isometry there are both elliptic and hyperbolic isome-

tries. In dimension 2, the unique fixed point at infinity becomes an interior fixed point or an invariant axis. Annular regions between horocycles develop into regions between equidistant curves.

Notice also that the geometry of equidistant curves varies continuously: the geodesic curvature is $\coth r$ at distance r from a point, $\tanh r$ at distance r from a geodesic axis, and $+1$ on any horocycle.

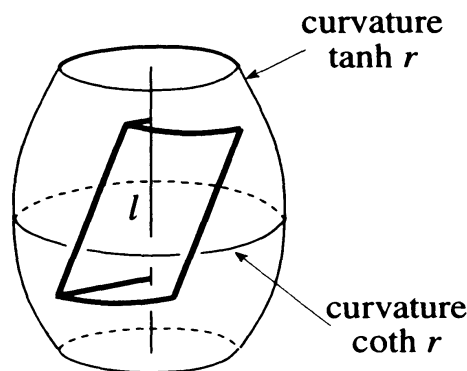


In dimension 3 both elliptic and hyperbolic isometries have an invariant axis. Commuting elements share the same axis. The *complex length*, \mathcal{L} , of such an element of $PSL(2, \mathbb{C})$ is defined to $\ell + i\theta$, where ℓ is the translation distance along the axis and θ is the angle of rotation around the axis. It satisfies $\text{tr} = 2 \cosh(\mathcal{L}/2)$. Thus, a group element is elliptic if and only if $|\text{tr}| < 2$ and tr is real. Further, the angle of rotation θ is related to the trace by $\text{tr} = 2 \cos(\theta/2)$.



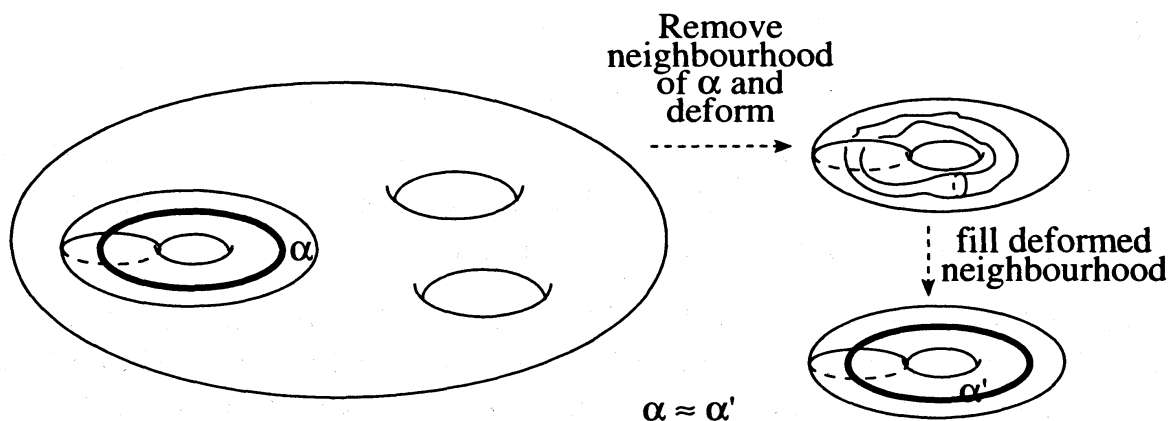
If one deforms the complete structure on the 3-manifold $M = Q - \Sigma$, the region between two parallel horocyclic tori will develop into a region between two equidistant surfaces. The quotient under the holonomy group of the torus will contain a foliation by equidistant tori. If the holonomies of the meridians are elliptic the tori can be filled in to a cone-manifold structure on X_Q .

Again the geometry of the equidistant surfaces varies continuously: the surface at distance r from an axis has an intrinsic flat metric with principal curvatures $\tanh r$ and $\coth r$, while all principal curvatures on a horosphere are $+1$.



Combining this discussion with the general deformation theory of section 5.3 shows that to find a nearby cone-manifold structure, it suffices to find a nearby holonomy representation for which the holonomy of the meridian is elliptic.

To see this, we remove a neighbourhood of the singular locus. The developing image of a neighbourhood of the boundary will lie in a neighbourhood of a nearby axis. We then fill the neighbourhood in to obtain a cone-manifold structure, with the new cone angle.



Exercise 5.4. Extend this discussion to the case where the singular locus contains trivalent vertices.

5.5 Representation spaces

We now want to show that one can always deform the holonomy representation, keeping the meridians elliptic and varying the cone angles independently. First we need to find non-trivial deformations of a hyperbolic structure on M . To do this we estimate the dimension of the representation variety, $\mathcal{R} = \text{Hom}(\pi_1(M), G)$, where $G = \text{PSL}(2, \mathbb{C})$.

Let $\mathcal{R} = \text{Hom}(\Gamma, G)$ denote the set of all representations (homomorphisms) from Γ into G . If Γ is a finitely generated group, and G is a Lie group this has the structure of a real analytic variety (an algebraic variety if G is an algebraic group). If $\gamma_1, \dots, \gamma_g$ is a set of generators for Γ then \mathcal{R} embeds in G^g , via the evaluation map

$$\mathcal{R} \rightarrow G^g, \quad \rho \mapsto (\rho(\gamma_1), \dots, \rho(\gamma_g)).$$

The image is the analytic subset of G^g satisfying the relations of Γ .

We'll be interested in representations $\rho : \Gamma \rightarrow \text{PSL}(2, \mathbb{C})$; in this case we get a *complex* analytic variety. (It becomes a complex algebraic variety, if we lift the representations into $SL(2, \mathbb{C})$ — see [24].)

Proposition 5.5. *If a group π is described in terms of g generators and r relations, the dimension of the variety of representations of π into a complex analytic Lie group G is at least*

$$(g - r) \dim G.$$

Sketch of proof. If π has a presentation

$$\pi = \langle a_1, \dots, a_g \mid w_1 = \dots = w_r = e \rangle,$$

define $\tau : G^g \rightarrow G^r$ by $\{\rho(a_i)\} \mapsto \{\rho(w_j)\}$. Then $\mathcal{R} = \tau^{-1}(e, e, \dots, e)$. Each equation $w_i = e$ is given locally by $\dim G$ complex analytic equations so reduces the complex dimension by at most $\dim G$. \square

For any 3-manifold M with $\partial M \neq \emptyset$, there is a deformation retraction of M to a 2-complex K (obtained by collapsing 3-cells away from free boundary faces, starting at the boundary). This gives a presentation of $\pi_1(M) \cong \pi_1(K)$ with g generators and r relations, where $g - r = 1 - \chi(K) = 1 - \chi(M)$.

For 3-dimensional hyperbolic structures we have $G = PSL(2, \mathbb{C})$ and $\dim_{\mathbb{C}} G = 3$, so we get the estimate

$$\dim_{\mathbb{C}} \mathcal{R} \geq 3(1 - \chi(M)).$$

If the holonomy representation ρ has trivial centralizer

$$Z(\rho) = \{g \in G : g\rho(\gamma)g^{-1} = \rho(\gamma) \text{ for all } \gamma \in \pi_1(M)\}$$

then conjugation determines a 3-complex dimensional subvariety of equivalent structures so we obtain:

$$\dim_{\mathbb{C}} \text{Def}(M) \geq -3(\chi(M)).$$

Exercise 5.6. Show that the centralizer is trivial for any holonomy representation of a finite volume hyperbolic structure.

Exercise 5.7. If M is a compact n -manifold with n odd, then $\chi(M) = \frac{1}{2}\chi(\partial M)$. [Hint: consider the double of M .]

Using this we obtain

$$\dim_{\mathbb{C}} \text{Def}(M) \geq -\frac{3}{2}\chi(\partial M).$$

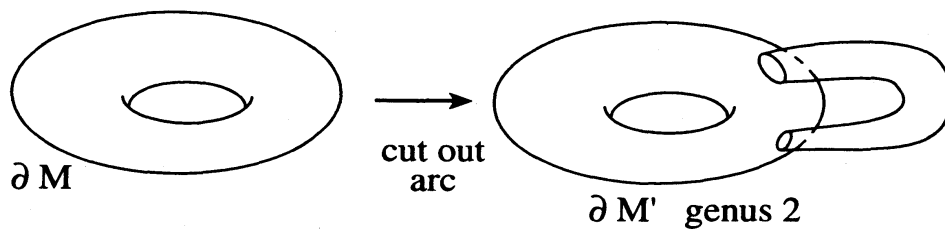
So if ∂M is a union of tori, this only gives 0 as a lower bound. A subtler argument of Thurston (see [80], [22]) gives:

Theorem 5.8. *Let $\rho : \pi_1(M) \rightarrow PSL(2, \mathbb{C})$ be an irreducible representation (i.e. $\rho(\pi_1(M))$ has no fixed point on S_{∞}^2). Then each irreducible component of $\mathcal{R} = \text{Hom}(\pi_1(M), PSL(2, \mathbb{C}))$ containing ρ has complex dimension $\geq 3 - \frac{3}{2}\chi(\partial M) + t$, where t is the number of torus boundary components.*

Hence,

$$\dim_{\mathbb{C}} \text{Def}(M) \geq -\frac{3}{2}\chi(\partial M) + t.$$

Idea of proof. If ∂M consists of a single torus, drill out a (suitable) properly embedded arc from M giving a new manifold M' with $\partial M'$ of genus 2. Then $\dim_{\mathbb{C}} \text{Def}(M') \geq 3$. Thurston shows that we can kill off the fundamental group of a 2-handle to obtain a representation of $\pi_1 M$ by adding just two complex relations: that *two* carefully chosen elements have trace equal to 2. \square



We also need to know the behaviour of holonomies for meridians as representations are deformed. If ∂M consists of t tori, T_i , and γ_i are meridian curves on T_i , define:

$$\text{Tr} : \text{Def}(M) \rightarrow \mathbb{C}^t$$

by

$$\text{Tr}(\rho) = (\text{tr}\rho(\gamma_1), \dots, \text{tr}\rho(\gamma_t))$$

By *Mostow-Prasad rigidity* there is a unique complete hyperbolic structure on M . It follows that the holonomy of the complete structure ρ_0 gives an isolated point in $\text{Tr}^{-1}(\pm 2, \dots, \pm 2)$. Hence the polynomial functions $\text{tr}\rho(\gamma_i)$ are non-constant near ρ_0 , and it can be shown that Tr gives an open mapping whose image contains a neighbourhood of $(\pm 2, \dots, \pm 2)$. (See [22], [23].) In fact, with some additional work it can be shown that $\dim_{\mathbb{C}} \text{Def}(M) = t$ near ρ_0

From the previous section, we conclude that all representations near ρ_0 whose meridians have traces in the open interval $(-2, 2)$ correspond to cone-manifold structures with cone angles α_i given by $\text{tr}\rho(\gamma_i) = 2 \cos(\alpha_i/2)$.

Corollary 5.9. *Suppose that M is a hyperbolic cone-manifold with singular locus a 1-manifold and whose holonomy is on the component of the representation variety containing ρ_0 . If the cone angles of M are α_i , there is $\epsilon > 0$ such that if $|\alpha'_i - \alpha_i| < \epsilon$ for all i then there is a hyperbolic cone-manifold structure on M close to the original structure and with these cone angles. Furthermore the holonomy of this structure is on the same component of the representation variety as ρ_0 .*

The proof of the Orbifold Theorem consists of a study of what can happen at the boundary of the realizable angle set.

Theorem 5.10. (Hodgson-Kerckhoff [45]) *Suppose that N is a closed 3-manifold and L is a closed 1-manifold in N . Finite volume hyperbolic cone-manifold structures on (N, L) (i.e. structures on N with singularities along L) are locally parametrized by the cone angles on the components of L when all cone angles are $\leq 2\pi$.*

The proof of this result uses quite different, analytic techniques: infinitesimal deformations give cohomology classes which can be represented by harmonic forms. These are studied by the use of a Bochner formula and a Fourier series type analysis of asymptotic behaviour of harmonic forms near the singular locus. A survey of this approach is given in [49].

Theorem 5.11. (Kojima [53]) *Suppose that N is a closed 3-manifold and L is a closed 1-manifold in N . Two hyperbolic cone-manifold structures on (N, L) with corresponding cone angles equal are isometric, provided all angles are $\leq \pi$,*

Sketch of proof. The proof uses part of the proof of the Orbifold Theorem, Mostow rigidity for the complete structure, and the local parametrization by cone angle of theorem 5.10. Given two structures, the idea is to decrease the angles to zero, giving complete hyperbolic structures on $N - L$. By Mostow-Prasad rigidity, these structures are equal. By the local parametrization theorem, they were equal throughout the deformation. \square

5.6 Hyperbolic Dehn filling

Given a 3-manifold M with boundary a union of tori T_1, \dots, T_k , and a choice $\gamma = (\gamma_1, \dots, \gamma_k)$ of a non-trivial simple closed curve γ_i on each T_i , one can do γ -Dehn filling on M by attaching a solid torus to each T_i so that γ_i bounds a disk. The result is denoted by $M(\gamma)$ as in section 1.11.

With a choice of generators for each $\pi_1(T_i)$, the γ_i correspond to pairs of relatively prime integers (p_i, q_i) .

In the following discussion, we *assume* there is a single boundary torus $T = T_1$ for simplicity. It is shown in [80], [68] that the complex translation length for elements in the boundary torus can be lifted to \mathbb{C} so that rotation angle is lifted from S^1 to \mathbb{R} , and each parabolic (at the complete structure) has complex length 0. If μ, λ are the complex lengths for a chosen set of generators for $\pi_1(T)$, then the complex length of the (p, q) -curve is $p\mu + q\lambda$. A solution to $p\mu + q\lambda = \alpha i$ near the complete structure gives a cone-manifold structure on $M(p, q)$ with cone angle α ; it is a smooth structure if $\alpha = 2\pi$.

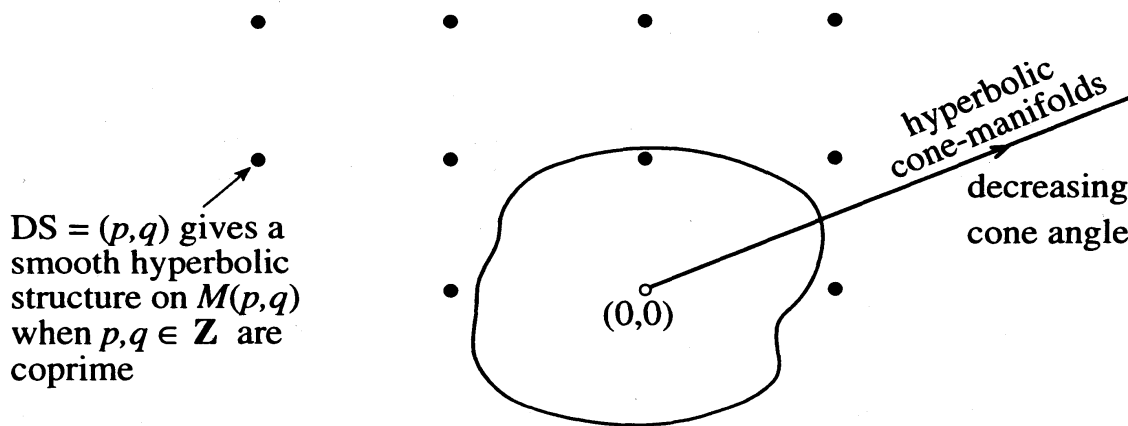
Define

$$DS : \text{Def}(M) \rightarrow (\mathbb{R}^2 \cup \infty) / \pm 1$$

by

$$DS(\rho) = (x, y) \text{ if } x\mu + y\lambda = 2\pi i.$$

Then points along lines of rational slope in $\mathbb{R}^2 \cup \infty$ correspond to hyperbolic cone-manifolds.



Theorem 5.12. (Thurston [80]) *DS maps onto a neighbourhood of ∞ in $\mathbb{R}^2 \cup \infty$. Thus all but finitely many Dehn fillings on M are hyperbolic.*

The number of the *exceptional* (non-hyperbolic) surgeries is not effectively computable from this proof. However, the computer program Snappea developed by Jeff Weeks [88] provides a powerful tool for studying examples, and can estimate the number of exceptional surgeries quickly.

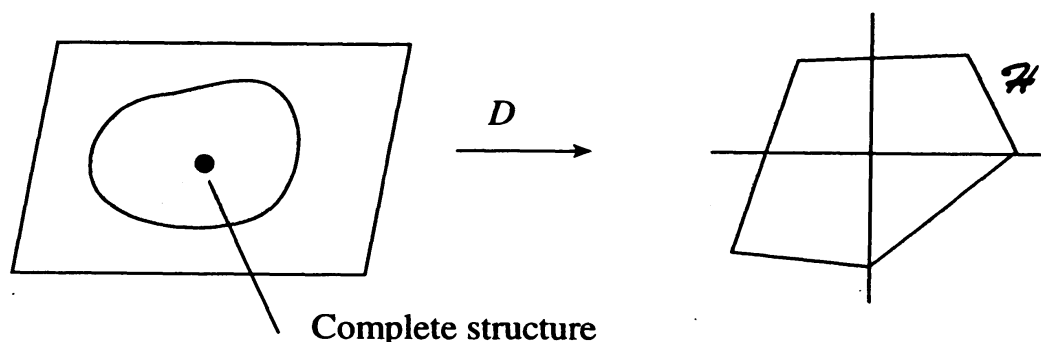
There are universal bounds on the number of Dehn surgeries without *negatively curved* metrics given by the “length 2π ” Theorem of Gromov-Thurston, (see [33], [6], [1]). The recent “length 6” theorem of Agol and Lackenby (see [1], [57]) gives new bounds on the number of surgeries giving manifolds whose fundamental group is not *word hyperbolic*.

Recently, Hodgson-Kerckhoff have obtained the first *universal* bounds on the number of *non-hyperbolic* surgeries (see [46], [49]).

Finally we mention some conjectures on the global structure of hyperbolic Dehn Surgery space.

Conjecture 1. The Dehn surgery coordinate map $DS : \text{Def}(M) \rightarrow (\mathbb{R}^2 \cup \infty) / \pm 1$ should be a (local) diffeomorphism onto its image.

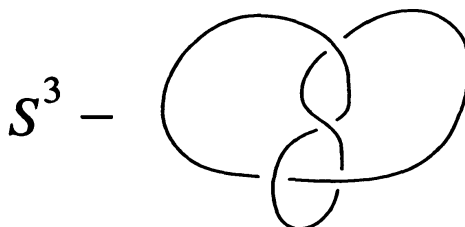
Conjecture 2. Hyperbolic Dehn surgery space, \mathcal{H} , should be star-like with respect to rays from infinity towards the origin. In particular, it should be a connected set.



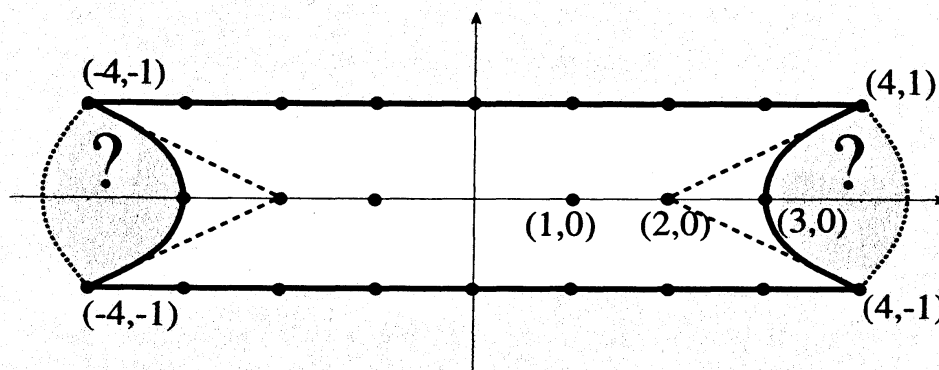
If both conjectures are true, then this implies global (Mostow-Kojima) rigidity for all hyperbolic cone-manifolds. (The proof of theorem 5.11 sketched above for cone angles $\leq \pi$, would again apply.)

5.7 Dehn surgery on the figure eight knot

Thurston's Princeton University notes [80] and Hodgson's thesis [43] include detailed studies of the hyperbolic Dehn surgery space for the complement $M = S^3 - K$ of the figure eight knot K in S^3 .



In the following discussion the Dehn surgery coordinate (p, q) refers to $p\mu + q\lambda$ where μ is a meridian and λ a standard longitude for the figure eight knot. Thurston shows that the lightly shaded region shown below consists of hyperbolic structures obtained by gluing together positively oriented ideal tetrahedra. Hodgson shows that the "algebraic volume" associated with representations into $PSL(2, \mathbb{C})$ goes to zero along the solid curve shown below. This curve consists of straight line segments corresponding to representations into $\text{Isom}(\mathbb{H}^2)$ and curves corresponding to representations into $SO(3)$. It is conjectured that this represents the true boundary of the hyperbolic Dehn surgery space, but currently hyperbolic structures with Dehn surgery type singularities are only known for some special points within the darkly shaded region.



On the boundary of the hyperbolic region degenerations of the following kinds occur.

(1) Dehn surgery coordinates $(m, 1)$, $-4 < m < 4$.

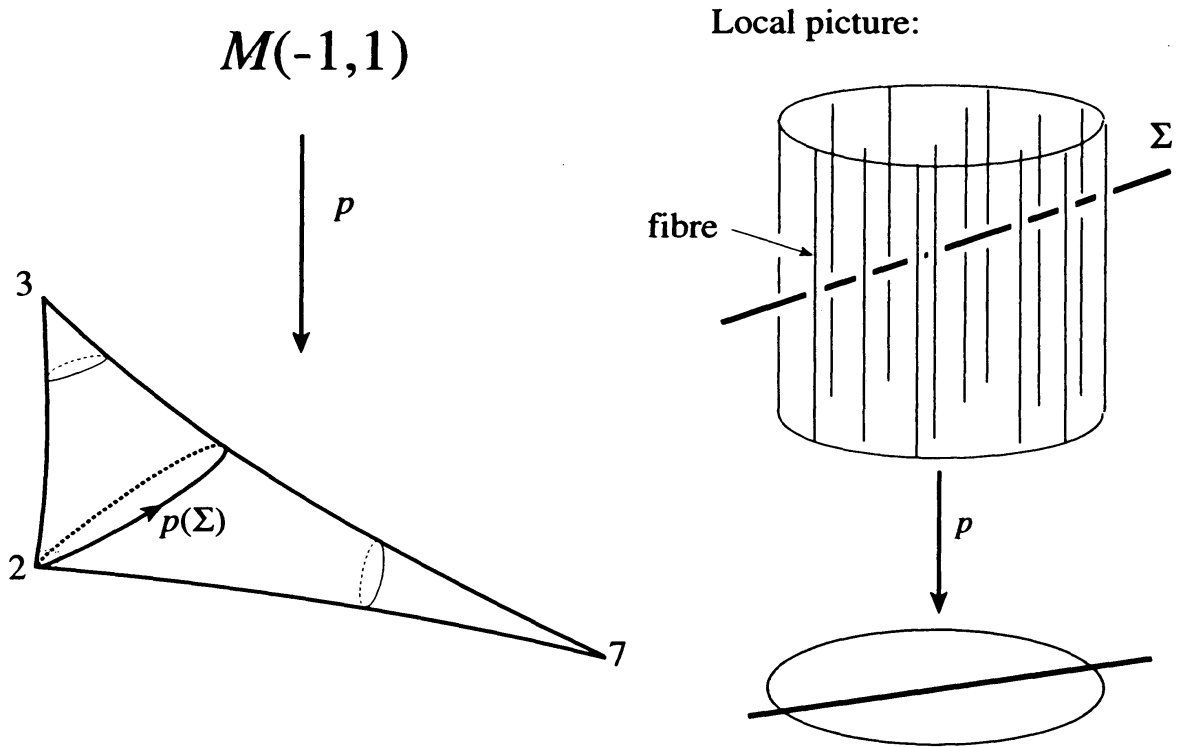
Here there are limiting representations $\pi_1(M) \rightarrow PSL(2, \mathbb{R})$, corresponding to foliations with transverse hyperbolic structures. The foliations can be seen directly, since we have two positively oriented simplices flattening out simultaneously. It was shown explicitly in [80] that this is part of the exact boundary of hyperbolic Dehn surgery space. The manifold points in the boundary are as follows. (Note that $M(p, q)$ and $M(-p, q)$ are oppositely oriented copies of the same manifold, since the figure eight knot has an orientation reversing symmetry.)

(a) The manifold $M(0, 1)$ is a torus bundle over S^1 with Anosov gluing map with matrix $\Phi = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$; this gives rise to a Solv geometry structure.

Here there are hyperbolic cone-manifold structures on $M(0, 1)$ for cone angles $\theta < 2\pi$ which collapse as $\theta \rightarrow 2\pi$ to a circle whose length is $\log \lambda$, where $\lambda > 1$ is the larger eigenvalue of the matrix Φ . (See [43], [41], [78].)

(b) Each manifold $M(n, 1)$ for $n = \pm 1, \pm 2, \pm 3$ is a Seifert fibre space over a hyperbolic 2-orbifold which is sphere with 3 cone points. There are hyperbolic cone-manifold structures on $M(n, 1)$ for cone angles $\theta < 2\pi$ which collapse as $\theta \rightarrow 2\pi$ to the 2-dimensional hyperbolic structure on the base orbifold. The singular locus is transverse to the fibres of the Seifert fibration and projects to a geodesic in the base orbifold.

For example, $M(\pm 1, 1)$ is the unit tangent bundle of the $(2, 3, 7)$ spherical orbifold (see example 2.39) and the singular locus Σ is the horizontal lift of a geodesic through the order 2 cone point (double covering a geodesic in base):



Similarly, $M(\pm 2, 1)$ is the unit tangent bundle to $S^2(2, 4, 5)$, and $M(\pm 3, 1)$ is the unit tangent bundle to $S^2(3, 3, 4)$.

Each manifold $M(\pm 4, 1)$ is a graph manifold containing an incompressible torus which splits the manifold into the union of a trefoil knot complement and the non-trivial I -bundle over the Klein bottle. In this case, the hyperbolic cone-manifold structures for cone angles $\theta < 2\pi$ split along an essential torus and collapse to give a limiting cusped Seifert fibred structure on the complementary pieces. (Compare example 3.4.)

Remark: These non-hyperbolic manifolds resulting from Dehn surgery on the figure eight knot can be identified using the Kirby calculus (see [43]), or via the “Montesinos trick” which divides out by a 180 degree rotational symmetry and studies the quotient orbifold. (For examples of this technique see [6].)

(2) There are orthogonal representations corresponding to Dehn surgery coordinates on a curve from $(3.618\dots, 0.809\dots)$ passing through $(3, 0)$ to $(3.618\dots, -0.809\dots)$.

Special case: The orbifold $M(3, 0)$ has a Euclidean structure described earlier (see 2.33). In this case there are hyperbolic cone-manifold structures for cone angles $\theta < 2\pi/3$ which collapse to a point as $\theta \rightarrow 2\pi/3$. After rescaling the hyperbolic metrics, these converge to the Euclidean orbifold structure with cone angle $2\pi/3$. Further, there are spherical cone-manifold

structures for $2\pi/3 < \theta \leq \pi$. All of these cone-manifold structures can be constructed directly from suitable polyhedra by identifying pairs of faces by isometries — see [41], [78].

Hodgson's work in [43] implies that near this point, the curve where $volume = 0$ corresponds to Euclidean structures with Dehn surgery singularities, and is *locally* the boundary of hyperbolic Dehn surgery space. (See also [67].) Furthermore, each Euclidean structure can also be approximated by spherical structures with Dehn surgery type singularities. If we consider the larger space $GS(M)$ of all constant curvature geometric structures on M with Dehn surgery type singularities; then we locally obtain a manifold. The Dehn surgery coordinates give a local diffeomorphism from geometric structures in $GS(M)$, up to rescaling of metrics, to a neighbourhood of $(3, 0)$ in \mathbb{R}^2 . The Euclidean structures correspond to the codimension one subspace, where $volume = 0$.

(3) Dehn surgery coordinates on the straight line from $(3.618\dots, .809\dots)$ to $(4, 1)$ and on the straight line from $(-3.618\dots, 0.809\dots)$ to $(-4, 1)$ also correspond to representations into $\text{Isom}(\mathbb{H}^2)$ since simplices are flat. But little is currently known about the geometric meaning of these representations.

Remark: In the proof of the orbifold theorem, we will encounter analogues of all the kinds of degeneration of hyperbolic structures described for the figure eight knot complement. However, the Seifert fibre spaces arising will be orbifolds which have both intervals and circles as fibres.