Chapter 4

Two-dimensional Cone-Manifolds

In this chapter, we discuss the classification of 2-dimensional geometric cone-manifolds. The main ingredients needed for the classification are the Dirichlet domains and Gauss-Bonnet theorem for cone-manifolds 3.15 discussed in the previous chapter, and the developing map and holonomy representation.

4.1 Developing map and holonomy

Let M be a constant curvature cone-manifold. Although M itself has singularities, if we remove the singular set, Σ, the remainder, $M - \Sigma$, has a smooth (G, X) structure. In particular, can still define the *developing map* on the universal cover $\widetilde{M - \Sigma}$ of $M - \Sigma$:

$$\text{dev} : \widetilde{M - \Sigma} \to X$$

There also exists a *holonomy* homomorphism

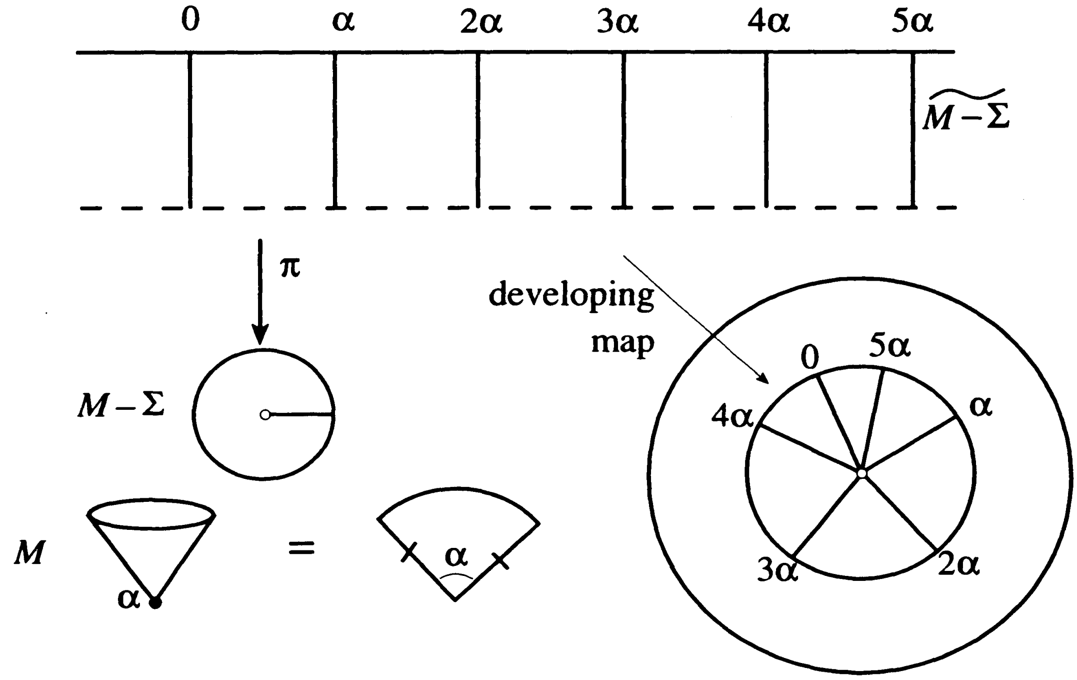
$$h : \pi_1(M - \Sigma) \to G$$

satisfying $\text{dev} \circ \gamma = h(\gamma) \circ \text{dev}$ for each deck transformation γ in $\pi_1(M - \Sigma)$.

However, the difference between this case and that for a compact *manifold* is that the holonomy group need not be discrete and the developing map need not be a diffeomorphism. In particular this means that neither M nor $M - \Sigma$ can be described as X/Γ, where Γ is the image of the holonomy homomorphism.

The key property that distinguishes the two cases is that of *completeness.* When N is a complete constant curvature manifold, then dev : $\tilde{N} \to X$ is a covering map, hence a diffeomorphism. However, if $N = M - \Sigma$, then it will be *incomplete.* Even though dev is a local diffeomorphism, it will not satisfy the unique path lifting property.

Example 4.1. Developing map for a cone with angle α.



The element of the holonomy group corresponding to a loop linking a component of the singular locus is a rotation by angle α, where α is the cone angle of that component.

4.2 Two-dimensional spherical cone-manifolds

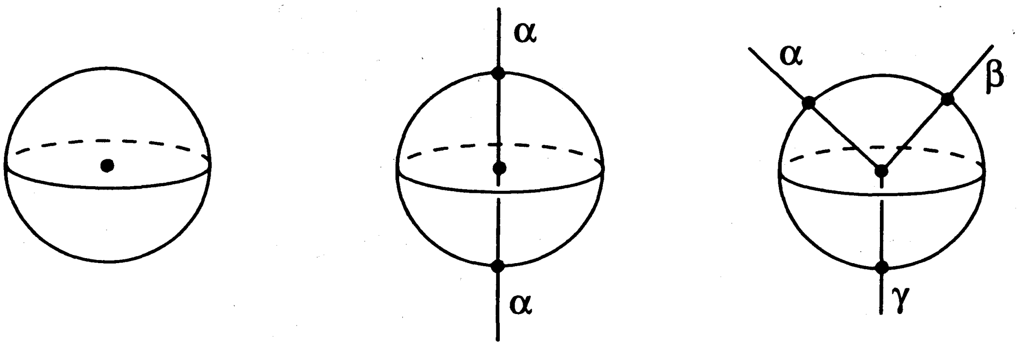
Proposition 4.2. *A 2-dimensional orientable spherical cone-manifold M with cone angles $\leq \pi$ is a 2-sphere with 0, 2 or 3 cone points. The metrics are described as follows:*

(i) 0 cone points: $M = S^2$

(ii) 2 cone points: A spherical football, *with two equal cone angles α. (M is the double of lune of angle $\alpha/2$)*

(iii) 3 cone points: A spherical turnover, *with cone angles α, β, γ such that $\alpha + \beta + \gamma > 2\pi$. (M is the double of a spherical triangle with angles $\alpha/2, \beta/2, \gamma/2$.)*

Note: The local models for 3-dimensional cone-manifolds with cone angles $\leq \pi$ are cones on these.



Proof. Note that a spherical cone-manifold with all cone angles less than 2π is compact. This follows from the fact that every minimal length geodesic has length no more than π ($\frac{\pi}{\sqrt{K}}$ for curvature K). Thus the diameter of the Dirichlet domain is at most 2π and its closure is compact.

The Gauss-Bonnet theorem 3.15 gives

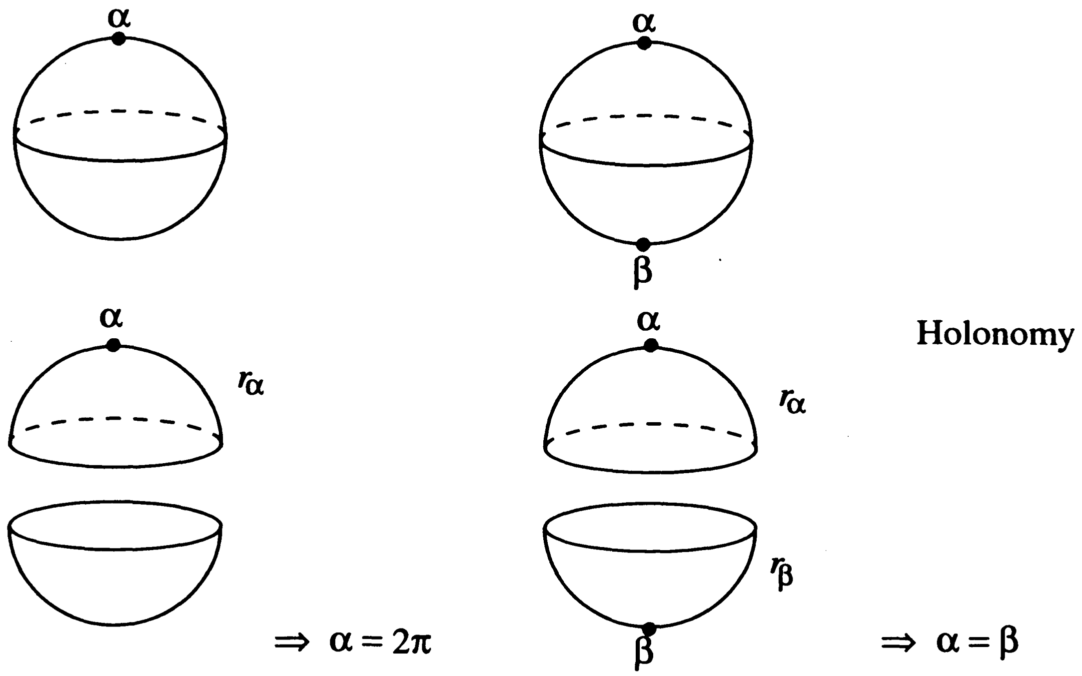
$$\int_M K dA + \sum_i (2\pi - \theta_i) = 2\pi\chi(M),$$

where $\theta_1, \dots, \theta_k$ are the cone angles. Since the curvature $K = 1$ and the cone angles are at most π it follows that $\chi(M) > 0$, thus $M = S^2$ and $\sum_i (2\pi - \theta_i) < 2\pi\chi(S^2) = 4\pi$. Since the cone angles are at most π, there are at most 3 cone points.

The developing map $\mathrm{dev} : (\widetilde{M - \Sigma}) \to S^2$ induces a holonomy representation $h : \pi_1(M - \Sigma) \to SO(3)$. The holonomy of a loop around a cone point is a rotation r_θ by an angle equal to the cone angle θ. Using this fact, one sees that

(i) it is impossible to have a single cone point.
(ii) if there are two cone points, then the two cone angles are equal.
(iii) if there are three cone angles α, β, γ, then the cone angles are twice the angles of some spherical triangle.

The figure below proves (i) and (ii). To see (iii), observe that $\pi_1(M - \Sigma)$ has a presentation $\langle A, B, C \mid AB = C \rangle$, where A, B, C are meridians. Given the fixed points of the rotations A and B we can construct the product $C = AB$ as follows. Write $hol(A) = R_1 \circ R_2$, $hol(B) = R_2 \circ R_3$, where R_i are reflections with axis(R_2) joining Fix(A) to Fix(B); axis(R_1) through Fix(A) at angle $\alpha/2$ to axis(R_2); and axis(R_3) through Fix(B) at angle $\beta/2$ to axis(R_2). Then one sees immediately that $hol(C) = R_1 \circ R_3$ is a rotation through twice the third angle of the spherical triangle formed by the axes.

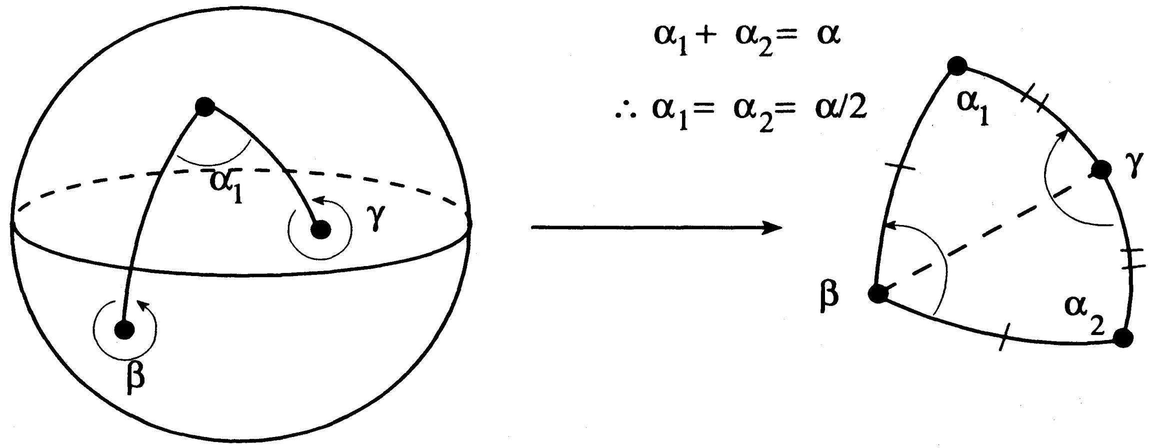


This shows that the *holonomy* has one of the forms indicated in the proposition, but we don't yet know the cone-manifold structure has the form claimed. To see this we show that M is a obtained from a convex spherical polygon and therefore has a "standard" structure.

If there are two cone points choose a geodesic joining them. Cut M along this geodesic to obtain D. Then D is simply connected so the developing map sends D to a region in the sphere bounded by two geodesics. The angle between these geodesics is the cone angle at each cone point. Since this cone angle is at most π the developing map embeds D. Hence D is isometric to a lune and M is a football.

If there are three cone angles α, β, γ, then take minimal geodesics from one of the singular points to each of the other two. These are disjoint; otherwise we can cut, paste and straighten to get a shorter geodesic.

The complement of these two geodesics is simply connected and thus maps into S^2 via the developing map. As before this map is an embedding. Its image in S^2 is bounded by two copies of each geodesic. From the figure we see that $\alpha_1 = \alpha_2$.



This implies that the region consists of two isometric triangles. The boundary of each triangle consists of the images of the two geodesics from x, together with another geodesic connecting the other two singular points. □

A 2-dimensional spherical cone-manifold is an orbifold if and only if the cone angle at each cone point is of the form $2\pi/n$ for some integer $n \geq 2$. Then it is the quotient of S^2 by a discrete group of isometries by theorem 2.26 (Poincaré's theorem).

Since a neighbourhood of a point in a 3-dimensional cone-manifold M is a cone on a 2-dimensional spherical cone-manifold it follows that the underlying topology of $(M, \Sigma(M))$ is the same as that of an orbifold when the cone angles in M are at most π. However this is no longer the case when cone angles larger than π are allowed. For example take the cone on the double of any spherical polygon.

Exercise 4.3. Extend proposition 4.2 to the non-orientable case.

4.3 Two-dimensional euclidean cone-manifolds

We can classify 2-dimensional Euclidean cone-manifolds by the techniques we used for spherical cone-manifolds.

Proposition 4.4. *A complete (orientable) 2-dimensional Euclidean cone-manifold C with cone angles $\leq \pi$ is one of the following:*
(a) compact
(i) a Euclidean turnover *(the double of an acute angled Euclidean triangle)*
(ii) a pillowcase *(the quotient of a Euclidean torus by an isometric involution with 4 fixed points as in example 2.1)*

   (iii) a torus.

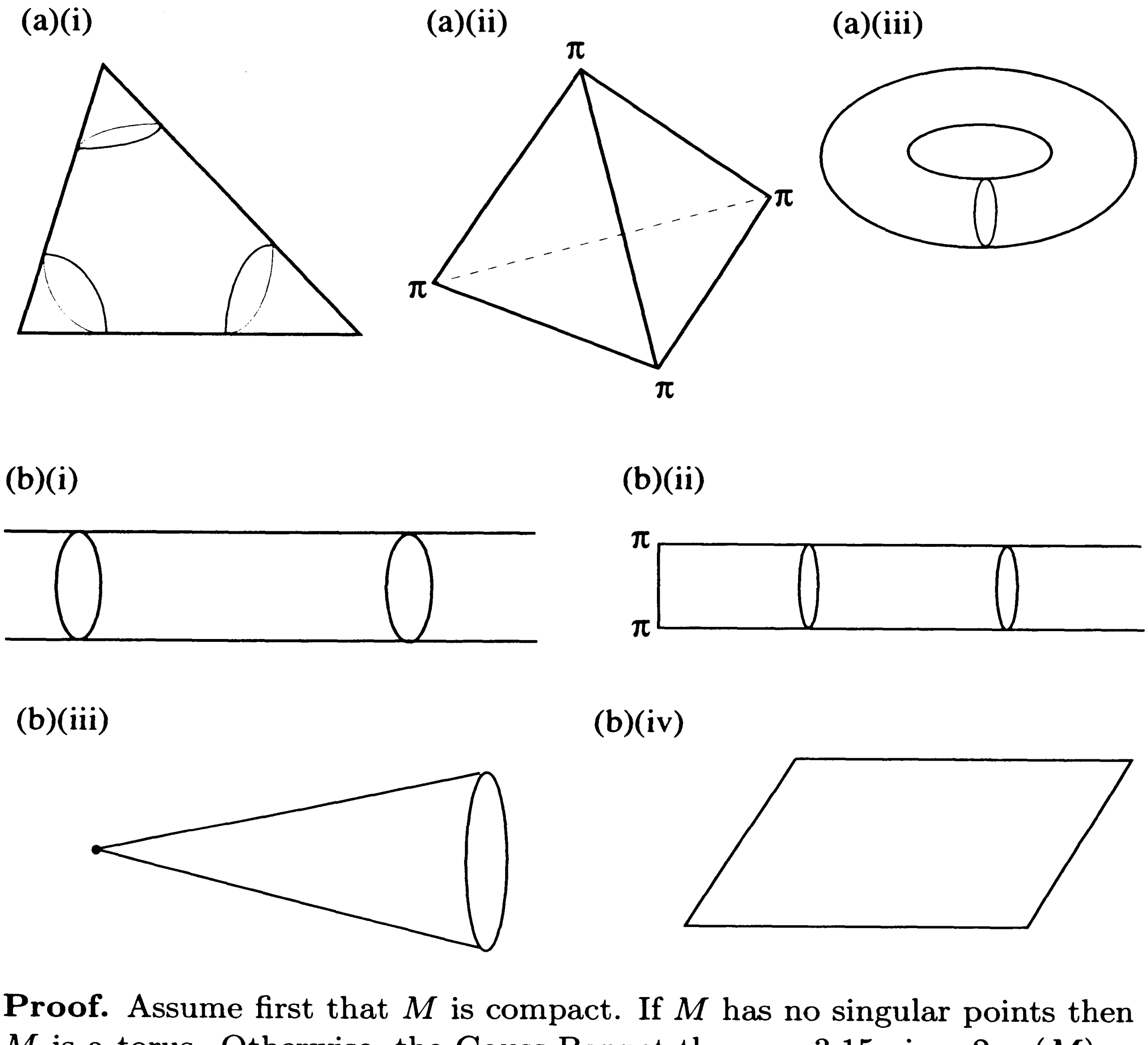
(b) non-compact

   (i) infinite cylinder,

   (ii) infinite pillowcase,

   (iii) infinite cone,

   (iv) plane.

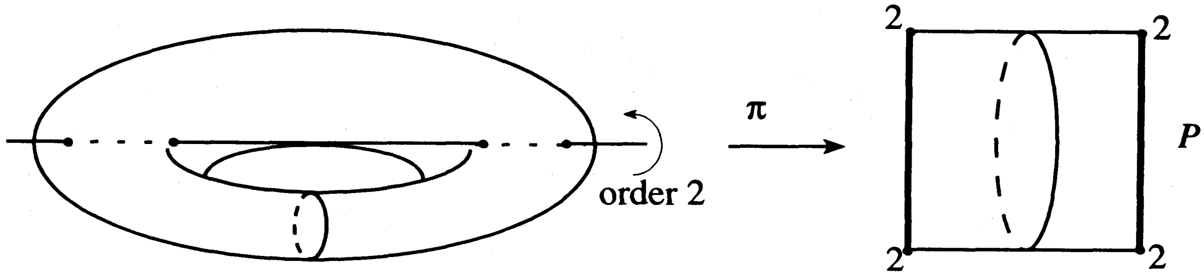


Proof. Assume first that M is compact. If M has no singular points then M is a torus. Otherwise, the Gauss-Bonnet theorem 3.15 gives $2\pi\chi(M) = \sum(2\pi -$ cone angle$)$. Hence $\chi(M) > 0$ so M is a sphere, and there are 3 or 4 cone points. When there are 3 cone points the same argument as in the spherical case shows that it is the double of a triangle.

When there are 4 cone points, they will all have angle π and M is a geometric orbifold with group $\mathbb{Z}_2$ at each singular point. The orbifold fundamental group has a presentation

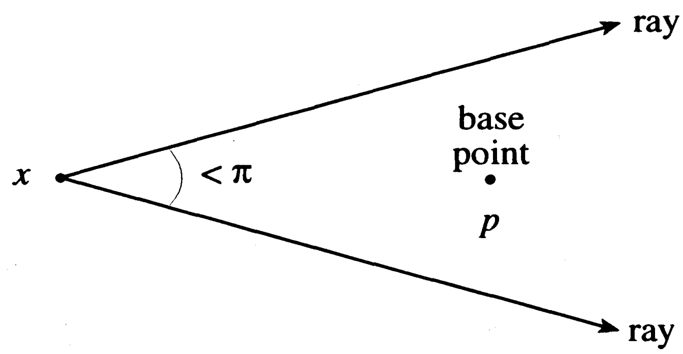
$$\langle a, b, c, d \mid a^2 = b^2 = c^2 = d^2 = abcd = 1 \rangle,$$

as in example 2.10. The kernel of the map to $\mathbb{Z}_2$ sending each generator to 1 determines a 2-fold regular orbifold covering space which is a torus.

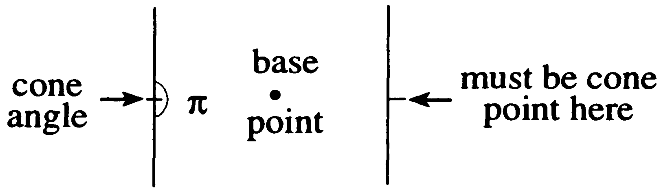


Suppose M is non-compact. If M has no cone points then M is an infinite cylinder or plane. If there is a cone point x, then since M is non-compact there is a (length minimizing) ray γ starting at x and going out to infinity. (Choose a sequence of points in in M going out to infinity. Join these points to the base point; then the limiting position of the line segments gives a ray.) We claim that by choosing a suitable basepoint, p, there is a Dirichlet domain P for M with γ contained in ∂P. To see this there is a neighbourhood U of x which is a standard cone. Then $U \cap \gamma$ is a geodesic. Choose p in U close to x and symmetric with respect to γ. Clearly the cut locus for p inside U is $\gamma \cap U$. If γ were not contained in ∂P this would contradict that γ is length minimizing.

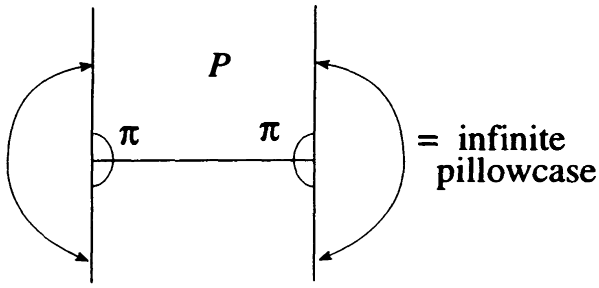
We now regard P as a convex subset of the plane $\mathbb{E}^2$. Observe that ∂P contains two copies of γ meeting at an angle equal to the cone angle at x.



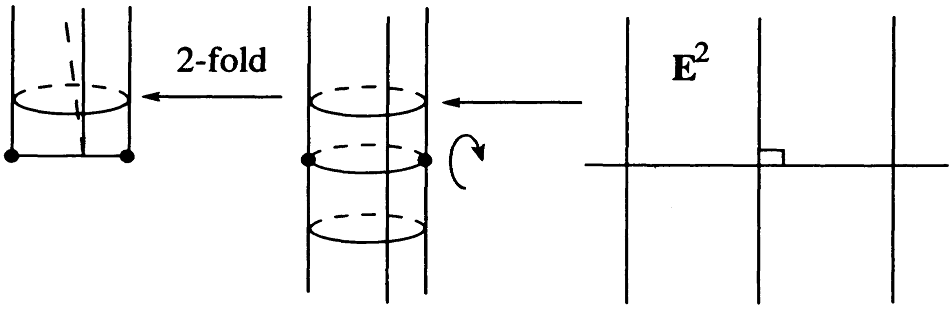
If the cone angle is $< \pi$, P contains the entire wedge between the rays by convexity; so M is a cone. If the angle is π, then ∂P contains a line L made up of two copies of γ. Hence either P is a half-plane and M is a cone, or P is an infinite strip lying between two parallel lines.



M is obtained from P by isometric identifications on ∂P. If P is a strip, both lines must be identified to themselves by an order 2 rotation. The order 2 points must both lie on a common perpendicular to the two parallel lines in ∂P, otherwise γ would not be length minimizing. Hence M is an infinite pillowcase. □



Remark: The infinite pillowcase has a geodesic connecting the cone points. It is the image of an affine subspace invariant under the holonomy group. The orthogonal affine subspaces are also invariant, forming the orbifold normal bundle to the geodesic.



Exercise 4.5. Extend proposition 4.4 to the non-orientable case.

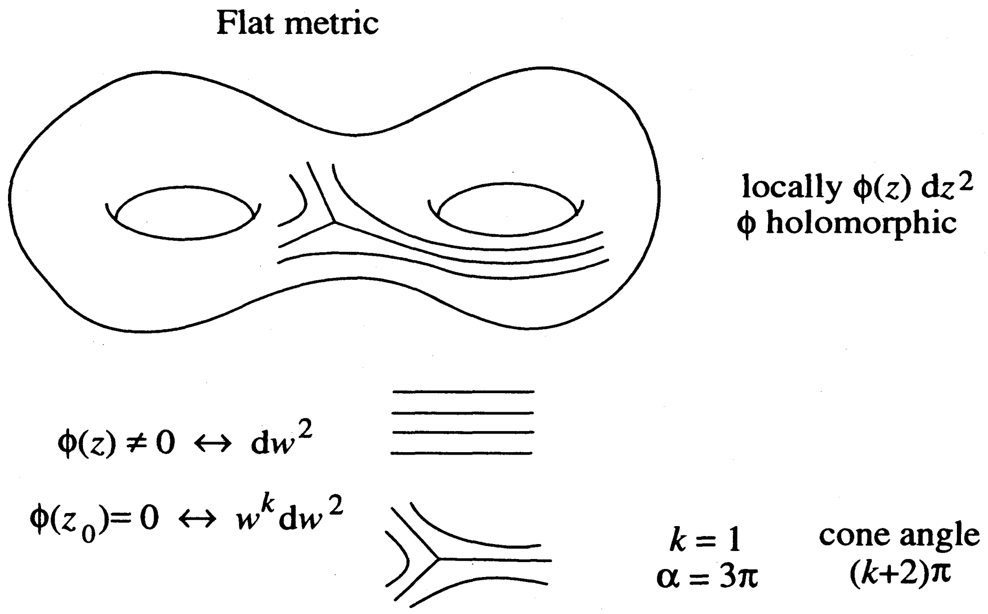
Exercise 4.6. Show that every pillowcase is isometric to some Euclidean tetrahedron which has opposite edges of equal length.

4.4 Euclidean examples with large cone angles

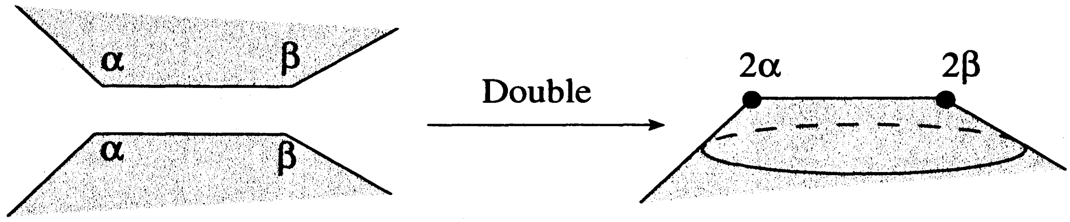
If the cone angles are allowed to be greater than 2π, the topology, even in the compact case, is uncontrolled, i.e. the genus can be anything. Such Euclidean metrics arise naturally from *quadratic differentials* on a Riemann surface of genus $g \geq 2$ (see Strebel [77, chapter 2]). A quadratic differential can be written locally as $\omega = \phi(z)\, dz^2$ where z is a local complex coordinate on the surface and ϕ is a holomorphic function. Then

$$ds = |\sqrt{\omega}| = |\phi(z)|^{1/2}|dz|$$

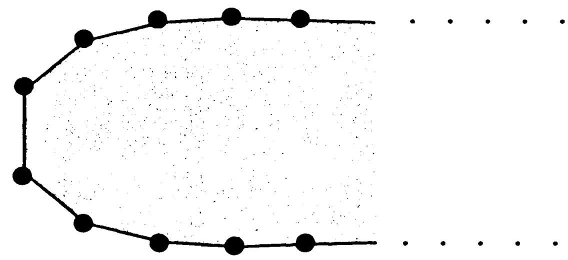
gives a well-defined Euclidean metric with cone points at the zeros of ϕ.



If the cone angles are allowed to be $\pi < \alpha < 2\pi$ and the space is compact then the Gauss-Bonnet theorem implies that the underlying space is still a sphere, but the number of singular points is unbounded.



In the non-compact Euclidean case, with $\pi < \alpha < 2\pi$, there are infinitely many examples with infinitely many cone points going out to infinity. These examples are not either a bundle or a cone.



4.5 Spaces of cone-manifold structures

The previous theorems describe the topological structure of certain cone-manifolds. What parameters are there in their geometric structures? Smooth (G, X)-structures on a manifold N are locally parametrized by their holonomy representations $\pi_1(N) \to G$, up to conjugacy. This will be discussed in more detail in chapter 5.

We can consider a geometric cone-manifold structure on M as an incomplete smooth structure on $M - \Sigma$ whose metric completion is M. The holonomy along a *meridian curve* linking one component of Σ determines the cone angle on that component, as in example 4.1.

Example 4.7. The footballs and spherical turnovers described in Proposition 4.2 are determined by the number of singular points and their cone angles. Similarly for the Euclidean turnovers. The corresponding holonomy representations are either abelian (2 singular points) or of the form

$$\langle A, B, C \mid AB = C \rangle$$

where A, B, C are all rotations of the model space X.

Example 4.8. The *non-compact* Euclidean cone-manifolds of Proposition 4.4 are parametrized by length of closed geodesic in cases (i) and (ii), or by angle in case (iii).

Example 4.9. Euclidean tori and pillowcases are parametrized by lattices in the plane (up to isometry). Representations of $\pi_1(torus) = \mathbb{Z} \oplus \mathbb{Z}$ are determined by two translation vectors; the pillowcase group is a $\mathbb{Z}_2$ extension of the torus group.

4.6 Two-dimensional hyperbolic cone-manifolds

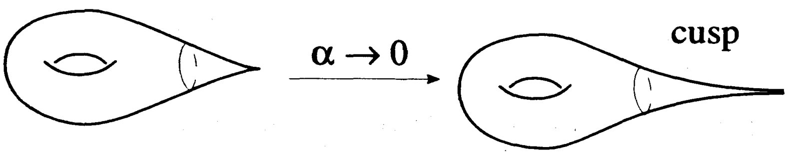
The uniformization theorem shows that compact two dimensional hyperbolic *surfaces* are parametrized by their underlying conformal structures.

For a closed surface of genus g, the space of these structures has dimension $6g - 6$.

Two dimensional hyperbolic *cone-manifolds* are parametrized by their underlying conformal structures (including position of singular points) and cone angles. Subject to the restriction on cone angles from the Gauss-Bonnet theorem 3.15, all possible angles and conformal structures occur. (See Troyanov [85], McOwen [60].) For fixed genus and number of cone points, the angle constraint determines a convex set of angles including the origin:

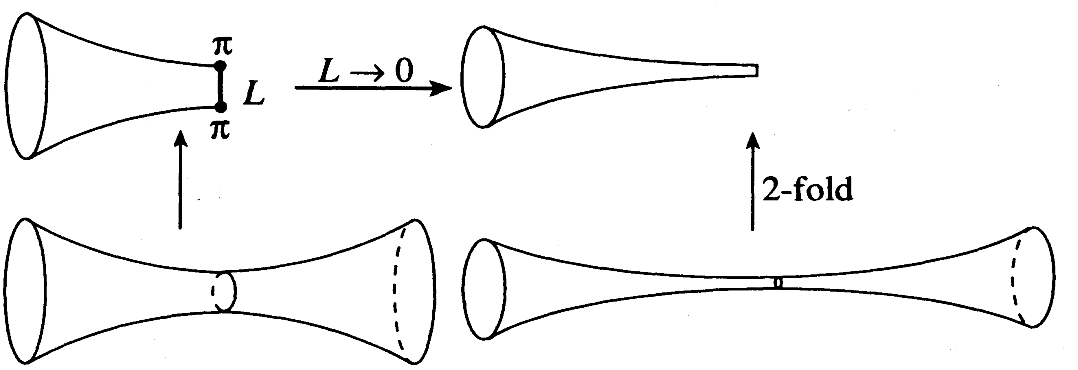
$$\sum_i (2\pi - \alpha_i) > 2\pi\chi.$$

Note that a *cusp* can arise as limit of cone points as cone angle $\alpha \to 0$.

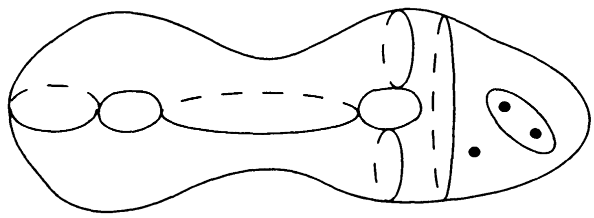


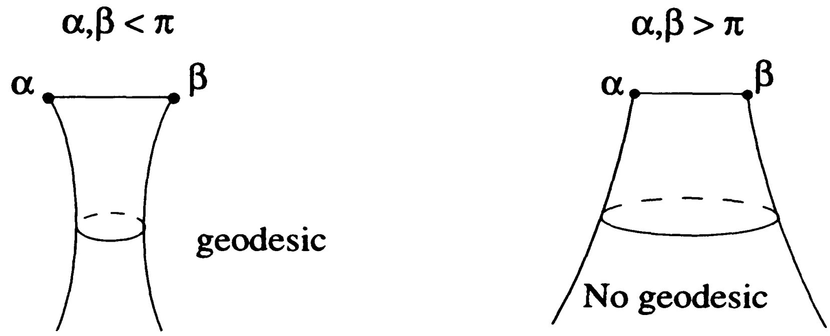
Finally we mention some of the complications that arise when cone angles $\geq \pi$ are allowed.

Cone points can't collide when $\alpha < \pi$. However, two angle π cone points can become arbitrarily close together; for example, approximating an infinite pillowcase cusp.



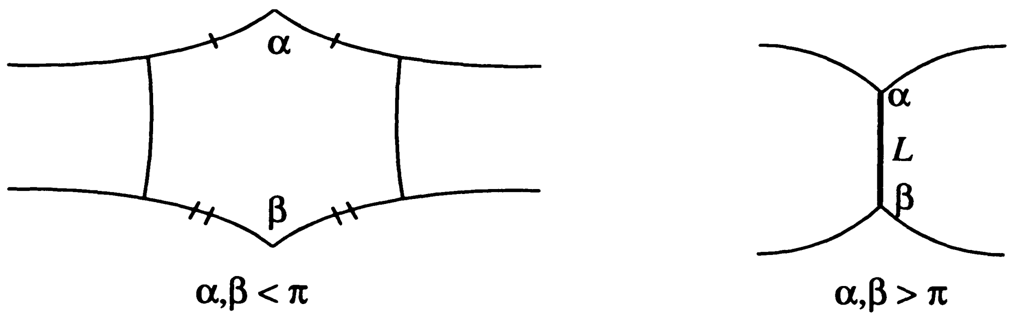
When all cone angles are $< \pi$, there is a pair of pants decomposition, obtained by surrounding pairs of singular points by a geodesic. (The resulting "pants" may have some boundary curves replaced by cone points.) When angles are between π and 2π, the shortest curve will pass through the singular points.

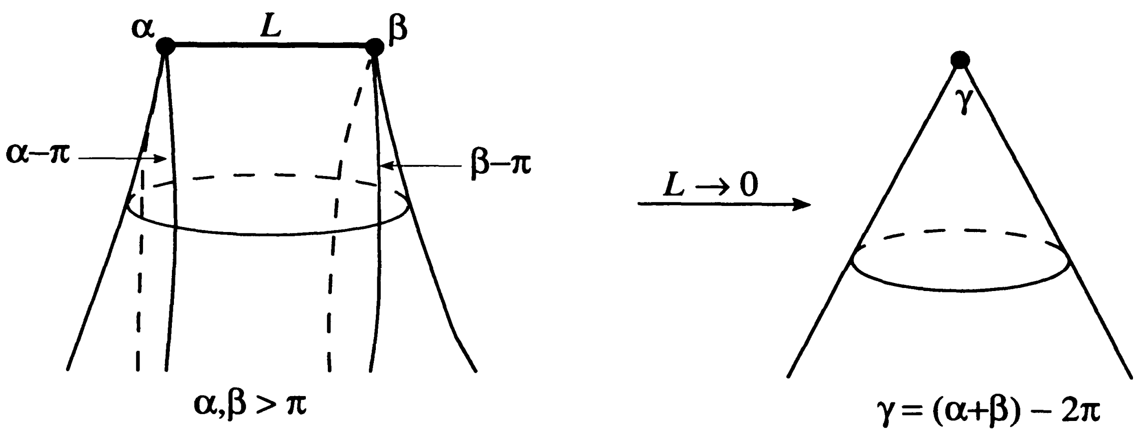




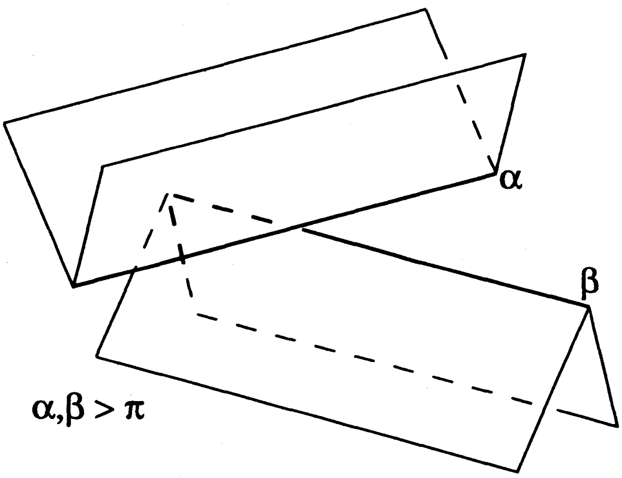
If cone angles are only restricted to the interval $(0, 2\pi)$, then two cone points of angles α, β can collide without creating a thin collar. They just become a single cone point of angle $\alpha + \beta - 2\pi$.

Fundamental domains





In dimension 3 the topology of such a collision is much more complicated (a link becomes a graph with vertices). This is a key reason why cone angles at most π are much easier to deal with than cone angles less than 2π.



In three dimensions two circle components of Σ can collide when cone angles are $\geq \pi$