Part IV

Appendix

A Vertex superalgebras

A.1 $\mathbb{Z}_2$-graded vector spaces

A vector space M with a direct sum decomposition $M = M_{\bar{0}} \oplus M_{\bar{1}}$ is called a $\mathbb{Z}_2$-graded vector space. Elements of $M_{\bar{0}}$ are called even whereas those of $M_{\bar{1}}$ odd. We set

$$p(v) = \begin{cases} 0, & \text{if } v \text{ is even,} \\ 1, & \text{if } v \text{ is odd.} \end{cases}$$

For any $v \in M$, let v' (resp. v'') be the even (resp. odd) part of v: $v = v' + v''$ where $p(v') = 0$ and $p(v'') = 1$. We will abbreviate $(-1)^{p(v)}$ by $(-1)^v$, $(-1)^{p(u)p(v)}$ by $(-1)^{uv}$ and so on.

For a $\mathbb{Z}_2$-graded vector space $M = M_{\bar{0}} \oplus M_{\bar{1}}$, the space $\operatorname{End} M$ is canonically $\mathbb{Z}_2$-graded by setting

$$(\operatorname{End} M)_{\bar{i}} = \{a \in \operatorname{End} M \mid a(M_{\bar{j}}) \subset M_{\bar{i}+\bar{j}}\},$$

namely, by setting $a'(v) = a(v')' + a(v'')''$ and $a''(v) = a(v')'' + a(v'')'$.

The supercommutator of $a, b \in \operatorname{End} M$ is defined by

$$[a, b] = (a'b' - b'a') + (a'b'' - b''a') + (a''b' - b'a'') + (a''b'' + b''a'').$$

We will simply write this as $[a, b] = ab - (-1)^{ab} ba$. Then we have $[b, a] = -(-1)^{ab}[a, b]$ and the Jacobi identity

$$[[a, b], c] = [a, [b, c]] - (-1)^{ab}[b, [a, c]],$$

which is also written as

$$(-1)^{ca}[a, [b, c]] + (-1)^{ab}[b, [c, a]] + (-1)^{bc}[c, [a, b]] = 0.$$

Let M and N be $\mathbb{Z}_2$-graded vector spaces. The tensor product $M \otimes N$ of vector spaces is canonically $\mathbb{Z}_2$-graded by setting

$$(M \otimes N)_{\bar{0}} = (M_{\bar{0}} \otimes N_{\bar{0}}) \oplus (M_{\bar{1}} \otimes N_{\bar{1}}), \quad (M \otimes N)_{\bar{1}} = (M_{\bar{0}} \otimes N_{\bar{1}}) \oplus (M_{\bar{1}} \otimes N_{\bar{0}}).$$

The tensor product of $a \in \operatorname{End} M$ and $b \in \operatorname{End} N$ is defined by

$$(a \otimes b)(u \otimes v) = (-1)^{bu} a(u) \otimes b(v).$$

A.2 Mutually local fields on a $\mathbb{Z}_2$-graded vector space

Let $M = M_{\bar{0}} \oplus M_{\bar{1}}$ be a $\mathbb{Z}_2$-graded vector space. Then the space of fields on M is $\mathbb{Z}_2$-graded according to the $\mathbb{Z}_2$-grading of $\operatorname{End} M$.

Turning to the $\mathbb{Z}_2$-graded case, we just replace the commutator by the supercommutator in the definition of the residue products and the locality. Thus the m-th residue product of fields $A(z)$ and $B(z)$ is defined by

$$A(z)_{(m)}B(z) = \operatorname*{Res}_{y=0}\,[A(y), B(z)](y-z)^m,$$

and two fields $A(z)$ and $B(z)$ are said to be mutually local if

$$[A(y), B(z)](y-z)^n = 0$$

holds for some nonnegative integer n. Then the statements in Section 1-3 remain valid provided the skew symmetry is replaced by

$$\left(B(z)_{(m)}A(z)\right) = (-1)^{A(z)B(z)}\sum_{i=0}^{\infty}(-1)^{m+i+1}\partial^{(i)}\left(A(z)_{(m+i)}B(z)\right)$$

and the Borcherds identity by

$$\begin{aligned}&\sum_{i=0}^{\infty}\binom{p}{i}\left(A(z)_{(r+i)}B(z)\right)_{(p+q-i)}C(z)\\ &= \sum_{i=0}^{\infty}(-1)^i\binom{r}{i}\Big(A(z)_{(p+r-i)}(B(z)_{(q+i)}C(z))\\ &\qquad\qquad -(-1)^{A(z)B(z)+r}B(z)_{(q+r-i)}(A(z)_{(p+i)}C(z))\Big).\end{aligned}$$

A.3 Vertex superalgebras

Definition A.3.1. A vertex superalgebra is a $\mathbb{Z}_2$-graded vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$ equipped with countably many binary operations

$$\begin{array}{ccc} V \times V & \longrightarrow & V \\ (a,b) & \longmapsto & a_{(n)}b, \quad (n \in \mathbb{Z}), \end{array}$$

such that $(V_{\bar{i}})_{(n)}(V_{\bar{j}}) \subset V_{\bar{i}+\bar{j}}, i, j \in \mathbb{Z}_2$, and a vector $\mathbf{1} \in V_{\bar{0}}$ subject to the following conditions

(S0) For each pair of vectors $a, b \in V$, there exists a nonnegative integer n_0 such that

$$a_{(n)}b = 0 \quad \text{for all} \quad n \geq n_0.$$

(S1) For all vectors $a, b, c \in V$ and all integers $p, q, r \in \mathbb{Z}$,

$$\sum_{i=0}^{\infty} \binom{p}{i} (a_{(r+i)}b)_{(p+q-i)}c$$
$$= \sum_{i=0}^{\infty} (-1)^i \binom{r}{i} \left(a_{(p+r-i)}(b_{(q+i)}c) - (-1)^{ab+r} b_{(q+r-i)}(a_{(p+i)}c) \right).$$

(S2) For any $a \in V$,

$$a_{(n)}\mathbf{1} = \begin{cases} 0, & (n \geq 0), \\ a, & (n = -1). \end{cases}$$

Here the conditions (S0) and (S2) are the same as (B0) and (B2) for a vertex algebra.

The results in section 4-6 hold for the $\mathbb{Z}_2$-graded case if the commutator is replaced by the supercommutator and sign factors are appropriately modified.

B Analytic method

In this section, we discuss the condition sufficient for justifying the argument of contour deformation in two-dimensional quantum field theory. We define the notion of admissible fields, and apply it to the Borcherds identity for local fields and that of a vertex algebra respectively.

In this section, we will work over the complex number field $\mathbb{C}$, while some of the statements make sense over any field of characteristic zero.

B.1 Admissible fields

Let M be a $\mathbb{C}$-vector space and M^* the dual space of M. We denote the canonical pairing by

$$\langle \, , \, \rangle : M^* \times M \longrightarrow \mathbb{C}.$$

We say that a subspace $M^\vee \subset M^*$ is *nondegenerate* if the condition $\langle M^\vee, u \rangle = 0$ implies $u = 0$.

Lemma B.1.1. *Let N_m, $(m \in \mathbb{Z})$, be subspaces of M such that $\cdots \subset N_m \subset N_{m+1} \subset \cdots$ and $\bigcap_{m\in\mathbb{Z}} N_m = \{0\}$. Then there exists a nondegenerate subspace $M^\vee \subset M^*$ such that for any $v^\vee \in M^\vee$, there exists an $m \in \mathbb{Z}$ such that $\langle v^\vee, N_m\rangle = 0$.*

Proof. Set $N = \bigcup_{m\in\mathbb{Z}} N_m$ and consider

$$N_m^\vee = \{ v^\vee \in N^* \mid \langle v^\vee, N_m\rangle = 0 \} \subset N^*.$$

Take a complement P of N in M so that $M = N \oplus P$ and define

$$M^\vee = \Big(\bigcup_{m\in\mathbb{Z}} N_m^\vee\Big) \oplus P^* \subset M^*.$$

Then this $M^\vee$ has the desired properties. □

Let us recall that a series $A(z) = \sum_{n\in\mathbb{Z}} A_n z^{-n-1}$ on M is called a field if for any $u \in M$, we have $A_n u = 0, (n \geq n_0)$, for some $n_0 \in \mathbb{Z}$. Then, a series $A(z)$ is a field if and only if

$$\bigcup_{n_0\in\mathbb{Z}} \Big(\bigcap_{n=n_0}^{\infty} \operatorname{Ker} A_n\Big) = M.$$

Now we define the dual notion as follows: A series $A(z)$ on M is said to be *cotruncated* if for any nonzero $u \in M$, we have $u \notin \operatorname{Im} A_n, (n < n_0)$, for some $n_0 \in \mathbb{Z}$. Then, a series $A(z)$ is cotruncated if and only if

$$\bigcap_{n_0\in\mathbb{Z}} \Big(\sum_{n=-\infty}^{n_0} \operatorname{Im} A_n\Big) = \{0\}.$$

If $A(z)$ is cotruncated, then by Lemma B.1.1, there exists a nondegenerate subspace $M^\vee$ such that, for any $v^\vee \in M^\vee$, there exists n_0 satisfying $\langle v^\vee, A_n u\rangle = 0, (n < n_0)$, for all $u \in M$.

More generally, we prepare the following notion: We say that series $A^1(z), \ldots, A^\ell(z)$ are *admissible* if, for any $u \in M$,

$$\bigcap_{m\in\mathbb{Z}} \Big(\sum_{p_1+\cdots+p_\ell=-\infty}^{m} \sum_{\sigma\in\mathfrak{S}_\ell} \mathbb{C}A_{p_1}^{\sigma(1)} \cdots A_{p_\ell}^{\sigma(\ell)} u\Big) = \{0\},$$

where $\mathfrak{S}_\ell$ is the symmetric group acting on $\{1, \ldots, \ell\}$.

Remark B.1.2. Suppose given a direct sum decomposition

$$M = \oplus_{\lambda\in\mathbb{C}} M_\lambda = \bigoplus_{[\lambda_0]\in\mathbb{C}/\mathbb{Z}} \left(\oplus_{n\in\mathbb{Z}} M_{\lambda_0+n}\right)$$

such that $A^i_n(M_\lambda) \subset M_{\lambda+h_i-n}$, $(h_i \in \mathbb{Z})$, for $A^1(z), \dots, A^\ell(z)$. If the set

$$\{\, n\in\mathbb{Z} \mid M_{\lambda_0+n} \neq 0 \,\}$$

is bounded below, then the series $A^1(z), \dots, A^\ell(z)$ are admissible fields. More generally, if there is a filtration

$$\mathcal{F}_0 M \subset \mathcal{F}_1 M \subset \cdots, \qquad \bigcup_{k=0}^{\infty} \mathcal{F}_k M = M,$$

such that $\mathcal{F}_k M = \oplus_{\lambda\in\mathbb{C}} (\mathcal{F}_k M) \cap M_\lambda$, the set $\{\, n\in\mathbb{Z} \mid (\mathcal{F}_k M)_{\lambda+n} \neq 0 \,\}$ is bounded below, and $\left(A^i_n\right)^{-1}(\mathcal{F}_k M) \subset \mathcal{F}_{k+r_i(k)} M$, then the series $A^1(z), \dots, A^\ell(z)$ are admissible fields.

Now, let $A^1(z), \dots, A^\ell(z)$ be admissible series. Then, for any $u \in M$, there exists a nondegenerate subspace $M^\vee_u \subset M^*$ such that for any $v^\vee \in M^\vee_u$ there exists an integer $m_0 \in \mathbb{Z}$ satisfying

$$\langle v^\vee, A^{\sigma(1)}_{p_1} \cdots A^{\sigma(\ell)}_{p_\ell} u \rangle = 0, \quad (p_1, \dots, p_\ell \in \mathbb{Z}, p_1 + \cdots + p_\ell < m_0, \sigma \in \mathfrak{S}_\ell).$$

We call such a $M^\vee_u \subset M^*$ a *restricted dual space* compatible with $A^1(z), \dots, A^\ell(z)$ with respect to $u \in M$.

Lemma B.1.3. *Let $A(z)$ and $B(z)$ be admissible fields. Then for any $u \in M$ and $v^\vee \in M^\vee_u$,*

$$\langle v^\vee, A(y)B(z)u \rangle \in \mathbb{C}((y^{-1}, z)), \quad \langle v^\vee, B(z)A(y)u \rangle \in \mathbb{C}((y, z^{-1})).$$

Proof. Since $B(z)$ is a field, $\langle v^\vee, A(y)B(z)u \rangle$ has only finitely many terms of negative degree in z. On the other hand, we have

$$\langle v^\vee, A_p B_q \rangle = 0 \quad \text{if} \quad p + q \leq m_0.$$

If $p \leq m_0 - q_0$, then we have $\langle v^\vee, A_p B_q u \rangle = 0$ since either $q \geq q_0$ or $p + q \leq m_0$ holds. Thus $\langle v^\vee, A(y)B(z)u \rangle \in \mathbb{C}((y^{-1}, z))$. Similarly we have $\langle v^\vee, B(z)A(y)u \rangle \in \mathbb{C}((y, z^{-1}))$. □

B.2 Borchérds identity for local fields

Now, under the admissibility, the locality is characterized as follows:

Proposition B.2.1. *Let $A(z)$ and $B(z)$ be admissible fields. Then, $A(z)$ and $B(z)$ are mutually local if and only if, for any $u \in M$ and $v^\vee \in M_u^\vee$, $\langle v^\vee, A(y)B(z)u\rangle$ and $\langle v^\vee, B(z)A(y)u\rangle$ are the expansions of the same rational function of the form*

$$\frac{P(y,z)}{(y-z)^m}, \quad P(y,z) \in \mathbb{C}[y, y^{-1}, z, z^{-1}],$$

to series convergent in $|y| > |z|$ and $|y| < |z|$ respectively.

Proof. Suppose $A(z)$ and $B(z)$ are local. Then,

$$\langle v^\vee, A(y)B(z)u\rangle (y-z)^m = \langle v^\vee, B(z)A(y)u\rangle (y-z)^m$$

for some $m \gg 0$. Since

$$\text{(B.2.1)} \qquad \begin{aligned} &\langle v^\vee, A(y)B(z)u\rangle (y-z)^m \in \mathbb{C}((y^{-1}, z)), \\ &\langle v^\vee, B(z)A(y)u\rangle (y-z)^m \in \mathbb{C}((y, z^{-1})), \end{aligned}$$

they are equal to a Laurent polynomial $P(y,z) \in \mathbb{C}[y, y^{-1}, z, z^{-1}]$. By (B.2.1), we have

(B.2.2)

$$\langle v^\vee, A(y)B(z)u\rangle = \left.\frac{P(y,z)}{(y-z)^m}\right|_{|y|>|z|}, \quad \langle v^\vee, B(z)A(y)u\rangle = \left.\frac{P(y,z)}{(y-z)^m}\right|_{|y|<|z|}$$

by Lemma 1.1.1.

Conversely, if (B.2.2) holds for any $u \in M, v^\vee \in M_u^\vee$, then

$$\langle v^\vee, \big(A(y)B(z)(y-z)^m - B(z)A(y)(y-z)^m\big)u\rangle = 0.$$

Therefore, since $M_u^\vee$ is nondegenerate,

$$\big(A(y)B(z)(y-z)^m - B(z)A(y)(y-z)^m\big)u = 0$$

and the fields $A(z)$ and $B(z)$ are local by definition. □

Theorem B.2.2. *Let $A^1(z), \dots, A^\ell(z)$ be admissible fields. If they are local, then for any $u \in M$ and $v^\vee \in M_u^\vee$,*

$$\langle v^\vee, A^1(z_1), \dots, A^\ell(z_\ell)u\rangle$$

and its permutations with respect to $A^1(z), \dots, A^\ell(z)$ *are the expansions of the same rational functions of the form*

$$\frac{P(z_1, \dots, z_\ell)}{\prod_{i<j}(z_i - z_j)^{m_{ij}}}, \quad P(z_1, \dots, z_\ell) \in \mathbb{C}[z_1, z_1^{-1}, \dots, z_\ell, z_\ell^{-1}],$$

into the series of convergent in the corresponding regions.

Proof. Consider $\langle v^\vee, A^1(z_1) \cdots A^\ell(z_\ell)u \rangle$. It has only finitely many terms of negative degree in z_ℓ because $A^\ell(z)$ is a field. On the other hand, by Proposition 2.1.6, there exists $m \in \mathbb{N}$ such that

$$A^2_{p_2} \cdots A^\ell_{p_\ell} u = 0, \quad (p_2 + \cdots + p_\ell \geq m),$$

and by the admissibility, there exists $n \in \mathbb{N}$ such that

$$\langle v^\vee, A^1_{p_1} \cdots A^\ell_{p_\ell} u \rangle = 0, \quad (p_1 + \cdots + p_\ell < n).$$

Therefore, $\langle v^\vee, A^1(z_1) \cdots A^\ell(z_\ell)u \rangle$ has only finitely many terms of positive degree in z_1. Similar statements holds for its permutations.

Now, take sufficiently large $n_{ij} \in \mathbb{N}, (i < j)$, and consider the series

$$\langle v^\vee, A^1(z_1) \cdots A^\ell(z_\ell)u \rangle \prod_{i<j}(z_i - z_j)^{n_{i\ell}}$$

and its permutations. Then by the locality, they are equal to each other. In particular, they are equal to the same Laurent polynomial $P(z_1, \dots, z_\ell)$. Therefore, we have by Lemma 1.1.1, for example,

$$\langle v^\vee, A^1(z_1) \cdots A^\ell(z_\ell)u \rangle = \left. \frac{P(z_1, \dots, z_\ell)}{\prod_{i<j}(z_i - z_j)^{n_{i\ell}}} \right|_{|z_1| > \cdots > |z_\ell|}.$$

□

In particular, if $A(z), B(z)$ and $C(z)$ are admissible mutually local fields, then the series $\langle v^\vee, A(x)B(y)C(z)u \rangle$ is analytically continued to its permutations as a rational function. Therefore, we are allowed to prove the Borcherds identity for local fields (Corollary 3.4.2) in the following way[16]:

$$
\begin{aligned}
&\sum_{i=0}^{\infty}\binom{p}{i}(A(z)_{(r+i)}B(z))_{(p+q-i)}C(z)\\
&=\sum_{i=0}^{\infty}\binom{p}{i}\oint_{C_z}\frac{dy}{2\pi\sqrt{-1}}\oint_{C_y}\frac{dx}{2\pi\sqrt{-1}}(x-y)^{r+i}(y-z)^{p+q-i}C(z)A(x)B(y)\\
&=\oint_{C_z}\frac{dy}{2\pi\sqrt{-1}}\oint_{C_y}\frac{dx}{2\pi\sqrt{-1}}(x-y)^{r}(y-z)^{q}(x-z)^{p}C(z)A(x)B(y)\\
&=\oint_{C_z}\frac{dy}{2\pi\sqrt{-1}}\oint_{C_{y,z}}\frac{dx}{2\pi\sqrt{-1}}(x-y)^{r}(y-z)^{q}(x-z)^{p}A(x)B(y)C(z)\\
&\quad-\oint_{C_z}\frac{dy}{2\pi\sqrt{-1}}\oint_{C_z}\frac{dx}{2\pi\sqrt{-1}}(x-y)^{r}(y-z)^{q}(x-z)^{p}B(y)A(x)C(z)\\
&=\oint_{C_z}\frac{dy}{2\pi\sqrt{-1}}\oint_{C_z}\frac{dx}{2\pi\sqrt{-1}}(x-y)^{r}(y-z)^{q}(x-z)^{p}A(x)B(y)C(z)\\
&\quad-\oint_{C_z}\frac{dy}{2\pi\sqrt{-1}}\oint_{C_z}\frac{dx}{2\pi\sqrt{-1}}(x-y)^{r}(y-z)^{q}(x-z)^{p}B(y)A(x)C(z)\\
&=\sum_{i=0}^{\infty}(-1)^{i}\binom{r}{i}\oint_{C_z}\frac{dy}{2\pi\sqrt{-1}}\oint_{C_z}\frac{dx}{2\pi\sqrt{-1}}(x-y)^{p+r-i}(y-z)^{q+i}A(x)B(y)C(z)\\
&-\sum_{i=0}^{\infty}(-1)^{r+i}\binom{r}{i}\oint_{C_z}\frac{dy}{2\pi\sqrt{-1}}\oint_{C_z}\frac{dx}{2\pi\sqrt{-1}}(x-y)^{p+r-i}(y-z)^{q+i}B(y)A(x)C(z)\\
&=\sum_{i=0}^{\infty}(-1)^{i}\binom{r}{i}(A(z)_{(p+r-i)}(B(z)_{(q+i)}C(z))-(-1)^{r}B(z)_{(q+r-i)}(A(z)_{(p+i)}C(z)))
\end{aligned}
$$

where C_y is a contour around y, C_z is around z, and $C_{y,z}$ is around both y and z. Here we have omitted writing $\langle v^\vee$, and $u\rangle$.

B.3 Borcherds identity of vertex algebra

Let V be a vector space and suppose given a map $Y : V \longrightarrow (\operatorname{End} V)[[z, z^{-1}]]$. We further assume that the series $Y(a, z)$ and $Y(b, z)$ are admissible fields for any $a, b \in V$. For each $a, b, c \in V$, we denote by $V^\vee_{abc}$ the restricted dual space compatible with $Y(a, z)$ and $Y(b, z)$ with respect c.

Consider the binary operations $a_{(n)}b$ defined by $Y(a, z)b = \sum_{n\in\mathbb{Z}} a_{(n)}bz^{-n-1}$. Then by Proposition B.2.1, the locality (4.3.4) holds if and only if, for any $c \in V$

[16] Special cases of such derivations are found in physics literatures, e.g., [BBS], [BS], [T].

and $v^\vee \in V_{abc}^\vee$,

$$\langle v^\vee, Y(a,y)Y(b,z)c\rangle = \left.\frac{Q(y,z)}{y^k z^\ell (y-z)^m}\right|_{|y|>|z|},$$

$$\langle v^\vee, Y(b,z)Y(a,y)c\rangle = \left.\frac{Q(y,z)}{y^k z^\ell (y-z)^m}\right|_{|y|<|z|}$$

holds for some polynomial $Q(y,z) \in \mathbb{C}[y,z]$.

Then, under the locality, the duality (4.3.5) means that

$$\begin{aligned}\langle v^\vee, Y(Y(a,x)b,z)c\rangle (x+z)^p &= \langle v^\vee, Y(a,x+z)Y(b,z)c\rangle|_{|x|<|z|}(x+z)^p \\ &= \left.\frac{Q(x+z,z)}{(x+z)^k z^\ell x^m}\right|_{|x|<|z|}(x+z)^p\end{aligned}$$

for the polynomial $Q(y,z)$ as above. Substituting $x = y-z$ and using Lemma 1.1.1, we rewrite this as

$$\langle v^\vee, Y(Y(a,y-z)b,z)c\rangle = \left.\frac{Q(y,z)}{y^k z^\ell (y-z)^m}\right|_{|y-z|<|z|}.$$

Thus we arrive at the following condition:

(R) For any $a,b,c \in V$ and any $v^\vee \in V_{abc}^\vee$, the series

$$\langle v^\vee, Y(a,y)Y(b,z)c\rangle, \quad |y|>|z|,$$
$$\langle v^\vee, Y(b,z)Y(a,y)c\rangle, \quad |y|<|z|,$$
$$\langle v^\vee, Y(Y(a,y-z)b,z)c\rangle, \quad |y-z|<|z|$$

are the expansions of the same rational function of the form

$$\frac{Q(y,z)}{y^k z^\ell (y-z)^m}, \quad Q(y,z) \in \mathbb{C}[y,z],$$

into series convergent in the respective regions.

Summarizing the consideration above, we have (cf. [FHL, Proposition 3.4.1])

Proposition B.3.1. *Let V be a vector space and suppose given a map $Y : V \longmapsto (\mathrm{End}\, V)[[z,z^{-1}]]$. If $Y(a,z)$ and $Y(b,z)$ are admissible fields for any $a,b \in V$, then the axiom (B1) is equivalent to the property (R).*

C List of expansions of $(x-y)^r(y-z)^q(x-z)^p$

Let $F(x,y,z) = (x-y)^r(y-z)^q(x-z)^p$. We will give the list of power series expansions of $F(x,y,z)$ in various regions.

C.1 The expansion in the region $|y-z| > |x-y|$

$$F(x,y,z) = \sum_{i=0}^{\infty} \binom{p}{i} (x-y)^{r+i}(y-z)^{p+q-i}$$

$$=^* \sum_{i,j,k=0}^{\infty} (-1)^{j+k} \binom{p}{i}\binom{r+i}{j}\binom{p+q-i}{k} x^{r+i-j} y^{p+q-i+j-k} z^k,$$

$$(|x| > |y| > |z|)$$

$$=^* \sum_{i,j,k=0}^{\infty} (-1)^{r+i+j+k} \binom{p}{i}\binom{r+i}{j}\binom{p+q-i}{k} x^j y^{p+q+r-j-k} z^k,$$

$$(|y| > |x| > |z|)$$

$$= \sum_{i,j,k=0}^{\infty} (-1)^{p+q+i+j+k} \binom{p}{i}\binom{r+i}{j}\binom{p+q-i}{k} x^{r+i-j} y^{j+k} z^{p+q-i-k},$$

$$(|z| > |x| > |y|)$$

$$= \sum_{i,j,k=0}^{\infty} (-1)^{j+k} \binom{p}{i}\binom{r+i}{j}\binom{p+q-i}{k} x^j y^{r+i-j+k} z^{p+q-i-k},$$

$$(|z| > |y| > |x|)$$

Here the expressions with $=^*$ do not make sense for $p < 0$.

C.2 The expansion in the region $|x-z| > |y-z|$

$$F(x,y,z) = \sum_{i=0}^{\infty}(-1)^i \binom{p}{i}(x-z)^{p+r-i}(y-z)^{q+i}$$

$$= \sum_{i,j,k=0}^{\infty} (-1)^{i+j+k}\binom{r}{i}\binom{p+r-i}{j}\binom{q+i}{k} x^{p+r-i-j}y^{q+i-k}z^{j+k},$$

$$(|x| > |y| > |z|)$$

$$= \sum_{i,j,k=0}^{\infty} (-1)^{p+r+i+j+k}\binom{r}{i}\binom{p+r-i}{j}\binom{q+i}{k} x^{p+r-i-j}y^{k}z^{j+q+i-k},$$

$$(|x| > |z| > |y|)$$

$$=^{**} \sum_{i,j,k=0}^{\infty} (-1)^{i+j+k}\binom{r}{i}\binom{p+r-i}{j}\binom{q+i}{k} x^{j}y^{q+i-k}z^{p+r-i-j+k},$$

$$(|y| > |z| > |x|)$$

$$=^{**} \sum_{i,j,k=0}^{\infty} (-1)^{p+q+r+i+j+k}\binom{r}{i}\binom{p+r-i}{j}\binom{q+i}{k} x^{j}y^{k}z^{p+q+r-j-k},$$

$$(|z| > |y| > |x|)$$

Here the expressions with $=^{**}$ do not make sense for $r < 0$.

C.3 The expansion in the region $|y - z| > |x - z|$

$$F(x,y,z) = \sum_{i=0}^{\infty}(-1)^{r+i}\binom{r}{i}(y-z)^{q+r-i}(x-z)^{p+i}$$

$$= \sum_{i,j,k=0}^{\infty} (-1)^{r+i+j+k}\binom{r}{i}\binom{q+r-i}{j}\binom{p+i}{k} x^{p+i-k}y^{q+r-i-j}z^{j+k},$$

$$(|y| > |x| > |z|)$$

$$= \sum_{i,j,k=0}^{\infty} (-1)^{p+r+j+k}\binom{r}{i}\binom{q+r-i}{j}\binom{p+i}{k} x^{k}y^{q+r-i-j}z^{p+i+j-k},$$

$$(|y| > |z| > |x|)$$

$$=^{**} \sum_{i,j,k=0}^{\infty} (-1)^{r+i+j+k}\binom{r}{i}\binom{q+r-i}{j}\binom{p+i}{k} x^{p+i-k}y^{j}z^{p+r-i-j+k},$$

$$(|x| > |z| > |y|)$$

$$=^{**} \sum_{i,j,k=0}^{\infty} (-1)^{p+q+i+j+k}\binom{r}{i}\binom{q+r-i}{j}\binom{p+i}{k} x^{k}y^{j}z^{p+q+r-j-k},$$

$$(|z| > |x| > |y|)$$

Here the expressions with $=^{**}$ do not make sense for $r < 0$.