Part III

Topics and examples

In Section 7, we will explain some particular classes of vertex algebras and describe the notion of an invariant bilinear form. Some relations to other algebraic objects such as Lie algebras, commutative algebras and associative algebras are described in Section 8. Section 9 is devoted to describing some famous examples: the vertex algebras associated to the free boson, to the β-γ system, to the affine Lie algebras, to the Virasoro algebra and to the $W_{1+\infty}$ algebra, and the lattice vertex algebra.

7 Summary of related notions

In this section, we will give a brief survey of various notions such as gradings, quasiconformal structures, conformal structures and invariant bilinear forms. We also summarize the notion and terminologies on vertex operator algebras.

7.1 Gradings of a vertex algebra

We first review the notion of a grading.

Definition 7.1.1. A *grading* of a vertex algebra V is a direct sum decomposition $V = \oplus_r V^r$, where r runs over a set of scalars, such that

$$V^r_{(n)} V^s \subset V^{r+s-n-1} \tag{7.1.1}$$

for all $n \in \mathbb{Z}$. A graded vertex algebra is a vertex algebra equipped with a grading.

A vector of a graded vertex algebra V is said to be homogeneous of degree r if it belongs to V^r. We denote the degree of a homogeneous vector a by $\Delta(a)$. (Whenever mentioning $\Delta(a)$, the vector a is implicitly supposed to be homogeneous).

For a graded vertex algebra V, we consider the operator $D : V \longrightarrow V, Da = \Delta(a)a$. Then the condition (7.1.1) is written as

$$D(a_{(n)}b) = (Da)_{(n)}b + a_{(n)}(Db) - (n+1)a_{(n)}b \tag{7.1.2}$$

which is equivalent to

$$[D, Y(a,z)] = z\partial Y(a,z) + \Delta(a)Y(a,z).$$

Conversely, for any semisimple operator $D : V \longrightarrow V$ satisfying (7.1.2), the eigenvalue decomposition $V = \oplus V^r, V^r = \{a \in V \mid Da = ra\}$, gives a grading of V.

Let V be a graded vertex algebra. Then substituting $a = b = \mathbf{1}$ and $n = -1$ in (7.1.2), we have $D(\mathbf{1}_{(-1)}\mathbf{1}) = (D\mathbf{1})_{(-1)}\mathbf{1} + \mathbf{1}_{(-1)}(D\mathbf{1})$. Hence

$$D\mathbf{1} = 0, \text{ i.e., } \mathbf{1} \in V^0. \tag{7.1.3}$$

Therefore, $\Delta(Ta) = \Delta(a_{(-2)}\mathbf{1}) = \Delta(a) + \Delta(\mathbf{1}) - (-2) - 1 = \Delta(a) + 1$, so we have $T(V^r) \subset V^{r+1}$. In other words, the operators $T, D : V \longrightarrow V$ satisfy

$$[D, T] = T. \tag{7.1.4}$$

Remark 7.1.2. Let V be a vertex algebra, and suppose given a vector $\omega \in V$ such that $\omega_{(0)} = T$ and $\omega_{(1)}$ is semisimple. Then $D = \omega_{(1)}$ gives rise to a grading of V. In fact, by (4.3.2) and (4.2.4),

$$\begin{aligned} D(a_{(n)}b) &= \omega_{(1)}(a_{(n)}b) \\ &= a_{(n)}(\omega_{(1)}b) + \sum_{i=0}^{\infty} \binom{1}{i} (\omega_{(i)}a)_{(n+1-i)}b \\ &= a_{(n)}(Db) + (Ta)_{(n+1)}b + (Da)_{(n)}b \\ &= (Da)_{(n)}b + a_{(n)}(Db) - (n+1)a_{(n)}b. \end{aligned}$$

Now, let $\omega = \sum \omega^r$ be the homogeneous decomposition. Comparing the degree of the both sides of $\omega_{(0)}a = Ta$ and $\omega_{(1)}a = \Delta(a)a$, we see that $\omega^r_{(0)}a = \omega^r_{(1)}a = 0$ if $r \neq 2$. Therefore, we must have $\omega^2_{(0)} = \omega_{(0)} = T$ and $\omega^2_{(1)} = \omega_{(1)} = D$. Thus we may suppose that ω is homogeneous of degree 2, i.e., $\omega_{(1)}\omega = 2\omega$, without affecting the grading.

7.2 Quasiconformal structure on a vertex algebra

Next we turn to the quasiconformal structure on a vertex algebra. Consider the Lie algebra sl_2 spanned by the matrices

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

with the Lie bracket given by the commutator. This is identified with the Lie algebra spanned by $\{L_1, L_0, L_{-1}\}$ with

$$[L_m, L_n] = (m-n)L_{m+n}, \quad (m, n = 1, 0, -1),$$

by letting

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \longmapsto L_1, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \longmapsto -2L_0, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \longmapsto -L_{-1}.$$

The action of L_m on an sl_2-module will be denoted by the same symbol for brevity.

Definition 7.2.1. A quasiconformal vertex algebra is a vertex algebra V equipped with an sl_2-module structure $L_1, L_0, L_{-1} : V \longrightarrow V$ such that
(Q1) For any $a, b \in V$ and $n \in \mathbb{Z}$

$$L_m(a_{(n)}b) = a_{(n)}(L_m b) + \sum_{i=0}^{2} \binom{m+1}{i} (L_{i-1}a)_{(m+n+1-i)}b, \quad (m = 1, 0, -1),$$

(Q2) L_{-1} coincides with T,
(Q3) L_0 is semisimple.

Here the condition (Q1) for $m = -1$ follows from (Q2), since $T(a_{(n)}b) = (Ta)_{(n)}b + a_{(n)}(Tb)$. Note that (Q1) is written as

$$[L_m, Y(a,z)] = \sum_{i=0}^{2} \binom{m+1}{i} z^{m+1-i} Y(L_{i-1}a, z)$$

in terms of the generating series.

Proposition 7.2.2. *A quasiconformal vertex algebra is graded by $D = L_0$.*

Proof. By (Q1) for $m = 0$ and (Q2),

$$L_0(a_{(n)}b) = (L_0 a)_{(n)}b + a_{(n)}(L_0 b) - (n+1)(a_{(n)}b).$$

Therefore, since L_0 is semisimple by (Q3), it gives a grading on V. □

Proposition 7.2.3. *For a quasiconformal vertex algebra, $L_m\mathbf{1} = 0$, $(m = 1, 0, -1)$.*

Proof. By (Q2), $L_{-1}\mathbf{1} = T\mathbf{1} = 0$. Since V is graded by $D = L_0$, we have $L_0\mathbf{1} = 0$ by (7.1.3). Substituting $a = b = \mathbf{1}$ and $n = -1$ in (Q1) for $m = 1$, we have $L_1\mathbf{1} = 0$. □

Remark 7.2.4. For a graded vertex algebra V, we set $L_{-1} = T$ and $L_0 = D$. Suppose given an operator $L_1 : V \longrightarrow V$. Then V becomes a quasiconformal vertex algebra by these operators if and only if L_1 satisfies (Q1) for $m = 1$ and $[L_0, L_1] = -L_1$. ($[L_1, L_{-1}] = 2L_0$ follows from (Q1).)

Proposition 7.2.5. *Let V be a vertex algebra and suppose given a vector $\omega \in V$ such that $\omega_{(0)} = T$ and $\omega_{(1)}$ is semisimple. We also suppose that $\omega_{(1)}\omega = 2\omega$. Then V becomes a quasiconformal vertex algebra by setting $L_m = \omega_{(m+1)}$, $(m = 1, 0, -1)$.*

Proof. By Remark 7.1.2, V is graded by $D = L_0 = \omega_{(1)}$, and we have

$$\begin{aligned} L_1(a_{(n)}b) &= \omega_{(2)}(a_{(n)}b) \\ &= a_{(n)}(\omega_{(2)}b) + \sum_{i=0}^{2} \binom{2}{i} (\omega_{(i)}a)_{(n+2-i)}b \\ &= a_{(n)}(L_1 b) + \sum_{i=0}^{2} \binom{2}{i} (L_{i-1}a)_{(n+2-i)}b \end{aligned}$$

and

$$\begin{aligned} [L_0, L_1] &= [\omega_{(1)}, \omega_{(2)}] \\ &= (\omega_{(0)}\omega)_{(3)} + (\omega_{(1)}\omega)_{(2)} = -\omega_{(2)} = -L_1. \end{aligned}$$

Hence it follows from Remark 7.2.4 that V is a quasiconformal vertex algebra. □

We finally note that a vector in the space

$$Q^{\Delta} = \{ a \in V^{\Delta} \mid L_1 a = 0 \}$$

is called a *quasiprimary state* of degree Δ.

7.3 Conformal structure on a vertex algebra

Now we turn to the conformal structure on a vertex algebra. Let $\mathcal{V}ir = \oplus_{n\in\mathbb{Z}}\mathbf{k}L_n \oplus \mathbf{k}C$ be the Virasoro algebra. The subspace spanned by $\{L_1, L_0, L_{-1}\}$ is a Lie subalgebra isomorphic to sl_2 as before.

Let c be a scalar.

Definition 7.3.1. A *Virasoro vector* of central charge c of a vertex algebra V is a vector $e \in V$ such that

(V) the map $L_n \longmapsto e_{(n+1)}$ form a representation of the Virasoro algebra of central charge c.

We note that the condition (V) is equivalent to

$$e_{(n)}e = \begin{cases} 0 & (n \geq 4), \\ (c/2)\mathbf{1} & (n = 3), \\ 0 & (n = 2), \\ 2e & (n = 1), \\ Te & (n = 0), \end{cases}$$

which further reduces to the condition

$$e_{(n)}e = \begin{cases} 0 & (n \geq 5), \\ (c/2)\mathbf{1} & (n = 3), \\ 2e & (n = 1) \end{cases}$$

by the skew symmetry.

A *conformal vector* of central charge c of a vertex algebra V is a vector $\omega \in V$ such that,
(C1) ω is a Virasoro vector of central charge c,
(C2) the action of L_{-1} coincides with T, i.e., $\omega_{(0)}a = a_{(-2)}\mathbf{1}$ for any $a \in V$,
(C3) the action of L_0 on V is semisimple.
A vertex algebra equipped with a conformal vector is called a *conformal vertex algebra*.

It follows from Proposition 7.2.5 that a conformal vertex algebra is a quasiconformal vertex algebra. Thus we have the implication

$$\text{conformal} \Longrightarrow \text{quasiconformal} \Longrightarrow \text{graded}.$$

For a conformal vertex algebra with the conformal vector ω, we set

$$P^{\Delta} = \{\, a \in V^{\Delta} \mid L_n a = 0,\ (n \geq 1)\,\}$$

whose elements are called *primary states* of conformal weight Δ.

Remark 7.3.2. For a conformal vertex algebra V, we have $\operatorname{Ker} T \subset P^0$ ([Li,p.292, Corollary 4.2]). To show this, suppose $Ta = 0$. Then since $\partial Y(a,z) = Y(Ta,z) = 0$, we have $a_{(n)} = 0$ for $n \neq 0$. Hence

$$L_n a = \omega_{(n+1)}a = \sum_{i=0}^{\infty} (-1)^{n+i;2} T^{(i)}(a_{(n+i+1)}\omega) = 0$$

for $n \geq -1$, so $a \in P^0$.

7.4 Vertex operator algebras and their modules

For reader's convenience, we summarize the terminologies in the theory of vertex operator algebras.

The notion of a vertex operator algebra (VOA) is defined in the literature as follows (cf. [FLM], [FHL], [DL]):

Definition 7.4.1. A VOA is a $\mathbb{Z}$-graded vector space $V = \oplus_{n\in\mathbb{Z}} V^n$ such that $\dim V^n < \infty$ and $V^n = 0$ for sufficiently small n, equipped with a linear map

$$\begin{array}{rccl} Y: & V & \longrightarrow & (\operatorname{End} V)[[z, z^{-1}]] \\ & a & \longmapsto & Y(a,z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1} \end{array}$$

and with distinguished vectors $\mathbf{1} \in V^0$ and $\omega \in V^2$ satisfying the following conditions

(1) $a_{(n)}b = 0$ for sufficiently large n;

(2) $Y(\mathbf{1}, z) = \mathrm{id}_V$;

(3) $Y(a,z)\mathbf{1} \in V[[z]]$ and $\lim_{z\to 0}(Y(a,z)\mathbf{1}) = a$.

(4)

$$\begin{aligned} &z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) Y(Y(a,z_0)b, z_2) \\ &\quad = z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y(a,z_1)Y(b,z_2) - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right) Y(b,z_2)Y(a,z_1); \end{aligned}$$

(5)

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0}c;$$

(6) $L_0 a = na$ for $a \in V^n$;

(7) $\partial Y(a,z) = Y(L_{-1}a, z)$;

Here $L_n = \omega_{n+1}$, and the delta function is defined by

$$\delta\left(\frac{z_1 - z_2}{z_0}\right) = \sum_{n\in\mathbb{Z}} \left(\frac{z_1 - z_2}{z_0}\right)^n \Bigg|_{|z_1|>|z_2|} = \sum_{n\in\mathbb{Z}}\sum_{i=0}^{\infty}(-1)^i \binom{n}{i} z_0^{-n} z_1^{n-i} z_2^i,$$

thus

$$\delta\left(\frac{z_2 - z_1}{-z_0}\right) = \sum_{n\in\mathbb{Z}}\sum_{i=0}^{\infty}(-1)^{n+i}\binom{n}{i} z_0^{-n} z_1^i z_2^{n-i},$$

$$\delta\left(\frac{z_1 - z_0}{z_2}\right) = \sum_{n\in\mathbb{Z}}\sum_{i=0}^{\infty}(-1)^i\binom{n}{i} z_0^i z_1^{n-i} z_2^{-n}.$$

The identity (4) is called the (*Cauchy-*) *Jacobi identity*, which is easily seen to be equivalent to the Borcherds identity by taking the coefficients to $z_0^{-r-1}z_1^{-p-1}z_2^{-q-1}$. Since the conditions (1), (4) and (3) are equivalent to (B0),(B1) and (B2) respectively, from which the condition (2) follows. Now the conditions (5),(6) and (7) show that ω is a conformal vector and $V = \oplus V^n$ is the associated grading. Therefore,

Proposition 7.4.2. *A vertex operator algebra is nothing but a conformal vertex algebra such that* $\dim V^n < \infty$ *for all* n *and* $V^n = 0$, $(n \le N)$, *for some* N.

Note 7.4.3. In the literature of vertex operator algebras, the conformal vector ω is called the *Virasoro element* and the central charge is called the *rank* of the vertex operator algebra.

Now we are in the position to give a definition of modules for a vertex operator algebra V.

Definition 7.4.4. A *weak module* for a VOA V is a vector space M equipped with a linear map

$$\begin{array}{rccl} Y_M: & V & \longrightarrow & (\operatorname{End} M)[[z, z^{-1}]] \\ & a & \longmapsto & Y_M(a,z) = \sum_{n\in\mathbb{Z}} a_{[n]}z^{-n-1}, a_{[n]} \in \operatorname{End} M \end{array}$$

satisfying

(W0) For each pair $(a, v) \in V \times M$, there exists a nonnegative integer n_0 such that $a_{[n]}v = 0$ for all $n \ge n_0$.

(W1) $Y_M(\mathbf{1}, z) = \mathrm{id}_M$.

(W2) For all $a, b \in V$ and $v \in M$,

$$z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) Y_M(Y(a, z_0)b, z_2)v$$
$$= z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y_M(a, z_1)Y_M(b, z_2)v - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right) Y_M(b, z_2)Y_M(a, z_1)v.$$

In other words, a weak module for a VOA V is just a module for the vertex algebra V. Then (4.6.1) says that

$$\partial Y_M(a, z) = Y_M(L_{-1}a, z)$$

holds for a weak module, and (4.6.2) applied to $a = b = \omega$ is nothing but

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0}c$$

where $Y_M(\omega, z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$ and c is the rank of V.

Definition 7.4.5. A weak module M for a VOA V is called $\mathbb{N}$-*gradable* if there exists a grading $M = \oplus_{n\in\mathbb{N}} M_n$ such that

$$\text{(N)} \qquad V^p_{[n]} M_q \subset M_{p+q-n-1}.$$

A weak module equipped with such a grading is called an *admissible* module (cf. [Li3], [DLM2]).

We finally show the definition of a module for a VOA:

Definition 7.4.6. A *module* for a VOA V is a weak module with a grading $M = \oplus_{\lambda\in\mathbf{k}} M_\lambda$ satisfying
(1) $\dim M_\lambda < \infty$ for all $\lambda \in \mathbf{k}$.
(2) For each $\lambda \in \mathbf{k}$, $M_{\lambda+n} = 0$ for $n \gg 0$.
(3) $L_0|_{M_\lambda} = \lambda \cdot \mathrm{id}$.

Note that a module for a VOA is always $\mathbb{N}$-gradable as a weak module.

7.5 Invariant bilinear form

Let V be a quasiconformal vertex algebra such that it is integrally graded by L_0, and suppose that L_1 is locally nilpotent on V. Then the expression

$$Y(e^{zL_1}(-z^{-2})^{L_0}a, z^{-1})$$

makes sense as an element of $(\mathrm{End}\, V)[[z, z^{-1}]]$.

Definition 7.5.1. A bilinear form $\langle\ |\ \rangle : V \times V \longrightarrow \mathbf{k}$ on V is called *invariant* if

$$\langle Y(a,z)b \,|\, c\rangle = \langle b \,|\, Y(e^{zL_1}(-z^{-2})^{L_0}a, z^{-1})c\rangle, \text{ and} \tag{7.5.1}$$

$$\langle L_n a \,|\, b\rangle = \langle a \,|\, L_{-n} b\rangle, \quad (n = -1, 0, 1) \tag{7.5.2}$$

hold for any $a, b, c \in V$.

If $\langle\ |\ \rangle$ is an invariant bilinear form, then

$$\begin{aligned}\langle Y(e^{zL_1}(-z^{-2})^{L_0}a, z^{-1})b \,|\, c\rangle &= \langle b \,|\, Y(e^{z^{-1}L_1}(-z^2)^{L_0} e^{zL_1}(-z^{-2})^{L_0}a, z)c\rangle \\ &= \langle b \,|\, Y(e^{z^{-1}L_1} e^{-z^{-1}L_1}a, z)c\rangle \\ &= \langle b \,|\, Y(a,z)c\rangle.\end{aligned}$$

Let $\langle\ |\ \rangle : V \times V \longrightarrow \mathbf{k}$ be an invariant bilinear form. If $L_1 a = 0$, then since

$$\begin{aligned} Y(e^{zL_1}(-z^{-2})^{L_0}a, z^{-1}) &= (-1)^{\Delta(a)} z^{-2\Delta(a)} Y(a, z^{-1}) \\ &= (-1)^{\Delta(a)} \sum_{n \in \mathbb{Z}} a_{(n)} z^{n-2\Delta(a)+1} \\ &= (-1)^{\Delta(a)} \sum_{n \in \mathbb{Z}} a_{(-n+2\Delta(a)-2)} z^{-n-1}, \end{aligned}$$

we have $\langle a_{(n)} b \,|\, c \rangle = (-1)^{\Delta(a)} \langle b \,|\, a_{(-n+2\Delta(a)-2)} c \rangle$. Therefore, setting

$$a_n = a_{(n+\Delta(a)-1)},$$

we have

$$\langle a_n b \,|\, c \rangle = \langle b \,|\, a_{-n} c \rangle.$$

For example, if V is a conformal vertex algebra with the conformal vector $\omega \in V$,

$$\langle \omega_{(n+1)} b \,|\, c \rangle = \langle b \,|\, \omega_{(-n+1)} c \rangle.$$

Hence, the condition (7.5.2) follows from (7.5.1) for a conformal vertex algebra.

The reader can find further information and applications in [FHL], [Li1], [Li3] and [S].

8 Relation to other algebraic structures

In this section we construct various algebraic structures out of the structure of a vertex algebra.

8.1 Lie algebras related to a vertex algebra

We first consider structures induced from the (0)-th product of a vertex algebra.

Let V be a vertex algebra and set

$$[a, b] = a_{(0)} b$$

for $a, b \in V$. Then the bilinear map

$$[\,,\,] : V \times V \longrightarrow V \tag{8.1.1}$$

satisfies the Jacobi identity

$$[[a,b],c] = [a,[b,c]] - [b,[a,c]],$$

but $[b,a] \neq -[a,b]$ in general, because

$$[b,a] = -[a,b] + \sum_{i=0}^{\infty}(-1)^{i+1}T^{(i)}(a_{(i)}b)$$

by the skew symmetry. Since $[Ta,b] = (Ta)_{(0)}b = 0$, and $[a,Tb] = a_{(0)}(Tb) = T(a_{(0)}b)$ for any $a,b \in V$, the map (8.1.1) induces a map

$$[\,,\,] : V/TV \times V/TV \longrightarrow V/TV$$

on the quotient space. Then we have

Proposition 8.1.1 (Borcherds). *For a vertex algebra V, the (0)-th product induces a Lie algebra structure on V/TV.*

Remark 8.1.2. Let V be a vertex algebra and let M be a V-module. Since $(Ta)_{[0]}v = 0$ for any $a \in V$ and $v \in M$, the map $(a,v) \longmapsto a.v = a_{[0]}v$ induces a well-defined action

$$V/TV \times M \longrightarrow M.$$

Since $[a,b]_{[0]}v = (a_{[0]}b)_{[0]}v = a_{[0]}(b_{[0]}v) - b_{[0]}(a_{[0]}v) = a.(b.v) - b.(a.v)$ for any $a,b \in V$ and $v \in M$, the above action gives a structure of a module over the Lie algebra V/TV on M.

We next suppose that $V = \oplus V^{\Delta}$ is a graded vertex algebra. Then $[V^1, V^{\Delta}] \subset V^{\Delta}$ and in particular, $[V^1, V^1] \subset V^1$.

Proposition 8.1.3. *For a graded vertex algebra $V = \oplus V^{\Delta}$, the (0)-th product induces a Lie algebra structure on V^1/TV^0.*

If $\dim V^0 = 1$, then $TV^0 = 0$ for V^0 is spanned by $\mathbf{1}$, so $V^1/TV^0 = V^1$. In this case, consider the bilinear form

$$(\,|\,) : V^1 \times V^1 \longrightarrow V^1$$

given by the (1) st product:

$$(a|b)\mathbf{1} = a_{(1)}b \quad \text{for} \quad a,b \in V^1.$$

Proposition 8.1.4. *If a graded vertex algebra $V = \oplus V^\Delta$ satisfies* $\dim V^0 = 1$, *then the* (0)*-th product equips* V^1 *with a Lie algebra structure and the* (1) *st product gives an invariant bilinear form, which is symmetric if* $TV \cap V^0 = \{0\}$.

Proof. The bilinear form $(\,|\,)$ is invariant since

$$\begin{aligned}([a,b]|c)\mathbf{1} &= -(b_{(0)}a)_{(1)}c \\ &= -b_{(0)}(a_{(1)}c) + a_{(1)}(b_{(0)}c) \\ &= -(a|c)b_{(0)}\mathbf{1} + (a|[b,c])\mathbf{1} \\ &= (a|[b,c])\mathbf{1}.\end{aligned}$$

If $TV \cap V^0 = \{0\}$, then the form is symmetric since

$$\begin{aligned}((a|b) - (b|a))\mathbf{1} &= a_{(1)}b - b_{(1)}a \\ &= \sum_{i=0}^{\infty}(-1)^i T^{(i)}(a_{(i+1)}b) \in TV \cap V^0.\end{aligned}$$

□

We next suppose that V is quasiconformal. Consider the space

$$Q^\Delta = \{a \in V^\Delta \mid L_1 a = 0\},$$

of quasiprimary states of conformal weight Δ. Then, for $a \in V^1$,

$$\begin{aligned}L_1[a,b] &= L_1(a_{(0)}b) \\ &= a_{(0)}(L_1 b) + (L_1 a)_{(0)}b + 2(L_0 a)_{(1)}b + (L_{-1}a)_{(2)}b \\ &= a_{(0)}(L_1 b) + (L_1 a)_{(0)}b \\ &= [a, L_1 b] + [L_1 a, b],\end{aligned}$$

by (Q1) since $2a_{(1)}b + (Ta)_{(2)}b = 0$. Hence $[Q^1, Q^\Delta] \subset Q^\Delta$. Therefore,

Proposition 8.1.5. *For a quasi-primary vertex algebra* V, (0)*-th product induces a Lie algebra structure on* $Q^1/(Q^1 \cap TV^0)$.

We finally consider the case of a conformal vertex algebra. Consider the space

$$P^\Delta = \{a \in V^\Delta \mid L_n a = 0, (n \geq 1)\}$$

of primary states of conformal weight Δ. Then, for $a \in V^1$,

$$\begin{aligned} L_n[a,b] &= \omega_{(n+1)}(a_{(0)}b) \\ &= a_{(0)}(\omega_{(n+1)}b) + \sum_{i=0}^{\infty} \binom{n+1}{i} (\omega_{(i)}a)_{(n+1-i)}b \\ &= [a, L_n b] + \sum_{i=1}^{\infty} \binom{n+1}{i+1} (L_i a)_{(n-i)} b, \end{aligned}$$

since $(L_{-1}a)_{(n+1)}b + (n+1)(L_0 a)_{(n)}b = 0$. Hence $[P^1, P^\Delta] \subset P^\Delta$, and the bilinear map$[\,,\,]$ induces a Lie algebra structure on $P^1/(P^1 \cap TV^0)$.

Lemma 8.1.6. *For a conformal vertex algebra V, $P^1 \cap TV^0 = TP^0$.*

Proof. (cf. [S,Proof of Proposition 2.5].) Let $Ta \in P^1$ where $a \in V^0$. Then, since $L_n(Ta) = 0$ for all $n \geq 1$, we have

(8.1.2) $$T(L_n a) = L_n(Ta) - (n+1)L_{n-1}a = -(n+1)L_{n-1}a, \quad (n \geq 1).$$

Since $L_n = \omega_{(n+1)}$, there exists an n_0 such that $L_n = 0$ for all $n \geq n_0$. Then, by induction on n using (8.1.2), we deduce $L_n a = 0$ for all $n \geq 0$. Hence $a \in P^0$, and we have $P^1 \cap TV^0 \subset TP^0$. The other inclusion easily follows from (8.1.2). □

Therefore ([B1, Section 5], cf. [S]),

Proposition 8.1.7 (Borcherds). *For a conformal vertex algebra V, the (0)-th product induces a Lie algebra structure on P^1/TP^0.*

We refer the reader to [B2], [Geb], [J], [S] for its very important relation to generalized Kac-Moody algebras.

8.2 Griess algebra

Let V be a graded vertex algebra such that

(8.2.1) $$V^n = 0,\ (n \leq -1), \quad V^0 = \mathbf{k}\mathbf{1} \quad \text{and} \quad V^1 = 0.$$

Let B denote the subspace V^2 of degree 2. Define bilinear maps $\cdot : B \times B \longrightarrow B$ and $(\,|\,) : B \times B \longrightarrow \mathbf{k}$ by setting

$$a \cdot b = \frac{1}{2} a_{(1)} b \quad \text{and} \quad (a|b)\mathbf{1} = 2a_{(3)}b$$

for $a, b \in B$.

Proposition 8.2.1. *Let V be a graded vertex algebra satisfying the condition (8.2.1). Then the bilinear map $\cdot$ defines a commutative nonassociative algebra structure on B such that $(\,|\,)$ is a symmetric invariant bilinear form.*

Proof. By the assumption (8.2.1) on the grading, we have $a_{(2)}b = 0, T(a_{(3)}b) = 0$, and $a_{(i)}b = 0$ for $i \geq 4$, since $\Delta(a_{(i)}b) = 2 + 2 - i - 1 = 3 - i$ for $a, b \in B = V^2$. Therefore, by the skew symmetry,

$$b_{(1)}a = a_{(1)}b + \sum_{i=1}^{\infty}(-1)^i T^{(i)}(a_{(i+1)}b) = a_{(1)}b,$$

hence $b \cdot a = a \cdot b$. Similarly, $b_{(3)}a = a_{(3)}b$ and we have $(b|a) = (a|b)$. Finally, Borcherds identity for $p = 1, q = 2, r = 1$ yields

$$\begin{aligned}(a_{(1)}b)_{(3)}c &= -(a_{(2)}b)_{(2)}c - a_{(1)}(b_{(3)}c) + b_{(3)}(a_{(1)}c) + a_{(2)}(b_{(2)}c) - b_{(2)}(a_{(2)}c)\\ &= -a_{(1)}(b_{(3)}c) + b_{(3)}(a_{(1)}c).\end{aligned}$$

Since $a_{(1)}(b_{(3)}c) = 0$ by $a_{(1)}\mathbf{1} = 0$, we have

$$(a \cdot b|c)\mathbf{1} = (a_{(1)}b)_{(3)}c = b_{(3)}(a_{(1)}c) = (b|a \cdot c).$$

□

The algebra B equipped with the invariant symmetric bilinear form is called the *Griess algebra* of V (cf. [FLM]).

Remark 8.2.2. An element $e \in B$ is an idempotent of the Griess algebra if and only if it is a Virasoro vector of central charge $c = (e|e)$. Two idempotents $e, f \in B$ are mutually orthogonal, i.e., $e \cdot f = 0$, if and only if $[Y(e, y), Y(f, z)] = 0$. If V is a conformal vertex algebra with the conformal vector $\omega \in V$, then ω is an identity element of the Griess algebra.

8.3 Commutative Poisson algebra $V/C_2(V)$

Let V be a vertex algebra and consider the (-1) st product:

$$\begin{array}{ccc} V \times V & \longrightarrow & V \\ (a, b) & \longmapsto & ab = a_{(-1)}b. \end{array}$$

Then we have

$$a_{(-1)}b - b_{(-1)}a = \sum_{i=0}^{\infty}(-1)^i T^{(i+1)}(a_{(i)}b) = \sum_{i=0}^{\infty}(-1)^i (a_{(i)}b)_{(-i-2)}\mathbf{1} \tag{8.3.1}$$

by the skew symmetry (4.2.6), and

$$(a_{(-1)}b)_{(-1)}c - a_{(-1)}(b_{(-1)}c) = \sum_{i=0}^{\infty}(a_{(-i-2)}(b_{(i)}c) + b_{(-i-2)}(a_{(i)}c)) \tag{8.3.2}$$

by the Borcherds identity (B1) for $p = 0, q = r = -1$.

Hence, if $V_{(n)}V = 0$ for all $n \geq 0$, then the (-1) st product endows V with a structure of a commutative associative algebra (cf. Note 4.2.2). Now, recall that $V_{(n-1)}V \subset V_{(n)}V$ if $n \neq 0$. Following Zhu [Z], consider the subspace

$$C_2(V) = V_{(-2)}V = \sum_{n \leq -2} V_{(n)}V.$$

It is an ideal with respect to the (-1) st product, for

$$(a_{(-2)}b)_{(-1)}c = \sum_{i=0}^{\infty}(-1)^i \binom{-2}{i}(a_{(-2-i)}(b_{(-1+i)}c) + b_{(-3-i)}(a_{(i)}c)) \in C_2(V)$$

by the Borcherds identity (B1) for $p = 0, q = -1, r = -2$, and

$$a_{(-1)}(b_{(-2)}c) = b_{(-2)}(a_{(-1)}c) + \sum_{i=0}^{\infty}\binom{-1}{i}(a_{(i)}b)_{(-3-i)}c \in C_2(V)$$

by (B1) for $p = -1, q = -2, r = 0$. Therefore, the (-1) st product induces a well-defined bilinear multiplication

$$\begin{array}{ccc} V/C_2(V) \times V/C_2(V) & \longrightarrow & V/C_2(V) \\ (u, v) & \longmapsto & uv \end{array}$$

on the quotient space $V/C_2(V) = V_{(-1)}V/V_{(-2)}V$. It is commutative and associative since (8.3.1) and (8.3.2) belong to $C_2(V)$.

Therefore,

Proposition 8.3.1. *For a vertex algebra V, the (-1) st product induces a commutative associative algebra structure on $V/C_2(V)$.*

We next consider the relation of this algebra structure and the Lie algebra structure induced by the (0)-th product.

First note that the subspace $C_2(V)$ is also an ideal with respect to the (0)-th product. In fact,

$$(a_{(-2)}b)_{(0)}c = \sum_{i=0}^{\infty}(-1)^i\binom{-2}{i}(a_{(-2-i)}(b_{(i)}c) - b_{(-2-i)}(a_{(i)}c)) \in C_2(V)$$

by (B1) for $p = q = 0, r = -2$ and

$$a_{(0)}(b_{(-2)}c) = b_{(-2)}(a_{(0)}c) + (a_{(-2)}b)_{(0)}c \in C_2(V)$$

by (B1) for $p = 0, q = -2, r = 0$, the (0)-th product induces a bilinear map

$$\{\,\} : V/C_2(V) \times V/C_2(V) \longrightarrow V/C_2(V).$$

Since $Ta = a_{(-2)}\mathbf{1}$, we have $TV \subset C_2(V)$. Hence the bracket $\{\,\}$ gives a Lie algebra structure on $V/C_2(V)$, which is a quotient Lie algebra of $\overline{V} = V/TV$.

Moreover, since

$$\begin{aligned}(a_{(-1)}b)_{(0)}c - a_{(-1)}(b_{(0)}c) - b_{(-1)}(a_{(0)}c)&\\ = \sum_{i=1}^{\infty}(a_{(-i-1)}(b_{(i)}c) + b_{(-i-1)}(a_{(i)}c)) &\in C_2(V)\end{aligned}$$

by (B1) for $p = q = 0, r = -1$, the bracket $\{\,\}$ and the multiplication induced by the (-1) st product satisfy the Leibniz rule:

$$\{\alpha\beta, \gamma\} = \alpha\{\beta, \gamma\} + \{\alpha, \gamma\}\beta$$

for $\alpha, \beta, \gamma \in V/C_2(V)$.

Summarizing:

Proposition 8.3.2 (Zhu). *For a vertex algebra V, the* (0)*-th product and the* (-1) *st product induce a commutative Poisson algebra structure on $V/C_2(V)$.*

This result is useful in verifying the finiteness of $\dim V/C_2(V)$, which is important in the study of modular invariance of the trace functions (cf. [Z]).

Remark 8.3.3. Though the (-1) st product is not associative on the original space V, it satisfies

$$(a_{(-1)}b)_{(-1)}c - (b_{(-1)}a)_{(-1)}c = a_{(-1)}(b_{(-1)}c) - b_{(-1)}(a_{(-1)}c)$$

as it immediately follows from (8.3.2). (cf. Subsection 1.4). We set

$$[\![a, b]\!] = \nabla_a b - \nabla_b a, \quad \text{where} \quad \nabla_a b = a_{(-1)} b.$$

Then the above relation is written as

$$[\nabla_a, \nabla_b] = \nabla_{[\![a,b]\!]},$$

from which it follows that the bracket $[\![\,]\!]$ gives a Lie algebra structure on V.

8.4 Zhu's algebra

Let V be a graded vertex algebra. Consider the binary operations

$$\underset{m}{\circ} : V \times V \longrightarrow V$$

defined by

$$a \underset{m}{\circ} b = \sum_{i=0}^{\infty} \binom{\Delta(a)}{i} a_{(m+i)} b$$

for $a, b \in V$. Here a is implicitly supposed to be homogeneous of degree $\Delta(a)$.

Lemma 8.4.1. *Let V be a graded vertex algebra and let $a \in V$ be homogeneous of degree $\Delta(a)$. Then*

$$(Ta) \underset{m}{\circ} b + (\Delta(a) + m + 1) a \underset{m}{\circ} b = -ma \underset{m-1}{\circ} b$$

holds for any $b \in V$ and $m \in \mathbb{Z}$.

Proof. Since $\Delta(Ta) = \Delta(a) + 1$,

$$\begin{aligned}
(Ta) \underset{m}{\circ} b &= \sum_{i=0}^{\infty} \binom{\Delta(a)+1}{i} (Ta)_{(m+i)} b \\
&= \sum_{i=0}^{\infty} \binom{\Delta(a)+1}{i} (-m-i) a_{(m+i-1)} b \\
&= -m \sum_{i=0}^{\infty} \binom{\Delta(a)}{i} a_{(m+i-1)} b - m \sum_{i=0}^{\infty} \binom{\Delta(a)}{i} a_{(m+i)} b \\
&\quad - (\Delta(a)+1) \sum_{i=0}^{\infty} \binom{\Delta(a)}{i} a_{(m+i)} b \\
&= -ma \underset{m-1}{\circ} b - (\Delta(a) + m + 1) a \underset{m}{\circ} b.
\end{aligned}$$

□

For example,

$$(Ta) \underset{0}{\circ} b = -(\Delta(a)+1) a \underset{0}{\circ} b, \quad (Ta) \underset{-1}{\circ} b = -\Delta(a) a \underset{-1}{\circ} b + a \underset{-2}{\circ} b.$$

Lemma 8.4.2. *Let V be a graded vertex algebra. Then*

$$(a \underset{n}{\circ} b) \underset{m}{\circ} c$$
$$= \sum_{k=0}^{\infty} \binom{-n-1}{k} \sum_{i=0}^{\infty} (-1)^i \binom{n}{i} (a \underset{n-i}{\circ} (b \underset{m+k+i}{\circ} c) - (-1)^n b \underset{m+n+k-i}{\circ} (a \underset{i}{\circ} c))$$

holds for any $a, b, c \in V$ and $m, n \in \mathbb{Z}$.

Proof. Using the binomial identities

$$\binom{\Delta(a)+\Delta(b)-i-n-1}{j} = \sum_{k=0}^{\infty} \binom{-n-1}{k} \binom{\Delta(a)+\Delta(b)-i}{j-k},$$
$$\binom{\Delta(a)}{i} \binom{\Delta(a)+\Delta(b)-i}{j} = \sum_{s,t \geq 0, s+t=i+j} \binom{\Delta(a)}{s} \binom{\Delta(b)}{t} \binom{s}{i},$$

and noting $\Delta(a_{(n+1)} b) = \Delta(a) + \Delta(b) - n - i - 1$, we have

$$(a \underset{n}{\circ} b) \underset{m}{\circ} c$$
$$= \sum_{i,j \geq 0} \binom{\Delta(a)}{i} \binom{\Delta(a)+\Delta(b)-i-n-1}{j} (a_{(n+i)} b)_{(m+j)} c$$
$$= \sum_{k=0}^{\infty} \binom{-n-1}{k} \sum_{i,j \geq 0} \sum_{s,t \geq 0, s+t=i+j-k} \binom{\Delta(a)}{s} \binom{\Delta(b)}{t} \binom{s}{i} (a_{(n+i)} b)_{(m+j)} c$$
$$= \sum_{k=0}^{\infty} \binom{-n-1}{k} \sum_{i=0}^{\infty} \sum_{s,t \geq 0} \binom{\Delta(a)}{s} \binom{\Delta(b)}{t} \binom{s}{i} (a_{(n+i)} b)_{(m+s+t+k-i)} c.$$

By the Borcherds identity (B1) for $p = s, q = m + t + k, r = n$, this becomes

$$\sum_{k=0}^{\infty} \binom{-n-1}{k} \sum_{s,t \geq 0} \binom{\Delta(a)}{s} \binom{\Delta(b)}{t}$$
$$\times \sum_{i=0}^{\infty} (-1)^i \binom{n}{i} (a_{(n+s-i)} (b_{(m+t+k+i)} c) - (-1)^n b_{(m+n+t+k-i)} (a_{(s+i)} c))$$
$$= \sum_{k=0}^{\infty} \binom{-n-1}{k} \sum_{i=0}^{\infty} (-1)^i \binom{n}{i} (a \underset{n-i}{\circ} (b \underset{m+k+i}{\circ} c) - (-1)^n b \underset{m+n+k-i}{\circ} (a \underset{i}{\circ} c)).$$

□

Now, for a graded vertex algebra V, we set

$$V \underset{n}{\circ} V = \operatorname{Span}\{a \underset{n}{\circ} b \,|\, a, b \in V\}.$$

Note that $V \underset{-1}{\circ} V = V$ since $a = a_{(-1)}\mathbf{1}$. It follows from Lemma 8.4.1 that

$$V \underset{m-1}{\circ} V \subset V \underset{m}{\circ} V \quad \text{if} \quad m \neq 0.$$

Consider the subspace

$$O(V) = V \underset{-2}{\circ} V = \sum_{n \leq -2} V \underset{n}{\circ} V$$

and set

$$A(V) = V/O(V) = V \underset{-1}{\circ} V / V \underset{-2}{\circ} V.$$

Proposition 8.4.3 (Zhu). *Let V be a graded vertex algebra. Then the operation $\underset{-1}{\circ}$ induces an associative algebra structure on $A(V)$.*

Proof. By Lemma 8.4.2,

$$\begin{aligned}(a \underset{-2}{\circ} b) \underset{-1}{\circ} c = &\sum_{i=0}^{\infty}(-1)^i\binom{-2}{i}(a \underset{-2-i}{\circ} (b \underset{-1+i}{\circ} c) + b \underset{-3-i}{\circ} (a \underset{i}{\circ} c)) \\ &+ \sum_{i=0}^{\infty}(-1)^i\binom{-2}{i}(a \underset{-2-i}{\circ} (b \underset{i}{\circ} c) + b \underset{-2-i}{\circ} (a \underset{i}{\circ} c)) \in O(V)\end{aligned}$$

and

$$a \underset{-1}{\circ} (b \underset{-2}{\circ} c) = (a \underset{-1}{\circ} b) \underset{-2}{\circ} c - \sum_{i=1}^{\infty} a \underset{-1-i}{\circ} (b \underset{-2+i}{\circ} c) - \sum_{i=0}^{\infty} b \underset{-3-i}{\circ} (a \underset{i}{\circ} c) \in O(V).$$

Thus $O(V)$ is an ideal with respect to $\underset{-1}{\circ}$. Hence $\underset{-1}{\circ}$ induces an operation on the quotient space $A(V)$, which is associative since

$$(a \underset{-1}{\circ} b) \underset{-1}{\circ} c - a \underset{-1}{\circ} (b \underset{-1}{\circ} c) = \sum_{i=1}^{\infty} a \underset{-1-i}{\circ} (b \underset{-1+i}{\circ} c) + \sum_{i=0}^{\infty} b \underset{-2-i}{\circ} (a \underset{i}{\circ} c) \in O(V).$$

□

Note that the image of the vacuum vector $\mathbf{1}$ in $A(V)$ is the unit of the algebra. The algebra $A(V)$ is called *Zhu's algebra* associated to V.

Note 8.4.4. Zhu's algebra plays a very important role in the representation theory of vertex operator algebras; There is a functor from the category of admissible V-modules to the category of $A(V)$-modules such that the set of equivalence classes of irreducible admissible V-modules are in one-to-one correspondence with the set of equivalence classes of irreducible $A(V)$-modules. See [Z] and [DLM2] for further information on this topic.

9 Examples

In this section, we will describe various examples of vertex algebras. Here we will restrict our attention to the existence of vertex algebra structure, and we refer the reader to appropriate references for further topics such as those we have briefly described in the preceding sections. We will work over the field $\mathbb{C}$ of complex numbers for simplicity.

9.1 Vertex algebra associated to the free boson

Let ν be a fixed nonzero scalar. Let $\mathcal{A} = \mathcal{A}^\nu$ be the associative algebra generated by $\{\alpha_n \mid n \in \mathbb{Z}\}$ subject to the defining relations

$$[\alpha_m, \alpha_n] = m\nu\delta_{m+n,0}$$

where $[\,,\,]$ denotes the commutator.

Consider the polynomial ring $\mathbb{C}[x_1, x_2, \dots]$ which we regard as an $\mathcal{A}$-module by

$$\alpha_n = \begin{cases} \nu n \dfrac{\partial}{\partial x_n} & (n \geq 1), \\ r & (n = 0), \\ x_{-n} & (n \leq -1) \end{cases}$$

where r is a scalar. This $\mathcal{A}$-module is called the Fock representation of charge r, which we will denote by $\mathcal{F}[r] = \mathcal{F}_\nu^\alpha[r]$. Since we have assumed that $\nu \neq 0$, the Fock space representation is irreducible. We set $|r\rangle = 1$, the unit of the polynomial ring. The Fock representation of charge r is characterized as an $\mathcal{A}$-module generated by a nonzero vector $|r\rangle$ satisfying

$$\alpha_n|r\rangle = \begin{cases} 0 & (n \geq 1), \\ r & (n = 0). \end{cases}$$

Such a vector $|r\rangle$ is called the vacuum of charge r. Consider the following generating series; $\alpha(z) = \sum_{n\in\mathbb{Z}} \alpha_n z^{-n-1}$. Then this is a field on $\mathcal{F}[r]$ which is local to itself with the OPE

$$\alpha(y)\alpha(z) \sim \frac{\nu}{(y-z)^2}.$$

In other words,

$$\alpha(z)_{(n)}\alpha(z) = \begin{cases} 0 & (n \geq 2), \\ \nu I(z) & (n = 1), \\ 0 & (n = 0). \end{cases} \tag{9.1.1}$$

Let $\mathcal{O}$ be the vertex algebra generated by the field $\alpha(z)$ on $M = \oplus_{r\in\mathbb{C}}\mathcal{F}[r]$. Then it becomes an $\mathcal{A}$-module via

$$\mathcal{A} \longrightarrow \mathrm{End}(\mathcal{O}), \quad \alpha_n \longmapsto \alpha(z)_{(n)}.$$

In fact, by (4.3.2) and (9.1.1), we have

$$\begin{aligned}[][\alpha(z)_{(m)}, \alpha(z)_{(n)}] &= \sum_{i=0}^{\infty} \binom{m}{i} (\alpha(z)_{(i)}\alpha(z))_{(m+n-i)} \\ &= m(\alpha(z)_{(1)}\alpha(z))_{(m+n-1)} \\ &= m\nu I(z)_{(m+n-1)} \\ &= m\nu\delta_{m+n,0}\end{aligned}$$

Now, the identity field $I(z)$ satisfies

$$\alpha_n I(z) = \alpha(z)_{(n)} I(z) = 0$$

not only for $n \geq 1$ but also for $n = 0$; it is the vacuum of charge 0. Since $\mathcal{O}$ is generated by $I(z)$ as an $\mathcal{A}$-module, it is isomorphic to $\mathcal{F}[0]$ by the unique homomorphism $\mathcal{F}[0] \longrightarrow \mathcal{O}$ of $\mathcal{A}$-modules that sends $|0\rangle$ to $I(z)$. Since $\mathcal{F}[0]$ is an irreducible $\mathcal{A}$-module, this homomorphism is an isomorphism of $\mathcal{A}$-modules, which provides us with a structure of a vertex algebra on $\mathcal{F}[0]$.

To be more precise, consider the state map

$$\begin{array}{rccl} s|_{\mathcal{O}}: & \mathcal{O} & \longrightarrow & M \\ & A(z) & \longmapsto & |A\rangle, \quad |A\rangle = A_{-1}|I\rangle. \end{array}$$

Then it is a homomorphism of $\mathcal{A}$-modules by Lemma 5.1.2. Since $s(I(z)) = I_{-1}|0\rangle = |0\rangle$, the map $s|_{\mathcal{O}}$ is injective and the image coincides with $\mathcal{F}[0]$. Therefore, by Theorem 5.2.3, we obtain

Proposition 9.1.1. *There exists a unique structure of a vertex algebra on* $V = \mathcal{F}[0]$ *with the vacuum vector* $|0\rangle$ *such that*

$$Y(|\alpha\rangle, z) = \alpha(z), \quad |\alpha\rangle = \alpha_{-1}|0\rangle,$$

which endows $\mathcal{F}[r]$ *with a structure of a* V*-module for any* $r \in \mathbb{C}$.

This is called the vertex algebra associated to the Fock representation of the free boson. The generating series $Y(a, z)$, ($a \in \mathcal{F}[0]$), are given by

$$Y(x_{i_1} \cdots x_{i_n}, z) = \,{}^{\circ}_{\circ}\, \partial^{(i_1-1)}\alpha(z) \cdots \partial^{(i_n-1)}\alpha(z) \,{}^{\circ}_{\circ}\,.$$

The OPEs of them are given by Wick's formula:

$$\begin{aligned} &{}^{\circ}_{\circ}\, \partial^{(p_1)}\alpha(y) \cdots \partial^{(p_m)}\alpha(y) \,{}^{\circ}_{\circ}\, {}^{\circ}_{\circ}\, \partial^{(q_1)}\alpha(z) \cdots \partial^{(q_n)}\alpha(z) \,{}^{\circ}_{\circ} \\ &= \sum_{d=0}^{\max(m,n)} \frac{1}{d!} \sum_{\substack{\phi\,:\,[1,d] \to [1,m], \\ \psi\,:\,[1,d] \to [1,n]}} \prod_{i=1}^{d} \langle \partial^{(p_{\phi(i)})}\alpha(y) \partial^{(q_{\psi(i)})}\alpha(z) \rangle \times \\ &\quad \times \,{}^{\circ}_{\circ} \prod_{j \in [1,d] \setminus \mathrm{Im}\ \phi} \partial^{(p_j)}\alpha(y) \prod_{j \in [1,d] \setminus \mathrm{Im}\ \psi} \partial^{(q_j)}\alpha(z) \,{}^{\circ}_{\circ} \end{aligned}$$

where the second summation is over all injective maps ϕ and ψ and

$$\langle \partial^{(p)}\alpha(y) \partial^{(q)}\alpha(z) \rangle = (-1)^p \frac{(p+q+1)!}{p!q!} \frac{1}{(y-z)^{p+q+2}} \nu.$$

Note 9.1.2. To supply the vertex algebra structure on $\mathcal{F}[0]$, we could have used the existence theorem. In fact, the map $T : \mathcal{F}[0] \longrightarrow \mathcal{F}[0]$ defined by $T = \sum_{n=1}^{\infty} n x_{n+1} \frac{\partial}{\partial x_n}$ has the desired properties $T|0\rangle = 0$, $[T, \alpha(z)] = \partial\alpha(z)$ to apply the theorem.

Note 9.1.3. For $\mu, \nu \neq 0$, the vertex algebras associated to $\mathcal{F}^{\alpha}_{\mu}[0]$ and $\mathcal{F}^{\alpha}_{\nu}[0]$ are isomorphic. In fact, the isomorphism

$$\mathcal{F}^{\alpha}_{\mu}[0] \longrightarrow \mathcal{F}^{\alpha}_{\nu}[0]$$

of polynomial rings defined by

$$x_n \longmapsto \sqrt{\nu/\mu}\, x_n$$

gives rise to an isomorphism of vertex algebras.

9.2 Vertex algebra associated to $\beta\gamma$-system

Let $\mathcal{A} = \mathcal{A}^{\beta\gamma}$ be the associative algebra generated by $\{\beta_n \,|\, n \in \mathbb{Z}\} \cup \{\gamma_n \,|\, n \in \mathbb{Z}\}$ subject to the defining relations

$$[\beta_m, \gamma_n] = -\delta_{m+n,0}, \quad [\beta_m, \beta_n] = [\gamma_m, \gamma_n] = 0$$

where $[\ ,\ ]$ is the commutator.

Let $\mathcal{F}$ denote the polynomial ring $\mathcal{F} = \mathbb{C}[\dots, x_{-1}, x_0, x_1, \dots]$. This becomes an $\mathcal{A}$-module by letting it act as

$$\beta_n = \begin{cases} -\frac{\partial}{\partial x_n}, & (n \geq 1), \\ x_n, & (n \leq 0), \end{cases} \qquad \gamma_n = \begin{cases} \frac{\partial}{\partial x_{-n}}, & (n \geq 0), \\ x_{-n}, & (n \leq -1). \end{cases}$$

We call it the Fock representation of $\mathcal{A}$, which is easily seen to be irreducible. We set $|0\rangle = 1$, the unit of the polynomial ring. Then the Fock representation is characterized as an $\mathcal{A}$-module generated by a nonzero vector $|0\rangle$ satisfying

$$\beta_k|0\rangle = 0, (k \geq 1), \quad \gamma_\ell|0\rangle = 0, (\ell \geq 0).$$

Consider the following generating series:

$$\beta(z) = \sum_{k \in \mathbb{Z}} \beta_k z^{-k}, \quad \gamma(z) = \sum_{\ell \in \mathbb{Z}} \gamma_\ell z^{-\ell-1}.$$

Then these are mutually local fields on $\mathcal{F}$ with the OPE

$$\beta(y)\gamma(z) \sim -\frac{1}{y-z}, \quad \beta(y)\beta(z) \sim 0, \quad \gamma(y)\gamma(z) \sim 0.$$

Further, the fields $\beta(z)$ and $\gamma(z)$ are creative with respect to $|0\rangle$. Now, since $\mathcal{F}$ is a polynomial ring, there exists a unique derivation $T : \mathcal{F} \longrightarrow \mathcal{F}$ such that

$$Tx_n = \begin{cases} nx_{n+1}, & (n \geq 1), \\ (-n+1)x_{n-1}, & (n \leq 0). \end{cases}$$

In other words.

$$T\beta_n|0\rangle = (-n+1)\beta_{n-1}|0\rangle, (n \leq 0), \quad T\gamma_n|0\rangle = -n\gamma_{n-1}|0\rangle, (n \leq -1).$$

The derivation T is explicitly written as

$$T = \sum_{n \geq 1} nx_{n+1}\frac{\partial}{\partial x_n} + \sum_{n \leq 0}(-n+1)x_{n-1}\frac{\partial}{\partial x_n} = -\sum_{k+\ell=-1} k\beta_k\gamma_\ell.$$

Then we easily see that

$$T|0\rangle = T1 = 0, \quad [T, \beta(z)] = \partial\beta(z), \quad [T, \gamma(z)] = \partial\gamma(z).$$

Therefore, by the existence theorem, there exists a unique structure of a vertex algebra on $\mathcal{F}$ with $|I\rangle = |0\rangle$ such that

$$Y(|\beta\rangle, z) = \beta(z), \quad Y(|\gamma\rangle, z) = \gamma(z)$$

where

$$|\beta\rangle = \beta_0|0\rangle = x_0, \quad |\gamma\rangle = \gamma_{-1}|0\rangle = x_1.$$

This vertex algebra is called the $\beta\gamma$-system. We note that $\beta_n = |\beta\rangle_{(n-1)}, \gamma_n = |\gamma\rangle_{(n)}$.

Now consider the vector

$$\begin{aligned} |J\rangle &= -|\beta\rangle_{(-1)}|\gamma\rangle \\ &= -\beta_0\gamma_{-1}|0\rangle = -x_0x_1. \end{aligned}$$

By (4.3.3), we have, for $n \geq 0$,

$$\begin{aligned} |J\rangle_{(n)}|\beta\rangle &= -(|\beta\rangle_{(-1)}|\gamma\rangle)_{(n)}|\beta\rangle \\ &= -\sum_{i=0}^{\infty}(-1)^i\binom{-1}{i}(|\beta\rangle_{(-1-i)}|\gamma\rangle_{(n+i)} + |\gamma\rangle_{(n-1-i)}|\beta\rangle_{(i)})|\beta\rangle \\ &= -\sum_{i=0}^{\infty}(\beta_{-i}\gamma_{i+n}\beta_0|0\rangle + \gamma_{-i+n-1}\beta_{i+1}\beta_0|0\rangle) \\ &= -\beta_0\gamma_n\beta_0|0\rangle \\ &= \begin{cases} -|\beta\rangle & (n = 0), \\ 0 & (n \geq 1). \end{cases} \end{aligned}$$

Similarly,

$$|J\rangle_{(n)}|\gamma\rangle = \begin{cases} |\gamma\rangle, & (n = 0), \\ 0, & (n \geq 1). \end{cases}$$

Therefore, by (4.3.2),

$$\begin{aligned}
|J\rangle_{(n)}|J\rangle &= -|J\rangle_{(n)}(|\beta\rangle_{(-1)}|\gamma\rangle) \\
&= -|\beta\rangle_{(-1)}(|J\rangle_{(n)}|\gamma\rangle) - \sum_{i=0}^{\infty}\binom{n}{i}(|J\rangle_{(i)}|\beta\rangle)_{(n-1-i)}|\gamma\rangle \\
&= -|\beta\rangle_{(-1)}(|J\rangle_{(n)}|\gamma\rangle) - (|J\rangle_{(0)}|\beta\rangle)_{(n-1)}|\gamma\rangle \\
&= \delta_{n,0}(-|\beta\rangle_{(-1)}|\gamma\rangle) + |\beta\rangle_{(n-1)}|\gamma\rangle \\
&= \delta_{n,0}(-|\beta\rangle_{(-1)}|\gamma\rangle + |\beta\rangle_{(-1)}|\gamma\rangle) + \delta_{n,1}(|\beta\rangle_{(0)}|\gamma\rangle) \\
&= \begin{cases} -|0\rangle, & (n=1), \\ 0, & (n=0, n\geq 2). \end{cases}
\end{aligned}$$

The corresponding field

$$J(z) = Y(|J\rangle, z) = -{:}\beta(z)\gamma(z){:}$$

is called the *number current*, and the relations are summarized as the OPE

$$\begin{aligned}
J(y)\beta(z) &\sim -\frac{\beta(z)}{y-z}, \\
J(y)\gamma(z) &\sim \frac{\gamma(z)}{y-z}, \\
J(y)J(z) &\sim -\frac{1}{(y-z)^2}.
\end{aligned}$$

Note 9.2.1. There exists an automorphism $\sigma^q : \mathcal{A} \longrightarrow \mathcal{A}$ such that

$$\sigma^q(\beta_n) = \beta_{n-q}, \quad \sigma^q(\gamma_n) = \gamma_{n+q}$$

for each integer q. Then the composition

$$\mathcal{A} \xrightarrow{\sigma^q} \mathcal{A} \longrightarrow \mathrm{End}(\mathcal{F})$$

gives another structure of an $\mathcal{A}$-module on $\mathcal{F}$, which we denote by $\mathcal{F}_q$. We set $|q\rangle = 1$. Then the module $\mathcal{F}_q$ is characterized as an $\mathcal{A}$-module generated by a nonzero vector $|q\rangle$ satisfying

$$\beta_n|q\rangle = 0, \ (n \geq q+1), \quad \gamma_n|q\rangle = 0, \ (n \geq -q).$$

Such a vector $|q\rangle$ is called the *q-vacuum*. The modules $\mathcal{F}_q$ are *not* isomorphic to each other.

9.3 Vertex algebras associated to affine algebras

Let $\mathfrak{g}$ be a Lie algebra equipped with a symmetric bilinear form $(\,|\,) : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{C}$ which is invariant (or associative): $([X,Y]|Z) = (X|[Y,Z])$ for all $X, Y, Z \in \mathfrak{g}$. Then the associated affine Lie algebra $\hat{\mathfrak{g}}$ is defined to be $\hat{\mathfrak{g}} = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$ with the Lie bracket

$$
\begin{aligned}
&[X \otimes t^m, Y \otimes t^n] = [X,Y] \otimes t^{m+n} + m(Y|Y)\delta_{m+n,0}K, \\
&[K, X \otimes t^m] = 0.
\end{aligned} \tag{9.3.1}
$$

Consider the Lie subalgebras

$$\mathfrak{g}_{\pm} = \mathfrak{g} \otimes \mathbb{C}[t^{\pm 1}]t^{\pm 1}, \quad \mathfrak{g}_0 = \mathfrak{g} \otimes \mathbb{C}1 \oplus \mathbb{C}K.$$

Then we have the triangular decomposition

$$U(\hat{\mathfrak{g}}) \cong U(\hat{\mathfrak{g}}_-) \otimes U(\hat{\mathfrak{g}}_0) \otimes U(\hat{\mathfrak{g}}_+)$$

where $U(\hat{\mathfrak{g}})$ etc denote the universal enveloping algebra of $\hat{\mathfrak{g}}$ etc. Let V_0 be a $\mathfrak{g}$-module. We regard it as a $\hat{\mathfrak{g}}_0 \oplus \hat{\mathfrak{g}}_+$-module by letting K act by a scalar k and $\hat{\mathfrak{g}}_+$ act trivially. The induced module

$$M_k^{\mathfrak{g}}(V_0) = U(\hat{\mathfrak{g}}) \otimes_{U(\hat{\mathfrak{g}}_0 \oplus \hat{\mathfrak{g}}_+)} V_0$$

is called the (generalized) Verma module associated to V_0 at level k.

For each $X \in \mathfrak{g}$, we set

$$X(z) = \sum_{n \in \mathbb{Z}} X_n z^{-n-1}, \quad X_n = X \otimes t^n.$$

Then $\mathcal{S}_k(V_0) = \{X(z) \mid X \in \mathfrak{g}\}$ is a set of pairwise mutually local fields on $M_k^{\mathfrak{g}}(V_0)$, with the OPE

$$X(y)Y(z) \sim \frac{(X|Y)k}{(y-z)^2} + \frac{[X,Y](z)}{y-z}.$$

In other words,

$$
X(z)_{(n)}Y(z) = \begin{cases} [X,Y](z), & (n = 0), \\ (X|Y)k, & (n = 1), \\ 0, & (n \geq 2). \end{cases} \tag{9.3.2}
$$

Now, consider the module $M_k^{\mathfrak{g}}(\mathbb{C})$ induced from the trivial $\mathfrak{g}$-module $\mathbb{C}$. We set $|0\rangle = 1 \otimes 1$. Let $\mathcal{O}_k^{\mathfrak{g}}$ denote the vertex algebra generated by the fields $\mathcal{S}_k(\mathbb{C})$. Then $\mathcal{O}_k^{\mathfrak{g}}$ has a $\hat{\mathfrak{g}}$-module structure by letting $X \otimes t^n$ act by $X(z)_{(n)}$. In fact, by (4.3.2) and (9.3.2), we have

$$[X(z)_{(m)}, Y(z)_{(n)}] = \sum_{i=0}^{\infty} \binom{m}{i} (X(z)_{(i)} Y(z))_{(m+n-i)}$$
$$= [X, Y](z)_{(m+n)} + m(X|Y)k\delta_{m+n,0},$$

which is nothing but (9.3.1). The identity field $I(z) \in \mathcal{O}_k^{\mathfrak{g}}$ satisfies

$$(X \otimes t^n)I(z) = X(z)_{(n)}I(z) = 0, \ (n \geq 0)$$

by (1.4.3), and the space $\mathcal{O}_k^{\mathfrak{g}}$ is generated by $I(z)$ as a $\hat{\mathfrak{g}}$-module. Therefore, by the universal property of the Verma module, there exists a unique surjective homomorphism $f : M_k^{\mathfrak{g}}(\mathbb{C}) \longrightarrow \mathcal{O}_k^{\mathfrak{g}}$ of $\hat{\mathfrak{g}}$-module such that $f(|0\rangle) = I(z)$.

Next we consider the state map

$$s : \mathcal{O}_k^{\mathfrak{g}} \longrightarrow M_k^{\mathfrak{g}}(\mathbb{C}), \quad A(z) \longmapsto A_{-1}|0\rangle.$$

Lemma 5.1.2 shows that it is a homomorphism of $\hat{\mathfrak{g}}$-modules such that $s(I(z)) = |0\rangle$. Therefore, the composition of $s \circ f$ is a homomorphism of $\hat{\mathfrak{g}}$-modules which sends $|0\rangle$ to $|0\rangle$. By the universal property again, $s \circ f$ must coincide with the identity map, which shows that the map f is injective. Hence the map f, as well as s, is an isomorphism of $\hat{\mathfrak{g}}$-modules.

Therefore, by Theorem 5.3.2 applied to $V = M_k^{\mathfrak{g}}(\mathbb{C})$ and $\mathcal{V} = \mathcal{O}_k^{\mathfrak{g}}$, we obtain ([FZ])

Proposition 9.3.1. *There exists a unique vertex algebra structure on $M_k^{\mathfrak{g}}(\mathbb{C})$ with the vacuum vector $\mathbf{1} = |0\rangle$ such that*

$$Y(X_{-1}|0\rangle, z) = X(z)$$

for any $X \in \mathfrak{g}$.

This is called the vertex algebra associated to the Verma module of the affine Lie algebra $\hat{\mathfrak{g}}$. Note that the vertex algebra associated to the free boson described in Subsection 9.1 is a particular case of this construction, for which $\mathfrak{g}$ is the one-dimensional abelian Lie algebra.

See [FZ], [Lian] for further information on this vertex algebra.

9.4 Vertex algebra associated to the Virasoro algebra

Recall that the Virasoro algebra $\mathcal{V}ir$ is, by definition, the Lie algebra

$$\mathcal{V}ir = (\oplus_{n\in\mathbb{Z}}\mathbb{C}L_n) \oplus \mathbb{C}C$$

with the Lie bracket

$$(9.4.1) \qquad [L_m, L_n] = (m-n)L_{m+n} + \frac{m^3-m}{12}\delta_{m+n,0}C, \qquad [C, L_n] = 0.$$

Consider the subalgebras

$$\mathcal{V}ir_+ = (\oplus_{n\geq 0}\mathbb{C}L_n) \oplus \mathbb{C}C, \quad \mathcal{V}ir_- = \oplus_{n<0}\mathbb{C}L_n.$$

Then we have the decomposition

$$U(\mathcal{V}ir) \cong U(\mathcal{V}ir_-) \otimes U(\mathcal{V}ir_+).$$

For each scalar c and h, consider the one-dimensional $\mathcal{V}ir_+$-module $\mathbb{C}|c,h\rangle$ with the action given by

$$L_n|c,h\rangle = 0, \ (n \geq 1), \quad L_0|c,h\rangle = h|c,h\rangle, \quad C|c,h\rangle = c|c,h\rangle.$$

The module $M_c(h) = U(\mathcal{V}ir) \otimes_{U(\mathcal{V}ir_+)} \mathbb{C}|c,h\rangle$ is called the Verma module with the conformal weight h of central charge c.

We set $T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$. This is a field on $M_c(h)$, which is mutually local itself, with OPE

$$T(y)T(z) \sim \frac{c/2}{(y-z)^4} + \frac{2T(z)}{(y-z)^2} + \frac{\partial T(z)}{y-z}.$$

In other words

$$T(z)_{(n)}T(z) = \begin{cases} \partial T(z) & (n=0), \\ 2T(z) & (n=1), \\ c/2 & (n=3), \\ 0 & (n=2, n\geq 4). \end{cases}$$

Now, consider the module $M_c(h)$ for $h=0$. Let $\mathcal{O}_c$ denote the vertex algebra generated by the field $T(z)$ on $M_c(0)$. We set $|0\rangle = 1\otimes 1$. Then $\mathcal{O}_c$ has a $\mathcal{V}ir$-module structure by letting L_n act by $T(z)_{(n+1)}$. In fact, we have

$$\begin{aligned}
&[T(z)_{(m+1)}, T(z)_{(n+1)}] \\
&= \sum_{i=0}^{\infty} \binom{m+1}{i} (T(z)_{(i)} T(z))_{(m+n+2-i)} \\
&= \partial T(z)_{(m+n+2)} + 2(m+1) T(z)_{(m+n+1)} + c/2 \binom{m+1}{3} I(z)_{(m+n-1)} \\
&= -(m+n+2) T(z)_{(m+n+1)} + 2(m+1) T(z)_{(m+n+1)} + \frac{m^3-m}{12} c I(z)_{(m+n-1)} \\
&= (m-n) T(z)_{(m+n+1)} + \frac{m^3-m}{12} \delta_{m+n,0} c,
\end{aligned}$$

which is nothing but (9.4.1). This time, the module $\mathcal{O}_c$ is not isomorphic to $M_c(0)$, since the identity field satisfies $L_n I(z) = 0$ not only for $n \geq 0$ but also for $n = -1$.

Since the vector $L_{-1}|c, h\rangle$ satisfies

$$\begin{aligned}
&L_n(L_{-1}|c,0\rangle) = (n+1)L_{n-1}|c,0\rangle + L_{-1}(L_n|c,0\rangle) = 0, \ (n \geq 1), \\
&L_0(L_{-1}|c,0\rangle) = L_{-1}|c,0\rangle,
\end{aligned}$$

it generates a proper submodule. Consider the quotient module

$$V_c = M_c(0)/U(\mathcal{V}ir)L_{-1}|c,0\rangle$$

and let $|0\rangle$ denote the image of $|c,0\rangle$. Then we have a unique surjective homomorphism $f: V_c \longrightarrow \mathcal{O}_c$ of $\mathcal{V}ir$-modules such that $f(|0\rangle) = I(z)$, which is shown to be the inverse to the state map

$$s: \mathcal{O}_c \longrightarrow V_c, \quad A(z) \longmapsto A_{-1}|0\rangle.$$

Here we have regarded $\mathcal{O}_c$ as a set of fields on V_c. Therefore ([FZ])

Proposition 9.4.1. *There exists a unique vertex algebra structure on V_c with $\mathbf{1} = |0\rangle$ such that*

$$Y(L_{-2}|0\rangle, z) = T(z).$$

This is called the vertex algebra associated to the Verma module of the Virasoro algebra of central charge c.

Now, let $J_c(0)$ be the unique proper maximal submodule of $M_c(0)$ and let J_c be the image of $J_c(0)$ by the projection $M_c(0) \longrightarrow V_c$. Then J_c is an ideal of the vertex algebra V_c. Let

$$L_c(0) = M_c(0)/J_c(0) = V_c/J_c.$$

Then ([FZ])

Proposition 9.4.2. *There exists a unique vertex algebra structure on* $L_c(0)$ *with* $\mathbf{1} = |0\rangle$ *such that*

$$Y(L_{-2}|0\rangle, z) = T(z).$$

Here we have denoted the image of $|0\rangle$ in $L_c(0)$ by the same symbol.

This is called the vertex algebra associated to the irreducible module of the Virasoro algebra of central charge c.

9.5 Lattice vertex algebras

Let L be an even lattice, i.e., a finitely generated free $\mathbb{Z}$-module equipped with a symmetric biadditive map $(\,|\,) : L \times L \longrightarrow \mathbb{Z}$ such that $(\lambda|\lambda) \in 2\mathbb{Z}$ for all $\lambda \in L$. We set $\mathfrak{h} = L \otimes_{\mathbb{Z}} \mathbb{C}$ and regard L as a subset of $\mathfrak{h}$. We extend $(\,|\,)$ to a symmetric bilinear form on $\mathfrak{h}$, and let $\hat{\mathfrak{h}}$ be the corresponding affine Lie algebra where $\mathfrak{h}$ is regarded as an abelian Lie algebra.

For each $\mu \in L$, let $\mathcal{F}[\mu]$ denote the Verma module at level 1 induced from the one-dimensional $\hat{\mathfrak{h}}_+$-module $\mathbb{C}|\mu\rangle$ with

$$h_n|\mu\rangle = \begin{cases} 0 & (n \geq 1), \\ (h|\mu)|\mu\rangle & (n = 0) \end{cases}$$

for all $h \in \mathfrak{h}$: $\mathcal{F} = U(\hat{\mathfrak{h}}) \otimes_{U(\hat{\mathfrak{h}}_+)} \mathbb{C}|\mu\rangle$, which is isomorphic to $U(\hat{\mathfrak{h}}_-) \cong S(\hat{\mathfrak{h}}_-)$ as a vector space, where $S(\hat{\mathfrak{h}}_-)$ denotes the symmetric algebra. We set $\mathcal{F}[L] = \oplus_{\mu \in L}\mathcal{F}[\mu]$. Then we have

$$\mathcal{F}[L] \cong S(\hat{\mathfrak{h}}_-) \otimes_{\mathbb{C}} \mathbb{C}[L]$$

as a vector space, where $\mathbb{C}[L]$ denotes the group ring of the abelian group L. The series $h(z) = \sum_{n \in \mathbb{Z}} h_n z^{-n-1}$, $h \in \mathfrak{h}$ are pairwise mutually local fields on $\mathcal{F}[L]$.

Now, associated to the lattice L, there exists a map $\varepsilon : L \times L \longrightarrow \{\pm 1\}$ satisfying

$$\varepsilon(\lambda,\mu)\varepsilon(\lambda+\mu,\nu) = \varepsilon(\mu,\nu)\varepsilon(\lambda,\mu+\nu), \quad \varepsilon(\lambda,0) = \varepsilon(0,\lambda) = 1 \tag{9.5.1}$$

such that

$$\varepsilon(\lambda,\mu)\varepsilon(\mu,\lambda) = (-1)^{(\lambda|\mu)}. \tag{9.5.2}$$

Let us fix such a map ε, and set

$$e^{\lambda}(P \otimes |\mu\rangle) = \varepsilon(\lambda, \mu) P \otimes |\lambda + \mu\rangle$$

for $P \otimes |\mu\rangle \in \mathcal{F}[\mu]$. Then it defines a map $e^{\lambda} : \mathcal{F}[\mu] \longrightarrow \mathcal{F}[\lambda + \mu]$, which gives us an operator on $\mathcal{F}[L]$. Then we have $[h_n, e^{\lambda}] = (h|\lambda)\delta_{n,0}e^{\lambda}$. Further, by (9.5.1) and (9.5.2), we easily see

$$e^{\lambda}e^{\mu} = \varepsilon(\lambda, \mu)e^{\lambda+\mu} = (-1)^{(\lambda|\mu)}e^{\mu}e^{\lambda}. \tag{9.5.3}$$

On the other hand, we set $z^{\lambda_0}(P \otimes |\mu\rangle) = z^{(\lambda|\mu)} P \otimes |\mu\rangle$. Then we have

$$z^{\lambda_0}e^{\mu} = z^{(\lambda|\mu)}e^{\mu}z^{\lambda_0}. \tag{9.5.4}$$

Now, consider the series $:e^{\phi^{\lambda}(z)}:$ defined by

$$:e^{\phi^{\lambda}(z)}: = \exp\left(\phi^{\lambda}_{-}(z)\right)\exp\left(\phi^{\lambda}_{+}(z)\right)e^{\lambda}z^{\lambda_0}$$

where

$$\phi^{\lambda}_{\pm}(z) = -\sum_{n\in\mathbb{Z}_{\pm}} \frac{\lambda_n}{n} z^{-n}$$

and where the exponential is defined by

$$\exp\left(a(z)\right) = \sum_{\ell=0}^{\infty} \frac{a(z)^{\ell}}{\ell!}.$$

Then the series $:e^{\phi^{\lambda}(z)}:$ is a field on $\mathcal{F}[L]$, called the *vertex operator*.

Let us discuss the locality of the fields $h(z)$ and $:e^{\phi^{\lambda}(z)}:$. First note that

$$\begin{aligned}
[h(z), \phi^{\lambda}_{\pm}(z)] &= -\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}_{\pm}} \frac{1}{n}[h_m, \lambda_n]y^{-m-1}z^{-n} \\
&= -\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}_{\pm}} \frac{(h|\lambda)m\delta_{m+n,0}}{n}y^{-m}z^{-n} \\
&= (h|\lambda)\sum_{n\in\mathbb{Z}_{\pm}} y^{n-1}z^{-n}.
\end{aligned}$$

Therefore, since $[h_m, \lambda_n]$ is central,

$$[h(y), \exp\left(\phi^{\lambda}(z)_{\pm}\right)] = (h|\lambda)\sum_{n\in\mathbb{Z}_{\pm}} y^{n-1}z^{-n}\exp\left(\phi^{\lambda}_{\pm}(z)\right).$$

We also have $[h(z), z^{\lambda_0}] = 0$ and $[h(z), e^{\lambda}] = (h|\lambda)z^{-1}e^{\lambda}$. Therefore, we have

$$[h(y), :e^{\phi^\lambda(z)}:] = (h|\lambda)\sum_{n\in\mathbb{Z}} y^{n-1}z^{-n} :e^{\phi^\lambda(z)}:$$
$$= (h|\lambda)\delta(y-z):e^{\phi^\lambda(z)}:.$$

Hence $h(z)$ and $:e^{\phi^\lambda(z)}:$ are local, with the OPE

$$h(z):e^{\phi^\lambda(z)}: \sim \frac{(h|\lambda)}{y-z}:e^{\phi^\lambda(z)}:.$$

We next consider the locality of $:e^{\phi^\lambda(z)}:$ and $:e^{\phi^\mu(z)}:$. Since $[\lambda_m, \mu_n]$ is central, we have

$$\begin{aligned}
&\exp\left(\phi_+^\lambda(y)\right)\exp\left(\phi_-^\lambda(z)\right)\\
&\quad= \exp\left[\phi_+^\lambda(y), \phi_-^\mu(z)\right]\exp\left(\phi_-^\mu(z)\right)\exp\left(\phi_+^\lambda(y)\right)\\
&\quad= \exp\left(-(\lambda|\mu)\sum_{m=1}^{\infty}\frac{1}{m}y^{-m}z^m\right)\exp\left(\phi_-^\mu(z)\right)\exp\left(\phi_+^\lambda(y)\right)\\
&\quad= \exp\left((\lambda|\mu)\log\left(1-z/y\right)\right)\exp\left(\phi_-^\mu(z)\right)\exp\left(\phi_+^\lambda(y)\right)\\
&\quad= (1-z/y)^{(\lambda|\mu)}\exp\left(\phi_-^\mu(z)\right)\exp\left(\phi_+^\lambda(y)\right),
\end{aligned}$$

where $(1-z/y)^{(\lambda|\mu)}$ is expanded into the series convergent in the region $|y| > |z|$. We also have $y^{\lambda_0}e^{\mu} = y^{(\lambda|\mu)}e^{\mu}y^{\lambda_0}$ by (9.5.4), and $e^{\lambda}e^{\mu} = \varepsilon(\lambda,\mu)e^{\lambda+\mu}$ by (9.5.3). Therefore, we have

$$:e^{\phi^\lambda(y)}::e^{\phi^\mu(z)}: = \varepsilon(\lambda,\mu)(y-z)^{(\lambda|\mu)}:e^{\phi^\lambda(y)+\phi^\mu(z)}:, \ (|y|>|z|)$$

where

$$:e^{\phi^\lambda(y)+\phi^\mu(z)}: = \exp\left(\phi_-^\lambda(y)+\phi_-^\mu(z)\right)\exp\left(\phi_+^\lambda(y)+\phi_+^\mu(z)\right)e^{\lambda+\mu}y^{\lambda_0}z^{\mu_0}.$$

Similarly,

$$:e^{\phi^\mu(z)}::e^{\phi^\lambda(z)}: = \varepsilon(\mu,\lambda)(z-y)^{(\lambda|\mu)}:e^{\phi^\lambda(y)+\phi^\mu(z)}:, \ (|y|<|z|).$$

Therefore, by (9.5.2), we see that

$$(y-z)^{|(\lambda|\mu)|}[:e^{\phi^\lambda(y)}:, :e^{\phi^\mu(z)}:] = 0.$$

Hence the fields $:e^{\phi^\lambda(z)}:$ and $:e^{\phi^\mu(z)}:$ are local.

Thus, we have seen that

$$\mathcal{S} = \{h(z) \mid h \in \mathfrak{h}\} \cup \{ :e^{\phi^\lambda(z)}: \mid \lambda \in L\}$$

is a set of pairwise mutually local fields on $\mathcal{F}[L]$. Then we have (cf. [B1], [FLM])

Proposition 9.5.1 (Borcherds). *There exists a unique vertex algebra structure on $\mathcal{F}[L]$ with $\mathbf{1} = |0\rangle$ such that*

$$Y(h_{-1}|0\rangle, z) = h(z), \quad \textit{and} \quad Y(|\lambda\rangle, z) = :e^{\phi^\lambda(z)}:.$$

Proof. Let $\mathcal{V}_L$ denote the vertex algebra generated by $\mathcal{S}$. The fields in $\mathcal{V}_L$ are creative with respect to $|0\rangle$, and it is easy to see that the state map $s : \mathcal{V}_L \longrightarrow \mathcal{F}[L]$ is surjective. Now, let $T : \mathcal{F}[L] \longrightarrow \mathcal{F}[L]$ be the operator defined by

$$T(P \otimes |\mu\rangle) = (TP + \mu_{-1}) \otimes |\mu\rangle,$$

where TP is defined by the vertex algebra structure on $\mathcal{F}$ as described in Subsection 9.3. Then we easily verify that

$$T|0\rangle = 0, \quad [T, h(z)] = \partial h(z), \quad [T, :e^{\phi^\lambda(z)}:] = \partial :e^{\phi^\lambda(z)}:.$$

Therefore, by the existence theorem (Corollary 6.4.1), we get the result. □

Note that the generating series is given by a linear combination of

$$Y(h^1_{(-i_1-1)} \cdots h^n_{(-i_n-1)}|\lambda\rangle, z) = {}^\circ_\circ \partial^{(i_1)} h^1(z) \cdots \partial^{(i_n)} h^n(z) e^{\phi^\lambda(z)} {}^\circ_\circ,$$

for $h^1, \ldots, h^n \in \mathfrak{h}, \lambda \in L$.

9.6 $W_{1+\infty}$ algebra

Let W be the vector space

$$W = \oplus_{k \geq 0, n \in \mathbb{Z}} \mathbb{C} L^k_n \oplus \mathbb{C}\, C'.$$

Introduce a Lie bracket $[\,,\,] : W \times W \longrightarrow W$ by setting

$$[L^k_m, L^\ell_n] = \sum_{i=0}^{\ell} \binom{\ell}{i} m^{\ell-i} L^{k+i}_{m+n} - \sum_{i=0}^{k} \binom{k}{i} n^{k-i} L^{\ell+i}_{m+n} + \sum_{i=1}^{m-1} (-i)^k (m-i)^\ell \delta_{m+n,0} C', \; (m \geq n), \tag{9.6.1}$$

$$[L^k_m, L^\ell_n] = -[L^\ell_n, L^k_m], \; (m < n),$$

$$[C', L^k_m] = 0.$$

In terms of the generating series

$$E_n(x) = \sum_{k=0}^{\infty} \frac{L_n^k}{k!} x^k,$$

the bracket (9.6.1) turns to the following simple form:

$$[E_m(x), E_n(y)] = -(e^{nx} - e^{my}) E_{m+n}(x+y) + \frac{e^{nx} - e^{my}}{1 - e^{x+y}} \delta_{m+n,0} C'.$$

By this expression, the Jacobi identity is easily verified, and we see that (9.6.1) endows W with a structure of a Lie algebra.

Consider the algebra $\mathcal{D} = \mathbb{C}[t, t^{-1}][\partial_t]$. Then the set $\{t^n D^k \mid k \in \mathbb{Z}_{\geq 0}, n \in \mathbb{Z}\}, D = t\partial_t$ is a basis of $\mathcal{D}$. Since the commutator is given by

$$[t^m e^{xD}, t^n e^{yD}] = t^{m+n}(e^{nx} - e^{my}) e^{(x+y)D},$$

the map $\pi : W \longrightarrow \mathcal{D}$,

$$L_n^k \longmapsto -t^n D^k, \quad C' \longmapsto 0$$

is a homomorphism of Lie algebras; W is a central extension of $\mathcal{D}$:

$$0 \longrightarrow \mathbb{C}C' \longrightarrow W \longrightarrow \mathcal{D} \longrightarrow 0,$$

of which the cocycle is given by

$$\Psi(t^m D^k, t^n D^\ell) = \sum_{i=1}^{m-1} (-i)^k (m-i)^\ell \delta_{m+n,0}, \quad (m \geq n).$$

Let us consider another basis $\{t^{n+k}\partial_t^k \mid k \in \mathbb{Z}_{\geq 0}, n \in \mathbb{Z}\}$ of $\mathcal{D}$. Note that

$$t^{k+n}\partial_t^k = t^n [D]_k, \quad [D]_k = D(D-1)\cdots(D-k+1).$$

Then the commutator is given by

$$[t^{k+m}\partial_t^k, t^{\ell+n}\partial_t^\ell] = \left(\sum_{i=0}^{k} \binom{k}{i} [n+\ell]_i - \sum_{i=0}^{\ell} \binom{\ell}{i} [k+m]_i \right) t^{k+\ell+m+n-i} \partial_t^{k+\ell-i}$$

and the cocycle turns out to be

$$\begin{aligned} \Psi(t^{k+m}\partial_t^k, t^{\ell+n}\partial_t^\ell) &= \Psi(t^m [D]_k, t^n [D]_\ell) \\ &= \sum_{i=1}^{m-1} [-i]_k [m-i]_\ell \delta_{m+n,0} \\ &= (-1)^k k! \ell! \binom{m-k}{k+\ell+1} \delta_{m+n,0}, \quad (m \geq n). \end{aligned}$$

Therefore, the Lie bracket of W is also described as

$$[J^k_m, J^\ell_n] = \left(\sum_{i=0}^{\ell} \binom{\ell}{i}[k+m]_i - \sum_{i=0}^{k} \binom{k}{i}[\ell+n]_i\right) J^{k+\ell-i}_{m+n} + (-1)^k k!\ell! \binom{m-k}{k+\ell+1} \delta_{m+n,0} C', \quad (m \geq n)$$

in term of the basis

$$J^k_n = \sum_{i=0}^{k} \gamma_i L^i_n$$

where $\sum_{i=0}^{k} \gamma_i D^i = [D]_i$ such that $\pi(J^k_n) = t^{k+n}\partial_t^k$. In terms of the generating series $J^k(z) = \sum_{n\in\mathbb{Z}} J^k_n z^{-k-n-1}$, the Lie bracket is written as

$$(9.6.2) \quad [J^k(y), J^\ell(z)] = -\sum_{i=0}^{\infty} \left(\binom{k}{i} J^{k+\ell-i}(y) - (-1)^i \binom{\ell}{i} J^{k+\ell-i}(z)\right) \delta^{(i)}(y-z) + (-1)^{k+1} \frac{k!\ell!}{(k+\ell+1)!} \delta^{(k+\ell+1)}(y-z).$$

We set $\mathcal{P} = \oplus_{k+n\geq 0} \mathbb{C} J^k_n$.

Now let us fix $c \in \mathbb{C}$ and consider the vacuum module

$$M_c = U(W) \otimes_{U(\mathcal{P}\oplus\mathbb{C}C')} \mathbb{C}|c, 0\rangle$$

where

$$J^k_n|c,0\rangle = 0, \ (k+n \geq 0), \quad C'|c,0\rangle = \frac{c}{2}|c,0\rangle.$$

Then the series $J^k(z)$ are fields on M_c which are pairwise mutually local to each other thanks to (9.6.2).

Proposition 9.6.1. *There exists a unique structure of a vertex algebra on M_c with $\mathbf{1} = |c, 0\rangle$ such that*

$$Y(J^k_{-k-1}|c,0\rangle, z) = J^k(z).$$

We refer the reader to [FKRW] for further information.