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A new method for computing the limiting tangent space of an isolated hypersurface singularity via algebraic local cohomology

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Abstract.

Limiting tangent hyperplanes associated with isolated hypersurface singularities are considered in the context of symbolic computation. A new effective method is proposed to compute the limiting tangent space of a given hypersurface. The key of the method is the concept of parametric local cohomology systems. The proposed method can provide the decomposition of the limiting tangent space by Milnor numbers of hyperplane sections of a given hypersurface. The resulting algorithm has been implemented in the computer algebra system Risa/Asir. Examples of the computation for some typical cases are given.

§1. Introduction

We introduce a new approach for studying limiting tangent spaces of an isolated hypersurface singularity. The limiting tangent spaces were introduced in 1965 by H. Whitney [21, 22] and have been extensively utilized in various ways in singularity theory, especially in problems that involve Whitney stratifications.

In pioneering works [7, 11] published in 1977 and 1979, J-P. G. Henry, Lê Dũng Tráng and B. Teissier studied the geometry of the limiting tangent space of a complex analytic surface, and they have highlighted in a series of papers the importance of limiting tangent spaces themselves ([8, 9, 10]). In general, limiting tangent spaces encode, as H. Whitney already showed in [21, 22], much more information of singularities than

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tangent cones. Note that A. G. Flores [2] recently generalized some results on limiting tangent spaces of [12].

In 1991, D. O'Shea gave computation methods of limiting tangent spaces in [17] by noticing the fact that only few computations of limiting tangent spaces had been made. The methods require Gröbner bases computation that involves elimination of many variables.

In this paper, we propose a new method for computing the limiting tangent space of a hypersurface with an isolated singularity. By utilizing the theory of algebraic local cohomology and the Grothendieck local duality theorem on residues, we have introduced, in previous papers [4, 5, 13, 14, 18], a concept of parametric local cohomology systems and have developed a framework for treating parametric problems in local rings.

Based on some results due to B. Teissier [19, 20], we derive in the present paper a new method of computing limiting tangent spaces by adapting the algorithms described in [14]. One of the advantage of the proposed method lies in the fact that the method is free from Gröbner bases computation and the main body consists of linear algebra computation. We emphasize here the fact that the resulting algorithm can provide a stratification of the limiting tangent space by Milnor numbers of hyperplane sections of a given hypersurface. This is another advantage.

This paper is organized as follows. Section 2 reviews algebraic local cohomology, parametric local cohomology systems, μ-stratifications, a definition of limiting tangent spaces and O'Shea's theorem. Section 3 provides a new method for computing limiting tangent spaces of isolated hypersurface singularities and gives some limiting tangent spaces.

§2. Preliminaries

Here we briefly recall the notions of algebraic local cohomology, parametric local cohomology systems and limiting tangent spaces and fix some notations. For details, we refer the reader to [4, 5, 13, 14, 18] for local cohomology, [7, 11, 17, 21, 22] for limiting tangent spaces. The set of natural numbers $\mathbb{N}$ includes zero. $\mathbb{C}$ is the field of complex numbers.

2.1. Algebraic local cohomology

Let X be an open neighborhood of the origin O of the n-dimensional complex space $\mathbb{C}^n$ with coordinates $x = (x_1, x_2, \ldots, x_n)$ and let $\mathcal{O}_X$ be the sheaf on X of holomorphic functions. Let $H^n_{[O]}(\mathcal{O}_X)$ denote the set of algebraic local cohomology classes, defined by $H^n_{[O]}(\mathcal{O}_X) =$

$\lim_{k\to\infty} \mathrm{Ext}^n_{\mathcal{O}_X}(\mathcal{O}_X/\langle x_1, x_2, \ldots, x_n\rangle^k, \mathcal{O}_X)$, where $\langle x_1, x_2, \ldots, x_n\rangle$ is the maximal ideal generated by $x_1, x_2, \ldots, x_n$.

We represent an algebraic local cohomology class as

$$\sum_\lambda c_\lambda \xi^\lambda = \sum c_\lambda \xi_1^{\lambda_1}\xi_2^{\lambda_2}\cdots\xi_n^{\lambda_n}$$

where $\xi^\lambda = \xi_1^{\lambda_1}\xi_2^{\lambda_2}\cdots\xi_n^{\lambda_n}$ and $\xi_1, \xi_2, \ldots, \xi_n$ correspond to $x_1, x_2, \ldots, x_n$ (see [18]). The multiplication is defined as follows:

$$x^\alpha * \xi^\lambda := \begin{cases} \xi^{\lambda-\alpha}, & \text{if } \lambda_i \geq \alpha_i, i = 1, \ldots, n, \\ 0, & \text{otherwise,} \end{cases}$$

where $x^\alpha = x_1^{\alpha_1}\cdots x_n^{\alpha_n} \in \mathbb{C}[x]$, $\xi^\lambda = \xi_1^{\lambda_1}\cdots\xi_n^{\lambda_n} \in H^n_{[O]}(\mathcal{O}_X)$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{N}^n$, and $\lambda - \alpha = (\lambda_1 - \alpha_1, \ldots, \lambda_n - \alpha_n)$. We use the symbol " $*$ " to represent the multiplication. The action of monomials on algebraic local cohomology classes is extended to polynomials by linearity. For example, let $f = 2x_1^2x_2 + x_2 \in \mathbb{C}[x_1, x_2]$ and $\psi = 3\xi_1^3\xi_2^2 + \xi_2 \in H^2_{[O]}(\mathcal{O}_X)$, where $X \subset \mathbb{C}^2$ with coordinates (x_1, x_2). Then,

$$\begin{aligned} f * \psi &= 2x_1^2x_2 * \psi + x_2 * \psi \\ &= (2x_1^2x_2 * 3\xi_1^3\xi_2^2 + 2x_1^2x_2 * \xi_2) + (x_2 * 3\xi_1^2\xi_2^2 + 3x_2 * \xi_2) \\ &= 6\xi_1\xi_2 + 0 + 3\xi_1^2\xi_2 + 1 \\ &= 3\xi_1^2\xi_2 + 6\xi_1\xi_2 + 1. \end{aligned}$$

Let f be a holomorphic function defined on X with an isolated singularity at the origin. We define a vector space $H_{J(f)}$ to be the set of algebraic local cohomology classes in $H^n_{[O]}(\mathcal{O}_X)$ that are annihilated by the Jacobi ideal $J(f) = \langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\rangle$:

$$H_{J(f)} := \left\{ \psi \in H^n_{[O]}(\mathcal{O}_X) \,\middle|\, \frac{\partial f}{\partial x_1} * \psi = \frac{\partial f}{\partial x_2} * \psi = \cdots = \frac{\partial f}{\partial x_n} * \psi = 0 \right\}.$$

It follows from the Grothendieck local duality theorem [4, 5] on residues that the vector space $H_{J(f)}$ is a dual space to the Milnor algebra $\mathcal{O}_{X,O}/J(f)$. Therefore $\dim_{\mathbb{C}}(H_{J(f)})$ is equal to the Milnor number $\mu(f)$ of the singularity.

It is known that, according to a result of M. Artin [1], a defining holomorphic function of any isolated hypersurface singularity can be represented by a polynomial. We can assume therefore that the defining holomorphic function is actually a polynomial.

In our previous works [13, 18], we introduced and implemented algorithms for computing bases of the vector space $H_{J(f)}$ for the case where f is a polynomial.

2.2. Parametric local cohomology systems

We turn to parametric cases of algebraic local cohomology. For details, we refer the reader to [13, 14].

We use the notation t as the abbreviation of m parameters $t_1, \dots, t_m$. For $g_1, \dots, g_k$ in $\mathbb{C}[t]$, $\mathbb{V}(g_1, \dots, g_k)$ denotes the affine variety of $g_1, \dots, g_k$, i.e., $\mathbb{V}(g_1, \dots, g_k) := \{t \in \mathbb{C}^m \mid g_1(t) = \cdots = g_k(t) = 0\}$. We consider a finite partition of $\mathbb{C}^m$ into disjoint algebraically constructible subsets of the form $\mathbb{V}(g_1, \dots, g_k) \setminus \mathbb{V}(g'_1, \dots, g'_{k'}) \subseteq \mathbb{C}^m$. For simplicity, we call those subsets *strata* and let notations $\mathbb{A}_1, \dots, \mathbb{A}_q$, $\mathbb{B}_1, \dots, \mathbb{B}_r$ stand for them.

We define $\mathbb{C}[t]_{\mathbb{A}}$, for a stratum $\mathbb{A} \subseteq \mathbb{C}^m$, as $\mathbb{C}[t]_{\mathbb{A}} = \{\frac{c}{b} | c, b \in \mathbb{C}[t], b(t) \neq 0 \text{ for } t \in \mathbb{A}\}$. Then, for every $\bar{a} \in \mathbb{A}$, we can define the canonical specialization homomorphism $\sigma_{\bar{a}} : \mathbb{C}[t]_{\mathbb{A}}[x] \to \mathbb{C}[x]$ (or $\sigma_{\bar{a}} : \mathbb{C}[t]_{\mathbb{A}}[\xi] \to \mathbb{C}[\xi]$) by putting $t = \bar{a}$. When we say that $\sigma_{\bar{a}}(h_t)$ makes sense for $h_t \in \mathbb{C}(t)[x]$, it has to be understood that $h_t \in \mathbb{C}[t]_{\mathbb{A}}[x]$ for some $\mathbb{A}$ with $\bar{a} \in \mathbb{A}$ where $\mathbb{C}(t)$ is the field of rational functions of t. For instance, let $h_t = t_1 x_1^3 x_2 + \frac{1}{t_2} x_1$ in $\mathbb{C}(t_1, t_2)[x_1, x_2]$ and $(2, 1), (0, \frac{2}{3}) \in \mathbb{C}^2 \setminus \mathbb{V}(t_2)$. Then, $\sigma_{(2,1)}(h_t) = 2x_1^3 x_2 + x_1$ and $\sigma_{(0, \frac{2}{3})}(h_t) = \frac{3}{2} x_1$.

Let h_t be a polynomial in $\mathbb{C}[x]$ with parameters $t = (t_1, t_2, \dots, t_m)$ which generically has an isolated singularity at the origin O, namely, there exists a Zariski open dense subset $U \subset \mathbb{C}^m$ such that for all t in U, h_t has an isolated singularity at the origin. The following notion is used to describe the parameter dependency of the structure of the vector space $H_{J(h_t)}$.

Definition 2.1. *Let $\mathbb{A}_1, \dots, \mathbb{A}_q, \mathbb{B}_1, \dots, \mathbb{B}_r$ be strata in $\mathbb{C}^m$ suth that $\mathbb{A}_1 \cup \cdots \cup \mathbb{A}_q \cup \mathbb{B}_1 \cup \cdots \cup \mathbb{B}_r = \mathbb{C}^m$, $S_1, \dots, S_q$ subsets of $\mathbb{C}(t)[\xi]$. Set $\mathcal{S} = \{(\mathbb{A}_1, S_1), \dots, (\mathbb{A}_q, S_q)\}$ and $\mathcal{W} = \{\mathbb{B}_1, \dots, \mathbb{B}_r\}$. Then, a pair $(\mathcal{S}, \mathcal{W})$ is called a* **parametric local cohomology system (PLCS)** *of $H_{J(h_t)}$ on the parameter space $\mathbb{C}^m$, if for all $i \in \{1, \dots, q\}$, $S_i \subset \mathbb{C}[t]_{\mathbb{A}_i}[\xi]$ and $\bar{a} \in \mathbb{A}_i$, $\sigma_{\bar{a}}(S_i)$ is a basis of the vector space $H_{J(\sigma_{\bar{a}}(h_t))}$ and for all $j \in \{1, \dots, r\}$ and $\bar{b} \in \mathbb{B}_j$, $\sigma_{\bar{b}}(h_t)$ does not define an isolated singularity at the origin. We call a pair $(\mathbb{A}_i, S_i)$ a* **segment** *of the PLCS of $H_{J(h_t)}$, for $1 \leq i \leq q$.*

In the papers [13, 14], we have introduced algorithms for computing a PLCS of $H_{J(h_t)}$ on the parameter space $\mathbb{C}^m$, which have been implemented in a computer algebra system Risa/Asir [16].

Definition 2.2. *Let $\mathbb{A}_1, \dots, \mathbb{A}_q$, $\mathbb{B}_1, \dots, \mathbb{B}_r$ strata in $\mathbb{C}^m$ such that $\mathbb{A}_1 \cup \dots \cup \mathbb{A}_q \cup \mathbb{B}_1 \cup \dots \cup \mathbb{B}_r = \mathbb{C}^m$ and $\mu_1, \dots, \mu_q$ natural numbers. Set $\mathcal{M} = \{(\mathbb{A}_1, \mu_1), \dots, (\mathbb{A}_q, \mu_q)\}$ and $\mathcal{W} = \{\mathbb{B}_1, \dots, \mathbb{B}_r\}$. Then, a pair $(\mathcal{M}, \mathcal{W})$ is called a* μ-**stratification** *of $\mathbb{C}^m$ for h_t if for all $i \in \{1, \dots, q\}$, μ_i is the Milnor number of h_t at the origin $O \in \mathbb{C}^n$ on $\mathbb{A}_i$, and for all $j \in \{1, \dots, r\}$ and $\bar{b} \in \mathbb{B}_j$, $\sigma_{\bar{b}}(h_t)$ does not define an isolated singularity at the origin. We call a pair $(\mathbb{A}_i, \mu_i)$ a* **segment** *of the μ-stratification of h_t, for $1 \le i \le q$.*

Example 1. *Let us consider $h_t = x_1^3 + t_1 x_1 x_2^3 + x_2^6 + t_2 x_2^4$ where x_1, x_2 are variables and t_1, t_2 are parameters. Then, our* Risa/Asir-*implementation outputs Table 1 as a PLCS of $H_{J(h_t)}$ and Milnor numbers μ.*

stratum	*basis of* $H_{J(h_t)}$	μ
$\mathbb{C}^2 \setminus \mathbb{V}(t_1 t_2)$	$\{1, \xi_1, \xi_2, \xi_1\xi_2, \xi_2^2, -\frac{3}{4}t_1\xi_2^3 + t_2\xi_1\xi_2 + \frac{1}{4}a^2\xi_1^2\}$	*6*
$\mathbb{V}(t_2) \setminus \mathbb{V}(t_1, t_2)$	$\{1, \xi_1, \xi_2, \xi_1\xi_2, \xi_2^2, \xi_2^3 - \frac{1}{3}a\xi_1^2, \xi_2^4 - \frac{1}{3}a\xi_1^2\xi_2\}$	*7*
$\mathbb{V}(t_1) \setminus \mathbb{V}(t_1, t_2)$	$\{1, \xi_1, \xi_2, \xi_1\xi_2, \xi_2^2, \xi_1\xi_2^2\}$	*6*
$\mathbb{V}(t_1, t_2)$	$\{1, \xi_1, \xi_2, \xi_1\xi_2, \xi_2^2, \xi_1\xi_2^2, \xi_2^3, \xi_1\xi_2^3, \xi_2^4, \xi_1\xi_2^4\}$	*10*

Table 1. A PLCS of $H_{J(h_t)}$ on $\mathbb{C}^2$

Since the dimension of the vector space $H_{J(h_t)}$ is equal to the Milnor number of h_t, μ-stratifications of parametric polynomials can be constructed by computing PLCS of $H_{J(h_t)}$.

2.3. Limiting tangent spaces

Here we recall a definition of the limiting tangent space for a hypersurface with an isolated singularity and O'Shea's method for computing limiting tangent spaces.

Let $f(x)$ be an holomorphic function with an isolated singularity at the origin and $S = \{x \in X | f(x) = 0\}$. If $x \in S - \{O\}$, we let $T(S, x)$ denote the tangent hyperplane to S at x in $\mathbb{C}^n$ translated so that it passes through the origin. If we identify a hyperplane $p_1x_1 + p_2x_2 + \dots + p_nx_n = 0$ with the conormal vector $[p_1, p_2, \dots, p_n]$ in projective space $\check{\mathbb{P}}^{n-1}$, then we can write the map

$$\mathrm{grad}(f) : x \longrightarrow \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n}\right] \in \check{\mathbb{P}}^{n-1}.$$

Consider the graph $\mathrm{graph}(\mathrm{grad}(f)) \subset S \times \check{\mathbb{P}}^{n-1}$ and the closure $\overline{\mathrm{graph}(\mathrm{grad}(f))}$. Projection onto the first factor $S \times \check{\mathbb{P}}^{n-1} \longrightarrow S$ induces the Nash blow up ν.

$$\begin{tikzcd} \overline{\mathrm{graph}(\mathrm{grad}(f))} \arrow[rr] \arrow[dr, "\nu"] & & S \times \check{\mathbb{P}}^{n-1} \arrow[dl] \\ & S & \end{tikzcd}$$

We denote the second factor of the fiber $\nu^{-1}(O)$ as $K(S, O)$. The space $K(S, O) \subset \check{\mathbb{P}}^{n-1}$ is called the **limiting tangent space** of S. We refer [6, 9, 10, 15, 21, 22] for the notion of limiting tangent spaces.

In general, the limiting tangent space of the tangent cone of S is a subset of the limiting tangent space of the hypersurface S, which means in particular that the limiting tangent space has more information than the tangent cone.

D. O'Shea proved the following theorem for computing limiting tangent spaces in [17].

Theorem 1 (D. O'Shea [17])**.** *Let f be polynomial in $\mathbb{C}[x_1, \ldots, x_n]$ with an isolated singularity at the origin and $S = \{x \in X | f(x) = 0\}$. Let $K(S, O) \subset \check{\mathbb{P}}^{n-1}$ be the limiting tangent space. Let A denote the ideal in $\mathbb{C}[x_1, \ldots, x_n, u, p_1, \ldots, p_n]$ given by setting*

$$A = \left\langle f, p_1 - u\frac{\partial f}{\partial x_1}, \ldots, p_n - u\frac{\partial f}{\partial x_n} \right\rangle.$$

Then, the ideal $\mathbb{I}(K(S, O))$ of $K(S, O)$ in $\check{\mathbb{P}}^{n-1}$ is the radical of the ideal in $\mathbb{C}[p_1, \ldots, p_n]$ given by eliminating u and setting $x_1, \ldots, x_n$ equal to zero. That is,

$$\mathbb{I}(K(S, O)) = \sqrt{A \cap \mathbb{C}[x_1, \ldots, x_n, p_1, \ldots, p_n]/\langle x_1, \ldots, x_n \rangle}.$$

(*For an ideal I, $\sqrt{I}$ is a radical ideal of I.*)

The theorem above implies that the limiting tangent space can be obtained by Gröbner bases computation of A w.r.t. the elimination term order.

§3. Main results

Here we see some results of B. Teissier and give a new computation method of limiting tangent spaces at the singular point O. The key ingredient of the method is the μ-stratification (i.e., PLCS).

3.1. A new computation method

Let $p = (p_1, p_2, \dots, p_n)$ be a non-zero vector and let $[p]$ be the corresponding point in the complex projective space $\check{\mathbb{P}}^{n-1}$. We identify the hyperplane

$$H_p = \{x \in \mathbb{C}^n \mid p_1x_1 + p_2x_2 + \cdots + p_nx_n = 0\}$$

with the point $[p]$ in $\check{\mathbb{P}}^{n-1}$.

Let $f(x)$ be a holomorphic function defined on X with an isolated singularity at the origin O and $S = \{x \in X | f(x) = 0\}$. If the restriction $f|_{H_p}$ of f on $H_p \cap X$ has an isolated singularity at the origin of H_p, we define $\mu^{(n-1)}(f|_{H_p})$ to be $\mu(S \cap H_p)$, the Milnor number at the origin of the hyperplane section $S \cap H_p$. Otherwise we define $\mu^{(n-1)}(f|_{H_p}) = \infty$.

Definition 3.1. *Let $f(x)$ be a holomorphic function defined on X with an isolated singularity at the origin. Then, $\mu^{(n-1)}(f)$ is defined by*

$$\mu^{(n-1)}(f) = \min_{[p]\in\check{\mathbb{P}}^{n-1}} \mu(f|_{H_p}).$$

B. Teissier proved the following important theorems [19, 20].

Theorem 2 (B. Teissier)**.** *Let $U = \{[p] \in \check{\mathbb{P}}^{n-1} \mid \mu^{(n-1)}(f|_{H_p}) = \mu^{(n-1)}(f)\}$. Then, U is Zariski open and a dense subset of $\check{\mathbb{P}}^{n-1}$.*

Theorem 3 (B. Teissier)**.** *For $[p] \in \check{\mathbb{P}}^{n-1}$, the following are equivalent:*

(i) $[p] \in K(S, O)$.
(ii) $\mu^{(n-1)}(f|_{H_p}) > \mu^{(n-1)}(f)$.

Let us recall the cell decomposition of the projective space $\check{\mathbb{P}}^{n-1}$ given by

$$\begin{aligned}\check{\mathbb{P}}^{n-1} &= (\check{\mathbb{P}}^{n-1} - \check{\mathbb{P}}^{n-2}) \cup (\check{\mathbb{P}}^{n-2} - \check{\mathbb{P}}^{n-3}) \cup \cdots \cup (\check{\mathbb{P}}^{1} - \check{\mathbb{P}}^{0}) \cup \check{\mathbb{P}}^{0} \\ &\cong \mathbb{C}^{n-1} \cup \mathbb{C}^{n-2} \cup \cdots \cup \mathbb{C} \cup \check{\mathbb{P}}^{0}\end{aligned}$$

where

$$\check{\mathbb{P}}^{n-i} = \{[p] \in \check{\mathbb{P}}^{n-1} \mid p_1 = p_2 = \cdots = p_i = 0\}, \quad i = 1, 2, ..., n-1.$$

Let $\kappa_i = (\check{\mathbb{P}}^{n-i} - \check{\mathbb{P}}^{n-i-1}) \cap K(S, O)$. Then,

$$\kappa_i = \{[p] \in \check{\mathbb{P}}^{n-i} - \check{\mathbb{P}}^{n-i-1} \mid \mu^{(n-1)}(f|_{H_p}) > \mu^{(n-1)}(f)\}.$$

Now, we are ready to describe a method to compute the limiting tangent space $K(S, O)$.

Method 1

Let $f(x_1, x_2, \ldots, x_n)$ be a polynomial with an isolated singularity at the origin.

Step 1: Let us assume $p_1 \neq 0$ and $[p_1, p_2, \ldots, p_n] \in \check{\mathbb{P}}^{n-1}$. Consider the hyperplane $x_1 = s_2x_2 + s_3x_3 + \cdots + s_nx_n$ where $s_i = -\frac{p_i}{p_1}$ and $i \in \{2, \ldots, n\}$. Compute a μ-stratification of $f(s_2x_2 + s_3x_3 + \cdots + s_nx_n, x_2, \ldots, x_n)$ on the cell $\mathbb{C}^{n-1}$ by the method that is described in section 2.2, where $s_2, s_3, \ldots, s_n$ are parameters. After that, compute the union of the strata whose Milnor number is not the minimum (or the open dense subset's one), and compute κ_1 by $s_i = -\frac{p_i}{p_1}$ and $p_1 \neq 0$. Then, $\kappa_1 \subset \check{\mathbb{P}}^{n-1}$ becomes a part of the limiting tangent space $K(S, O)$ by Theorem 2 and Theorem 3.

Step 2: Next, we consider the case $p_1 = 0$, $p_2 \neq 0$ and $x_2 = s_3x_3 + \cdots + s_nx_n$ where $s_i = -\frac{p_i}{p_2}$ and $i \in \{3, \ldots, n\}$. Compute a μ-stratification of $f(x_1, s_3x_3 + \cdots + s_nx_n, x_3, \ldots, x_n)$ on $\mathbb{C}^{n-2}$ where $s_3, \ldots, s_n$ are parameters. Compute the union of the strata whose Milnor number is not $\mu^{(n-1)}(f)$ and compute κ_2 by $s_i = -\frac{p_i}{p_2}, p_1 = 0$ and $p_2 \neq 0$. The set κ_2 becomes a part of the limiting tangent space $K(S, O)$.

Repeat: Repeat the same procedure until $x_n = 0$ i.e., $(p_1, \ldots, p_{n-1}, p_n) = (0, \ldots, 0, p_n)$ with $p_n \neq 0$. Then,

$$K(S, O) = \kappa_1 \cup \kappa_2 \cup \cdots \cup \kappa_n.$$

Remark that an inequality $\mu^{(n)}(f) \geq \mu^{(n-1)}(f)$ holds [20]. Thus, if a number of elements of a basis of algebraic local cohomology classes associated with $f|_{H_p}$ becomes bigger than $\mu(f)$ on a stratum $\mathbb{A}$ as the halfway result in the computation of a PLCS, then we can stop the computation on the stratum $\mathbb{A}$. That is, $\mathbb{A} \subset K(S, O)$. This technique have been implemented in our implementation of μ-stratifications.

Theorem 4. *Method 1 returns the limiting tangent space of S correctly and terminates.*

Proof. Since the algorithm for computing a μ-stratification always terminates by the remark above and the algorithm for computing PLCSs [14], this method terminates. The correctness also follows from the algorithm for computing a μ-stratification, Theorem 2 and Theorem 3.

Q.E.D.

We can compute a limiting tangent space of S via PLCSs. We give an example to facilitate the method.

Example 2. *Let us consider* $f(x_1, x_2, x_3) = x_1^2 x_3 + x_2^3 + x_3^4 + x_2 x_3^3$ *(*Q_{10}*-singularity) with an isolated singularity at the origin.* $(S = \{x \in X | f(x) = 0\})$

Set a hyperplane H_p *with the point* $[p_1, p_2, p_3]$ *in* $\check{\mathbb{P}}^2$*. First, we consider the case* $p_1 \neq 0$*. Set* $x_1 = s_2 x_2 + s_3 x_3$ *where* $s_2 = -\frac{p_2}{p_1}$ *and* $s_3 = -\frac{p_3}{p_1}$*. Let compute a* μ*-stratification of* $h_{(s_2,s_3)}(x_2, x_3) = f(s_2 x_2 + s_3 x_3, x_2, x_3)$ *where* s_2 *and* s_3 *are parameters. Then, our implementation for computing* μ*-stratifications, returns Table 2.*

strata	μ
$\mathbb{C}^2 \setminus \mathbb{V}(s_2 s_3(4s_2^3 - 27s_3))$, $\mathbb{V}(s_2) \setminus \mathbb{V}(s_2, s_3)$	*4*
$\mathbb{V}(s_3) \setminus \mathbb{V}(s_2, s_3)$, $\mathbb{V}(4s_2^3 - 27s_3) \setminus \mathbb{V}(4s_3^3 - s_3, s_2 - 3s_3)$	*5*
$\mathbb{V}(4s_3^3 - s_3, s_2 - 3s_3) \setminus \mathbb{V}(s_2, s_3)$, $\mathbb{V}(s_2, s_3)$	*6*

Table 2. μ-stratification of $h_{(s_2,s_3)}$

Since the stratum $\mathbb{C}^2 \setminus \mathbb{V}(s_2 s_3(4s_2^3 - 27s_3))$ *is open and dense,* $\mu^{(2)}(f) = 4$*. (Obviously 4 is the minimum.) Thus, the union of strata whose Milnor number is not 4, is*

$$(\mathbb{V}(s_3) \setminus \mathbb{V}(s_2, s_3)) \cup \big(\mathbb{V}(4s_2^3 - 27s_3) \setminus \mathbb{V}(4s_3^3 - s_3, s_2 - 3s_3)\big) \\ \cup \big(\mathbb{V}(4s_3^3 - s_3, s_2 - 3s_3) \setminus \mathbb{V}(s_2, s_3)\big) \cup \mathbb{V}(s_2, s_3) = \mathbb{V}(s_3(4s_2^3 - 27s_3)).$$

As $s_2 = -\frac{p_2}{p_1}$, $s_3 = -\frac{p_3}{p_1}$ *and* $p_1 \neq 0$*, the stratum defined by* $s_3(4s_2^3 - 27s_3) = 0$ *on* $\mathbb{C}^2$ *can be written as* $\kappa_1 = \mathbb{V}(p_3(4p_2^3 - 27p_1^2 p_3)) \setminus \mathbb{V}(p_1, p_2 p_3)$*.*

Second, we consider the case $p_1 = 0$ *and* $p_2 \neq 0$*, i.e.,* $x_2 = -\frac{p_3}{p_2} x_3$*. Set* $x_2 = t_3 x_3$ *where* $t_3 = -\frac{p_3}{p_2}$*. Let compute a* μ*-stratification of* $h_{(t_3)}(x_1, x_3) = f(x_1, t_3 x_3, x_3)$ *where* t_3 *is a parameter. Our implementation returns Table 3.*

strata	μ
$\mathbb{C}^2 \setminus \mathbb{V}(t_3^2 + t^3)$, $\mathbb{V}(t_3 + 1)$	*4*
$\mathbb{V}(t_3)$	*5*

Table 3. μ-stratification of $h_{(t_3)}$

As $\mu^{(2)}(f) = 4$, we require $t_3 = -\frac{p_3}{p_1} = 0$ as a part of $K(S,O)$. Thus, in this case, $p_3 = 0$ corresponds to the limiting tangent hyperplane $H_{\{(0,p_2,0)\}} = \{p_2x_2|p_2 \neq 0\}$. Hence, we have $\kappa_2 = \mathbb{V}(p_1,p_3)\setminus\mathbb{V}(p_1,p_2,p_3)$.

Finally, we consider the case $p_1 = 0$, $p_2 = 0$ and $p_3 \neq 0$, i.e., $x_3 = 0$. Then, the Milnor number of $f(x_1,x_2,0) = x_2^3$ is infinity (∞). Thus, $H_{\{(0,0,p_3)\}} = \{p_3x_3|p_3 \neq 0\}$ is a limiting tangent hyperplane. Hence, we have $\kappa_3 = \mathbb{V}(p_1,p_2)\setminus\mathbb{V}(p_1,p_2,p_3)$. Since

$$\big(\mathbb{V}(p_3(4p_2^3-27p_1^2p_3))\setminus\mathbb{V}(p_1)\big)\cup(\mathbb{V}(p_1,p_3)\setminus\mathbb{V}(p_2))\cup(\mathbb{V}(p_1,p_2)\setminus\mathbb{V}(p_3))$$
$$= \mathbb{V}(p_3(4p_2^3-27p_1^2p_3))\setminus\mathbb{V}(p_1,p_2,p_3),$$

$p_3(4p_2^3-27p_1^2p_3) = 0$ is the limiting tangent space of S. (As $(p_1,p_2,p_3) \neq (0,0,0)$, we omit $\mathbb{V}(p_1,p_2,p_3)$.)

It is easy to see that the limiting tangent space of the tangent cone $x_1^2x_3 + x_2^3 = 0$ of the hypersurface S is $4p_2^3 - 27p_1^2p_3 = 0$.

Let M_S be the set of all Milnor numbers at the origin of hyperplane sections of S: $M_S = \{\mu^{(n-1)}(f|_{H_p}) \in \mathbb{N}\cup\{\infty\} \mid [p] \in \check{\mathbb{P}}^{n-1}\}$. Then, Method 1 can also compute

$$\tau_\mu = \{p \in \check{\mathbb{P}}^{n-1}\,|\mu^{(n-1)}(f|_{H_p}) = \mu\}, \quad \mu \in M_S.$$

The next example is a parametric case.

Example 3. *Let us consider $f(x_1,x_2,x_3) = x_1^2x_2+x_1x_3^3+x_2^2x_3+ax_3^5$ (S_{12} singularity) with an isolated singularity at the origin where "a" is a deformation parameter.*

First, set $h_{(s_2,s_3)}(x_2,x_3) = f(s_2x_2 + s_3x_3, x_2, x_3)$ with parameters s_2 and s_3. Then, we can obtain a μ-stratification of $h_{(s_2,s_3)}$ by our implementation, that is Table 4.

strata	μ
$\mathbb{C}^3\setminus\mathbb{V}(s_3(4s_2s_3+1)a)$, $\mathbb{V}(a)\setminus\mathbb{V}(s_3(4s_2s_3+1),a)$	*4*
$\mathbb{V}(s_3)\setminus\mathbb{V}(s_2(4a-s_2^2)a,s_3)$, $\mathbb{V}(s_2,s_3)\setminus\mathbb{V}(a,s_2,s_3)$ $\mathbb{V}(s_3,a)\setminus\mathbb{V}(a,s_2,s_3)$	*6*
$\mathbb{V}(4s_2s_3+1)\setminus\mathbb{V}(a,4s_2s_3+1)$, $\mathbb{V}(4s_2s_3+1,a)$	*5*
$\mathbb{V}(4a-s_2^2,s_3)\setminus\mathbb{V}(a,s_2,s_3)$	*7*
$\mathbb{V}(a,s_2,s_3)$	∞

Table 4. μ-stratification of $h_{(s_1,s_2)}$

Since the stratum $\mathbb{C}^3 \setminus \mathbb{V}(s_3(4s_2s_3+1)a)$ *is open and dense,* $\mu^{(2)}(f) = 4$. *Thus, the union of strata whose Milnor number is not 4, is* $\mathbb{V}(s_3(4s_2s_3 +1))$. *As* $s_2 = -\frac{p_2}{p_1}$, $s_3 = -\frac{p_3}{p_1}$ *and* $p_1 \neq 0$, *the stratum* $s_3(4s_2s_3+1) = 0$ *on* $\mathbb{C}^3$ *can be written as*

$$\kappa_1 = \mathbb{V}(p_3(4p_2p_3 + p_1^2)) \setminus \mathbb{V}(p_1, p_2p_3).$$

Second, set $h_{(t_3)}(x_1, x_3) = f(x_1, t_3x_3, x_3)$ *with a parameter* t_3. *A* μ*-stratification of* $h_{(t_3)}$ *is Table 5.*

strata	μ
$\mathbb{C}^2 \setminus \mathbb{V}(t_3(10at_3 - 3)(4at_3 - 1))$, $\mathbb{V}(10at_3 - 3)$, $\mathbb{V}(4at_3 - 1)$	*4*
$\mathbb{V}(t_3)$	∞

Table 5. μ-stratification of $h_{(t_3)}$

As $t_3 = -\frac{p_3}{p_2}$, $p_1 = 0$ *and* $p_2 \neq 0$, *the stratum* $t_3 = 0$ *can be written as* $\kappa_2 = \mathbb{V}(p_1, p_3) \setminus \mathbb{V}(p_1, p_2, p_3)$.

Finally, we consider the case $p_1 = 0$, $p_2 = 0$ *and* $p_3 \neq 0$, *i.e.,* $x_3 = 0$. *Then, the Milnor number of* $h(x_1, x_2) = f(x_1, x_2, 0) = x_1^2x_2$ *is infinity* (∞). *Thus,* $\kappa_3 = \mathbb{V}(p_1, p_2) \setminus \mathbb{V}(p_1, p_2, p_3)$.

Hence,

$$\kappa_1 \cup \kappa_2 \cup \kappa_3 = \mathbb{V}(p_3(4p_2p_3 + p_1^2)).$$

Therefore, $p_3(4p_2p_3 + p_1^2) = 0$ *is the limiting tangent space of* S *that does not depend on the parameter* a.

Notice that, if $a = 0$, *then* $M_S = \{4, 5, 6, \infty\}$, *and if* $a \neq 0$ *then* $M_S = \{4, 5, 6, 7, \infty\}$ *holds. Furthermore,* $\tau_7 = \mathbb{V}(4a - s_2^2, s_3) \setminus \mathbb{V}(a, s_2, s_3)$. *Namely, the* μ*-stratification depends on the deformation parameter "*a*" (see* [3]*).*

3.2. Comparisons

Here we give the results of benchmark tests and some limiting tangent spaces. Table 6 shows a comparison of the Risa/Asir implementation of Method 1 with our Risa/Asir implementation of O'Shea's method (Theorem 1) in computation time (CPU time). x, y, z are variables and the hyperplane is $p_1x + p_2y + p_3z = 0$. The term order is the block term order $\{x, y, z\} \gg \{p_1, p_2, p_3\}$ with the total degree reverse lexicographic term order on each block.

All results of limiting tangent spaces in this paper have been computed on a PC with [OS: Windows 7 (64bit), CPU: Intel(R) Core i-7-2600 CPU @ 3.40 GHz 3.40 GHz, RAM: 4 GB]. The time is given in second. In Table 6, $> 2h$ means it takes more than 2 hours.

We use the following typical polynomials that define an isolated singularity.

$$
\begin{aligned}
&Q_{11} : x^3 + y^2z + xz^3 + z^5,\\
&Q_{12} : x^3 + y^5 + yz^2 + xy^4,\\
&S_{11} : x^4 + y^2z + xz^2 + x^3z,\\
&S_{12} : x^2y + y^2z + xz^3 + z^5,\\
&U_{12} : x^3 + y^3 + z^4 + xyz^2,\\
&S_{16} : x^2z + yz^2 + xy^4 + y^6,\\
&S_{17} : x^2z + yz^2 + y^6 + y^4z,\\
&Q_{16} : x^3 + yz^2 + y^7 + xy^5,\\
&Q_{17} : x^3 + yz^2 + xy^5 + y^8,\\
&Q_{18} : x^3 + yz^2 + y^8 + xy^6,\\
&U_{16} : x^3 + xz^2 + y^5 + x^2y^2,\\
&U_{1,3} : x^3 + xz^2 + xy^3 + y^3z^2 + y^4z^2 \ (\mu = 17),\\
&Q_{2,3} : x^3 + yz^2 + x^2y^2 + y^9 + y^{10} \ (\mu = 17),\\
&S_{1,3} : x^2z + yz^2 + x^2y^2 + y^8 + y^9 \ (\mu = 17).\\
&V^{\sharp}_{1,10} : x^2y + z^3 + y^2z^2 + y^3z^2 + y^4 + z^9 + z^{10} \ (\mu = 25).
\end{aligned}
$$

Singularity	O'Shea	Method 1	Limiting tangent space
Q_{11}	0.6084	0.0156	$p_3(4p_1^3 - 27p_2^2p_3) = 0$
Q_{12}	2.184	0.0312	$p_2(4p_1^3 - 27p_2p_3^2) = 0$
S_{11}	1.466	0.0624	$p_1(4p_1p_3 + p_2^2) = 0$
S_{12}	1.279	0.0312	$p_3(p_1^2 + 4p_2p_3) = 0$
U_{12}	0.4056	0.0312	$p_3 = 0$
S_{16}	3.869	0.1092	$p_2(p_1^2 + 4p_2p_3) = 0$
S_{17}	0.5772	0.0468	$p_2(p_1^2 + 4p_2p_3) = 0$
Q_{17}	345.8	0.0468	$p_2(4p_1^3 - 27p_2p_3^2) = 0$
Q_{18}	5.756	0.156	$p_2(4p_1^3 - 27p_2p_3^2) = 0$
U_{16}	1.966	0.0312	$p_3 = 0$
$U_{1,3}$	70.68	0.1248	$p_2 = 0$
$Q_{2,3}$	>2h	0.078	$p_2(4p_1^3 - 27p_2p_3^2) = 0$
$S_{1,3}$	> 2h	0.0624	$p_2(p_1^2 + 4p_2p_3) = 0$
$V^{\sharp}_{1,10}$	>2h	5.881	$p_2(27p_2p_1^2 - 4p_3^3) = 0$

Table 6. comparisons of Method 1 with O'Shea's method

As is evident from Table 6, Method 1 results in better performance in contrast to O'Shea's method. O'Shea's method requires Gröbner bases computation whose computational complexity is quite big (double exponential). In contrast, the new method use PLCSs computation that mainly consists of linear algebra computation. This is the big advantage. That's why the new method results in better performance.

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