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Evolutes of curves in the Lorentz-Minkowski plane

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Abstract.

We can use a moving frame, as in the case of regular plane curves in the Euclidean plane, in order to define the arc-length parameter and the Frenet formula for non-lightlike regular curves in the Lorentz-Minkowski plane. This leads naturally to a well defined evolute associated to non-lightlike regular curves without inflection points in the Lorentz-Minkowski plane. However, at a lightlike point the curve shifts between a spacelike and a timelike region and the evolute cannot be defined by using this moving frame. In this paper, we introduce an alternative frame, the *lightcone frame*, that will allow us to associate an evolute to regular curves without inflection points in the Lorentz-Minkowski plane. Moreover, under appropriate conditions, we shall also be able to obtain globally defined evolutes of regular curves with inflection points. We investigate here the geometric properties of the evolute at lightlike points and inflection points.

§1. Introduction

The evolute of a regular plane curve is a classical subject of differential geometry on Euclidean plane which is defined to be the locus of the centres of the osculating circles of the curve (cf. [3, 7, 8]). It is useful to recognize a vertex of a regular plane curve as a singularity (generically, a 3/2 cusp singularity) of the evolute. Recently, the evolutes have been considered in other spaces, such as hyperbolic, de Sitter, anti de Sitter and Minkowski space, as an application of singularity theory, see [4, 9, 10, 11, 13, 14, 15, 16, 17].

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For a non-lightlike regular curve in the Lorentz-Minkowski plane, we can use a moving frame along the curve and define the arc-length parameter and the Frenet formula. This leads to the definition of the curvature and the evolute of a non-lightlike regular curves without inflection points in the Lorentz-Minkowski plane, see [14] for the definition and properties of the evolute of a non-lightlike regular curves without inflection points. On the other hand, we can consider the caustics of a regular curve, which is defined even at the lightlike points of the curve. Then the caustics of a non-lightlike regular curve without inflection points coincides the evolute.

The lightlike points occur when of the curve moves between spacelike and timelike regions and it can be seen that closed curves in the Lorentz-Minkowski plane must have at least four lightlike points. Hence we can not define the evolute globally by using the standard moving frame. In this paper, we introduce an alternative frame, composed of lightlike vector directions at each point, that we shall call the *lightcone frame*. This allows us to define not only an evolute for the regular curves without inflection points, but also for regular curves with inflection points under certain conditions in the Lorentz-Minkowski plane. We can see that the evolute of a regular curve with lightlike points is a completion of the evolute of a non-lightlike regular curve.

In §2, we introduce the Frenet formula for non-lightlike curves and the evolute of a non-lightlike regular curves without inflection points. In order to consider the lightlike points, we introduce to the lightcone frame in §3. We obtain a kind of a curvature for a regular curve in the Lorentz-Minkowski plane and prove the corresponding existence and uniqueness theorems. In §4, we see that the evolute of a regular curve without inflection points can be regarded not only as a front (a wavefront) but also as a caustic. Furthermore, we describe the behaviour of the evolute at a lightlike point. In §5, we define the evolute of a regular curve with inflection points under appropriate conditions. We show with some examples that the evolutes obtained in this way for the Lorentz-Minkowski geometry happen to be quite different from the corresponding ones in the well known case of the Euclidean geometry.

All maps and manifolds considered here are differentiable of class C^∞.

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§2. Preliminaries

The *Lorentz-Minkowski plane* $\mathbb{R}^2_1$ is the plane $\mathbb{R}^2$ endowed with the metric induced by the pseudo-scalar product $\langle \boldsymbol{u}, \boldsymbol{v}\rangle = -u_0v_0 + u_1v_1$, where $\boldsymbol{u} = (u_0, u_1)$ and $\boldsymbol{v} = (v_0, v_1)$.

We say that a non-zero vector $\boldsymbol{u} \in \mathbb{R}^2_1$ is *spacelike* if $\langle \boldsymbol{u}, \boldsymbol{u}\rangle > 0$, *lightlike* if $\langle \boldsymbol{u}, \boldsymbol{u}\rangle = 0$, and *timelike* if $\langle \boldsymbol{u}, \boldsymbol{u}\rangle < 0$ respectively. The *norm* of a vector $\boldsymbol{u} = (u_0, u_1) \in \mathbb{R}^2_1$ is defined by $||\boldsymbol{u}|| = \sqrt{|\langle \boldsymbol{u}, \boldsymbol{u}\rangle|}$ and the vector $\boldsymbol{u}^\perp$ is given by $\boldsymbol{u}^\perp = (u_1, u_0)$. By definition, $\langle \boldsymbol{u}, \boldsymbol{u}^\perp\rangle = 0$ and $||\boldsymbol{u}|| = ||\boldsymbol{u}^\perp||$. We have $\boldsymbol{u}^\perp = \pm\boldsymbol{u}$ if and only if $\boldsymbol{u}$ is lightlike, and $\boldsymbol{u}^\perp$ is timelike (respectively, spacelike) if and only if $\boldsymbol{u}$ is spacelike (respectively, timelike).

We have the pseudo-circle in $\mathbb{R}^2_1$ with centre $\boldsymbol{v} \in \mathbb{R}^2_1$ and $a \in \mathbb{R}$,

$$PS(\boldsymbol{v}, a) = \{\boldsymbol{u} \in \mathbb{R}^2_1 \mid \langle \boldsymbol{u} - \boldsymbol{v}, \boldsymbol{u} - \boldsymbol{v}\rangle = a\}.$$

We can classify the pseudo-circles with centre $\boldsymbol{v} \in \mathbb{R}^2_1$ and radius $r > 0$ into the following types:

$$\begin{aligned} S^1_1(\boldsymbol{v}, r) &= \{\boldsymbol{u} \in \mathbb{R}^2_1 \mid \langle \boldsymbol{u} - \boldsymbol{v}, \boldsymbol{u} - \boldsymbol{v}\rangle = r^2\}, \\ LC^*(\boldsymbol{v}, 0) &= \{\boldsymbol{u} \in \mathbb{R}^2_1 \mid \langle \boldsymbol{u} - \boldsymbol{v}, \boldsymbol{u} - \boldsymbol{v}\rangle = 0\}, \\ H^1(\boldsymbol{v}, -r) &= \{\boldsymbol{u} \in \mathbb{R}^2_1 \mid \langle \boldsymbol{u} - \boldsymbol{v}, \boldsymbol{u} - \boldsymbol{v}\rangle = -r^2\}. \end{aligned}$$

We denote by $S^1_1(r), LC^*$ and $H^1(-r)$ the pseudo-circles centred at the origin in $\mathbb{R}^2_1$.

Let $\gamma : I \to \mathbb{R}^2_1$ be a smooth curve, where I is an interval of $\mathbb{R}$. We say that γ is *spacelike* (respectively, *timelike*) if $\dot\gamma(t) = (d\gamma/dt)(t)$ is a spacelike (respectively, timelike) vector for any $t \in I$. Moreover, a point $\gamma(t)$ (or, t) is called a *spacelike* (respectively, *lightlike*, *timelike*) point if $\dot\gamma(t)$ is a spacelike (respectively, lightlike, timelike) vector.

Let $\gamma : I \to \mathbb{R}^2_1$ be a spacelike or a timelike curve. In this case, we may take the arc-length parameter s of γ. It follows that $||\gamma'(s)|| = 1$ for all $s \in I$, where $\gamma'(s) = (d\gamma/ds)(s)$. We denote by $\boldsymbol{t}(s)$ the unit tangent vector and $\boldsymbol{n}(s)$ the unit normal vector to $\gamma(s)$ such that $\{\boldsymbol{t}(s), \boldsymbol{n}(s)\}$ is oriented anti-clockwise. Actually, $\boldsymbol{t}(s) = \gamma'(s)$ and $\boldsymbol{n}(s) = (-1)^{\omega+1}\gamma'(s)^\perp$, where $\omega = 1$ if γ is timelike and $\omega = 2$ if γ is spacelike. Then we have the Frenet formula:

$$\begin{pmatrix} \boldsymbol{t}'(s) \\ \boldsymbol{n}'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) \\ \kappa(s) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{t}(s) \\ \boldsymbol{n}(s) \end{pmatrix},$$

where $\kappa(s)$ is defined to be the curvature of γ. Thus,

$$\kappa(s) = \frac{\langle \boldsymbol{t}'(s), \boldsymbol{n}(s)\rangle}{\langle \boldsymbol{n}(s), \boldsymbol{n}(s)\rangle} = (-1)^{\omega+1}\langle \boldsymbol{t}'(s), \boldsymbol{n}(s)\rangle = \langle \gamma''(s), \gamma'(s)^{\perp}\rangle.$$

Even if γ is not parametrised by the arc-length and t denotes the parameter, then the unit tangent and the unit normal vectors to $\gamma(t)$ such that $\{\boldsymbol{t}(t), \boldsymbol{n}(t)\}$ is oriented anti-clockwise are given by

$$\boldsymbol{t}(t) = \frac{\dot\gamma(t)}{||\dot\gamma(t)||},\ \boldsymbol{n}(t) = (-1)^{\omega+1}\frac{\dot\gamma(t)^{\perp}}{||\dot\gamma(t)||}.$$

It follows that

$$\begin{pmatrix} \dot{\boldsymbol{t}}(t) \\ \dot{\boldsymbol{n}}(t) \end{pmatrix} = \begin{pmatrix} 0 & ||\dot\gamma(t)||\kappa(t) \\ ||\dot\gamma(t)||\kappa(t) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{t}(t) \\ \boldsymbol{n}(t) \end{pmatrix}$$

and the curvature is given by $\kappa(t) = \langle \ddot\gamma(t), \dot\gamma(t)^{\perp}\rangle / ||\dot\gamma(t)||^3$.

We call a point $\gamma(t_0)$ (or, t_0) an *inflection point* if $\langle \ddot\gamma(t_0), \dot\gamma(t_0)^{\perp}\rangle = 0$. An inflection point of a spacelike, or a timelike regular curve γ is a point $\gamma(t)$ such that $\kappa(t) = 0$.

The *evolute* of a curve γ without inflection points is defined to be the curve in $\mathbb{R}^2_1$ given by

$$e(t) = \gamma(t) - \frac{1}{\kappa(t)}\boldsymbol{n}(t). \tag{1}$$

The properties of the evolute of a spacelike or a timelike curve are given in [14].

We cannot consider the evolute (1) at a lightlike point, since the curvature is not well defined at it. In this paper, we introduce another frame and define the evolutes of regular curves, both without inflection points and with inflection points under appropriate conditions, in the Lorentz-Minkowski plane.

§3. Lightcone frame

We denote $\mathbb{L}^+ = (1,1)$ and $\mathbb{L}^- = (1,-1)$. By definition, $\mathbb{L}^+$ and $\mathbb{L}^-$ are independent lightlike vectors and $\langle \mathbb{L}^+, \mathbb{L}^-\rangle = -2$. We call $\{\mathbb{L}^+, \mathbb{L}^-\}$ a *lightcone frame* on $\mathbb{R}^2_1$.

Let $\gamma : I \to \mathbb{R}^2_1$ be a regular curve (with lightlike points). There exists a smooth function $(\alpha, \beta) : I \to \mathbb{R}^2 \setminus \{0\}$ such that

$$\dot\gamma(t) = \alpha(t)\mathbb{L}^+ + \beta(t)\mathbb{L}^- \tag{2}$$

for all $t \in I$. We say that a *regular curve* γ *with the lightlike tangential data* (α, β) if the condition (2) holds. Then we have $\dot{\gamma}(t)^{\perp} = \alpha(t)\mathbb{L}^{+} - \beta(t)\mathbb{L}^{-}$. Since $\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle = -4\alpha(t)\beta(t)$, $\gamma(t)$ is a spacelike (respectively, lightlike or timelike) point if and only if $\alpha(t)\beta(t) < 0$ (respectively, $= 0$ or > 0).

Theorem 1. (The Existence Theorem) *Let $(\alpha, \beta) : I \to \mathbb{R}^2 \setminus \{0\}$ be a smooth mapping. There exists a regular curve $\gamma : I \to \mathbb{R}^2_1$ with the lightlike tangential data (α, β).*

Proof. Let $\gamma : I \to \mathbb{R}^2_1$ be

$$\gamma(t) = \left(\int (\alpha(t) + \beta(t)) \; dt, \int (\alpha(t) - \beta(t)) \; dt \right).$$

By a direct calculation, γ is a regular curve and satisfies the condition (2). Q.E.D.

Proposition 1. *If γ and $\widetilde{\gamma} : I \to \mathbb{R}^2_1$ are regular curves with the same lightlike tangential data (α, β), then there exists a constant $c \in \mathbb{R}^2_1$ such that $\widetilde{\gamma}(t) = \gamma(t) + c$.*

Proof. Since $\dot{\gamma}(t) = \dot{\widetilde{\gamma}}(t)$ for all $t \in I$, we have the result. Q.E.D.

The condition of Proposition 1 seems to be strong. We consider a mild condition for the uniqueness as a Lorentz motion.

Definition 1. Let γ and $\widetilde{\gamma} : I \to \mathbb{R}^2_1$ be regular curves. We say that γ and $\widetilde{\gamma}$ are *congruent through a Lorentz motion* if there exist a matrix A and a constant $c \in \mathbb{R}^2_1$ such that $\widetilde{\gamma}(t) = A(\gamma(t)) + c$ for all $t \in I$, where A is given by

$$A = \begin{pmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{pmatrix} \text{ or } A = -\begin{pmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{pmatrix}$$

for some $\theta \in \mathbb{R}$.

Proposition 2. *Let γ and $\widetilde{\gamma} : I \to \mathbb{R}^2_1$ be regular curves with the lightlike tangential data (α, β) and $(\widetilde{\alpha}, \widetilde{\beta})$ respectively. Suppose that γ and $\widetilde{\gamma}$ are congruent through a Lorentz motion, that is, there exist a matrix*

$$A = \begin{pmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{pmatrix} \left(\text{or, } A = -\begin{pmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{pmatrix} \right)$$

and a constant $c \in \mathbb{R}^2_1$ such that $\widetilde{\gamma}(t) = A(\gamma(t)) + c$. Then

$$\widetilde{\alpha}(t) = (\cosh\theta - \sinh\theta)\alpha(t), \; \widetilde{\beta}(t) = (\cosh\theta + \sinh\theta)\beta(t)$$

(*or,* $\widetilde{\alpha}(t) = -(\cosh\theta - \sinh\theta)\alpha(t)$, $\widetilde{\beta}(t) = -(\cosh\theta + \sinh\theta)\beta(t)$).

Proof. Suppose that $\widetilde{\gamma}(t) = A(\gamma(t)) + c$. Since

$$\begin{aligned}\dot{\widetilde{\gamma}}(t) &= A(\dot{\gamma}(t)) = A\left(\alpha(t)\mathbb{L}^+ + \beta(t)\mathbb{L}^-\right) = \alpha(t)A(\mathbb{L}^+) + \beta(t)A(\mathbb{L}^-)\\ &= \alpha(t)(\cosh\theta - \sinh\theta)\mathbb{L}^+ + \beta(t)(\cosh\theta + \sinh\theta)\mathbb{L}^-,\end{aligned}$$

we have the result. Q.E.D.

Note that $\cosh\theta - \sinh\theta = e^{-\theta}$ and $\cosh\theta + \sinh\theta = e^{\theta}$.

Theorem 2. (The Uniqueness Theorem) *Let γ and $\widetilde{\gamma} : I \to \mathbb{R}^2_1$ be regular curves with the lightlike tangential data (α, β) and $(\widetilde{\alpha}, \widetilde{\beta})$ respectively. Suppose that the lightlike points of γ and $\widetilde{\gamma}$ are isolated. If*

$$\alpha(t)\beta(t) = \widetilde{\alpha}(t)\widetilde{\beta}(t)$$

and

$$\dot{\alpha}(t)\beta(t) - \alpha(t)\dot{\beta}(t) = \dot{\widetilde{\alpha}}(t)\widetilde{\beta}(t) - \widetilde{\alpha}(t)\dot{\widetilde{\beta}}(t)$$

for all $t \in I$, then γ and $\widetilde{\gamma}$ are congruent through a Lorentz motion.

Proof. We fix a non-lightlike point $\gamma(t_0)$ of γ and $\widetilde{\gamma}(t_0)$ of $\widetilde{\gamma}$. Then $\alpha(t_0)\beta(t_0) = \widetilde{\alpha}(t_0)\widetilde{\beta}(t_0) > 0$ or < 0. There exists a Lorenz motion, namely, a matrix $A = \begin{pmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{pmatrix}$ and a constant $c \in \mathbb{R}^2_1$, such that

$$\widetilde{\gamma}(t_0) = \pm A(\gamma(t_0)) + c, \quad \dot{\widetilde{\gamma}}(t_0) = \pm A\dot{\gamma}(t_0).$$

By differentiating $\alpha(t)\beta(t) = \widetilde{\alpha}(t)\widetilde{\beta}(t)$, we have

$$\dot{\alpha}(t)\beta(t) + \alpha(t)\dot{\beta}(t) = \dot{\widetilde{\alpha}}(t)\widetilde{\beta}(t) + \widetilde{\alpha}(t)\dot{\widetilde{\beta}}(t).$$

It follows from the second condition

$$\dot{\alpha}(t)\beta(t) - \alpha(t)\dot{\beta}(t) = \dot{\widetilde{\alpha}}(t)\widetilde{\beta}(t) - \widetilde{\alpha}(t)\dot{\widetilde{\beta}}(t)$$

that $\dot{\alpha}(t)\beta(t) = \dot{\widetilde{\alpha}}(t)\widetilde{\beta}(t)$ and $\alpha(t)\dot{\beta}(t) = \widetilde{\alpha}(t)\dot{\widetilde{\beta}}(t)$. Thus we have

$$\begin{pmatrix} \alpha(t) & \widetilde{\alpha}(t) \\ \dot{\alpha}(t) & \dot{\widetilde{\alpha}}(t) \end{pmatrix} \begin{pmatrix} \beta(t) \\ -\widetilde{\beta}(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a non-lightlike point $\gamma(t)$, we have $\alpha(t) \neq 0$ and $\beta(t) \neq 0$. Therefore $\alpha(t)\dot{\widetilde{\alpha}}(t) - \dot{\alpha}(t)\widetilde{\alpha}(t) = 0$ for non-lightlike points. It follows that $(d/dt)(\widetilde{\alpha}(t)/\alpha(t)) = 0$ and hence there is a constant $b \in \mathbb{R}$ such that $\widetilde{\alpha}(t) = b\alpha(t)$. Since $\gamma(t_0)$ is a non-lightlike point and $\widetilde{\alpha}(t_0) = b\alpha(t_0)$, we have $b = \pm e^{-\theta}$. Moreover, $\widetilde{\beta}(t) = (1/b)\beta(t)$ for non-lightlike points. Since lightlike points of γ and $\widetilde{\gamma}$ are isolated, we have $\widetilde{\alpha}(t) = b\alpha(t)$ and $\widetilde{\beta}(t) = (1/b)\beta(t)$ on I. Thus,

$$\widetilde{\alpha}(t) = \pm(\cosh\theta - \sinh\theta)\alpha(t),\ \widetilde{\beta}(t) = \pm(\cosh\theta + \sinh\theta)\beta(t).$$

It follows that $(d/dt)(\widetilde{\gamma}(t) \mp A(\gamma(t))) = 0$. By $\widetilde{\gamma}(t_0) = \pm A(\gamma(t_0)) + c$, we have $\widetilde{\gamma}(t) = \pm A(\gamma(t)) + c$. Therefore, γ and $\widetilde{\gamma}$ are congruent through the Lorentz motion. Q.E.D.

Remark 1. Let $\gamma(t) = (t,t)$ and $\widetilde{\gamma}(t) = (t,-t)$. Since $(\alpha(t),\beta(t)) = (1,0)$ and $(\widetilde{\alpha}(t),\widetilde{\beta}(t)) = (0,1)$, the conditions $\alpha(t)\beta(t) = \widetilde{\alpha}(t)\widetilde{\beta}(t)$ and $\dot{\alpha}(t)\beta(t) - \alpha(t)\dot{\beta}(t) = \dot{\widetilde{\alpha}}(t)\widetilde{\beta}(t) - \widetilde{\alpha}(t)\dot{\widetilde{\beta}}(t)$ in Theorem 2 are satisfied. However, $\mathbb{L}^+$ and $\mathbb{L}^-$ are not congruent through a Lorentz motion by Proposition 2.

§4. Evolutes of regular curves without inflection points

Let $\gamma : I \to \mathbb{R}^2_1$ be a regular curve with the lightlike tangential data (α,β). Since $\langle\ddot{\gamma}(t),\dot{\gamma}(t)^\perp\rangle = 2(\dot{\alpha}(t)\beta(t) - \alpha(t)\dot{\beta}(t))$, $\gamma(t_0)$ is an inflection point of γ if and only if

$$\dot{\alpha}(t_0)\beta(t_0) - \alpha(t_0)\dot{\beta}(t_0) = 0. \tag{3}$$

We define an *evolute* $Ev(\gamma) : I \to \mathbb{R}^2_1$ of $\gamma : I \to \mathbb{R}^2_1$ with the lightlike tangential data (α,β) by

$$Ev(\gamma)(t) = \gamma(t) - \frac{2\alpha(t)\beta(t)}{\dot{\alpha}(t)\beta(t) - \alpha(t)\dot{\beta}(t)}\left(\alpha(t)\mathbb{L}^+ - \beta(t)\mathbb{L}^-\right) \tag{4}$$

without inflection points.

Suppose that γ is a spacelike (or, timelike) regular curve. We have the following expression for the curvature κ in terms of the lightlike tangential data (α,β) of γ.

Proposition 3. *Let $\gamma : I \to \mathbb{R}^2_1$ be a spacelike (or, timelike) regular curve with the lightlike tangential data (α,β). The curvature κ of γ is given by*

$$\kappa(t) = \frac{\dot{\alpha}(t)\beta(t) - \alpha(t)\dot{\beta}(t)}{4|\alpha(t)\beta(t)|\sqrt{|\alpha(t)\beta(t)|}}.$$

Proof. Since $\dot{\gamma}(t) = \alpha(t)\mathbb{L}^+ + \beta(t)\mathbb{L}^-$, we have $\ddot{\gamma}(t) = \dot{\alpha}(t)\mathbb{L}^+ + \dot{\beta}(t)\mathbb{L}^-$ and $\dot{\gamma}(t)^\perp = \alpha(t)\mathbb{L}^+ - \beta(t)\mathbb{L}^-$. It follows that

$$\kappa(t) = \frac{\langle\ddot{\gamma}(t),\dot{\gamma}(t)^\perp\rangle}{||\dot{\gamma}(t)||^3} = \frac{\dot{\alpha}(t)\beta(t) - \alpha(t)\dot{\beta}(t)}{4|\alpha(t)\beta(t)|\sqrt{|\alpha(t)\beta(t)|}}.$$

Q.E.D.

Remark 2. By Proposition 3, the conditions of Theorem 2 say that the curvatures of spacelike (or, timelike) congruent regular curves are the same.

Since

$$\boldsymbol{n}(t) = (-1)^{\omega}\frac{\dot{\gamma}(t)^{\perp}}{||\dot{\gamma}(t)||} = (-1)^{\omega}\frac{\alpha(t)\mathbb{L}^{+} - \beta(t)\mathbb{L}^{-}}{2\sqrt{|\alpha(t)\beta(t)|}}$$

and Proposition 3, the evolute (1) of a regular non-lightlike curve is given by

$$\begin{aligned} e(t) &= \gamma(t) - \frac{1}{\kappa(t)}\boldsymbol{n}(t) \\ &= \gamma(t) + (-1)^{\omega}\frac{2|\alpha(t)\beta(t)|}{\dot{\alpha}(t)\beta(t) - \alpha(t)\dot{\beta}(t)}(\alpha(t)\mathbb{L}^{+} - \beta(t)\mathbb{L}^{-}). \end{aligned}$$

If γ is spacelike (respectively, timelike), then $\omega = 2$ and $\alpha(t)\beta(t) < 0$ (respectively, $\omega = 1$ and $\alpha(t)\beta(t) > 0$). It follows that

$$e(t) = \gamma(t) - \frac{2\alpha(t)\beta(t)}{\dot{\alpha}(t)\beta(t) - \alpha(t)\dot{\beta}(t)}\left(\alpha(t)\mathbb{L}^{+} - \beta(t)\mathbb{L}^{-}\right) = Ev(\gamma)(t).$$

Therefore, the evolute $Ev(\gamma)(t)$ is a generalization of the evolute $e(t)$.

Remark 3. If $\gamma(t_0)$ is a lightlike point of γ, then $\alpha(t_0) = 0$ and $\beta(t_0) \neq 0$, or $\alpha(t_0) \neq 0$ and $\beta(t_0) = 0$. Thus, we have $Ev(\gamma)(t_0) = \gamma(t_0)$.

We see next that the evolute $Ev(\gamma)(t)$ of γ without inflection points can be regarded not only as a front (a wavefront), but also as a caustic.

Let $\gamma : I \to \mathbb{R}^2_1$ be a regular curve with the lightlike tangential data (α, β) and without inflection points. We consider two families of functions:

$$F : I \times \mathbb{R}^2_1 \to \mathbb{R}$$

is given by

$$F(t, \boldsymbol{v}) = \langle \gamma(t) - \boldsymbol{v}, \dot{\gamma}(t) \rangle,$$

and

$$D : I \times \mathbb{R}^2_1 \to \mathbb{R}$$

is given by

$$D(t, \boldsymbol{v}) = \langle \gamma(t) - \boldsymbol{v}, \gamma(t) - \boldsymbol{v} \rangle.$$

Given $\boldsymbol{v} \in \mathbb{R}^2_1$, we denote $f_{\boldsymbol{v}}(t) = F(t, \boldsymbol{v})$ and $d_{\boldsymbol{v}}(t) = D(t, \boldsymbol{v})$.

Proposition 4. (1) $f_{\boldsymbol{v}}(t) = 0$ *if and only if there exists* $\lambda \in \mathbb{R}$ *such that* $\boldsymbol{v} = \gamma(t) - \lambda\dot{\gamma}(t)^{\perp}$.

(2) $f_{\boldsymbol{v}}(t) = \dot{f}_{\boldsymbol{v}}(t) = 0$ *if and only if*

$$\boldsymbol{v} = \gamma(t) - \Big(2\alpha(t)\beta(t)/(\dot{\alpha}(t)\beta(t) - \alpha(t)\dot{\beta}(t))\Big)\,\dot{\gamma}(t)^{\perp}.$$

Proof. (1) $\langle \gamma(t) - \boldsymbol{v}, \dot{\gamma}(t)\rangle = 0$ if and only if there exists $\lambda \in \mathbb{R}$ such that $\gamma(t) - \boldsymbol{v} = \lambda\dot{\gamma}(t)^{\perp}$ if and only if $\boldsymbol{v} = \gamma(t) - \lambda\dot{\gamma}(t)^{\perp}$.

(2) Since $\dot{f}_{\boldsymbol{v}}(t) = \langle \dot{\gamma}(t), \dot{\gamma}(t)\rangle + \langle \gamma(t) - \boldsymbol{v}, \ddot{\gamma}(t)\rangle = -4\alpha(t)\beta(t) + 2\lambda(\dot{\alpha}(t)\beta(t) - \alpha(t)\dot{\beta}(t)) = 0$, we have $\lambda = 2\alpha(t)\beta(t)/(\dot{\alpha}(t)\beta(t) - \alpha(t)\dot{\beta}(t))$. The converse also holds. Q.E.D.

Clearly, we have the following relationship between $f_{\boldsymbol{v}}$ and $d_{\boldsymbol{v}}$: $\dot{d}_{\boldsymbol{v}}(t) = 2f_{\boldsymbol{v}}(t)$. Then, as a consequence of Proposition 4, we obtain the following result.

Proposition 5. (1) $\dot{d}_{\boldsymbol{v}}(t) = 0$ *if and only if there exists* $\lambda \in \mathbb{R}$ *such that* $\boldsymbol{v} = \gamma(t) - \lambda\dot{\gamma}(t)^{\perp}$.

(2) $\dot{d}_{\boldsymbol{v}}(t) = \ddot{d}_{\boldsymbol{v}}(t) = 0$ *if and only if*

$$\boldsymbol{v} = \gamma(t) - \Big(2\alpha(t)\beta(t)/(\dot{\alpha}(t)\beta(t) - \alpha(t)\dot{\beta}(t))\Big)\,\dot{\gamma}(t)^{\perp}.$$

We refer to [1, 2, 11, 12, 13, 18] for the definitions of Morse families in the theories of Legendre and Lagrange singularities. In particular, we shall follow the notations in [11, 12, 13].

Proposition 6. *The map* $F : I \times \mathbb{R}^2_1 \to \mathbb{R}$ *is a Morse family of hypersurfaces, namely,*

$$\left(F, \frac{\partial F}{\partial t}\right) : I \times \mathbb{R}^2_1 \to \mathbb{R} \times \mathbb{R}$$

is non-singular.

Proof. We denote $\gamma(t) = (x(t), y(t))$ and $\boldsymbol{v} = (x, y)$. It is enough to show that

$$\mathrm{rank}\begin{pmatrix} \partial F/\partial t & \partial^2 F/\partial t^2 \\ \partial F/\partial x & \partial^2 F/\partial t\partial x \\ \partial F/\partial y & \partial^2 F/\partial t\partial y \end{pmatrix}(t, \boldsymbol{v}) = 2.$$

Since $F(t, \boldsymbol{v}) = \langle \gamma(t) - \boldsymbol{v}, \dot{\gamma}(t)\rangle = -(x(t) - x)\dot{x}(t) + (y(t) - y)\dot{y}(t)$, we have

$$\frac{\partial F}{\partial x}(t, \boldsymbol{v}) = \dot{x}(t),\ \ \frac{\partial F}{\partial y}(t, \boldsymbol{v}) = -\dot{y}(t),$$

$$\frac{\partial^2 F}{\partial t\partial x}(t, \boldsymbol{v}) = \ddot{x}(t),\ \ \frac{\partial^2 F}{\partial t\partial y}(t, \boldsymbol{v}) = -\ddot{y}(t).$$

It follows that $-\dot{x}(t)\ddot{y}(t)+\ddot{x}(t)\dot{y}(t)=-\langle\ddot{\gamma}(t),\dot{\gamma}(t)^{\perp}\rangle\neq 0$. Q.E.D.

The discriminant set of F is given by

$$\Sigma(F)=\left\{(t,\boldsymbol{v})\in I\times\mathbb{R}^2_1 \mid f_{\boldsymbol{v}}(t)=f'_{\boldsymbol{v}}(t)=0\right\}.$$

We consider the projective cotangent bundle $\pi : PT^*\mathbb{R}^2_1\to\mathbb{R}^2_1$ over $\mathbb{R}^2_1$. By Proposition 6, we have that $\Sigma(F)$ is a 1-dimensional submanifold and

$$\mathcal{L}_F:\Sigma(F)\to PT^*\mathbb{R}^2_1;\ (t,\boldsymbol{v})\mapsto\left(\boldsymbol{v},\left[\frac{\partial F}{\partial x}(t,\boldsymbol{v}):\frac{\partial F}{\partial y}(t,\boldsymbol{v})\right]\right)$$

is a Legendre immersion with respect to the canonical contact structure on $PT^*\mathbb{R}^2_1$. Now, it follows from Proposition 4 that $\pi\circ\mathcal{L}_F(\Sigma(F))$ coincides with the evolute of γ. Therefore, we get that the evolute $Ev(\gamma)$ can be interpreted as the front (wavefront) of the $\mathcal{L}_F$.

Proposition 7. *The map $D : I\times\mathbb{R}^2_1\to\mathbb{R}$ is a Morse family of functions, namely,*

$$\frac{\partial D}{\partial t}:I\times\mathbb{R}^2_1\to\mathbb{R}$$

is a non-singular.

Proof. We use the same notations as in the proof of Proposition 6. Since $(\partial D/\partial t)(t,\boldsymbol{v})=2F(t,\boldsymbol{v})$, it is enough to show that the gradient vector of F is non-zero. $(\partial F/\partial x)(t,\boldsymbol{v})=\dot{x}(t),(\partial F/\partial y)(t,\boldsymbol{v})=-\dot{y}(t)$ and since γ is a regular curve, we have the conclusion. Q.E.D.

The catastrophe set and the bifurcation set of D are respectively given by

$$C(D)=\left\{(t,\boldsymbol{v})\in I\times\mathbb{R}^2_1 \mid \dot{d}_{\boldsymbol{v}}(t)=0\right\}$$

and

$$B_D=\left\{\boldsymbol{v}\in\mathbb{R}^2_1 | \text{ there exists } t\in I \text{ such that } (t,\boldsymbol{v})\in C(D),\ \ddot{d}_{\boldsymbol{v}}(t)=0\right\}.$$

We consider the cotangent bundle $\widetilde{\pi} : T^*\mathbb{R}^2_1\to\mathbb{R}^2_1$ over $\mathbb{R}^2_1$. By Proposition 7, $C(D)$ is a smooth 2-dimensional submanifold and

$$L(D):C(D)\to T^*\mathbb{R}^2_1;\ (t,\boldsymbol{v})\mapsto\left(\boldsymbol{v},\frac{\partial D}{\partial x}(t,\boldsymbol{v}),\frac{\partial D}{\partial y}(t,\boldsymbol{v})\right)$$

is a Lagrange immersion with respect to the canonical symplectic structure on $T^*\mathbb{R}^2_1$. By Proposition 5, the critical value set of $\widetilde{\pi}\circ L(D)$ is the bifurcation set of D. Therefore, the evolute $Ev(\gamma)$ is the caustic of $L(D)$.

Example 1. Let $\gamma : [0, 2\pi) \to \mathbb{R}^2_1$ be a circle $\gamma(t) = (r\cos t, r\sin t)$ in the Minkowski plane, where $r > 0$. Since

$$\begin{aligned}\dot\gamma(t) &= (-r\sin t, r\cos t)\\ &= \frac{1}{2}(-r\sin t + r\cos t)\,\mathbb{L}^+ + \frac{1}{2}(-r\sin t - r\cos t)\,\mathbb{L}^-,\end{aligned}$$

we have

$$\alpha(t) = \frac{1}{2}(-r\sin t + r\cos t)\,,\ \beta(t) = \frac{1}{2}(-r\sin t - r\cos t)\,.$$

It follows that the evolute of the circle is given by

$$Ev(\gamma)(t) = \big(r(1-\sin^2 t + \cos^2 t)\cos t, r(1+\sin^2 t - \cos^2 t)\sin t\big)\,,$$

see Figure 1.

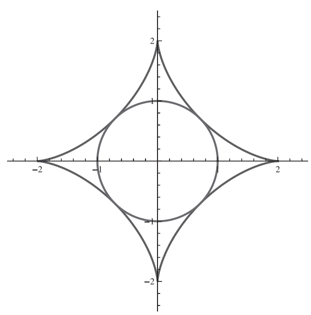


Figure 1. the circle with $r = 1$ and the evolute.

Remark 4. It is worth noting that the evolute of circles in the Euclidean plane is a point. Therefore the evolute in the Lorenz-Minkowski plane is different from the evolute in the Euclidean plane.

A point t (or, $\gamma(t)$) is called a *vertex* for a non-lightlike regular curve γ if $\dot\kappa(t) = 0$. The following result has been given in [14].

Proposition 8. ([14, Proposition 3.2]) *Let $\gamma : I \to \mathbb{R}^2_1$ be a non-lightlike regular curve without inflection points.*

(1) *The evolute of a spacelike* (*respectively, timelike*) *curve is a time-like* (*respectively, spacelike*) *curve.*

(2) *The evolute of γ is singular precisely at the vertices of γ.*

We consider now the case of lightlike points.

Proposition 9. *Let $\gamma : I \to \mathbb{R}^2_1$ be a regular curve with the lightlike tangential data (α, β) and without inflection points.*

(1) *If $\gamma(t_0)$ is a lightlike point of γ, then $Ev(\gamma)(t_0)$ is also a lightlike point of $Ev(\gamma)$.*

(2) *If $\gamma(t_0)$ is a lightlike point of γ, then $Ev(\gamma)(t_0)$ is a regular point of $Ev(\gamma)$.*

Proof. (1) By definition of the evolute of γ, we have

$$\dot{Ev}(\gamma)(t) = \alpha(t)\mathbb{L}^+ + \beta(t)\mathbb{L}^- - \frac{2\alpha(t)\beta(t)}{\dot\alpha(t)\beta(t) - \alpha(t)\dot\beta(t)}\left(\dot\alpha(t)\mathbb{L}^+ - \dot\beta(t)\mathbb{L}^-\right) - \frac{d}{dt}\left(\frac{2\alpha(t)\beta(t)}{\dot\alpha(t)\beta(t) - \alpha(t)\dot\beta(t)}\right)\left(\alpha(t)\mathbb{L}^+ - \beta(t)\mathbb{L}^-\right).$$

Moreover, $(d/dt)(2\alpha(t)\beta(t)/\dot\alpha(t)\beta(t) - \alpha(t)\dot\beta(t))$ is given by

$$2\frac{\dot\alpha^2(t)\beta^2(t) - \alpha^2(t)\dot\beta^2(t) - \alpha(t)\beta(t)(\ddot\alpha(t)\beta(t) - \alpha(t)\ddot\beta(t))}{\left(\dot\alpha(t)\beta(t) - \alpha(t)\dot\beta(t)\right)^2}.$$

If $\alpha(t_0) = 0$ and $\beta(t_0) \neq 0$, then $\dot{Ev}(\gamma)(t_0) = 3\beta(t_0)\mathbb{L}^-$. On the other hand, if $\beta(t_0) = 0$ and $\alpha(t_0) \neq 0$, then $\dot{Ev}(\gamma)(t_0) = 3\alpha(t_0)\mathbb{L}^+$. Hence $Ev(\gamma)(t_0)$ is also a lightlike point of $Ev(\gamma)$.

(2) By the same calculation of (1), $\dot{Ev}(\gamma)(t_0) \neq 0$ at a lightlike point $\gamma(t_0)$ of the curve. Q.E.D.

If we denote $\dot{Ev}(\gamma)(t) = \alpha_{Ev}(t)\mathbb{L}^+ + \beta_{Ev}(t)\mathbb{L}^-$, then $\alpha_{Ev}(t) =$

$$\alpha(t)\left(\frac{-3\dot\alpha^2(t)\beta^2(t) + 3\alpha^2(t)\dot\beta^2(t) + 2\alpha(t)\beta(t)(\ddot\alpha(t)\beta(t) - \alpha(t)\ddot\beta(t))}{(\dot\alpha(t)\beta(t) - \alpha(t)\dot\beta(t))^2}\right),$$

$\beta_{Ev}(t) =$

$$-\beta(t)\left(\frac{-3\dot\alpha^2(t)\beta^2(t) + 3\alpha^2(t)\dot\beta^2(t) + 2\alpha(t)\beta(t)(\ddot\alpha(t)\beta(t) - \alpha(t)\ddot\beta(t))}{(\dot\alpha(t)\beta(t) - \alpha(t)\dot\beta(t))^2}\right).$$

As a corollary of Propositions 8 and 9, we have the following result.

Corollary 1. *Let $\gamma : I \to \mathbb{R}^2_1$ be a regular curve with lightlike tangential data (α, β) and without inflection points.*

(1) *Suppose that $Ev(\gamma)$ is a regular curve. Then γ is a spacelike (respectively, lightlike or timelike) curve if and only if $Ev(\gamma)$ is a timelike (respectively, lightlike or spacelike) curve.*

(2) *The evolute $Ev(\gamma)$ is singular precisely at the vertices of γ.*

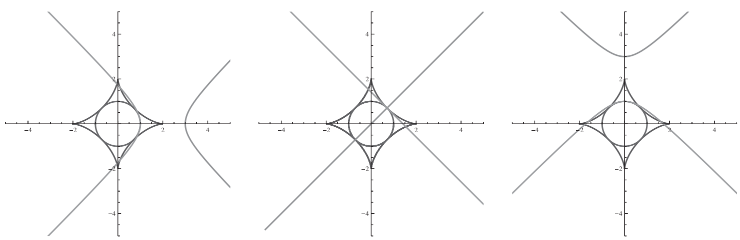
The singularities of $d_{\boldsymbol{v}}$ estimate the contact of γ with the pseudo circles. By Proposition 5, the evolute is given by the locus of the centres of the pseudo circles of at least second order contact with γ at t_0. This pseudo circle is given by its centre $\boldsymbol{v} = Ev(\gamma)(t_0)$ and radius $r = ||\gamma(t_0) - \boldsymbol{v}||$, namely,

$$\begin{aligned}&PS(\boldsymbol{v}, \langle \gamma(t_0) - \boldsymbol{v}, \gamma(t_0) - \boldsymbol{v}\rangle)\\ &= \{(x,y) \in \mathbb{R}^2_1 \mid \langle (x,y) - \boldsymbol{v}, (x,y) - \boldsymbol{v}\rangle = \langle \gamma(t_0) - \boldsymbol{v}, \gamma(t_0) - \boldsymbol{v}\rangle\}.\end{aligned}$$

By a direct calculation, we have

$$\langle \gamma(t_0) - \boldsymbol{v}, \gamma(t_0) - \boldsymbol{v}\rangle = 4\left(\frac{2\alpha(t_0)\beta(t_0)}{\dot{\alpha}(t_0)\beta(t_0) - \alpha(t_0)\dot{\beta}(t_0)}\right)^2 \alpha(t_0)\beta(t_0).$$

Since $\gamma(t_0)$ is a timelike (respectively, lightlike, or spacelike) point of $\gamma(t)$ if and only if $\alpha(t_0)\beta(t_0) > 0$ (respectively, $= 0$ or < 0), the pseudo circle is $S^1_1(\boldsymbol{v}, r)$ (respectively, $LC^*(\boldsymbol{v}, 0)$ or $H^1(\boldsymbol{v}, -r)$), see Figure 2.



$t = 0$ $t = \pi/4$ $t = \pi/2$

Figure 2. The pseudo circles and the evolute of the circle in Example 1.

§5. Evolutes of regular curves with inflection points

In the Euclidean plane, we cannot define the evolutes of regular curves and fronts at their inflection points (cf. [3, 5, 7, 8]). On the other hand, under appropriate conditions in the Euclidean plane, we can define an evolute at the inflection points of a frontal (cf. [6]).

In the Lorentz-Minkowski plane, the lightlike points play the role of the singular points. We may define the evolute of a regular curve at its inflection points under appropriate conditions. It follows that the situation in both cases, the Euclidean geometry and the Lorentz-Minkowski geometry, appears to be quite different.

Let $\gamma : I \to \mathbb{R}^2_1$ be a regular curve with inflection points, having lightlike tangential data (α, β). We may define an evolute under the following existence and uniqueness conditions:

Definition 2. The *evolute* $Ev(\gamma) : I \to \mathbb{R}^2_1$ *of* γ is given by

$$Ev(\gamma)(t) = \gamma(t) + \lambda(t)(\alpha(t)\mathbb{L}^+ - \beta(t)\mathbb{L}^-), \tag{5}$$

if there exists a unique smooth function $\lambda : I \to \mathbb{R}$ such that

$$-2\alpha(t)\beta(t) = \lambda(t)(\dot\alpha(t)\beta(t) - \alpha(t)\dot\beta(t)).$$

In such case, we say that *the evolute* $Ev(\gamma)$ *exists.*

The uniqueness condition is well-known as a topological condition.

Lemma 1. *Suppose that there exists a continuous function* $\lambda : I \to \mathbb{R}$ *such that* $\lambda(t) = -2\alpha(t)\beta(t)/(\dot\alpha(t)\beta(t) - \alpha(t)\dot\beta(t))$ *on* $\Lambda = \{t \in I \mid \dot\alpha(t)\beta(t) - \alpha(t)\dot\beta(t) \neq 0\}$*. Then the function* λ *is a unique if and only if* Λ *is a dense subset of* I*.*

Remark 5. If the inflection points are isolated, then the condition that Λ is a dense subset of I is satisfied.

In this section, we assume that $\Lambda = \{t \in I \mid \dot\alpha(t)\beta(t) - \alpha(t)\dot\beta(t) \neq 0\}$ is a dense subset of I. Then we have that if such a smooth function λ exists, the uniqueness condition is guaranteed by Lemma 1.

Observe that provided the evolute $Ev(\gamma)$ exists at an inflection point, then this must be a lightlike point of γ. Since γ is a regular curve, the function D is a Morse family of functions. Hence $Ev(\gamma)$ is still a caustic of $L(D)$. However, the function F is not a Morse family of hypersurface.

We can now prove an extension of Proposition 9 including the inflection points case.

Proposition 10. *Let* $\gamma : I \to \mathbb{R}^2_1$ *be a regular curve with the lightlike tangential data* (α, β)*. Suppose that the evolute* $Ev(\gamma)$ *exists and* $-2\alpha(t)\beta(t) = \lambda(t)(\dot\alpha(t)\beta(t) - \alpha(t)\dot\beta(t))$*.*

(1) *If* $\gamma(t_0)$ *is an inflection point of* γ *and a regular point of* $Ev(\gamma)$*, then* $Ev(\gamma)(t_0)$ *is a lightlike point of* $Ev(\gamma)$*. Moreover,* $Ev(\gamma)(t_0)$ *is an inflection point of* $Ev(\gamma)$*.*

(2) *Suppose that* $\gamma(t_0)$ *is a lightlike point. Then* $Ev(\gamma)(t_0)$ *is a singular point of* $Ev(\gamma)$ *if and only if one of the following condition holds.*

(*i*) $\alpha(t_0) = \lambda(t_0) = 1 - \dot\lambda(t_0) = 0$ *and* $\beta(t_0) \neq 0$,

(*ii*) $\beta(t_0) = \lambda(t_0) = 1 + \dot\lambda(t_0) = 0$ *and* $\alpha(t_0) \neq 0$,

(*iii*) $\alpha(t_0) = \dot\alpha(t_0) = (1-\dot\lambda(t_0))\beta(t_0) - \lambda(t_0)\dot\beta(t_0) = 0$ *and* $\beta(t_0) \neq 0$,

(*iv*) $\beta(t_0) = \dot\beta(t_0) = (1+\dot\lambda(t_0))\alpha(t_0) + \lambda(t_0)\dot\alpha(t_0) = 0$ *and* $\alpha(t_0) \neq 0$.

Proof. (1) By differentiating the evolute

$$Ev(\gamma)(t) = \gamma(t) + \lambda(t)(\alpha(t)\mathbb{L}^+ - \beta(t)\mathbb{L}^-),$$

we have

$$\frac{d}{dt}Ev(\gamma)(t) = \Big((1+\dot\lambda(t))\alpha(t)+\lambda(t)\dot\alpha(t)\Big)\mathbb{L}^+ + \Big((1-\dot\lambda(t))\beta(t)-\lambda(t)\dot\beta(t)\Big)\mathbb{L}^-.$$

It follows that $\alpha_{Ev}(t) = (1+\dot\lambda(t))\alpha(t)+\lambda(t)\dot\alpha(t)$ and $\beta_{Ev}(t) = (1-\dot\lambda(t))\beta(t)-\lambda(t)\dot\beta(t)$. Since $\gamma(t_0)$ is an inflection point of γ, it holds that $\gamma(t_0)$ is a lightlike point of γ. It follows that $\alpha(t_0)=\dot\alpha(t_0)=0, \beta(t_0)\neq 0$ or $\beta(t_0)=\dot\beta(t_0)=0, \alpha(t_0)\neq 0$. Therefore, we have $\alpha_{Ev}(t_0)=0$ or $\beta_{Ev}(t_0)=0$. If $Ev(\gamma)(t_0)$ is a regular point of $Ev(\gamma)$, then $Ev(\gamma)(t_0)$ is a lightlike point of $Ev(\gamma)$.

By differentiating $-2\alpha(t)\beta(t)=\lambda(t)(\dot\alpha(t)\beta(t)-\alpha(t)\dot\beta(t))$, we have

$$\begin{aligned}&-2(\dot\alpha(t)\beta(t)+\alpha(t)\dot\beta(t))\\&\quad=\dot\lambda(t)(\dot\alpha(t)\beta(t)-\alpha(t)\dot\beta(t))+\lambda(t)(\ddot\alpha(t)\beta(t)-\alpha(t)\ddot\beta(t)).\end{aligned}$$

Moreover,

$$\begin{aligned}&\dot\alpha_{Ev}(t)\beta_{Ev}(t)-\alpha_{Ev}(t)\dot\beta_{Ev}(t)\\&\quad=\Big(\ddot\lambda(t)\alpha(t)+(1+2\dot\lambda(t))\dot\alpha(t)+\lambda(t)\ddot\alpha(t)\Big)\beta_{Ev}(t)\\&\qquad-\Big(-\ddot\lambda(t)\beta(t)+(1-2\dot\lambda(t))\dot\beta(t)-\lambda(t)\ddot\beta(t)\Big)\alpha_{Ev}(t).\end{aligned}$$

If $\alpha(t_0)=\dot\alpha(t_0)=0, \beta(t_0)\neq 0$, then $\alpha_{Ev}(t_0)=0$ and $\lambda(t_0)\ddot\alpha(t_0)=0$. Also, if $\beta(t_0)=\dot\beta(t_0)=0, \alpha(t_0)\neq 0$, then $\beta_{Ev}(t_0)=0$ and $\lambda(t_0)\ddot\beta(t_0)=0$. Both cases, we have $\dot\alpha_{Ev}(t_0)\beta_{Ev}(t_0)-\alpha_{Ev}(t_0)\dot\beta_{Ev}(t_0)=0$. Hence $Ev(\gamma)(t_0)$ is an inflection points of $Ev(\gamma)$.

(2) Since $\gamma(t_0)$ is a lightlike point of γ, we have $\lambda(t_0)=0$ or $\gamma(t_0)$ is an inflection point of γ. By definition, $Ev(\gamma)(t_0)$ is a singular point of $Ev(\gamma)$ if and only if $\alpha_{Ev}(t_0)=\beta_{Ev}(t_0)=0$.

First we assume that $\lambda(t_0)=0$. If $\alpha(t_0)=0$ and $\beta(t_0)\neq 0$, then $Ev(\gamma)(t_0)$ is a singular point of $Ev(\gamma)$ if and only if $1-\dot\lambda(t_0)=0$. Also if $\beta(t_0)=0$ and $\alpha(t_0)\neq 0$, then $Ev(\gamma)(t_0)$ is a singular point of $Ev(\gamma)$ if and only if $1+\dot\lambda(t_0)=0$.

Next, we assume that $\gamma(t_0)$ is an inflection point of γ. By the proof of (1), $Ev(\gamma)(t_0)$ is a singular point of $Ev(\gamma)$ if and only if $\alpha(t_0)=\dot\alpha(t_0)=0, \beta(t_0)\neq 0$ and $\beta_{Ev}(t_0)=0$, or $\beta(t_0)=\dot\beta(t_0)=0, \alpha(t_0)\neq 0$ and $\alpha_{Ev}(t_0)=0$. This completes the proof. Q.E.D.

Remark 6. We can use the same definition (5) in order to define the evolute of γ with singular points. In this case, α and β vanish simultaneously at the singular points. Moreover, a singular point of γ is also an inflection point of γ.

Example 2. Let $\gamma : \mathbb{R} \to \mathbb{R}^2_1$ be a graph of a smooth function f, that is, $\gamma(t) = (t, f(t))$. Then we have $\alpha(t) = (1+\dot{f}(t))/2, \beta(t) = (1-\dot{f}(t))/2$. It follows that

$$\alpha(t)\beta(t) = \frac{1}{4}(1+\dot{f}(t))(1-\dot{f}(t)),\ \dot{\alpha}(t)\beta(t) - \alpha(t)\dot{\beta}(t) = \frac{\ddot{f}(t)}{2}.$$

Hence if there exists a unique smooth function λ such that

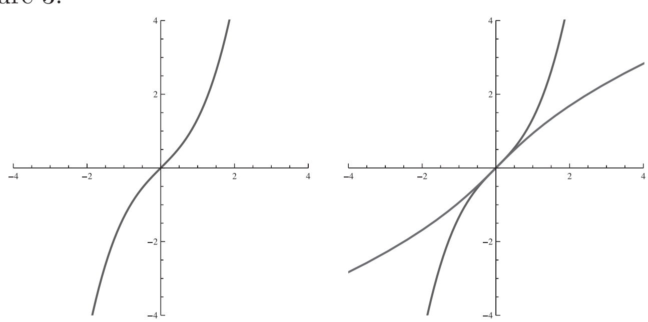
$$-(1+\dot{f}(t))(1-\dot{f}(t)) = \lambda(t)\ddot{f}(t),$$

then we have the evolute $Ev(\gamma)(t) = \gamma(t) + \lambda(t)(\alpha(t)\mathbb{L}^+ - \beta(t)\mathbb{L}^-)$ of $\gamma(t)$.

For example, let $f(t) = t + t^3$. Note that $\gamma(0)$ is an inflection point of γ. Then $\alpha(t) = (2+3t^2)/2, \beta(t) = -(3/2)t^2, \alpha(t)\beta(t) = -3t^2(2+3t^2)/4$ and $\dot{\alpha}(t)\beta(t) - \alpha(t)\dot{\beta}(t) = 6t$. It follows that we have $\lambda(t) = (1/2)t(2+3t^2)$ and the evolute $Ev(\gamma)$ is given by

$$Ev(\gamma)(t) = \left(t + \frac{1}{2}t(2+3t^2)(1+3t^2), 2t + \frac{5}{2}t^3\right),$$

see Figure 3.



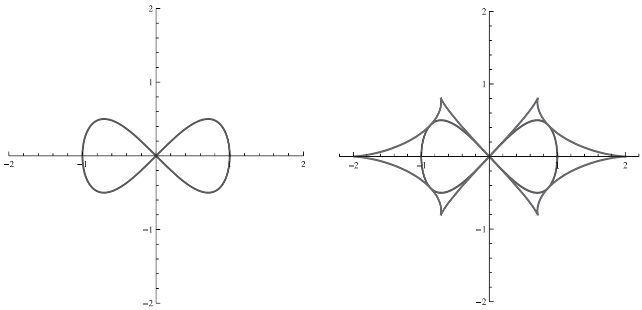
$\gamma(t) = (t, t+t^3)$ $\qquad$ $\gamma(t)$ and the evolute $Ev(\gamma)$

Figure 3.

Example 3. Let $\gamma : [0, 2\pi) \to \mathbb{R}, \gamma(t) = (\cos t, \sin t \cos t)$ be an eight figure. Then $\alpha(t) = (\cos 2t - \sin t)/2, \beta(t) = -(\sin t + \cos 2t)/2$, $\alpha(t)\beta(t) = -(\cos 2t - \sin t)(\cos 2t + \sin t)/4$ and $\dot{\alpha}(t)\beta(t) - \alpha(t)\dot{\beta}(t) = \cos t(1+2\sin^2 t)/2$. It follows that we have $\lambda(t) = \cos t(4\cos^2 t - 3)/(1+2\sin^2 t)$ and the evolute $Ev(\gamma)$ is given by $Ev(\gamma)(t) =$

$$\left(\cos t\left(1 + \frac{(4\cos^2 t - 3)\cos 2t}{1+2\sin^2 t}\right), \sin t \cos t\left(1 - \frac{4\cos^2 t - 3}{1+2\sin^2 t}\right)\right),$$

see Figure 4. Note that $\gamma(t)$ for $t = \pi/2$ and $t = 3\pi/2$ are inflection points.



$\gamma(t) = (\cos t, \sin t \cos t)$ $\gamma(t)$ and the evolute $Ev(\gamma)$

Figure 4.

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