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Asymptotic directions on a surface in a 4-dimensional metric Lie group

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Abstract.

In this paper we introduce the notion of asymptotic directions of a surface in a 4-dimensional riemannian manifold, and study the special case of a surface in a 4-dimensional metric Lie group. It appears that this notion depends on the left invariant metric in general.

§1. Introduction

The aim of this paper is to introduce the notion of asymptotic directions of a surface in a general 4-dimensional riemannian manifold, and to study their first properties.

If M is a surface in a 4-dimensional riemannian manifold $\tilde{M}$, we define the asymptotic directions of M at a point x as the directions of T_xM which annulate a natural quadratic form $\delta : T_xM \to \mathbb{R}$: this quadratic form is constructed from the generalized Gauss map $\varphi : M \to \Lambda^2 T\tilde{M}$ of the surface by the formula

$$\delta(X) := -\frac{1}{2}\nabla_X\varphi \wedge \nabla_X\varphi$$

for all $X \in T_xM$, where ∇ denotes the Levi-Civita connection on $\Lambda^2 T\tilde{M}$; it measures the complexity of the first variation of the tangent planes of the surface at x. The construction is explained in details in the first section of the paper. We show that these directions may be equivalently defined in terms of the second fundamental form, or in terms of the curvature ellipses of the surface; it appears that these directions

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are also the directions of higher contact of the surface with some 3-dimensional totally geodesic spaces. So, the definition extends to a general 4-dimensional riemannian manifold the notion of asymptotic directions for a surface in $\mathbb{R}^4$.

We then study the special case of a surface in a Lie group equipped with a left invariant metric. The case of a surface which is also a subgroup is particularly interesting, since the asymptotic directions satisfy a purely algebraic equation in that case. We then give an example showing that the notion of asymptotic directions generally depends on the invariant metric on the group; this is in contrast with the notion of asymptotic directions on a surface in $\mathbb{R}^4$, or more generally in a Lie group if we consider only metrics which are bi-invariant. We then finish the paper with the special case of a surface in a 3-dimensional subgroup of a metric Lie group, and especially study the case of a surface in the 4-dimensional hyperbolic space $\mathbb{H}^4$, which is regarded as a Lie group with a left invariant metric.

We quote some related papers: the notion of asymptotic directions of a surface in $\mathbb{R}^4$ has been widely studied, especially in relation to the theory of singularities, see e.g. [5, 7]. It has been extended to other pseudo-euclidian 4-dimensional spaces in [2, 3, 4].

The outline of the paper is as follows: we introduce the generalized Gauss map and the asymptotic directions of a surface in a 4-dimensional riemannian manifold in Section 2; we also show in the same section that the principal properties of the asymptotic directions of a surface in $\mathbb{R}^4$ hold in this general setting. We then study in Section 3 the notion of asymptotic directions on a surface in a 4-dimensional metric Lie group, especially when the surface is also a subgroup, or belongs to a 3-dimensional subgroup. We finally recall the fundamental equations of Gauss, Codazzi and Ricci in a short appendix, at the end of the paper.

§2. Asymptotic directions on a surface in a 4-dimensional riemannian manifold

We assume here that $\tilde{M}$ is a 4-dimensional riemannian manifold, with a given orientation, and denote by ∇ its Levi-Civita connection. We suppose that $F : M \to \tilde{M}$ is the immersion of an oriented surface into $\tilde{M}$. The purpose of the section is to define in that context the notion of asymptotic directions of M into $\tilde{M}$, and study their first properties.

2.1. The generalized Gauss map

We consider, in a neighborhood $\mathcal{U}$ of $x_o \in M$, a positively oriented and orthonormal frame (e_1, e_2) of TM. The tangent planes of M in $\mathcal{U}$

may be represented by $e_1 \wedge e_2$, a local section of $F^*\Lambda^2 T\tilde{M}$, the bundle of the bivectors of $T\tilde{M}$, induced on M : we set

$$\begin{array}{rccl} \varphi: & \mathcal{U} & \rightarrow & F^*\Lambda^2 T\tilde{M} \\ & x & \mapsto & e_1 \wedge e_2\ (x); \end{array}$$

this map is a natural generalization of the Gauss map. The connection ∇ on $T\tilde{M}$ naturally induces a connection on $F^*\Lambda^2 T\tilde{M}$, still denoted by ∇.

Lemma 2.1. *For $X \in \Gamma(T\mathcal{U})$, $|X| = 1$, we have the formula*

$$\nabla_X \varphi = B(X,X) \wedge X^\perp + X \wedge B(X, X^\perp), \tag{1}$$

where $B : TM \times TM \rightarrow E$ is the second fundamental form of M in $\tilde{M}$ (E denotes the normal bundle of M in $\tilde{M}$) and $X^\perp$ is the tangent vector obtained from X by a rotation of angle $+\pi/2$ in TM.

Proof. We assume that (e_1, e_2) is a positively oriented and orthonormal frame in $\mathcal{U} \subset M$ such that $\nabla^M e_1 = \nabla^M e_2 = 0$ at x_o (∇^M is the Levi-Civita connection in M), and we compute

$$\begin{aligned} \nabla_X \varphi &= \{\nabla_X e_1\} \wedge e_2 + e_1 \wedge \{\nabla_X e_2\} \\ &= B(X, e_1) \wedge e_2 + e_1 \wedge B(X, e_2), \end{aligned}$$

since $B(X, e_i) = \nabla_X e_i$ at x_o. We finally choose (e_1, e_2) such that $e_1 = X$ and $e_2 = X^\perp$ at x_o. Q.E.D.

Remark 1. *Formula* (1) *has the following interpretation*: *at $x_o \in M$, $\nabla_X \varphi$ represents the infinitesimal rotation of the tangent planes of M in the direction X in the 4-space $T_{x_o}\tilde{M}$, and* (1) *is its decomposition in infinitesimal rotations in two 3-spaces*: *the first term $B(X,X) \wedge X^\perp$ represents an infinitesimal rotation in the 3-space $T_{x_o}M \oplus B(X,X)$, around the tangent direction $X^\perp$, and the second term $X \wedge B(X, X^\perp)$ an infinitesimal rotation in the 3-space $T_{x_o}M \oplus B(X, X^\perp)$, around the tangent direction X.*

2.2. Asymptotic directions

We note that $\Lambda^4 T_{x_o}\tilde{M} \simeq \mathbb{R}$, since $\tilde{M}$ is 4-dimensional (by fixing a positively oriented and orthonormal frame of $T_{x_o}\tilde{M}$). This allows the following definition:

Definition 2.1. *Let us consider the quadratic map*

$$\begin{array}{rccl} \delta: & T_{x_o}M & \rightarrow & \mathbb{R} \\ & X & \mapsto & -\dfrac{1}{2}\nabla_X \varphi \wedge \nabla_X \varphi. \end{array} \tag{2}$$

We will say that $X \in T_{x_o}M$, $X \neq 0$, *defines an asymptotic direction of* M *at* x_o *if* $\delta(X) = 0$.

Remark 2. *By Lemma 2.1, for all* $X \in T_{x_o}M$, $|X| = 1$,

$$\delta(X) = X \wedge X^{\perp} \wedge B(X, X) \wedge B(X, X^{\perp}), \tag{3}$$

and X *is an asymptotic direction if and only if*

$$B(X, X) \wedge B(X, X^{\perp}) = 0.$$

An asymptotic direction has the following interpretation: by Remark 1, $\nabla_X \varphi$ *represents an infinitesimal rotation of the tangent planes of* M *in the direction* X, *and* (1) *is its decomposition in two infinitesimal rotations in 3-spaces; we readily see that* X *is an asymptotic direction of* M *if and only if the two 3-spaces in the decomposition* (1) *coincide: the infinitesimal rotation of the tangent planes of* M *in the direction* X *takes place in a 3-dimensional space* $\subset T_{x_o}\tilde{M}$, *instead of in the whole 4-space* $T_{x_o}\tilde{M}$; *this 3-dimensional space is an osculating 3-space of* M *in the direction* X. *We will precise this fact in the next section.*

Remark 3. *The notion of asymptotic directions does not really depend on the metric but rather on the associated connection: this notion is in fact defined for a surface in a general 4-dimensional manifold equipped with a linear connection. For example, in* $\mathbb{R}^4$, *the notion of asymptotic lines does not depend on the choice of the metric invariant by translation: we have a unique notion in* $\mathbb{R}^4$, $\mathbb{R}^{3,1}$ *or* $\mathbb{R}^{2,2}$, *since the Levi-Civita connection is the canonical connection in all these cases. However, we will see that in a general Lie group we may have different notions of asymptotic lines, since different left invariant metrics give different Levi-Civita connections in general.*

Similarly to the euclidean case $\mathbb{R}^4$ [5], we obtain the following:

Proposition 2.2. *If* $\tilde{R}$ *is the curvature tensor of* $\tilde{M}$ *and* (e_1, e_2) *and* (n_1, n_2) *are positively oriented and orthonormal bases of the tangent and the normal planes of* M *at* x_o, *we set*

$$\tilde{K}_N := \langle \tilde{R}(e_1, e_2)(n_2), n_1 \rangle.$$

The trace of δ *is*

$$tr_g \delta = K_N - \tilde{K}_N$$

where $K_N = \langle R^N(e_1, e_2)(n_2), n_1 \rangle$ *is the normal curvature of* M.

We moreover define

$$\Delta := det_g \delta.$$

By definition, M has an asymptotic direction at x_o if and only if $\Delta \leq 0$; it has two distinct asymptotic directions if and only if $\Delta < 0$.

Proof. We set $B_i = \langle B, n_i \rangle$, $i = 1, 2$. Let $(X, X^\perp)$ be an orthonormal basis of TM. The Ricci equation (26) in the appendix reads

$$K_N - \tilde{K}_N = \langle B(X, B^*(X^\perp, n_2)), n_1 \rangle - \langle B(X^\perp, B^*(X, n_2)), n_1 \rangle. \tag{4}$$

We compute the right-hand side terms: since $(X, X^\perp)$ is an orthonormal basis of TM, we have

$$\begin{aligned} B^*(X^\perp, n_2) &= \langle B^*(X^\perp, n_2), X \rangle X + \langle B^*(X^\perp, n_2), X^\perp \rangle X^\perp \\ &= \langle B(X^\perp, X), n_2 \rangle X + \langle B(X^\perp, X^\perp), n_2 \rangle X^\perp \\ &= B_2(X^\perp, X) X + B_2(X^\perp, X^\perp) X^\perp \end{aligned}$$

and thus

$$\begin{aligned} \langle B(X, B^*(X^\perp, n_2)), n_1 \rangle &= B_1(X, B^*(X^\perp, n_2)) \\ &= B_1(X, X) B_2(X^\perp, X) \\ &\quad + B_1(X, X^\perp) B_2(X^\perp, X^\perp). \end{aligned}$$

Similarly,

$$\begin{aligned} B^*(X, n_2) &= \langle B^*(X, n_2), X \rangle X + \langle B^*(X, n_2), X^\perp \rangle X^\perp \\ &= \langle B(X, X), n_2 \rangle X + \langle B(X, X^\perp), n_2 \rangle X^\perp \\ &= B_2(X, X) X + B_2(X, X^\perp) X^\perp \end{aligned}$$

and

$$\begin{aligned} \langle B(X^\perp, B^*(X, n_2)), n_1 \rangle &= B_1(X^\perp, B^*(X, n_2)) \\ &= B_1(X^\perp, X) B_2(X, X) \\ &\quad + B_1(X^\perp, X^\perp) B_2(X, X^\perp). \end{aligned}$$

The Ricci equation (4) thus reads

$$\begin{aligned} K_N - \tilde{K}_N &= B_1(X, X^\perp) \left(B_2(X^\perp, X^\perp) - B_2(X, X) \right) \\ &\quad + B_2(X^\perp, X) \left(B_1(X, X) - B_1(X^\perp, X^\perp) \right). \end{aligned}$$

On the other hand, we note that

$$\begin{aligned}\delta(X) &= X\wedge X^{\perp}\wedge B(X,X)\wedge B(X,X^{\perp})\\ &= X\wedge X^{\perp}\wedge (B_1(X,X)n_1+B_2(X,X)n_2)\wedge (B_1(X,X^{\perp})n_1\\ &\quad +B_2(X,X^{\perp})n_2)\\ &= X\wedge X^{\perp}\wedge n_1\wedge n_2(B_1(X,X)B_2(X,X^{\perp})\\ &\quad -B_2(X,X)B_1(X,X^{\perp}))\\ &\simeq B_1(X,X)B_2(X,X^{\perp})-B_2(X,X)B_1(X,X^{\perp}),\end{aligned}$$

and similarly

$$\delta(X^{\perp})=-B_1(X^{\perp},X^{\perp})B_2(X^{\perp},X)+B_2(X^{\perp},X^{\perp})B_1(X^{\perp},X).$$

We thus obtain

$$tr_g\delta=\delta(X)+\delta(X^{\perp})=K_N-\tilde{K}_N.$$

Q.E.D.

Remark 4. *Since the metric g is positive definite, we have*

$$(tr_g\delta)^2\geq 4\ det_g\delta,$$

and thus the inequality

$$(K_N-\tilde{K}_N)^2\geq 4\Delta.$$

We may also interpret the invariants of δ in terms of the curvature ellipse at x_o, which is an ellipse in the plane normal to the surface at x_o : the curvature ellipse of M at x_o is classically defined as

$$\mathcal{E}_{x_o}:=\{B(X,X),\ X\in T_{x_o}M,\ |X|=1\}\qquad\subset\ E_{x_o}.$$

Since $B(X,X^{\perp})$ is tangent to the ellipse at $B(X,X)$, the direction X is an asymptotic direction of M at x_o if and only if the line $(0,B(X,X))\subset E_{x_o}$ is tangent to the ellipse $\mathcal{E}_{x_o}$. Moreover, the sign of Δ has the following interpretation: $\Delta>0$ if and only if the null vector $0\in E_{x_o}$ belongs to the interior of the ellipse $\mathcal{E}_{x_o}$, $\Delta=0$ if and only if 0 belongs to $\mathcal{E}_{x_o}$, and $\Delta<0$ if and only 0 is exterior to the ellipse $\mathcal{E}_{x_o}$. We omit the proofs since they are similar to the case $\tilde{M}=\mathbb{R}^4$.

2.3. Asymptotic directions and height functions

If ν is a vector in $T_{x_o}\tilde{M}$, we may define the height function on a neighborhood $\mathcal{U}$ of x_o in M by

$$\begin{array}{rccl} h_\nu : & \mathcal{U} & \to & \mathbb{R} \\ & x & \mapsto & \langle \nu, \exp_{x_o}^{-1}(x)\rangle, \end{array}$$

where $\exp_{x_o}$ is the riemannian exponential map at x_o (this is a local diffeomorphism between a neighborhood of 0 in $T_{x_o}\tilde{M}$ and a neighborhood of x_o in $\tilde{M}$). The function h_ν represents the "height" of $\mathcal{U} \subset M$ with respect to the totally geodesic 3-dimensional submanifold $exp_{x_o}(\nu^\perp) \subset \tilde{M}$. We have the following:

Lemma 2.3. *$dh_\nu = 0$ at x_o if and only if ν belongs to the normal plane E_{x_o}. In that case, the hessian of h_ν is*

$$\nabla^M dh_\nu = B_\nu, \tag{5}$$

where B_ν is the quadratic form on $T_{x_o}M$ such that

$$B_\nu(X,Y) = \langle B(X,Y), \nu\rangle$$

for all $X, Y \in T_{x_o}M$. Here ∇^M stands for the Levi-Civita connection on M.

Proof. Let us set, for x belonging to a neighborhood $\tilde{\mathcal{U}}$ of x_o in $\tilde{M}$,

$$\tilde{h}_\nu(x) = \langle \nu, \exp_{x_o}^{-1}(x)\rangle;$$

the function h_ν is the restriction of $\tilde{h}_\nu$ to $\mathcal{U} \subset M$. Let $\gamma : (-\varepsilon, \varepsilon) \to \tilde{M}$ be the geodesic of $\tilde{M}$ such that $\gamma(0) = x_o$ and $\gamma'(0) = v \in T_{x_o}M$. By definition of the exponential map, $\gamma(t) = \exp_{x_o}(tv)$. Thus

$$\tilde{h}_\nu(\gamma(t)) = t\langle \nu, v\rangle$$

and

$$dh_\nu(v) = d\tilde{h}_\nu(v) = \frac{d}{dt}_{|t=0}\tilde{h}_\nu(\gamma(t)) = \langle \nu, v\rangle.$$

Thus $dh_\nu = 0$ at x_o if and only if $\langle \nu, v\rangle = 0$ for all $v \in T_{x_o}M$, that is ν belongs to E_{x_o}. We now assume that u and v are vector fields defined on $\tilde{\mathcal{U}}$, whose restrictions to $\mathcal{U}$ are tangent to M. By definition,

$$\nabla^M dh_\nu(u,v) = \partial_u\{dh_\nu(v)\} - dh_\nu(\nabla_u^M v)$$

and

$$\nabla d\tilde{h}_\nu(u,v) = \partial_u\{d\tilde{h}_\nu(v)\} - d\tilde{h}_\nu(\nabla_u v),$$

and thus

$$\begin{aligned}\nabla^M dh_\nu(u,v) - \nabla d\tilde{h}_\nu(u,v) &= -d\tilde{h}_\nu(\nabla^M_u v - \nabla_u v)\\ &= d\tilde{h}_\nu(B(u,v))\\ &= \langle \nu, B(u,v)\rangle\\ &= B_\nu(u,v).\end{aligned}$$

Now, if γ is a geodesic of $\tilde{M}$,

$$\frac{d^2}{dt^2}\ \tilde{h}_\nu(\gamma(t)) = \nabla d\tilde{h}_\nu(\gamma'(t),\gamma'(t)).$$

For $\gamma(t) = \exp_{x_o}(tv)$ we get $\tilde{h}_\nu(\gamma(t)) = t\langle \nu, v\rangle$, and deduce that $\nabla d\tilde{h}_\nu = 0$. Identity (5) follows. Q.E.D.

The following proposition shows that an asymptotic direction corresponds to a direction of higher contact between M and some totally geodesic 3-dimensional submanifold:

Proposition 2.4. *M admits an asymptotic direction X at x_o if and only if there exists $\nu \in E_{x_o}$, $|\nu| = 1$, such that $X \in Ker\ \nabla^M dh_\nu$.*

Proof. We suppose that ν is a unit vector belonging to E_{x_o}; by Lemma 2.3, $\nabla^M dh_\nu$ is degenerate if and only if so is B_ν, i.e. there exists $X \in T_{x_o}M$, $|X| = 1$, such that $B_\nu(X,.) : T_{x_o}M \to \mathbb{R}$ is the null form. But the later is equivalent to

$$B_\nu(X,X) = B_\nu(X,X^\perp) = 0,$$

i.e. the normal vectors $B(X,X)$ and $B(X,X^\perp)$ are both orthogonal to ν. We thus get the following: there exists a unit vector $\nu \in E_{x_o}$ such that $\nabla^M dh_\nu$ is degenerate in the direction X if and only if $B(X,X)$ and $B(X,X^\perp)$ are colinear, that is if and only if X is an asymptotic direction of M at x_o. Q.E.D.

The direction ν appearing in the proposition is traditionally called *a binormal direction* of the surface M at the point x_o; see [7] for surfaces in $\mathbb{R}^4$.

Remark 5. *We may interpret Proposition 2.4 as follows: if X is an asymptotic direction of M at x_o, then, for some open subset $\mathcal{V} \subset T_{x_o}M \oplus \mathbb{R}B(X,X) \subset T_{x_o}\tilde{M}$ containing 0, the totally geodesic 3-dimensional manifold $\exp_{x_o}(\mathcal{V})$ has a contact with M of order ≥ 2 in the direction X.*

§3. Asymptotic directions on a surface in a metric Lie group

We suppose here that $\tilde{M}$ is a Lie group G, and denote by $\mathcal{G}$ its Lie algebra: $\mathcal{G}$ is the space of the left invariant vector fields on G, equipped with the Lie bracket $[.,.]$ and is identified to the linear space tangent to G at the identity. We consider the Maurer-Cartan form $\omega \in \Omega^1(G,\mathcal{G})$ defined by

$$\omega_g(v) = L_{g^{-1}}{}_*(v) \qquad \in \ \mathcal{G}$$

for all $v \in T_gG$, where $L_{g^{-1}}$ denotes the left multiplication by g^{-1} on G and $L_{g^{-1}}{}_* : T_gG \to \mathcal{G}$ is its differential. This form induces a bundle isomorphism

$$\begin{array}{rcl} TG & \to & G \times \mathcal{G} \\ (g,v) & \mapsto & (g,\omega_g(v)). \end{array} \tag{6}$$

We note that a vector field $X \in \Gamma(TG)$ is left invariant if $\omega(X) : G \to \mathcal{G}$ is a constant map. We consider the canonical connection ∇^o on G defined by

$$\omega(\nabla^o_X Y) = \partial_X \left\{\omega(Y)\right\}$$

for all $X, Y \in \Gamma(TG)$, where ∂_X stands for the usual derivative in the direction X; it is left invariant, and such that $\nabla^o_X Y = 0$ if X, Y are left invariant vector fields.

3.1. Left invariant metrics and Levi-Civita connections

We now assume that a left invariant metric $\langle .,. \rangle$ is given on G, and denote by ∇ its Levi-Civita connection. Since ∇ is also left invariant, there exists a left invariant tensor Γ belonging to $T^*G \otimes T^*G \otimes TG$ such that, for all $X, Y \in \Gamma(TG)$,

$$\nabla_X Y = \nabla^o_X Y + \Gamma(X,Y). \tag{7}$$

Since Γ is left invariant, we may alternatively consider Γ as a bilinear map

$$\begin{array}{rrcl} \Gamma: & \mathcal{G} \times \mathcal{G} & \to & \mathcal{G} \\ & (X,Y) & \mapsto & \Gamma(X,Y); \end{array}$$

it is such that, for all $X, Y \in \mathcal{G}$

$$\nabla_X Y = \Gamma(X,Y). \tag{8}$$

By the Koszul formula, Γ is determined by the metric as follows: for all $X, Y, Z \in \mathcal{G}$,

$$\langle \Gamma(X,Y), Z\rangle = \frac{1}{2}\langle [X,Y], Z\rangle + \frac{1}{2}\langle [Z,X], Y\rangle - \frac{1}{2}\langle [Y,Z], X\rangle. \tag{9}$$

Since ∇ is without torsion, we have, for all $X, Y \in \mathcal{G}$,

$$\Gamma(X,Y) - \Gamma(Y,X) = [X,Y]. \tag{10}$$

If we consider Γ as a map

$$\begin{array}{rcl} \mathcal{G} & \to & \Lambda^2\mathcal{G} \subset End(\mathcal{G}) \\ X & \mapsto & \Gamma(X) : \; Y \mapsto \Gamma(X,Y) \end{array}$$

(note that $\Gamma(X) : \mathcal{G} \to \mathcal{G}$ is skew-symmetric since ∇ is compatible with the metric), the curvature of ∇ is given by

$$R(X,Y) = [\Gamma(X), \Gamma(Y)] - \Gamma([X,Y]) \qquad \in \Lambda^2\mathcal{G} \tag{11}$$

for all $X, Y \in \mathcal{G}$, where the first bracket in the right-hand side stands for the commutator of the endomorphisms.

3.2. A basic example: the Lie group $\mathbb{H}^n$

Here we briefly describe the group structure on $\mathbb{H}^n$, and refer to [6] for further details. Let us set

$$\mathbb{H}^n = \{a = (a', a_n) \; \in \mathbb{R}^n : \; a_n > 0\},$$

and, for $a \in \mathbb{H}^n$, *the similarity* of $\mathbb{R}^{n-1}$ (by a similarity we mean an homothety composed by a translation)

$$\begin{array}{rrcl} \varphi_a : & \mathbb{R}^{n-1} & \to & \mathbb{R}^{n-1} \\ & x & \mapsto & a_n x + a'. \end{array}$$

The similarities of $\mathbb{R}^{n-1}$ naturally form a group under composition, and the bijection

$$\begin{array}{rrcl} \varphi : & \mathbb{H}^n & \to & \{\text{similarities } \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}\} \\ & a & \mapsto & \varphi_a \end{array}$$

induces a group structure on $\mathbb{H}^n$: it is such that

$$ab = (a_n b' + a', a_n b_n) \tag{12}$$

for all $a, b \in \mathbb{H}^n$; the identity element is $e = (0,1) \in \mathbb{H}^n$. Let us denote by $(e_1, e_2, \ldots, e_n)$ the canonical basis of $T_e\mathbb{H}^n = \mathbb{R}^n$ and keep the same letters to denote the corresponding left invariant vector fields on $\mathbb{H}^n$. The Lie bracket may be easily seen to be given by

$$[e_i, e_j] = 0 \quad \text{and} \quad [e_n, e_i] = e_i$$

for $i, j = 1, \ldots, n-1$. This may also be written in the form

$$[X, Y] = l(X)Y - l(Y)X$$

for all $X, Y \in \mathbb{R}^n$, where $l : \mathbb{R}^n \to \mathbb{R}$ is the linear form such that $l(e_i) = 0$ if $i \leq n-1$ and $l(e_n) = 1$. This property implies that every left invariant metric on $\mathbb{H}^n$ has constant negative curvature $-|l|^2$ [8, 6]. We now suppose that a left invariant metric $\langle ., . \rangle$ is given on $\mathbb{H}^n$ and define

$$g_{ij} = \langle e_i, e_j \rangle$$

for all $i, j = 1, \ldots, n$; the structure constants Γ_{ij}^k, $1 \leq i, j, k \leq n$ associated to the Levi-Civita connection are defined by

$$\Gamma(e_i, e_j) = \sum_{k=1}^{n} \Gamma_{ij}^k e_k$$

for all $i, j = 1, \ldots, n$, and are easily computed using the Koszul formula (9): we have

$$\Gamma_{ij}^k = g^{kn} g_{ij} - \delta_{ik}\delta_{jn} \tag{13}$$

for all $i, j, k = 1, \ldots, n$, where $(g^{ij})_{1 \leq i,j \leq n} = {(g_{ij})_{1 \leq i,j \leq n}}^{-1}$.

3.3. The Gauss map and the quadratic form δ of a surface in a metric Lie group

Since the Lie algebra $\mathcal{G}$ of the group is supposed to be equipped with a scalar product, so is $\Lambda^2\mathcal{G}$. Let us still denote this scalar product by $\langle ., . \rangle$. There is an other symmetric and bilinear form on $\Lambda^2\mathcal{G}$

$$\begin{array}{rccc} \wedge : & \Lambda^2\mathcal{G} \times \Lambda^2\mathcal{G} & \to & \mathbb{R} \\ & (\eta, \eta') & \mapsto & \eta \wedge \eta', \end{array}$$

where a fixed positively oriented and orthonormal basis of $\mathcal{G}$ is used to identify $\Lambda^4\mathcal{G}$ to $\mathbb{R}$. The Hodge operator

$$\begin{array}{rccc} * : & \Lambda^2\mathcal{G} & \to & \Lambda^2\mathcal{G} \\ & \eta & \mapsto & *\eta \end{array}$$

is the symmetric operator on $\Lambda^2\mathcal{G}$ associated to the symmetric bilinear form $\wedge$: it is defined by the relation

$$\eta \wedge \eta' = \langle *\eta, \eta' \rangle \tag{14}$$

for all $\eta, \eta' \in \Lambda^2\mathcal{G}$. We set

$$\mathcal{Q} := \{\eta \in \Lambda^2\mathcal{G} : \ \eta \wedge \eta = 0 \ \text{ and } \ \langle \eta, \eta \rangle = 1\}.$$

$\mathcal{Q}$ is the set of the oriented 2-planes in $\mathcal{G}$. It is well known that $\mathcal{Q}$ is naturally isometric to the product of spheres $S^2(\sqrt{2}/2) \times S^2(\sqrt{2}/2)$.

We now use the trivialization $TG \simeq G \times \mathcal{G}$ given in (6) to identify vector fields on G with smooth maps $G \to \mathcal{G}$, and sections of $\Lambda^2 TG$ with smooth maps $G \to \Lambda^2\mathcal{G}$. If M is an oriented surface immersed in G, and (e_1, e_2) is a positively oriented and orthonormal frame of M defined in some open set $\mathcal{U} \subset M$, identifying the vector fields e_1 and e_2 to maps $\mathcal{U} \to \mathcal{G}$, the generalized Gauss map introduced in Section 2.1 reads

$$\begin{array}{rccccl} \varphi : & M & \to & \mathcal{Q} & \subset & \Lambda^2\mathcal{G} \\ & x & \mapsto & e_1 \wedge e_2\,(x). & & \end{array} \tag{15}$$

If $\varphi = e_1 \wedge e_2$ belongs to $\Lambda^2\mathcal{G}$, we set, for all $X \in \mathcal{G}$,

$$\Gamma(X)(\varphi) = \Gamma(X)(e_1) \wedge e_2 + e_1 \wedge \Gamma(X)(e_2). \tag{16}$$

This formula extends in fact $\Gamma(X)$ to a skew-symmetric operator $\Lambda^2\mathcal{G} \to \Lambda^2\mathcal{G}$. If M is a surface in G, the quadratic form δ defined in Section 2.2 reads as follows: for $X \in T_{x_o}M$,

$$\begin{aligned} \delta(X) &= -\frac{1}{2}\nabla_X\varphi \wedge \nabla_X\varphi \\ &= -\frac{1}{2}\left\{\partial_X\varphi + \Gamma(X)(\varphi)\right\} \wedge \left\{\partial_X\varphi + \Gamma(X)(\varphi)\right\}. \end{aligned}$$

Note that it is distinct to the form $\delta^o(X) = -\frac{1}{2}\partial_X\varphi \wedge \partial_X\varphi$ (this form is associated to the canonical connection of G, rather than to the Levi-Civita connection).

3.4. Asymptotic directions on a 2-dimensional subgroup

Our purpose here is to study the special case of a 2-dimensional Lie subgroup in G; this is a surface which admits a pair of left invariant and orthonormal vector fields. We are principally concerned with the question of the existence of the asymptotic directions on such a surface, and their dependence on the left invariant metric. This is indeed the simplest case in a Lie group since we will see that it reduces to an algebraic problem; it moreover furnishes the first basic examples.

We suppose that H is a 2-dimensional Lie subgroup of G, and denote by $\mathcal{H}$ its Lie algebra: there exist two orthonormal and left invariant vector fields u_1, u_2 on H; since $\varphi = \pm u_1 \wedge u_2$ we get $\partial_X \varphi = 0$ and the form δ of H simply reads

$$\delta(X) = -\frac{1}{2}\Gamma(X)(\varphi) \wedge \Gamma(X)(\varphi)$$

for all $X \in \mathcal{H}$. By definition, the tangent direction X is an asymptotic direction if $\delta(X) = 0$, which is an algebraic condition since Γ is moreover determined by the algebraic relation (9).

3.4.1. *A first negative result*: *the case of a bi-invariant metric.*

Proposition 3.1. *If the metric on G is bi-invariant, then the quadratic form δ on H is identically zero*: *all the tangent directions of H are asymptotic directions.*

We note that this result generalizes the trivial case of a plane through 0 in $\mathbb{R}^4$.

Proof. Since the metric is left invariant and H is a subgroup, it is sufficient to show that $\delta = 0$ at the origin $e \in H$. Since the metric is moreover bi-invariant, then $\Gamma(X, Y) = \frac{1}{2}[X, Y]$ (see e.g. [1] p. 61). Let us fix an orthonormal basis (u_1, u_2) of T_eH. Then, for all $X \in T_eH$,

$$\begin{aligned} \delta(X) &= u_1 \wedge u_2 \wedge \Gamma(X)(u_1) \wedge \Gamma(X)(u_2) \\ &= \frac{1}{4}\, u_1 \wedge u_2 \wedge [X, u_1] \wedge [X, u_2]. \end{aligned}$$

Writing $X = X_1 u_1 + X_2 u_2$, we see that $\delta(X) = 0$. Q.E.D.

3.4.2. *An intermediate case*: *the group* $\mathbb{H}^4$. We consider here the group $\mathbb{H}^4$ with a left invariant metric $\langle ., . \rangle$. The metric is not bi-invariant, since the group $\mathbb{H}^4$ is not unimodular. We keep the notation of Section 3.2, and introduce the vector $U_o \in T_e\mathbb{H}^4$ such that

$$l(X) = \langle U_o, X \rangle$$

for all $X \in T_e\mathbb{H}^4$. By the Koszul formula (9) the Levi-Civita connection is easily seen to be given by the map

$$\Gamma(X)(Y) = -\langle Y, U_o \rangle X + \langle X, Y \rangle U_o \tag{17}$$

for all $X, Y \in T_e\mathbb{H}^4$. We note that an arbitrary linear plane in $T_e\mathbb{H}^4$ is also a sub-algebra of the Lie algebra of $\mathbb{H}^4$, and thus generates a 2-dimensional subgroup. The following proposition shows that the 2-dimensional subgroups are umbilic surfaces:

Proposition 3.2. *Let H be a 2-dimensional subgroup of $\mathbb{H}^4$. The second fundamental form of H in $\mathbb{H}^4$ is given by*

$$B(X,Y) = \langle X, Y\rangle U_o^{\perp}$$

for all $X, Y \in T_eH$, where $U_o^{\perp}$ denotes the orthogonal projection of U_o onto the plane normal to H at e. Moreover, H is a totally geodesic surface if and only if $U_o^{\perp} = 0$ that is U_o belongs to T_eH.

In particular, on a 2-dimensional subgroup of $\mathbb{H}^4$ the quadratic differential δ vanishes identically.

Proof. Let us extend $X, Y \in T_eH$ to left invariant vector fields on H. The Levi-Civita connection of $\mathbb{H}^4$ on left invariant vector fields is given by (17); the first result follows since the second fundamental form is by definition the component normal to the surface of the covariant derivative. Finally, $\delta = 0$ by formula (3). Q.E.D.

The previous discussion contains the special case of the 4-dimensional hyperbolic space with its usual metric, if the left invariant metric is given at e by the canonical metric in $T_e\mathbb{H}^4 = \mathbb{R}^4$ (ie $g_{ij} = \delta_{ij}$ in Section 3.2). Note that in that case the vector U_o is the last vector of the canonical basis e_4.

3.4.3. *A first positive example*: *the group* $\mathbb{H}^2 \times \mathbb{H}^2$. We consider

$$\mathbb{H}^2 \times \mathbb{H}^2 = \{(x_1, x_3),\ x_1 \in \mathbb{R},\ x_3 > 0\} \times \{(x_2, x_4),\ x_2 \in \mathbb{R},\ x_4 > 0\}$$

with the product

$$\begin{aligned}((x_1, x_3), (x_2, x_4)) \cdot ((x_1', x_3'), (x_2', x_4')) \quad = \quad & ((x_3x_1' + x_1, x_3x_3'), \\ & (x_4x_2' + x_2, x_4x_4'));\end{aligned}$$

this is the natural structure on the product (recall that the group structure on $\mathbb{H}^2$ is given by (12), with $n = 2$). We also consider the subgroup

$$H = \{((x_1, 1), (x_2, 1)),\ x_1, x_2 \in \mathbb{R}\},$$

and denote by

$$e_1 = ((1, 0), (0, 0)) \quad \text{and} \quad e_2 = ((0, 0), (1, 0))$$

the natural basis of $T_eH \subset T_e(\mathbb{H}^2 \times \mathbb{H}^2)$. We moreover suppose that a left invariant metric is given on $\mathbb{H}^2 \times \mathbb{H}^2$.

Proposition 3.3. *The subgroup H has two asymptotic directions at every point*; *moreover the two asymptotic directions depend on the left invariant metric $\langle ., . \rangle$. Especially, e_1, e_2 are the two asymptotic directions at $e \in H$ if and only if the left invariant metric is such that $\langle e_1, e_2\rangle = 0$.*

Proof. Let us also consider the vectors

$$e_3 = ((0,1),(0,0)) \quad \text{and} \quad e_4 = ((0,0),(0,1)).$$

The Lie algebra structure of $\mathbb{H}^2 \times \mathbb{H}^2$ is given by

$$[e_3, e_1] = -[e_1, e_3] = e_1, \ [e_4, e_2] = -[e_2, e_4] = e_2$$

and the other brackets are zero. The Koszul formula (9) implies that Γ is given by the following formula: for $i, j \in \{1, 2\}$,

$$\Gamma(e_i, e_j) = \frac{1}{2} g_{ij} \sum_{k=1}^{4} \left(g^{k(i+2)} + g^{k(j+2)} \right) e_k. \tag{18}$$

Since the metric is left invariant and H is a subgroup, we only have to do the computations at the origin $e \in H$. Let us fix an orthonormal basis (u_1, u_2) of $T_e H$. Then, for all $X \in T_e H$,

$$\delta(X) = u_1 \wedge u_2 \wedge \Gamma(X)(u_1) \wedge \Gamma(X)(u_2).$$

Since $u_1 \wedge u_2$ is proportional to $e_1 \wedge e_2$, and $\Gamma(X)(u_1) \wedge \Gamma(X)(u_2)$ is proportional to $\Gamma(X)(e_1) \wedge \Gamma(X)(e_2)$, $\delta(X) = 0$ if and only if

$$\Gamma(X)(e_1) \wedge \Gamma(X)(e_2) = \sum_{1 \le i < j \le 4} c_{ij}(X) \ e_i \wedge e_j$$

is such that $c_{34}(X) = 0$. A straightforward computation using (18) then shows that, for $X = X_1 e_1 + X_2 e_2$,

$$c_{34}(X) = \frac{1}{2} \left(g^{33} g^{44} - (g^{34})^2 \right) \left(g_{12} g_{11} X_1^2 + 2 g_{11} g_{22} X_1 X_2 + g_{12} g_{22} X_2^2 \right).$$

Thus $\delta(X) = 0$ if and only if

$$g_{12} g_{11} X_1^2 + 2 g_{11} g_{22} X_1 X_2 + g_{12} g_{22} X_2^2 = 0. \tag{19}$$

The discriminant of this quadratic form is

$$g_{11} g_{22} \left(g_{12}^2 - g_{11} g_{22} \right) < 0,$$

which proves that there always exist two distinct asymptotic directions; by (19) they clearly depend on the metric, and e_1, e_2 are these asymptotic directions if and only if $g_{12} = \langle e_1, e_2 \rangle = 0$. Q.E.D.

Remark 6. *Note that, by contrast, the notion of asymptotic directions for a general surface in* $\mathbb{R}^4$ *does not depend on the metric since the Levi-Civita connection of a left invariant metric always coincide with the canonical connection in that case* (*the Koszul formula* (9) *with* $[.,.] = 0$ *yields* $\Gamma = 0$, *whatever the invariant metric is*).

3.5. Asymptotic directions on a surface in a 3-dimensional subgroup

We consider here a 4-dimensional metric Lie group G, and a 3-dimensional subgroup H of G. Our purpose is to study the asymptotic directions of a surface belonging to H. Let $\mathcal{G}$ and $\mathcal{H}$ denote the Lie algebras of G and H. We fix $(\underline{u}_1,\ \underline{u}_2,\ \underline{u}_3,\ \underline{u}_4)$ an orthonormal basis of $\mathcal{G}$ such that $(\underline{u}_1,\ \underline{u}_2,\ \underline{u}_3)$ is a basis of $\mathcal{H}$. If M is a surface in H, then its Gauss map is of the form

$$\varphi = \sum_{1\leq i<j\leq 3} a_{ij}\ \underline{u}_i \wedge \underline{u}_j \qquad \in \Lambda^2\mathcal{H}$$

where a_{ij}, $1 \leq i < j \leq 3$ are smooth functions on M, and, for $X \in TM$,

$$\partial_X\varphi = \sum_{1\leq i<j\leq 3} \partial_X a_{ij}\ \underline{u}_i \wedge \underline{u}_j \qquad \in \Lambda^2\mathcal{H}.$$

Thus $\partial_X\varphi \wedge \partial_X\varphi = 0$ and

$$\delta(X) = -\partial_X\varphi \wedge \Gamma(X)\varphi - \frac{1}{2}\Gamma(X)\varphi \wedge \Gamma(X)\varphi \tag{20}$$

for all $X \in TM$. We note that $\Lambda^2\mathcal{H} \simeq \mathcal{H}$ since $\dim\mathcal{H} = 3$, and the Gauss map φ is equivalent to a map

$$\begin{array}{rccl} N: & M & \to & S^2 \subset \mathcal{H} \\ & x & \mapsto & N(x), \end{array}$$

where $N(x)$ is a unit vector normal to M at x, translated to the Lie algebra $\mathcal{H}$ by left multiplication: precisely, if $* : \Lambda^2\mathcal{G} \to \Lambda^2\mathcal{G}$ stands for the Hodge operator, we have

$$\varphi = *(N \wedge \underline{u}_4).$$

We note that, as it is usual, $dN : TM \to TS^2$ may be regarded as an operator of TM; it is related to ∇N by the formula

$$\nabla_X N = dN(X) + \Gamma(X)N \tag{21}$$

for all $X \in TM$ (see (7)).

3.5.1. *The group* $\mathbb{R}^4$. If $G = \mathbb{R}^4$, then $H = \mathbb{R}^3 \subset \mathbb{R}^4$, $\Gamma = 0$, $\delta = 0$, and we recover the well known fact that all the tangent directions of a surface in $\mathbb{R}^3 \subset \mathbb{R}^4$ are asymptotic directions.

3.5.2. *The group* $\mathbb{H}^4$. In that case, the situation is different:

Proposition 3.4. *Let G be the group $\mathbb{H}^4$, equipped with a left invariant metric, and let H be a 3-dimensional subgroup of G. Recall that $U_o \in \mathcal{G}$ is such that* (17) *holds. If M is a surface belonging to H, then we have the following*:

(1) *If U_o belongs to $\mathcal{H}$ then all the directions in TM are asymptotic directions of M.*

(2) *If U_o does not belong to $\mathcal{H}$, then $X \in TM$ is an asymptotic direction of M if and only if it is a principal direction of M, i.e. is such that*

$$\nabla_X N = \lambda X \tag{22}$$

for some λ belonging to $\mathbb{R}$.

Remark 7. *The connection ∇ in* (22) *is the Levi-Civita connection of G; let us note that ∇N also coincides with the covariant derivative of N with respect to the Levi-Civita connection of H, since, by* (21) *and* (17), *it is in fact tangent to H. Thus $\nabla N : TM \to TM$ is a symmetric operator and* (22) *implies that the asymptotic directions are two orthogonal directions in the case* (2) (*if moreover ∇N is not an homothety of TM*).

Remark 8. *If $\mathbb{H}^4$ is equipped with the left invariant metric such that the canonical basis (e_1, e_2, e_3, e_4) of $T_e\mathbb{H}^4 = \mathbb{R}^4$ is orthonormal* (*see Section 3.2*)*, then $\mathbb{H}^4$ is the usual hyperbolic space, and a 3-dimensional sub-algebra $\mathcal{H}$ of $T_e\mathbb{H}^4$ may be assumed to be generated by vectors of the form $e_1, e_2, \lambda e_3 + \mu e_4$, with $\lambda, \mu \neq (0, 0)$. In that case $U_o = e_4$, and U_o belongs to $\mathcal{H}$ if and only if $\lambda = 0$. Thus, by the proposition, denoting by H the subgroup generated by $\mathcal{H}$, if $\lambda = 0$ then all the tangent directions of a surface in H are asymptotic directions, and if $\lambda \neq 0$ then the asymptotic directions of a surface in H coincide with its principal directions.*

Proof. We first note that, for all $X \in TM$,

$$\Gamma(X)\varphi = p(U_o) \wedge X^{\perp} \tag{23}$$

where $p(U_o)$ is the orthogonal projection of U_o onto the normal plane of M. Indeed, if $\varphi = e_1 \wedge e_2$ and $X = X_1 e_1 + X_2 e_2$, a straightforward computation using (16) and (17) gives

$$\begin{aligned} \Gamma(X)\varphi &= X_1 \left(U_o - \langle e_1, U_o\rangle e_1\right) \wedge e_2 - X_2 \left(U_o - \langle e_2, U_o\rangle e_2\right) \wedge e_1 \\ &= X_1\, p(U_o) \wedge e_2 - X_2\, p(U_o) \wedge e_1, \end{aligned}$$

since

$$p(U_o) = U_o - \langle e_1, U_o\rangle e_1 - \langle e_2, U_o\rangle e_2.$$

This gives (23) since

$$X_1 e_2 - X_2 e_1 = X^\perp.$$

Thus $\Gamma(X)\varphi \wedge \Gamma(X)\varphi = 0$ and (20) reads

$$\delta(X) = -\partial_X\varphi \wedge p(U_o) \wedge X^\perp.$$

Since $\partial_X\varphi = *(\partial_X N \wedge \underline{u}_4)$ and writing $p(U_o) = \alpha N + \beta\underline{u}_4$, with $\alpha, \beta \in \mathbb{R}$, we get

$$\delta(X) \quad = \quad -\alpha \ *(\partial_X N \wedge \underline{u}_4) \wedge N \wedge X^\perp - \beta \ *(\partial_X N \wedge \underline{u}_4) \wedge \underline{u}_4 \wedge X^\perp.$$

The first right-hand side term is zero since the bivectors $*(\partial_X N \wedge \underline{u}_4)$ and $N \wedge X^\perp$ belong to $\Lambda^2\mathcal{H}$ and $\dim \mathcal{H} = 3$. For the second term, we note that by the very definition (14) of the Hodge operator $*$ and since $** = id_{\Lambda^2\mathcal{G}}$, we have

$$*\eta \wedge \eta' \simeq \langle \eta, \eta'\rangle$$

for all $\eta, \eta' \in \Lambda^2\mathcal{G}$, which implies in particular

$$\begin{aligned} *(\partial_X N \wedge \underline{u}_4) \wedge \underline{u}_4 \wedge X^\perp &= \langle \partial_X N \wedge \underline{u}_4, \underline{u}_4 \wedge X^\perp\rangle \\ &= -\langle \partial_X N, X^\perp\rangle. \end{aligned}$$

Observing that

$$\nabla_X N - \partial_X N = \Gamma(X)N = -\langle N, U_o\rangle X$$

(by (21) and (17)), we finally get

$$\delta(X) = \beta\langle \nabla_X N, X^\perp\rangle \tag{24}$$

for all $X \in TM$.

If $\beta = \langle U_o, \underline{u}_4\rangle$ is zero, then $\delta = 0$, which gives the first claim of the proposition. If now $\beta \neq 0$, then $\delta(X) = 0$ if and only if $\nabla_X N$ is orthogonal to $X^\perp$, i.e. is colinear to X; this is the second claim of the proposition. Q.E.D.

§Appendix A. The fundamental equations

We recall here the equations of Gauss, Ricci and Codazzi for an immersion of a submanifold M into a riemannian manifold $\tilde{M}$: let us denote by $\tilde{R}$ the curvature tensor of $\tilde{M}$, R^T and R^N the curvature tensors of the connections on TM and on the normal bundle E, $B : TM \times TM \to$

E the second fundamental form and $B^* : TM \times E \to TM$ the bilinear map such that for all $X, Y \in \Gamma(TM)$ and $N \in \Gamma(E)$

$$\langle B(X,Y), N\rangle = \langle Y, B^*(X,N)\rangle;$$

then we have, for all $X, Y, Z \in \Gamma(TM)$ and $N \in \Gamma(E)$,

(1) the Gauss equation

$$(\tilde{R}(X,Y)Z)^T = R^T(X,Y)Z - B^*(X, B(Y,Z)) + B^*(Y, B(X,Z)), \tag{25}$$

(2) the Ricci equation

$$(\tilde{R}(X,Y)N)^N = R^N(X,Y)N - B(X, B^*(Y,N)) + B(Y, B^*(X,N)), \tag{26}$$

(3) the Codazzi equation

$$(\tilde{R}(X,Y)Z)^N = \tilde{\nabla}_X B(Y,Z) - \tilde{\nabla}_Y B(X,Z); \tag{27}$$

in the last equation, $\tilde{\nabla}$ denotes the natural connection on $T^*M \otimes T^*M \otimes E$.

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