

## The hyperbolic modular double and the Yang-Baxter equation

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### Abstract.

We construct a hyperbolic modular double – an algebra lying in between the Faddeev modular double for  $U_q(sl_2)$  and the elliptic modular double. The intertwining operator for this algebra leads to an integral operator solution of the Yang-Baxter equation associated with a generalized Faddeev-Volkov lattice model introduced by the second author. We describe also the L-operator and finite-dimensional R-matrices for this model.

### §1. Introduction

The representation theory is intimately related to special functions. The quantum groups and Yang-Baxter equation (YBE) provide a wide class of novel functions that do not appear in the classical representation theory of Lie groups. These functions possess a number of peculiar properties and satisfy many intricate identities which do not have classical counterparts. The noncompact (or modular) quantum dilogarithm [22, 23] is a remarkable special function significant for a large class of quantum integrable systems. In particular, it plays a prominent role in the space-time discretization of the Liouville model and in the construction of the lattice Virasoro algebra [26, 49], as well as in the investigations of the XXZ spin chain model in a particular regime [29, 9].

The observation that there exist two mutually commuting Weyl pairs led Faddeev [25] to the notion of a modular double of the quantum algebra  $U_q(sl_2)$ . It is formed by two copies of  $U_q(sl_2)$  with different deformation parameters whose generators mutually (anti)commute with each other. This doubling enables unambiguous fixing of the representation space of the algebra.

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The elliptic modular double introduced by the second author in [43] carry over the idea of doubling to the Sklyanin algebra [38]. This doubling is extremely useful. The symmetry constraints with respect to the extended algebra are much more powerful as compared to the initial algebra. They enable again unambiguous description of the relevant objects.

The Faddeev-Volkov model [50] is a solvable two-dimensional lattice model of statistical mechanics [5]. In contrast to the Ising model, its spin variables take continuous values. The Boltzmann weights are expressed in terms of the modular quantum dilogarithm. In [6] the free energy per edge of this model was derived in the thermodynamic limit using a particular form of the star-triangle relation.

A generalization of the Faddeev-Volkov model has been proposed by the second author in [44]. In this extension the Boltzmann weights are expressed in terms of the modular quantum dilogarithm as well, but they have more involved form as compared to the original model. The corresponding star-triangle relation is a degeneration of the elliptic beta integral evaluation formula [41]. The star-triangle relation associated with the latter integral appeared first in the operator form as main identity behind the integral Bailey lemma discovered in [42] (see also [20] for a detailed discussion) and later it was formulated in the functional form in [7].

In the present work we study an algebraic structure underlying the generalized Faddeev-Volkov model of [44] and related quantum integrable systems. First we consider a contraction of the Sklyanin algebra described in [27], which is more general than  $U_q(sl_2)$ . Then we show that this symmetry algebra can be enhanced using the doubling construction. So, we will supplement the algebra with a dual set of generators (anti)commuting with the initial generators. We baptize the resulting algebra as *the hyperbolic modular double*, following the terminology of [37] for the modular quantum dilogarithm considered as a generalization of the Euler gamma function. It lies in between the elliptic modular double and the modular double of  $U_q(sl_2)$  in the sense that the three algebras are arranged in a sequence of contractions. We will pass naturally from the language of lattice models to the standard YBE and find the most general solution of YBE having the symmetry of the hyperbolic modular double. It is an integral operator acting on a pair of infinite-dimensional spaces which are representation spaces of the latter algebra. An integral operator solution of the YBE (at the plain non-deformed level) was constructed for the first time in [16]. The factorization property of the corresponding R-operator was noticed later

in [15], which resulted in a powerful almost purely algebraic techniques of building general R-operators [10, 17, 19, 20].

Previously in [13, 14] we described finite-dimensional solutions of YBE for the elliptic modular double, modular double of  $U_q(\mathfrak{sl}_2)$  and the Lie group  $SL(2, \mathbb{C})$  in a concise form and elucidated their factorization property [11]. One of the principal aims of this paper is to find all finite-dimensional solutions of YBE having the symmetry of the hyperbolic modular double with generic deformation parameter. However, a detailed construction of the representation theory of the latter algebra together with the analysis of special functions associated with that, following the considerations of [34] or [27] and references therein, is left aside. We do not discuss modular doubling of quantum affine algebras as well, the subject which was considered recently in [33].

The plan of the paper is as follows. We start in Sect. 2 with a description of the hyperbolic gamma function (modular quantum dilogarithm) and state its basic properties. Then we present the solvable model of statistical mechanics generalizing the Faddeev-Volkov model and the corresponding hyperbolic star-triangle relation. In Sect. 3 we construct an integral R-operator in terms of the Boltzmann weights of this solvable vertex model and show that it satisfies the Yang-Baxter equation. We also rewrite it in the factorized form. In Sect. 4 we describe an algebra emerging as a degeneration of the Sklyanin algebra and construct the corresponding intertwining operator of equivalent representations. As a natural extension of this quantum algebra we introduce the hyperbolic modular double. Then we study finite-dimensional irreducible representations of the hyperbolic modular double. In Sect. 5 we reduce the integral R-operator to a finite-dimensional invariant subspace in one of its infinite-dimensional spaces. In this way we obtain an explicit formula for finite-dimensional solutions of the YBE with the symmetry of the hyperbolic modular double. In Sect. 6 we apply the reduction formula in the simplest nontrivial setting. We choose the fundamental representation in one of the spaces and recover the L-operator from the integral R-operator. It automatically takes the factorized form. Finally, in Sect. 7 the reduction formula is elaborated further on by a simplification to finite-dimensional matrix solutions of YBE such that all of them take a factorized form.

## §2. A solvable lattice model

Gamma functions are the main building blocks in the construction of special functions of hypergeometric type. The hierarchy of hyperbolic gamma functions is formed by particular combinations of two multiple

Barnes gamma functions [4] (the standard Jackson's  $q$ -gamma function [2] is a combination of two Barnes gamma functions of the second order as well, but we do not consider this function here). The hyperbolic gamma function of the second order is a homogeneous function of  $u, \omega_1, \omega_2 \in \mathbb{C}$ . For  $\operatorname{Re}(\omega_1), \operatorname{Re}(\omega_2) > 0$  and  $0 < \operatorname{Re}(u) < \operatorname{Re}(\omega_1 + \omega_2)$  it has the form

$$(1) \quad \gamma^{(2)}(u; \omega_1, \omega_2) := \exp\left(-\frac{\pi i}{2} B_{2,2}(u; \omega_1, \omega_2) - \int_{\mathbb{R}+i0} \frac{e^{ux}}{(1-e^{\omega_1 x})(1-e^{\omega_2 x})} \frac{dx}{x}\right),$$

where  $B_{2,2}$  is a multiple Bernoulli polynomial of the second order

$$B_{2,2}(u; \omega_1, \omega_2) = \frac{1}{\omega_1 \omega_2} \left( \left( u - \frac{\omega_1 + \omega_2}{2} \right)^2 - \frac{\omega_1^2 + \omega_2^2}{12} \right).$$

Denoting  $q = e^{2\pi i \omega_1 / \omega_2}$  and  $\tilde{q} = e^{-2\pi i \omega_2 / \omega_1}$  and assuming that  $|q| < 1$ , one can write

$$\exp\left(-\int_{\mathbb{R}+i0} \frac{e^{ux}}{(1-e^{\omega_1 x})(1-e^{\omega_2 x})} \frac{dx}{x}\right) = \frac{(e^{2\pi i u / \omega_1} \tilde{q}; \tilde{q})_\infty}{(e^{2\pi i u / \omega_2}; q)_\infty},$$

where  $(t; q)_\infty = \prod_{k=0}^{\infty} (1 - tq^k)$ .

The modular quantum dilogarithm is usually defined as a function obtained from (1) by removing the exponential factor involving  $B_{2,2}(u; \omega_1, \omega_2)$ , the shift  $u \rightarrow u + (\omega_1 + \omega_2)/2$ , and some renormalization of the variables  $u$  and  $\omega_{1,2}$ . In particular, in the context of  $2d$  conformal field theory it is accepted to denote  $\omega_1 = b$ ,  $\omega_2 = b^{-1}$ . We shall use the following representation

$$(2) \quad \gamma(z) := \gamma(z; b) := \exp\left(-\frac{i\pi}{2} \left(z - \frac{b + b^{-1}}{2}\right)^2 + \frac{i\pi}{24} (b^2 + b^{-2}) + \int_{-\infty}^{+\infty} \frac{dt}{4t} \frac{e^{t(2z - b - b^{-1})}}{\sin(ib t) \sin(ib^{-1} t)}\right),$$

where the integration contour goes above the singularity at  $t = 0$ . One can easily restore the original function

$$\gamma^{(2)}(z; \omega_1, \omega_2) = \gamma(z / \sqrt{\omega_1 \omega_2}; \sqrt{\omega_1 / \omega_2}).$$

This integral representation is valid for  $\operatorname{Re}(b) > 0$  and  $0 < \operatorname{Re}(z) < \operatorname{Re}(b + b^{-1})$ . The analytic continuation enables one to extend the definition (2) to a wider range of parameters.

The inverse of this special function is called also the double sine-function and denoted either  $S(u; \omega_1, \omega_2)$  [29] or  $S_b(z)$  [9]. The notation  $\gamma^{(2)}(z; \omega_1, \omega_2)$  is taken from [44]. Here we use the terminology suggested in [37]. Interrelations between main known modifications of function (2) are described in Appendix A of [44]. Various identities for the quantum dilogarithm can be found in [26, 48] and some other papers.

The definition (2) implies that  $\gamma(z)$  is invariant under the swap  $b \rightleftharpoons b^{-1}$ . It satisfies two linear difference equations of the first order

$$(3) \quad \gamma(z + b) = 2 \sin(\pi b z) \gamma(z), \quad \gamma(z + b^{-1}) = 2 \sin(\pi b^{-1} z) \gamma(z).$$

Let us introduce the crossing parameter

$$\eta := -\frac{b + b^{-1}}{2}.$$

Then the reflection formula can be written as follows

$$(4) \quad \gamma(z) \gamma(-2\eta - z) = 1.$$

The function  $\gamma(z)$  is meromorphic. It has a double series of zeros

$$(5) \quad z = b(n + 1) + b^{-1}(m + 1), \quad n, m \in \mathbb{Z}_{\geq 0},$$

and a double series of poles

$$(6) \quad z = -bn - b^{-1}m, \quad n, m \in \mathbb{Z}_{\geq 0}.$$

In the following we deal with multiple products of the hyperbolic gamma function. In order to avoid lengthy formulae we adopt the convention

$$\gamma(\pm x + g) := \gamma(x + g) \gamma(-x + g), \quad \gamma(\pm x \pm y) := \gamma(\pm x + y) \gamma(\pm x - y).$$

In [44] a nontrivial generalization of the Faddeev-Volkov model [49, 50] has been proposed. Such models of statistical mechanics are defined on the square lattice. The continuous spin variables sit in the lattice vertices. The rapidity variables are associated with the medial graph built with the help of pairs of directed lines crossing edges in the middle with the inclination of 45 or 135 degrees (see, e.g., a picture on Fig. 2). Self-interaction of spins and interactions between the nearest neighboring spins are allowed. The Boltzmann weight  $W(\alpha - \beta; x, y)$  is assigned to a horizontal edge connecting a pair of vertices with spins  $x, y$  that is

crossed by a pair of medial graph lines carrying the rapidities  $\alpha$  and  $\beta$ . Similarly the Boltzmann weight  $\overline{W}(\alpha - \beta; x, y)$  is assigned to a vertical edge connecting a pair of vertices with the spins  $x, y$  that is crossed by a pair of medial graph lines carrying rapidities  $\alpha$  and  $\beta$ . The self-interaction at a vertex  $z$  contributes the Boltzmann weight  $\rho(z)$  to the partition function. The edge Boltzmann weights  $W$  and  $\overline{W}$  of the model [44] are given by fourfold products of hyperbolic gamma functions and the vertex Boltzmann weight  $\rho$  is a product of two hyperbolic gamma functions

$$(7) \quad \begin{aligned} W(\alpha; x, y) &= \gamma(\alpha - \eta \pm ix \pm iy), \\ \overline{W}(\alpha; x, y) &= \gamma(-\alpha \pm ix \pm iy), \quad \rho(z) = \frac{1}{2\gamma(\pm 2iz)}. \end{aligned}$$

The edge Boltzmann weights depend on the difference of rapidities and they are symmetric in the spin variables  $W(\alpha; x, y) = W(\alpha; y, x)$ ,  $\overline{W}(\alpha; x, y) = \overline{W}(\alpha; y, x)$ . Let us recall that the Boltzmann weights depend on  $b$  which is a temperature-like parameter.

Physical interpretation of  $W$  and  $\overline{W}$  as true Boltzmann weights requires their positivity. This is possible in two regimes of the key parameter  $b$ : 1)  $b$  is real and  $0 < b < 1$ ; 2)  $|b| = 1$ ,  $\text{Im}(b^2) > 0$ . In both regimes  $\eta < 0$  and  $\gamma(z)^* = \gamma(z^*)$ . As a result one should demand that the spin variables are real. As to the rapidities, one can set  $\eta < \alpha < -\eta$  for  $W(\alpha)$  and  $0 < -\alpha < -2\eta$  for  $\overline{W}(\alpha)$ . These constraints should correspond to unitary representations of the hyperbolic modular double to be described below.

The Boltzmann weights possess a crossing symmetry, i.e. the horizontal and vertical edge weights are related as follows

$$(8) \quad \overline{W}(\alpha; x, y) = W(\eta - \alpha; x, y).$$

We note that in view of the reflection formula (4) and quasiperiodicity (3) the vertex Boltzmann weight  $\rho(z)$  can be rewritten solely in terms of the trigonometric functions, i.e. its expression in terms of the hyperbolic gamma functions is overcomplicated. In contrast, the edge Boltzmann weights  $W$  and  $\overline{W}$  are genuine products of hyperbolic gamma functions and the number of such functions in the products cannot be reduced.

The formulated model is solvable because of the star-triangle relation depicted in Fig. 1. This relation equates the partition functions of

two elementary cells,

$$(9) \quad \int_{-\infty}^{+\infty} \rho(z) \overline{W}(\alpha - \beta; x, z) W(\alpha - \gamma; y, z) \overline{W}(\beta - \gamma; w, z) dz = \chi(\alpha, \beta, \gamma) W(\alpha - \beta; y, w) \overline{W}(\alpha - \gamma; x, w) W(\beta - \gamma; x, y),$$

up to a normalization constant  $\chi$ . Using this example one can see that the horizontal edge Boltzmann weights  $W(\alpha - \beta; x, y)$  depends on the difference  $\alpha - \beta$ , where  $\alpha$  is the rapidity of the upward directed median line of 45 degrees and  $\beta$  is the rapidity of the upward directed median line of 135 degrees. For the vertical edge weights  $\overline{W}(\alpha - \beta; x, y)$  the situation is similar -  $\alpha$  is the rapidity of the line going to the right of the edge and  $\beta$  - of the line going to the left. We call identity (9) with the weights (7) the hyperbolic star-triangle relation. Corresponding normalization constant has the following form

$$\chi(\alpha, \beta, \gamma) = \gamma(2\beta - 2\alpha) \gamma(2\gamma - 2\beta) \gamma(2\alpha - 2\gamma - 2\eta).$$

This local relation enables one to construct a family of commuting row-

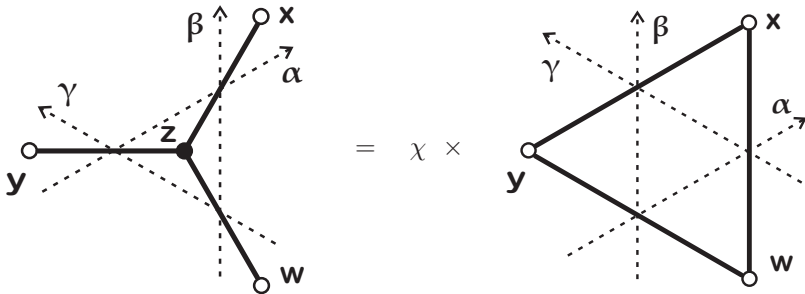


Fig. 1. The star-triangle relation.

to-row transfer matrices and then to calculate the partition function of the model using the machinery of QISM [5]. The free energy per edge of the model in the thermodynamical limit has been calculated in [44] following the method from [6, 7].

Let us substitute in (9)

$$(10) \quad W(\alpha; x, y) = m(\alpha)W_r(\alpha; x, y), \quad \overline{W}(\alpha; x, y) = m(\eta - \alpha)\overline{W}_r(\alpha; x, y)$$

and choose the function  $m(\alpha)$  in such a way that the normalization constant  $\chi$  on the right-hand side of (9) disappears,

$$\frac{m(\alpha - \beta)m(\eta - \alpha + \gamma)m(\beta - \gamma)}{m(\eta - \alpha + \beta)m(\alpha - \gamma)m(\eta - \beta + \gamma)} \chi(\alpha, \beta, \gamma) = 1.$$

Ascribe now to the edges the renormalized weights  $W_r(\alpha; x, y)$  and  $\overline{W}_r(\alpha; x, y)$ . Then, denoting the total number of edges in the infinitely growing lattice as  $N$ , one finds that the free energy per edge

$$\beta f_{edge} = - \lim_{N \rightarrow \infty} \frac{\log Z^{(r)}}{N} = 0,$$

where  $Z^{(r)}$  is the total partition function for the model with renormalized Boltzmann weights.

Equivalently, we can keep the original Boltzmann weights intact and compute the contribution of the renormalizing factors in the asymptotics of the partition function. Take the finite rectangular lattice with  $N$  spins along the horizontal line and  $M$  spins along the vertical line. Such lattice has  $(N - 1)M$  horizontal edges and  $N(M - 1)$  vertical edges. Therefore the indicated renormalization of the Boltzmann weights yields a scaling of the partition function

$$Z_{N,M} = Z_{N,M}^{(r)} m(\alpha)^{(N-1)M} m(\eta - \alpha)^{N(M-1)},$$

i.e. the free energy per edge of the original model is

$$\text{free energy per edge} = - \lim_{N,M \rightarrow \infty} \frac{\log Z_{N,M}}{NM} = - \log m(\alpha)m(\eta - \alpha).$$

It is easy to see that the needed normalization constant  $m(\alpha)$  is found from the equation

$$(11) \quad m(\alpha + \eta) = \gamma(2\alpha; b)m(-\alpha).$$

As shown in [44], the solution of this equation satisfying the unitarity relation  $m(\alpha)m(-\alpha) = 1$  is given by the ratio of two hyperbolic gamma functions of the third order  $\gamma^{(3)}(u; \omega_1, \omega_2, \omega_3)$  for a special choice of the quasiperiods  $\omega_i$ . In the current notation it has the following integral representation

$$(12) \quad m(\alpha) = \exp \left( - \pi i \left( \alpha^2 + \frac{1}{24}(1 - 2(b + b^{-1})^2) \right) + \frac{1}{8} \int_{\mathbb{R}+i0} \frac{e^{4\alpha t}}{\sin(ibt) \sin(ib^{-1}t) \cos(i(b + b^{-1})t)} \frac{dt}{t} \right).$$



It happens that this result coincides with a similar normalization factor  $m(\alpha)$  for the original Faddeev-Volkov model derived in [6].

The hyperbolic star-triangle relation (9) can be written as the integral identity

$$(13) \quad \int_{-\infty}^{+\infty} \prod_{k=1}^6 \gamma(g_k \pm iz) \frac{dz}{2\gamma(\pm 2iz)} = \prod_{1 \leq j < k \leq 6} \gamma(g_j + g_k),$$

where parameters  $g_k, k = 1, \dots, 6$ , satisfy the constraints  $\text{Re}(g_k) > 0$  and the balancing condition

$$\sum_{k=1}^6 g_k = -2\eta.$$

Note that the condition  $\text{Re}(g_k) > 0$  restricts the domain of complex values of the rapidities and spins as follows

$$\text{Re}(\beta - \alpha \pm ix), \text{Re}(\alpha - \gamma - \eta \pm iy), \text{Re}(\gamma - \beta \pm iw) > 0.$$

In principle these restrictions can be relaxed by the analytical continuation (e.g., it can be reached by a deformation of the integration contour).

The first mathematically rigorous proof of relation (13), and, so, of the star-triangle relation (9), was obtained in [47]. However, this identity is a special limiting case of the elliptic beta integral evaluation formula rigorously established in [41]. Let us note that the exactly computable integral (13) is a non-compact (hyperbolic) analogue of the Rahman (trigonometric)  $q$ -beta integral [36].

Another important property of the described Boltzmann weights is the unitarity relation. For real values of the spins  $x$  and  $y$  one has

$$(14) \quad \int_{-\infty}^{+\infty} \rho(z) \overline{W}(\alpha - \beta; x, z) \overline{W}(\beta - \alpha; y, z) dz = \frac{1}{2\rho(x)} \gamma(2\alpha - 2\beta) \gamma(2\beta - 2\alpha) (\delta(x - y) + \delta(x + y)).$$

This identity can be rigorously obtained by taking the limit  $\gamma \rightarrow \alpha$  in (9). In the computation of partition functions of three-dimensional supersymmetric field theories on the squashed spheres such relations indicate the chiral symmetry breaking phenomenon [45]. The Boltzmann weights  $W$  and  $\overline{W}$  satisfy the reflection equation that is an evident consequence of Eq. (4),

$$(15) \quad W(\alpha - \beta; x, y) W(\beta - \alpha; x, y) = 1.$$

### §3. From the lattice model to the integral R-operator

The star-triangle relations imply integrability of the two-dimensional lattice models, similar to the case outlined in the previous section. Another wide class of integrable systems is associated with the quantum spin chains. Formulation of the latter models in the framework of QISM [24, 32] requires definition of the R-matrix solving the YBE. In this section we show how to construct such R-matrices associated with the hyperbolic star-triangle relation (9). Since the spin variables sitting in vertices of the two-dimensional lattice take continuous values (in contrast to the discrete spins of the Ising and chiral Potts models), on the spin chains side we deal with integral operators instead of the finite-dimensional R-matrices. In other words, the quantum spaces of the relevant spin chain are infinite-dimensional functional spaces. Therefore we call solutions of the corresponding YBE the R-operators to emphasize this aspect. We will indicate in Sect. 5 that the integral R-operators represent in some sense the most general YBE solutions, since they embrace all finite-dimensional R-matrices. Our presentation below follows to some extent the original construction of [18].

Thus we are interested in the integral operator  $\mathbb{R}_{12}(u|g_1, g_2)$  defined on the tensor product of two infinite-dimensional function spaces that are representation spaces with labels (“spins”)  $g_1$  and  $g_2$  (arbitrary complex numbers) of some algebra. The symmetry algebra underlying the hyperbolic star-triangle relation will be introduced in Sect. 4. The R-operator depends on a complex number  $u$  – the spectral parameter and satisfies the YBE

$$(16) \quad \begin{aligned} & \mathbb{R}_{12}(u-v|g_1, g_2) \mathbb{R}_{13}(u|g_1, g_3) \mathbb{R}_{23}(v|g_2, g_3) \\ & = \mathbb{R}_{23}(v|g_2, g_3) \mathbb{R}_{13}(u|g_1, g_3) \mathbb{R}_{12}(u-v|g_1, g_2). \end{aligned}$$

Instead of the integral operator  $\mathbb{R}_{12}(u|g_1, g_2)$ , we can consider first a more general notation operator  $\mathbb{R}_{12}(u_1, u_2|v_1, v_2)$  depending on four complex parameters and satisfying the equation

$$(17) \quad \begin{aligned} & \mathbb{R}_{12}(u_1, u_2|v_1, v_2) \mathbb{R}_{13}(u_1, u_2|w_1, w_2) \mathbb{R}_{23}(v_1, v_2|w_1, w_2) \\ & = \mathbb{R}_{23}(v_1, v_2|w_1, w_2) \mathbb{R}_{13}(u_1, u_2|w_1, w_2) \mathbb{R}_{12}(u_1, u_2|v_1, v_2). \end{aligned}$$

This equation is rather similar to (16). An operator solution of Eq. (17) can be easily constructed in terms of the lattice model formulated in the previous section. The parameters  $u_1, u_2, v_1, v_2, w_1, w_2$  are identified with the rapidities. The kernel of the integral operator is a product of four edge Boltzmann weights (two horizontal and two vertical) and two

vertex Boltzmann weights,

(18)

$$[\mathbb{R}_{12}(u_1, u_2|v_1, v_2)\Phi](z_1, z_2) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \rho(x_1)\rho(x_2)W(u_1 - v_2; z_1, z_2) \times \\ \overline{W}(u_1 - v_1; z_1, x_2)\overline{W}(u_2 - v_2; z_2, x_1)W(u_2 - v_1; x_1, x_2)\Phi(x_1, x_2)dx_1dx_2.$$

Thus the kernel of  $\mathbb{R}_{12}(u_1, u_2|v_1, v_2)$  is the partition function of an elementary square cell, see Fig. 2.

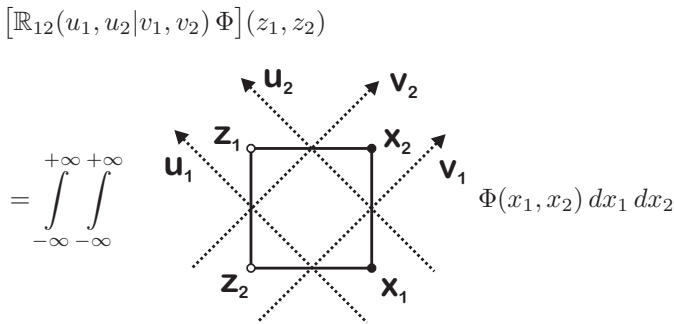


Fig. 2. The kernel of the integral R-operator is the partition function of an elementary square cell.

Taking into account explicit expressions for the Boltzmann weights we rewrite Eq. (18) as follows

$$[\mathbb{R}_{12}(u_1, u_2|v_1, v_2)\Phi](z_1, z_2) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{dx_1 dx_2 \Phi(x_1, x_2)}{4\gamma(\pm 2ix_1)\gamma(\pm 2ix_2)} \times \\ \gamma(\pm iz_1 \pm iz_2 + u_1 - v_2 - \eta) \gamma(\pm ix_1 \pm iz_2 + v_1 - u_1) \times \\ (19) \quad \gamma(\pm ix_2 \pm iz_1 + v_2 - u_2) \gamma(\pm ix_1 \pm ix_2 + u_2 - v_1 - \eta).$$

This R-operator corresponds to the generalized Faddeev-Volkov model of [44]. It can be derived from the elliptic hypergeometric R-operator constructed in [20] after taking a particular limit in the parameters, but the rigorous proof of this fact would require quite intricate techniques. Remind that the latter R-operator intertwines representations of the Sklyanin algebra.

Let us recall that the vertex Boltzmann weight  $\rho$  is in fact a trigonometric function. The integrand function in the expression (19) is a genuine product of 16 hyperbolic gamma functions (modulo a trigonometric multiplier) and their number cannot be reduced. In [10] the R-operator associated with the Faddeev modular double of  $U_q(sl_2)$  has been constructed in a similar form. It corresponds to the Faddeev-Volkov lattice model itself. However, the integrand function of the corresponding integral operator solution of YBE is a product of only 8 hyperbolic gamma functions (up to a trigonometric multiplier). One can obtain this R-operator rigorously from expression (19) by taking appropriate limits in the parameters such that the  $\gamma$ -functions depending on the sum  $x_1 + x_2$  disappear. E.g., such a limiting procedure is described in [44] for the reduction of the hyperbolic star-triangle relation down to the original Faddeev-Volkov model case.

Similar to the very first integral R-operator considerations in [18], it is easy to check that the operator  $\mathbb{R}_{12}(u_1, u_2|v_1, v_2)$ , Eq. (18), solves Eq. (17). In order to demonstrate how it works we resort to graphical representation of the integral R-operators. In Fig. 3 at the right top we depicted convolution of the kernels of the integral operators from the left-hand side expression in Eq. (17) and at the right bottom we depicted convolution of the kernels from the RHS of Eq. (17). External points are marked by numbers 1, 2, 3. They denote three quantum spaces. Integration over internal points is assumed (convolution of the kernels). The dotted lines are the rapidity lines. The left-hand and right-hand side expressions in Eq. (17) are connected to each other by a sequence of moves. Each move is the application of the star-triangle transformation, Eq. (9). Thus the YBE (17) boils down to a combination of the star-triangle relations (9). Keeping track of the normalization factors  $\chi$  arising at each step one can check that they eventually drop out.

However, we still need to understand the algebraic meaning of solution (18), which is the primary goal of the present paper. In other words, we have to give an algebraic interpretation of the rapidities  $u_1, u_2, v_1, v_2$ . As we will see later they can be chosen as linear combinations of the spectral parameters  $u, v$  and the representation labels  $g_1, g_2$  of a certain algebraic structure which is studied in the next section,

$$(20) \quad u_1 = \frac{u + g_1}{2}, \quad u_2 = \frac{u - g_1}{2}, \quad v_1 = \frac{v + g_2}{2}, \quad v_2 = \frac{v - g_2}{2}.$$

This relation yields a solution of the YBE in the form (16),

$$\mathbb{R}_{12}(u - v|g_1, g_2) = \mathbb{R}_{12}(u_1, u_2|v_1, v_2).$$

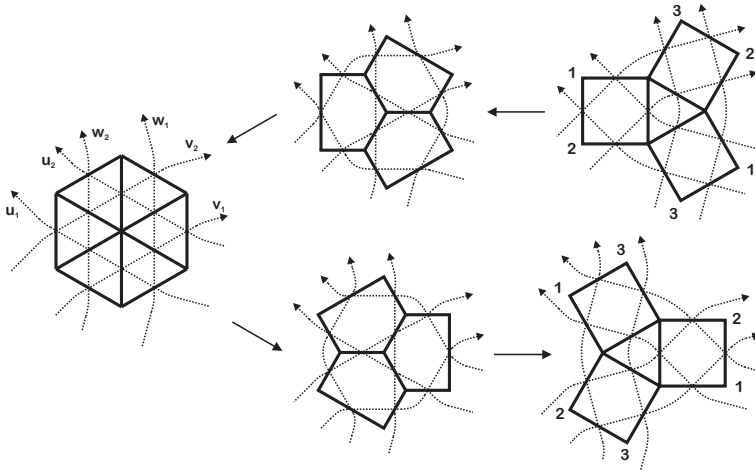


Fig. 3. The sequence of star-triangle moves transforming the left-hand side expression in (17) to its right-hand form.

Our considerations below indicate that this is in fact the most general solution of YBE compatible with certain quantum algebra. The most strong argument follows from the fact that Eq. (18) embraces all finite-dimensional solutions of the YBE (16) associated with this algebra.

Factorization of the kernel of the integral operator (18) implies factorization of the operator itself. Indeed, it can be written as a product of five elementary operators

$$(21) \quad \mathbb{R}_{12}(u_1, u_2|v_1, v_2) = P_{12} S(u_1 - v_2) M_2(u_2 - v_2) M_1(u_1 - v_1) S(u_2 - v_1).$$

Here  $P_{12}$  is a permutation operator of two tensor factors, i.e.  $P_{12}\Phi(z_1, z_2) = \Phi(z_2, z_1)$ .  $S(u)$  is an operator of multiplication by a particular function,

$$(22) \quad S(u) = W(u; z_1, z_2) = \gamma(\pm iz_1 \pm iz_2 + u - \eta).$$

$M_1$  and  $M_2$  are two copies of the integral operator

$$(23) \quad [M(g)\Phi](z) = \frac{1}{\gamma(-2g)} \int_{-\infty}^{+\infty} \rho(x) \overline{W}(g; z, x) \Phi(x) dx$$

acting in the first and second quantum spaces, respectively. This operator is a degeneration of an elliptic hypergeometric integral operator introduced in [42]. Owing to the chosen definition (23), normalizations of the R-operators (21) and (19) differ by the multiplicative numerical factor  $\gamma(2v_1 - 2u_1)\gamma(2v_2 - 2u_2)$ . This renormalization of the R-operator removes certain divergences appearing during the reduction we describe below and makes this procedure smooth.

At this point we should specify an appropriate function space for the operator M, Eq. (23). Firstly, the kernel of M is invariant with respect to the reflections  $x \rightarrow -x$  and  $z \rightarrow -z$ . Consequently, M projects out odd functions and maps onto the space of even functions. Moreover, calculating the asymptotics of the kernel at  $x \rightarrow \pm\infty$  we obtain a restriction on the asymptotic behavior of  $\Phi(x)$ . Thus we assume that  $\Phi$  is an even function, i.e.  $\Phi(-x) = \Phi(x)$ , and  $e^{4\pi i x g} \Phi(x)$  is rapidly decaying at  $x \rightarrow +\infty$ .

In view of the reflection equation (15) and unitarity, Eq. (14), of the Boltzmann weights the operators S and M satisfy (in the space of even functions) the inversion relations

$$(24) \quad S(u)S(-u) = 1, \quad M(g)M(-g) = 1.$$

The second relation is a degeneration of the inversion formula proved in [46] for the elliptic hypergeometric integral operator of [42]. The inversion relation (24) for  $S(u)$  is valid for generic values of  $g \in \mathbb{C}$ , but for  $M(g)$  it is violated for particular discrete lattice points on  $\mathbb{C}$ . In the latter case a nontrivial null-space of  $M(g)$  appears which is described in the next section. Let us note that for generic parameters the operators S,  $M_1$ ,  $M_2$  provide a twisted representation of generators of the permutation group  $S_4$  satisfying the Coxeter relations. More precisely, the star-triangle relation (9) can be reformulated as the cubic Coxeter relations for S,  $M_1$ ,  $M_2$ , whereas the inversion relations (24) represent quadratic Coxeter relations. For more details on this interpretation, see the end of Sect. 6 and [15, 17, 19, 20] where the allied constructions are elaborated in detail. Finally we note that the identities (24) lead to the unitarity-like relation for the R-operator (21),

$$(25) \quad \mathbb{R}_{12}(u|g_1, g_2) \mathbb{R}_{12}(-u|g_2, g_1) = 1.$$

#### §4. The intertwining operator of a degenerate Sklyanin algebra

In this section we study in detail the operator M, Eq. (23), in order to infer its algebraic meaning. We rewrite Eq. (23) explicitly in terms

of the hyperbolic gamma functions

$$(26) \quad [M(g) \Phi](z) = \frac{1}{2} \int_{-\infty}^{+\infty} \frac{\gamma(-g \pm iz \pm ix)}{\gamma(\pm 2ix)\gamma(-2g)} \Phi(x) dx.$$

For certain discrete values of  $g$  the operator  $M(g)$  simplifies considerably. First of all at the origin  $g = 0$  it is the identity operator,  $M(0) = 1$ , which can be seen by taking the limit  $g \rightarrow 0$  after a simple residue calculus. Moreover one can easily check that  $M(g)$  respects a pair of the contiguous relations involving the shifts of  $g$  by  $\frac{b}{2}$  and  $\frac{1}{2b}$ ,

$$(27) \quad -\frac{i}{\sin(2\pi ibz)} \sin(\frac{b}{2}\partial_z) M(g) = M(g + \frac{b}{2}),$$

$$(28) \quad -\frac{i}{\sin(2\pi ib^{-1}z)} \sin(\frac{1}{2b}\partial_z) M(g) = M(g + \frac{1}{2b}).$$

Elliptic analogues of these relations can be found in [12, 21]. Applying recurrences (27), (28) for constructing  $M(g)$  in the discrete quarter-infinite lattice of points  $g = \frac{nb}{2} + \frac{m}{2b}$ ,  $n, m \in \mathbb{Z}_{\geq 0}$ , we find that the integral operator (26) is converted to a product of  $n + m$  finite-difference operators of the first order

$$(29) \quad M(\frac{nb}{2} + \frac{m}{2b}) = \left[ \frac{-i}{\sin(2\pi ibz)} \sin(\frac{b}{2}\partial_z) \right]^n \left[ \frac{-i}{\sin(2\pi ib^{-1}z)} \sin(\frac{1}{2b}\partial_z) \right]^m.$$

We have already mentioned above that the R-operator (18) is related to a certain quantum algebra. Let us now specify it explicitly. It is a contraction of the Sklyanin algebra [38] that has been introduced in [27] and then investigated in [40, 3, 8]. This degenerate Sklyanin algebra is formed by four generators  $A, B, C, D$  which respect the following commutation relations

$$(30) \quad \begin{aligned} CA &= e^{i\pi b^2} AC, \quad DC = e^{i\pi b^2} CD, \\ [A, D] &= -2i \sin^3 \pi b^2 C^2, \quad [B, C] = \frac{A^2 - D^2}{2i \sin \pi b^2}, \\ AB - e^{i\pi b^2} BA &= e^{i\pi b^2} DB - BD = \frac{i}{2} \sin 2\pi b^2 (CA - DC). \end{aligned}$$

It has a pair of Casimir operators

$$(31) \quad \begin{aligned} K_0 &= e^{i\pi b^2} AD - \sin^2 \pi b^2 C^2, \\ K_1 &= \frac{e^{-i\pi b^2} A^2 + e^{i\pi b^2} D^2}{4 \sin^2 \pi b^2} - BC - \frac{1}{2} \cos \pi b^2 C^2 \end{aligned}$$

commuting with all four generators. This algebra is different from the conventional quantum deformation of the rank 1 Lie algebra,  $U_q(\mathfrak{sl}_2)$  [30], in particular, it does not obey the Hopf algebra structure. As shown in [27], the spectral problem for a special quadratic combination of the generators of this algebra reproduces the eigenvalue problem for the Askey-Wilson polynomials. Therefore this algebra comprises the Zhedanov algebra as well [51], which was constructed precisely with the aim of interpreting Askey-Wilson polynomials as a representation space of some quadratic algebra.

The Sklyanin algebra possesses a representation by finite-difference operators with elliptic coefficients which depend on an arbitrary complex parameter  $g$  labeling representations [39]. Particular linear combinations of this algebra generators in a degeneration limit, such that the elliptic nome goes to zero and the elliptic functions reduce to trigonometric ones, take the form (the details of this procedure can be found in [27], see also [43])

$$(32) \quad A(g) = \frac{ie^{\frac{i\pi}{2}b^2}}{2} \frac{e^{-\pi ibg}}{\sin 2\pi ibz} \left[ e^{2\pi bz} e^{\frac{ib}{2}\partial_z} - e^{-2\pi bz} e^{-\frac{ib}{2}\partial_z} \right],$$

$$(33) \quad B(g) = -\frac{1}{2} \cos \pi b^2 C(g) - \frac{1}{4 \sin \pi b^2} \frac{1}{\sin 2\pi ibz} \times \\ \left[ \cos \pi b(2g + 4iz - b) e^{\frac{ib}{2}\partial_z} - \cos \pi b(2g - 4iz - b) e^{-\frac{ib}{2}\partial_z} \right],$$

$$(34) \quad C(g) = \frac{1}{2 \sin \pi b^2} \frac{1}{\sin 2\pi ibz} \left[ e^{\frac{ib}{2}\partial_z} - e^{-\frac{ib}{2}\partial_z} \right],$$

$$(35) \quad D(g) = -\frac{ie^{-\frac{i\pi}{2}b^2}}{2} \frac{e^{\pi ibg}}{\sin 2\pi ibz} \left[ e^{-2\pi bz} e^{\frac{ib}{2}\partial_z} - e^{2\pi bz} e^{-\frac{ib}{2}\partial_z} \right].$$

These operators satisfy defining relations (30). In this representation the Casimir operators (31) take the values

$$K_0(g) = e^{i\pi b^2}, \quad K_1(g) = \frac{\cos 2\pi bg}{2 \sin^2 \pi b^2}.$$

We can construct Verma module representations of the algebra (30) following an analogy with the  $\mathfrak{sl}_2$  algebra. We choose  $|0\rangle = 1$  to be a lowest weight vector in the representation annihilated by the lowering operator  $C$ ,  $C|0\rangle = 0$ . In order to obtain the basis of the Verma module we act by the raising operator  $B$  on the lowest weight vector a number



of times:  $|k\rangle := B^k |0\rangle$ ,  $k \in \mathbb{Z}_{\geq 0}$ . Using relations (30) one can check that

$$\begin{aligned}
 A |k\rangle &= \sum_{l=0}^{[k/2]} a_{k,l}(g) |k-2l\rangle, \quad D |k\rangle = \sum_{l=0}^{[k/2]} d_{k,l}(g) |k-2l\rangle, \\
 (36) \quad C |k\rangle &= \sum_{l=0}^{[\frac{k-1}{2}]} c_{k,l}(g) |k-1-2l\rangle,
 \end{aligned}$$

where  $[k/2]$  is the integral part of  $k/2$  and  $a_{k,l}$ ,  $d_{k,l}$ ,  $c_{k,l}$  are some functions of  $g$ . Contrary to the familiar situation of  $sl_2$ , the lowering operator  $C$  acting on the vector  $|k\rangle$  produces not  $|k-1\rangle$ , but a linear combination of vectors with descending weights  $k-1, k-3, k-5, \dots$ . Similarly, the operators  $A$  and  $D$  are not diagonal contrary to their counterparts in  $sl_2$ . Acting on the vector  $|k\rangle$  they mix it with the vectors having descending weights  $k-2, k-4, k-6, \dots$ . For the chosen trigonometric polynomial realization, the vector  $|k\rangle$  is a linear combination of  $\cos(2\pi i j b z)$ , where  $j = k, k-2, k-4, \dots, 1$  (or 0).

For the generic values of  $g$  the representation is infinite-dimensional. However, if  $g = (n+1)\frac{b}{2}$ ,  $n \in \mathbb{Z}_{\geq 0}$ , the situation drastically changes. Then  $|n+1\rangle$  is a linear combination of  $\{|k\rangle\}_{k=0}^n$ . Acting by powers of the raising operator  $B$  on the vector  $|n\rangle$  we do not get out of the  $n$ -dimensional space. In order to avoid misunderstanding we note that  $C |n+1\rangle \neq 0$ , unlike the  $sl_2$  case.

Since representations with the labels  $g$  and  $-g$  have the same values of the Casimir operators, they are equivalent. Indeed, they are intertwined by the operator  $M(g)$ , Eq. (26), as follows from the relations

$$\begin{aligned}
 (37) \quad M(g) A(g) &= A(-g) M(g), \quad M(g) B(g) = B(-g) M(g), \\
 M(g) C(g) &= C(-g) M(g), \quad M(g) D(g) = D(-g) M(g),
 \end{aligned}$$

which can be checked by an explicit calculation.

The operator  $M(g)$  is invariant under the swap  $b \leftrightarrow b^{-1}$ . Therefore it is natural to introduce the second set of generators  $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$  respecting the commutation relations (30) with the replacement  $b \rightarrow b^{-1}$ , i.e.

$$\begin{aligned}
 (38) \quad \tilde{C} \tilde{A} &= e^{\frac{i\pi}{b^2}} \tilde{A} \tilde{C}, \quad \tilde{D} \tilde{C} = e^{\frac{i\pi}{b^2}} \tilde{C} \tilde{D}, \\
 [\tilde{A}, \tilde{D}] &= -2i \sin^3 \frac{\pi}{b^2} \tilde{C}^2, \quad [\tilde{B}, \tilde{C}] = \frac{\tilde{A}^2 - \tilde{D}^2}{2i \sin \frac{\pi}{b^2}}, \\
 \tilde{A} \tilde{B} - e^{\frac{i\pi}{b^2}} \tilde{B} \tilde{A} &= e^{\frac{i\pi}{b^2}} \tilde{D} \tilde{B} - \tilde{B} \tilde{D} = \frac{i}{2} \sin \frac{2\pi}{b^2} (\tilde{C} \tilde{A} - \tilde{D} \tilde{C}).
 \end{aligned}$$

We also need to specify the commutation relations for generators from different sets. The generators  $A, D$  anticommute with  $\tilde{B}, \tilde{C}$ ; the generators  $B, C$  anticommute with  $\tilde{A}, \tilde{D}$ ; the generators  $A, D$  commute with  $\tilde{A}, \tilde{D}$ ; and the generators  $B, C$  commute with  $\tilde{B}, \tilde{C}$ .

An explicit realization of the generators  $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$  by finite-difference operators is given by the formulae (32)–(35), where  $b$  should be replaced by  $b^{-1}$ . New Casimir operators have the form

$$(39) \quad \begin{aligned} \tilde{K}_0 &= e^{i\pi/b^2} \tilde{A} \tilde{D} - \sin^2 \pi/b^2 \tilde{C}^2 = e^{i\pi/b^2}, \\ \tilde{K}_1 &= \frac{e^{-i\pi/b^2} \tilde{A}^2 + e^{i\pi/b^2} \tilde{D}^2}{4 \sin^2 \pi/b^2} - \tilde{B} \tilde{C} - \frac{1}{2} \cos \pi/b^2 \tilde{C}^2 = \frac{\cos 2\pi g/b}{2 \sin^2 \pi/b^2}. \end{aligned}$$

Taken together, two sets of generators  $A, B, C, D$  and  $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$  form the algebra which we call *the hyperbolic modular double*. It lies in between the Faddeev's modular double of  $U_q(sl_2)$  [25] and the elliptic modular double [43] in the sense that these algebraic structures are related by a sequence of contractions

$$\begin{aligned} &\text{Elliptic modular double} \\ &\quad \rightarrow \text{Hyperbolic modular double} \\ &\quad \quad \rightarrow \text{Modular double of } U_q(sl_2). \end{aligned}$$

Using particular combinations of the generators of this algebra similar to the one considered in [27], it is possible to construct a modular double of the Zhedanov algebra [51] as well.

Evidently,  $M(g)$  works as the intertwining operator for the second set of generators as well,

$$(40) \quad \begin{aligned} M(g) \tilde{A}(g) &= \tilde{A}(-g) M(g), & M(g) \tilde{B}(g) &= \tilde{B}(-g) M(g), \\ M(g) \tilde{C}(g) &= \tilde{C}(-g) M(g), & M(g) \tilde{D}(g) &= \tilde{D}(-g) M(g). \end{aligned}$$

Irreducible representations of the hyperbolic modular double are fixed by one complex number  $g$ , with the  $g$  and  $-g$  label representations being equivalent. Realization of the generators of this algebra in the space of analytical functions is unique (up to a multiplication by a numerical factor), because solutions of a system of finite-difference equations with the shifts by  $b$  and  $b^{-1}$  (which can be taken real and incommensurate) are determined up to multiplication by a number. Relations (37) and (40) are natural extensions of the intertwining relations for the  $U_q(sl_2)$  algebra and its modular double derived by Ponsot and Tschner [35]. In the limit described in [44], when the hyperbolic R-matrix is degenerated

to that of Faddeev and Volkov, the integral operator  $M(g)$  passes to the intertwining operator of [35].

Intertwining operators are quite useful, since they enable one to get insight to the structure of representations of the corresponding algebra. Indeed, the null-space of  $M(g)$ ,  $\text{Ker } M(g)$ , and the image of  $M(-g)$ ,  $\text{Im } M(-g)$ , are invariant spaces of the representation with the label  $g$  that follows from (37) and (40).

The inversion formula (24) implies that

$$M_z(g)\gamma(g \pm ix \pm iz) = 0, \text{ where } g = \frac{nb}{2} + \frac{m}{2b}, \quad n, m \in \mathbb{Z}_{\geq 0}, \quad (n, m) \neq (0, 0),$$

since the normalization factor  $1/\gamma(2g)$  of  $M(-g)$  is divergent at the specified values of  $g$  (see Eq. (5)). Here the subindex in the operator  $M_z$  indicates that  $z$  is used as the integration variable. Hence, expanding  $\gamma(g \pm ix \pm iz)$  in  $x$  we recover the null-space of  $M_z(g)$ . Moreover, the function  $\gamma(g \pm ix \pm iz)$  is proportional to the integrand of the operator  $M(-g)$ . Consequently, its expansion in  $x$  lies in the image of  $M(-g)$ . Let us remind that for these values of  $g$  the integral operator  $M(g)$  turns to the finite-difference operator (29).

In the following we will be interested in irreducible finite-dimensional representations of the hyperbolic modular double at

$$(41) \quad g_{n,m} = \frac{b}{2}(n+1) + \frac{1}{2b}(m+1), \quad n, m \in \mathbb{Z}_{\geq 0},$$

which have the dimension  $(n+1)(m+1)$ . They are realized in the invariant space

$$\text{Ker } M(g_{n,m}) \cap \text{Im } M_{ren}(-g_{n,m}),$$

where  $M_{ren}(-g) = \gamma(2g)M(-g)$ .

All basis vectors of the finite-dimensional irreducible representation are embraced by the generating function

$$\gamma(\pm ix \pm iz + g_{n,m}),$$

where  $x$  is an auxiliary parameter. Indeed, owing to Eqs. (3) and (4), it turns into the finite product of trigonometric functions

$$(42) \quad \gamma(\pm ix \pm iz + g_{n,m}) = \prod_{r=0}^{n-1} 2 \sin \pi b (\pm ix + iz + \frac{b}{2}(n-1-2r) + \frac{1}{2b}(m+1)) \times \prod_{s=0}^{m-1} 2 \sin \frac{\pi}{b} (\pm ix + iz + \frac{1}{2b}(m-1-2s) - \frac{b}{2}(n-1)).$$

From the latter formula we extract the natural basis of the finite-dimensional representation

$$\cos(2j\pi ibz) \cos(2l\pi iz/b), \quad j = 0, 1, \dots, n, \quad l = 0, 1, \dots, m.$$

Note that the generating function coincides with the edge Boltzmann weight (7).

### §5. Reductions of the integral R-operator

In this section we show that the integral operator solution, Eq. (18), of the YBE (16) enables one to recover all finite-dimensional solutions of YBE as well. In order to do it we apply the operator  $\mathbb{R}_{12}(u|g_1, g_2)$  to the function  $\gamma(\pm iz_1 \pm iz_3 + u_1 - u_2)\Phi(z_2)$ , where  $z_3$  is an auxiliary parameter and  $\Phi(z_2)$  is an arbitrary function from the second space. For  $g_1 = g_{n,m}$  the first factor turns to the generating function, Eq. (42). Temporarily we assume  $g_1$  to be generic. Computation of the result of this action is pictorially presented in Fig. 4, where we slightly changed the graphical rules. Now all edges represent the Boltzmann weights  $\overline{W}$ , Eq. (7). The black blob corresponds to the vertex Boltzmann weight  $\rho$ . We omit rapidity lines and indicate corresponding differences of the rapidities explicitly.

At the first step we apply the star-triangle relation, Eq. (9), implementing integration at the vertex  $x_1$ . Thus the only integration left is at the vertex  $x_2$ . It corresponds to the integral operator  $M_1(u_2 - v_2)$  with the kernel  $\rho(x_2) \overline{W}(u_2 - v_2; z_1, x_2)$  which acts on the product  $\overline{W}(v_1 - u_1 + \eta; z_1, z_3) \overline{W}(u_1 - u_2 + \eta; z_1, z_2) \Phi(z_1)$ . At the second step we just rearrange the factors such that we gain the integral operator  $M_2(u_1 - u_2 + \eta)$  with the kernel  $\rho(x_2) \overline{W}(u_1 - u_2 + \eta; z_2, x_2)$  which acts on the product  $\overline{W}(u_1 - v_2; z_1, z_2) \overline{W}(v_1 - u_1 + \eta; z_2, z_3) \Phi(z_2)$ . Then we note that the

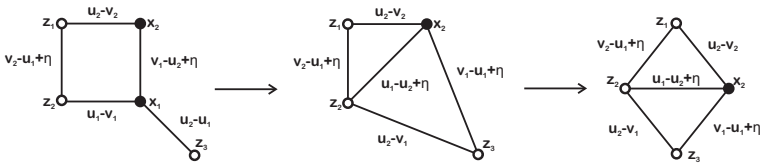


Fig. 4. The sequence of transformations converting the integral operator  $\mathbb{R}_{12}(u|g_1, g_2)$  at  $g_1 = g_{n,m}$  to a finite-dimensional matrix in the first space.

remaining integral operator  $M_2(u_1 - u_2 + \eta) = M_2(g_1 + \eta)$  for  $g_1 = g_{n,m}$

turns to the finite-difference operator  $M_2(\frac{nb}{2} + \frac{m}{2b})$ , Eq. (29). Thus we have obtained the reduction formula which encompasses all solutions of the YBE (16) that have the symmetry of the hyperbolic modular double and are realized on the tensor product of the finite-dimensional representation with the label  $g_{n,m}$ , Eq. (41), in the first space, and arbitrary infinite-dimensional representation with the label  $g$  in the second one,

$$\begin{aligned}
 (43) \quad & \mathbb{R}_{12}(u|g_{n,m}, g) \gamma(\pm iz_1 \pm iz_3 + \frac{b}{2}(n+1) + \frac{1}{2b}(m+1)) \Phi(z_2) \\
 &= c \cdot \frac{\gamma(\pm iz_2 \pm iz_3 + \frac{-u+g_{n,m}+g}{2})}{\gamma(\pm iz_1 \pm iz_3 + \frac{-u-g_{n,m}-g-2\eta}{2})} \times \\
 & \quad M_2(\frac{nb}{2} + \frac{m}{2b}) \frac{\gamma(\pm iz_1 \pm iz_2 + \frac{-u+g_{n,m}-g}{2})}{\gamma(\pm iz_2 \pm iz_3 + \frac{-u-g_{n,m}+g-2\eta}{2})} \Phi(z_2),
 \end{aligned}$$

where

$$c = \frac{1}{\gamma(u + g_{n,m} \pm g)}.$$

The same result can be obtained using the fusion following the procedure described in [13, 14].

Expanding both sides of this formula in the auxiliary parameter  $z_3$  we recover the reduced R-operator that is a matrix whose entries are some finite-difference operators acting in the second space, i.e. we have the L-operator. One can straightforwardly reduce further the L-operator (43) to R-matrices which are finite-dimensional in both spaces. In order to achieve it we just need to force the representation label  $g$  of the second space to lie on the second copy of the lattice (41).

Remarkably, the factorized form of the integrand function of the integral operator (18) is inherited by the reduced R-operator. We will see in Sect. 6 and 7 that the reduced solution of YBE (43) can be further arranged to the factorized product of matrices form. In [13, 14] an analogous reduction formula has been derived for the integral R-operators in the following three cases: 1) the R-operator with the symmetry group  $SL(2, \mathbb{C})$ ; 2) solutions of YBE with the symmetry of the modular double of  $U_q(sl_2)$ ; 3) the most general known R-operator obeying the symmetry of the elliptic modular double (and of the Sklyanin algebra, of course).

### §6. The fundamental representation L-operator and its factorization

Let us show how formula (43) works in practice. We consider the simplest nontrivial representation in the first space  $g_1 = g_{1,0} = b + \frac{1}{2b}$

(see Eq. (41)), i.e. the fundamental representation of the  $(A, B, C, D)$ -generated part of the hyperbolic double and trivial representation for the  $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ -part. Corresponding solution of the YBE is known as the (spin  $\frac{1}{2}$ ) L-operator. The generating function in this case has the following form (see Eq. (42))

$$\gamma(\pm iz_1 \pm iz_3 + b + \frac{1}{2b}) = 2 \cos 2\pi ibz_1 + 2 \cos 2\pi ibz_3 = \mathbf{e}_1 2 \cos 2\pi ibz_3 + \mathbf{e}_2,$$

where the basis of the 2-dimensional representation in the first space  $\mathbb{C}^2$  is formed by  $\mathbf{e}_1 = 1$  and  $\mathbf{e}_2 = 2 \cos 2\pi ibz_1$ . The intertwining operator from (43) simplifies to  $M_2(\frac{b}{2}) = c \cdot \frac{1}{\sin 2\pi ibz_2} \left( e^{\frac{ib}{2}\partial_2} - e^{-\frac{ib}{2}\partial_2} \right)$  (see Eq. (29)). To simplify the formulae we shift the spectral parameter  $u \rightarrow u + \frac{b}{2}$ . Now we wish to rewrite formula (43) in a matrix form. In the formula (43) we pull the hyperbolic gamma functions depending on  $\pm iz_1 \pm iz_2$  to the left and the hyperbolic gamma functions depending on  $\pm iz_2 \pm iz_3$  to the right. Then we simplify them by means of Eqs. (3) and (4). Thus the right-hand side expression in (43) takes the form

$$\begin{aligned} & \left[ 2 \cos 2\pi ibz_1 - 2 \cos \pi b(2iz_2 + u + g) \right] \times \\ & \quad e^{\frac{ib}{2}\partial_2} \left[ 2 \cos 2\pi ibz_3 - 2 \cos \pi b(2iz_2 - u + g) \right] \\ & - \left[ 2 \cos 2\pi ibz_1 - 2 \cos \pi b(2iz_2 - u - g) \right] \times \\ (44) \quad & \quad e^{-\frac{ib}{2}\partial_2} \left[ 2 \cos 2\pi ibz_3 - 2 \cos \pi b(2iz_2 + u - g) \right]. \end{aligned}$$

Now it is straightforward to rewrite the reduced R-operator  $\mathbb{R}_{12}(u + \frac{b}{2}|g_{1,0}, g) =: L(u|g)$  in a matrix form in the basis  $\{\mathbf{e}_1, \mathbf{e}_2\}$  of  $\mathbb{C}^2$ , using the definition of matrix elements  $L(u|g) \mathbf{e}_k := \sum_i \mathbf{e}_i [L(u|g)]_{i,k}$ ,

$$(45) \quad L(u|g) = \frac{1}{\sin 2\pi ibz} \begin{pmatrix} -2 \cos 2\pi b(iz + u_1) & -2 \cos 2\pi b(iz - u_1) \\ 1 & 1 \end{pmatrix} \times \begin{pmatrix} e^{\frac{ib}{2}\partial} & 0 \\ 0 & -e^{-\frac{ib}{2}\partial} \end{pmatrix} \begin{pmatrix} 1 & -2 \cos 2\pi b(iz - u_2) \\ 1 & -2 \cos 2\pi b(iz + u_2) \end{pmatrix}.$$

Here we substituted  $z_2 \rightarrow z$ . The rapidities  $u_1, u_2$  are defined as  $u_1 = \frac{u+g}{2}$  and  $u_2 = \frac{u-g}{2}$  (recall Eq. (20)). We stress that the L-operator is automatically obtained in the factorized form, Eq. (45). This might be expected since the initial formula (43) obeys similar factorization.

Choosing in the L-operator the second space representation label as  $g = g_{1,0} = b + \frac{1}{2b}$ , i.e. restricting it to the fundamental representation as well, we recover a  $4 \times 4$  matrix solution of the YBE (16). This

solution differs from the standard trigonometric R-matrix with 6 nonzero entries [31]. It is the R-matrix of the 7-vertex model [3, 1].

This factorized representation is analogous to the one found in [3, 28] (compare Eq. (45) with the normal-ordered factorized L-operator Eq. (2.20) in [3]). There the lateral matrices are identified with the trigonometric intertwining vectors that provide the vertex-face correspondence between the 7-vertex model and a trigonometric SOS model.

The L-operator, Eq. (45), can be written in terms of the degenerate Sklyanin algebra generators as well,

$$(46) \quad L(u|g) = 2 \left( \begin{array}{c} -e^{-i\pi bu} A(g) - e^{i\pi bu} D(g) \\ \sin \pi b^2 C(g) \\ -4 \sin \pi b^2 B(g) - 2 \sin \pi b^2 (\cos 2\pi bu + \cos \pi b^2) C(g) \\ e^{i\pi bu} A(g) + e^{-i\pi bu} D(g) \end{array} \right).$$

In [3] this L-operator (46) has been identified with the quantum L-operator for the 2-particle trigonometric Ruijsenaars model.

Taking  $g_1 = g_{0,1} = \frac{b}{2} + \frac{1}{b}$  one recovers in the same way the second L-operator  $\tilde{L}(u|g)$  whose entries are generators of the second half of the hyperbolic modular double.

Thus we see that our construction is self-consistent. A particular reduction of the integral R-operator results in the generators of the hyperbolic modular double. Consequently the latter quantum algebra is indeed the symmetry algebra of the integral R-operator (19). The algebraic interpretation of the rapidities stated above (20) is correct. The infinite-dimensional spaces in the construction of the integral R-operator are equipped with the structure of representations of the hyperbolic modular double.

Implementing the reduction condition  $g_3 = g_{1,0}$  in the YBE (16) we obtain an RLL-relation,

$$(47) \quad \begin{aligned} &\mathbb{R}_{12}(u-v|g_1, g_2) L_1(u|g_1) L_2(v|g_2) \\ &= L_2(v|g_2) L_1(u|g_1) \mathbb{R}_{12}(u-v|g_1, g_2). \end{aligned}$$

Here the integral R-operator (19) acts in a pair of infinite-dimensional spaces with the representation labels  $g_1$  and  $g_2$  and intertwines the matrix product of two L-operators. The lower index (1 or 2) of the L-operator enumerates the infinite-dimensional spaces where it acts non-trivially, i.e. the entries of  $L_i$  (see Eq. (45)) are some difference operators in the variable  $z_i$ . The RLL-relation (47) can be rewritten in terms of

the rapidities as well in a full analogy with Eq. (17),

$$(48) \quad \begin{aligned} & \mathbb{R}_{12}(u_1, u_2|v_1, v_2) L_1(u_1, u_2) L_2(v_1, v_2) \\ & = L_2(v_1, v_2) L_1(u_1, u_2) \mathbb{R}_{12}(u_1, u_2|v_1, v_2), \end{aligned}$$

where  $L(u_1, u_2) := L(u|g_1)$ ,  $L(v_1, v_2) := L(v|g_2)$  (recall Eq. (20)).

Now we can give a natural interpretation of the operator  $S(u)$ , Eq. (22), which is one of the factors of the R-operator (21). It implements the permutation of rapidities  $(u_1, u_2, v_1, v_2) \mapsto (u_1, v_1, u_2, v_2)$  in the matrix product of two L-operators, i.e.

$$S(u_2 - v_1) L_1(u_1, u_2) L_2(v_1, v_2) = L_1(u_1, v_1) L_2(u_2, v_2) S(u_2 - v_1).$$

This statement can be checked by a straightforward calculation. On the other hand, the R-operator itself implements the permutation of the rapidities  $(u_1, u_2, v_1, v_2) \mapsto (v_1, v_2, u_1, u_2)$  in Eq. (48). For more details of such permutation of parameters in various models, see [17, 19, 20].

## §7. Factorized finite-dimensional solutions of the YBE

In the previous section we have shown that the reduction formula (43) produces the L-operator in the factorized form from the fundamental representation in the first space for the integral R-operator. Now we are going to demonstrate that the same pattern persists for all finite-dimensional representations, i.e. we show that the higher-spin solutions of the YBE can be factorized as well. Finite-dimensional representations of the hyperbolic modular double naturally factorize to products of finite-dimensional representations of its two halves. Therefore without loss of generality we can consider nontrivial representations for only one of its halves. Thus, we choose  $g_1 = g_{n,0} = \frac{b}{2}(n+1) + \frac{1}{2b}$ ,  $n \in \mathbb{Z}_{\geq 0}$ , (recall Eq. (41)) in the reduction formula (43).

The generating function of the  $(n+1)$ -dimensional representation of interest takes the form (recall Eq. (42))

$$(49) \quad \begin{aligned} \gamma(\pm iz \pm ix + g_{n,0}) &= \prod_{r=0}^{n-1} \left[ 2 \cos 2\pi ibz + 2 \cos \pi b(2ix + b(n-1-2r)) \right] \\ &= \sum_{j=1}^{n+1} \psi_{n+2-j}^{(n)}(x) \varphi_j^{(n)}(z) = \sum_{j=1}^{n+1} \varphi_{n+2-j}^{(n)}(x) \psi_j^{(n)}(z), \end{aligned}$$

where

$$\varphi_j^{(n)}(z) := (2 \cos 2\pi ibz)^{j-1}, \quad j = 1, 2, \dots, n+1.$$



The second equality in (49) is used to define the dual basis  $\psi^{(n)}(x)$ , whereas the third equality follows from the invariance of the generating function under the permutation of  $x$  and  $z$ . Thus the generating function produces two natural bases  $\{\mathbf{e}_j\}_{j=1}^{n+1}$  and  $\{\mathbf{f}_j\}_{j=1}^{n+1}$  of  $\mathbb{C}^{n+1}$ ,

$$\mathbf{e}_j = \varphi_j^{(n)}(z) \quad , \quad \mathbf{f}_j = \psi_j^{(n)}(z) \quad , \quad j = 1, 2, \dots, n + 1.$$

Expanding both sides of Eq. (43) as linear combinations of  $\varphi^{(n)}(z_3)$ , we obtain a matrix form of the reduced R-operator written in the indicated pair of bases, i.e.

$$\mathbb{R}_{12}(u|g_n, g) \psi_j^{(n)}(z_1) = \varphi_l^{(n)}(z_1) [\mathbb{R}_{12}(u|g_n, g)]_{lj}.$$

In the previous section we did not have such a subtlety, since for  $n = 1$  (the fundamental representation) both bases coincide.

Similar to the pattern given in the previous section, the direct calculation yields the following factorization formula for the reduced R-operator

$$(50) \quad \mathbb{R}_{12}(u|g_{n,0}, g) = V(u + g, z) D(z, \partial) \mathbf{C} V^T(u - g, z) \mathbf{C},$$

consisting of the product of five matrices. Here we substituted  $z_2 \rightarrow z$  for brevity.  $\mathbf{C}$  is a numerical matrix with the unities on the antidiagonal, i.e.  $(\mathbf{C})_{lj} = \delta_{n+2-l,j}$ . Entries of the diagonal matrix

$$(51) \quad [D(z, \partial)]_{lj} := \delta_{lj} \beta_l^{(n)}(z) e^{(n+2-2l)\frac{i\eta}{2}\partial_z}.$$

are the shift operators determined by the expansion of  $M(\frac{nb}{2})$  of the form

$$(52) \quad M(\frac{nb}{2}) = \sum_{l=1}^{n+1} \beta_l^{(n)}(z) e^{(n+2-2l)\eta\partial_z}.$$

Entries of the matrix  $V$ ,  $[V(u, z)]_{jl} = V_{jl}^{(n)}(u, z)$ , are some trigonometric functions. They are defined by the relations

$$(53) \quad \sum_{j=1}^{n+1} \varphi_j^{(n)}(x) V_{jl}^{(n)}(u, z) := \prod_{r=0}^{l-2} \left[ 2 \cos 2\pi i b x - 2 \cos \pi b (2iz - u - 2\eta - g_{n,0} + 2br) \right] \times \prod_{r=0}^{n-l} \left[ 2 \cos 2\pi i b x - 2 \cos \pi b (-2iz - u - 2\eta - g_{n,0} + 2br) \right].$$

It is easy to see that  $V_{ji}^{(n)}(u, -z) = V_{j,n+2-i}^{(n)}(u, z)$ , i.e.  $V(u, -z) = V(u, z) \mathbf{C}$ .

Let us recall that for the factorized L-operator the lateral matrices are composed out of the trigonometric intertwining vectors providing the vertex-face correspondence [3]. Then, in the case of the  $(n + 1)$ -dimensional representation, the lateral matrices  $V$  are constructed out of the fused trigonometric intertwining vectors (see Eq. (53)) providing the vertex-face correspondence for the higher-spin models. In [11] an analogous factorization formula has been derived for finite-dimensional R-operators with the symmetry algebras  $sl_2$ ,  $U_q(sl_2)$ , and the Sklyanin algebra.

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