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Cusp singularities and quasi-polyhedral sets

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Abstract.

We study cusp singularities from the viewpoint of locally polyhedral sets and reflection groups. Following the definition of quasi-polyhedral sets by Grünbaum, we consider a special kind of rational quasi-polyhedral sets with group action and refer to the relation with cusp singularities. We give also some examples of such quasi-polyhedral sets by using discrete groups generated by reflections.

§ Introduction

In [T], Tsuchihashi defined toric type isolated cusp singularities by generalizing cusp singularities in dimension two. This class of singularities includes the cusp singularities of Hilbert modular varieties in general dimension, and more generally, the cusp singularities appearing in Satake compactifications of $\mathbf{Q}$-rank one tube domains (cf. [SO, 2.1]). However, besides these arithmetic cusp singularities, it is hard to give examples. Although Tsuchihashi gave some examples is dimension three, we do not know whether there are many toric type cusp singularities in general dimension or not.

In this note, we study cusp singularities from the viewpoint of locally polyhedral sets and reflection groups. Following the definition of quasi-polyhedral sets by Grünbaum, we consider a special kind of rational quasi-polyhedral sets with group action and refer to the relation with cusp singularities. In order to give examples of such quasi-polyhedral sets with group action, discrete groups generated by reflections will play an important role. In Sections 3 and 4, we will introduce some examples of groups generated by reflections.

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§1. Quasi-polyhedral sets

Let r be a non-negative integer and M a free $\mathbf{Z}$-module of rank r. We consider convex sets in the real space $M_{\mathbf{R}} = M \otimes_{\mathbf{Z}} \mathbf{R}$ of dimension r. As a subset of $M_{\mathbf{R}}$, M is called the lattice, and points of $M_{\mathbf{Q}} = M \otimes_{\mathbf{Z}} \mathbf{Q}$ are called rational points. A non-empty subset C of $M_{\mathbf{R}}$ is called a *cone* if $ax \in C$ for any $x \in C$ and $a \geq 0$. A non-empty C is a convex cone if and only if $ax + by \in C$ for any $x, y \in C$ and $a, b \geq 0$. By an *open cone*, we mean an open convex set C such that $C \cup \{0\}$ is a cone. For a convex cone C, we set $L(C) = C \cap (-C)$. $L(C)$ is the largest linear subspace included in C. A closed convex cone C is said to be *strongly convex* if $L(C) = \{0\}$.

For any subset S of $M_{\mathbf{R}}$, the cone generated by S is the set of finite linear combinations of S with non-negative coefficients, which is $\{0\}$ if S is empty. The cone C generated by a finite set S is called a *polyhedral cone*, which is always a closed convex cone. The polyhedral cone is said to be *rational* if S is a subset of $M_{\mathbf{Q}}$. A rational polyhedral cone is written as $C = \mathbf{R}_0 x_1 + \cdots + \mathbf{R}_0 x_s$ for a subset $\{x_1, \ldots, x_s\} \subset M$, where $\mathbf{R}_0 = \{c \in \mathbf{R} \,;\, c \geq 0\}$.

We set $N = \mathrm{Hom}_{\mathbf{Z}}(M, \mathbf{Z})$ and $N_{\mathbf{R}} = N \otimes_{\mathbf{Z}} \mathbf{R}$. Polyhedral cones of $N_{\mathbf{R}}$ are defined similarly. A strongly convex rational polyhedral cone in $N_{\mathbf{R}}$, which is used in the definition of affine toric variety, is called simply a cone, and usually denoted by a Greek lowercase letter such as σ, τ, and so on. Namely, if we write "a cone σ in $N_{\mathbf{R}}$", then σ is a strongly convex rational polyhedral cone in $N_{\mathbf{R}}$.

Let $\langle\ ,\ \rangle : M_{\mathbf{R}} \times N_{\mathbf{R}} \to \mathbf{R}$ be the natural $\mathbf{R}$-bilinear map. For an element $x \in M_{\mathbf{R}}$, we set

$$(x \geq 0) = \{u \in N_{\mathbf{R}} \,;\, \langle x, u \rangle \geq 0\}$$

and

$$(x = 0) = \{u \in N_{\mathbf{R}} \,;\, \langle x, u \rangle = 0\}\ .$$

A subset ρ of a cone σ in $N_{\mathbf{R}}$ is called a *face* and expressed as $\rho \prec \sigma$ if there exists x with $\sigma \subset (x \geq 0)$ and $\rho = \sigma \cap (x = 0)$. The set of faces of a cone σ is denoted by $F(\sigma)$. The *dual cone* $\sigma^{\vee}$ in $M_{\mathbf{R}}$ is defined by

$$\sigma^{\vee} = \{x \in M_{\mathbf{R}} \,;\, \sigma \subset (x \geq 0)\}\ ,$$

while

$$\sigma^{\perp} = \{x \in M_{\mathbf{R}} \,;\, \sigma \subset (x = 0)\}\ .$$

Note that each $x \in \sigma^{\vee}$ defines the face $\sigma \cap (x = 0)$ of σ.

For a cone σ in $N_{\mathbf{R}}$, the *relative interior* of σ, which is denoted by $\mathrm{rel.\,int}\,\sigma$, is the interior of σ in the minimal linear subspace containing it. It is known that $u \in \sigma$ is in the relative interior if and only if $\sigma^\vee \subset (u \geq 0)$ and $\sigma^\vee \cap (u = 0) = \sigma^\perp$ (cf. [O, Lemma A.4]). The relative interior of a polyhedral cone in $M_{\mathbf{R}}$ is defined similarly.

Lemma 1.1. *Let σ be a (strongly convex rational polyhedral) cone of $N_{\mathbf{R}}$. Then we have the following.*

(1) *σ itself and the zero cone $\mathbf{0} = \{0\}$ are in $F(\sigma)$, and each element of $F(\sigma)$ is a cone.*

(2) *The dual cone $\sigma^\vee$ is r-dimensional, and contains the $(r - \dim \sigma)$-dimensional linear subspace $\sigma^\perp$ as a face. $\sigma^\vee$ is strongly convex if and only if $\dim \sigma = r$. The dual cone of $\sigma^\vee$ in $N_{\mathbf{R}}$ is equal to σ.*

(3) *$F(\sigma)$ and $F(\sigma^\vee)$ are finite sets. There exists a bijection defined by mapping $\rho \in F(\sigma)$ to $\sigma^\vee \cap \rho^\perp \in F(\sigma^\vee)$.*

(4) *If $x \in \sigma^\vee$ defines a face $\rho \in F(\sigma)$, then x is in $\operatorname{rel.int}(\sigma^\vee \cap \rho^\perp)$ and $\sigma^\vee + \mathbf{R}x$ is equal to $\rho^\vee$. A point $u \in \operatorname{rel.int} \rho$ defines the faces $\rho^\perp$ and $\sigma^\vee \cap \rho^\perp$ of the r-dimensional cones $\rho^\vee$ and $\sigma^\vee$, respectively.*

(5) *For any $x \in \sigma^\vee$, $\sigma^\vee$ is a closed neighborhood of x in $\sigma^\vee + \mathbf{R}x$.*

Proof. We prove only (5). See [O, Appendix] for the others. Set $\rho = \sigma \cap (x = 0)$. Assume $\{u_1, \ldots, u_t\}$ generates σ, and $u_1, \ldots, u_s \in \rho$ and $u_{s+1}, \ldots, u_t \notin \rho$. In this case, ρ is generated by $u_1, \ldots, u_s$, and $\sigma^\vee + \mathbf{R}x = \rho^\vee = (u_1 \geq 0) \cap \cdots \cap (u_s \geq 0)$. Then $\rho^\vee \cap (u_{s+1} > 0) \cap \cdots \cap (u_t > 0)$ is an open neighborhood of x in $\rho^\vee$ included in $\sigma^\vee$. Q.E.D.

A non-empty closed convex subset $P \subset M_{\mathbf{R}}$ is called a *quasi-polyhedral set* if $P - x = \{y - x \ ; \ y \in P\}$ is equal to a polyhedral cone C_x in a neighborhood of the origin for all $x \in P$ (cf. [G, p. 36]). In this case, C_x is equal to the cone generated by $P - x$. A quasi-polyhedral set P is said to be *rational* if C_x is rational and $x + L(C_x)$ is a rational affine subspace for every $x \in P$, *non-degenerate* if P contains an interior point and *strongly convex* if P contains no line.

For an element $u \in N_{\mathbf{R}}$ and a real number a, we set

$$(u \geq a) = \{x \in M_{\mathbf{R}} \ ; \ \langle x, u \rangle \geq a\}$$

and

$$(u = a) = \{x \in M_{\mathbf{R}} \ ; \ \langle x, u \rangle = a\} .$$

$(u \geq 0)$ is a closed half space and $(u = a)$ is a hyperplane if $u \neq 0$. A non-empty subset $Q \subset P$ is said to be a *face* of P if there exist u and a such that $P \subset (u \geq a)$ and $Q = P \cap (u = a)$. A face Q

is also a quasi-polyhedral set and it is non-degenerate in the minimal affine space containing it. However, when we consider rational points or lattice points for Q, we have to assume suitable conditions on the affine subspace. A 0-dimensional face is called a *vertex*. Any face of a face of P is also a face of P, and any quasi-polyhedral set has a proper face if it is not an affine subspace. In particular, a minimal face of a quasi-polyhedral set is an affine subspace. Hence, a strongly convex quasi-polyhedral set has at least one vertex.

For a non-empty closed convex set D of $M_{\mathbf{R}}$, the *characteristic cone* $\mathrm{cc}(D)$ is defined by

$$\mathrm{cc}(D) = \{y \in M_{\mathbf{R}}\ ;\ x + \mathbf{R}_0 y \subset D\}$$

for $x \in D$. Since we assume D closed, this definition does not depend on the choice of x, and $\mathrm{cc}(D)$ is a closed convex cone (cf. [G, p. 24]). D is bounded if and only if $\mathrm{cc}(P) = \{0\}$ by [G, 2.5.1]. For closed convex sets $D_1, D_2 \subset M_{\mathbf{R}}$ with $D_1 \cap D_2 \neq \emptyset$, we have $\mathrm{cc}(D_1 \cap D_2) = \mathrm{cc}(D_1) \cap \mathrm{cc}(D_2)$. In particular, $\mathrm{cc}(D_1) \subset \mathrm{cc}(D_2)$ if $D_1 \subset D_2$.

We set $\text{cc-dim}(D) = \dim \mathrm{cc}(D)$. The closed convex cone $\mathrm{cc}(D)$ is not a polyhedral cone in general even if D is a quasi-polyhedral set. Here, we prove the following elementary lemma for reader's convenience.

Lemma 1.2. *Let $C \subset M_{\mathbf{R}}$ be a closed convex cone. Then $u \in C^\vee$ is in* $\mathrm{int}(C^\vee)$ *if and only if $C \cap (u = 0) = \{0\}$.*

Proof. We fix an Euclidean metric on $M_{\mathbf{R}}$. Let $S \subset M_{\mathbf{R}}$ be the unit sphere with the center at the origin. Then $u \in N_{\mathbf{R}}$ is in $C^\vee$ if and only if it is non-negative on $C \cap S$. If $C \cap (u = 0) = \{0\}$, then the linear function u restricted to $C \cap S$ has the positive minimum since $C \cap S$ is compact. Hence $u' \in N_{\mathbf{R}}$ sufficiently near u is positive on the compact set $C \cap S$. This means that u is in the interior of $C^\vee$.

Next assume that there exists non-zero $x \in C$ with $\langle x, u \rangle = 0$. We may assume $x \in C \cap S$. Since x is not zero, there exists $v \in N_{\mathbf{R}}$ with $\langle x, v \rangle < 0$. Then for any positive a, we have $\langle x, u + av \rangle < 0$, and $u + av$ is outside $C^\vee$. Hence the limit u is not in $\mathrm{int}(C^\vee)$. Q.E.D.

We need the following lemma for the proof of Theorem 1.4.

Lemma 1.3. *Let P be a non-degenerate strongly convex rational quasi-polyhedral set. For $x, y \in P$, we have $C_x^\vee \subset (y - x \geq 0)$, $C_y^\vee \subset (x - y \geq 0)$, and the equalities*

$$C_x^\vee \cap C_y^\vee = C_{(x+y)/2}^\vee = C_x^\vee \cap (y - x = 0) = C_y^\vee \cap (y - x = 0)$$

hold.

Proof. If $x = y$, then the assertions are obvious. Hence we assume $x \neq y$. The former two inclusions follow from $y - x \in C_x$ and $x - y \in C_y$. By these inclusions, we have $C_x^\vee \cap C_y^\vee \subset (y - x = 0)$. Since $x - (x+y)/2 = (x-y)/2$ and $y - (x+y)/2 = -(x-y)/2$ are in $C_{(x+y)/2}$, we know $C_{(x+y)/2}^\vee \subset (y - x = 0)$. In general, we have $C_w^\vee = \bigcap_{z \in P}(z - w \geq 0)$ for $w \in P$. Since the linear functions $z - x$, $z - y$, $z - (x+y)/2$ are equal in the hyperplane $(y - x = 0)$, these four cones in $(y - x = 0)$ are equal. Q.E.D.

For a non-empty quasi-polyhedral set P, we define

$$\Sigma(P) = \{C_x^\vee \; ; x \in P\} .$$

A non-empty set Σ of cones in $N_{\mathbf{R}}$ is said to be a *fan* if (1) $\sigma \in \Sigma$ and $\rho \prec \sigma$ imply $\rho \in \Sigma$, and (2) if $\sigma, \tau \in \Sigma$, then $\sigma \cap \tau$ is a common face of σ and τ. The *support* $|\Sigma|$ of a fan Σ is defined by $|\Sigma| = \bigcup_{\sigma \in \Sigma} \sigma$. The support is not necessarily closed if Σ is infinite.

Theorem 1.4. *Let P be a non-degenerate strongly convex rational quasi-polyhedral set. Then $\Sigma(P)$ is a fan of $N_{\mathbf{R}}$ such that*

$$\mathrm{int}(\mathrm{cc}(P)^\vee) \subset |\Sigma(P)| \subset \mathrm{cc}(P)^\vee .$$

$\Sigma(P)$ is locally finite at each point of $\mathrm{int}(\mathrm{cc}(P))$.

Proof. Since P is non-degenerate, $\Sigma(P)$ is a non-empty set of strongly convex cones.

For any $\sigma \in \Sigma(P)$, there exists $x \in P$ with $C_x^\vee = \sigma$. If ρ is a face of σ, then $\rho = \sigma \cap y^\perp$ for $y \in \mathrm{rel.\,int}(\sigma^\vee \cap \rho^\perp) \subset C_x$. We take $\epsilon > 0$ sufficiently small so that $x + 2\epsilon y \in P$. Then $(x + (x + 2\epsilon y))/2 = x + \epsilon y \in P$ and, by Lemma 1.3, $C_{x+\epsilon y}^\vee = C_x^\vee \cap (2\epsilon y = 0)$, which is equal to ρ. Hence $\rho \in \Sigma(P)$. If $\sigma = C_x^\vee$ and $\tau = C_y^\vee$ for $x, y \in P$, then $\sigma \cap \tau$ is a common face of σ and τ defined by $y - x$ and $x - y$, respectively, by Lemma 1.3. Hence $\Sigma(P)$ is a fan. Since $P - x \subset C_x$, we have $\mathrm{cc}(P) \subset C_x$ and $\sigma = C_x^\vee \subset \mathrm{cc}(P)^\vee$. Since σ is arbitrary, we have $|\Sigma(P)| \subset \mathrm{cc}(P)^\vee$.

Let u be in $\mathrm{int}(\mathrm{cc}(P)^\vee)$. We take $a \in \mathbf{R}$ such that $P' = P \cap (u \leq a)$ is non-empty. Then P' is bounded since

$$\mathrm{cc}(P \cap (u \leq a)) = \mathrm{cc}(P) \cap (u \leq 0) = \{0\}$$

by Lemma 1.2. By the compactness of P', there exists $x \in P' \subset P$ such that the linear function u has the minimum value. We have $u \in C_x^\vee \subset |\Sigma(P)|$ for this x. Hence $\mathrm{int}(\mathrm{cc}(P)^\vee)$ is contained in $|\Sigma(P)|$.

Consider the face $Q = P \cap (u = \langle x, u \rangle)$ of P for the above u and x. Q is bounded since it is contained in P'. Let $x_1, \ldots, x_s$ be the set of vertices of Q. We set $\sigma_i = C_{x_i}^\vee$ for $i = 1, \ldots, s$. For each i, u is in σ_i and $Q - x_i$ is equal to the face $C_{x_i} \cap u^\perp$ of C_{x_i} in a neighborhood of the origin. We set

$$D_i = \{v \in N_{\mathbf{R}} \; ; \langle x_i, v\rangle \leq \langle x_j, v\rangle, j = 1, \ldots, s\}$$

for each $i = 1, \ldots, s$. Then, since $v \in D_i$ for i with $\langle x_i, v\rangle$ minimum, $N_{\mathbf{R}}$ is the union of $D_1, \ldots, D_s$. Since $\{x_j - x_i \; ; j = 1, \ldots, s\}$ generates the cone $C_{x_i} \cap (u = 0)$, D_i is equal to $(C_{x_i} \cap (u = 0))^\vee = \sigma_i + \mathbf{R}u$ for every i. Since each σ_i is a neighborhood of u in D_i by Lemma 1.1 (5), the finite union $\sigma_1 \cup \cdots \cup \sigma_s$ is a neighborhood of u in $N_{\mathbf{R}}$. Let Σ_x be the subfan of $\Sigma(P)$ consisting of $\sigma_1, \ldots, \sigma_s$ and their faces. Then Σ_x is a finite fan and the support is a neighborhood of u. Hence, $\Sigma(P)$ is locally finite at $u \in \mathrm{int}(\mathrm{cc}(P))$. Q.E.D.

For a face Q of a non-degenerate quasi-polyhedral set P, the *relative interior* of Q, which is denoted by $\mathrm{rel.int}\, Q$, is the interior of Q in the minimal affine subspace containing it. For each $x \in P$, the face Q defined by $(u = \langle x, u\rangle)$ for an element $u \in \mathrm{rel.int}\, C_x^\vee$ is the unique face of P which contains x in its relative interior.

Lemma 1.5. *Let Q be a face of a non-degenerate quasi-polyhedral set P and x a point in* $\mathrm{rel.int}\, Q$. *Then a point $y \in P$ is in Q if and only if $C_y \subset C_x$, and y is in* $\mathrm{rel.int}\, Q$ *if and only if $C_x = C_y$.*

Proof. Since C_x is a polyhedral cone, we can take non-zero $u_1, \ldots, u_s \in N_{\mathbf{R}}$ such that $C_x = (u_1 \geq 0) \cap \cdots \cap (u_s \geq 0)$. Namely, there exist $a_1, \ldots, a_s \in \mathbf{R}$ such that x is in $L = (u_1 = a_1) \cap \cdots \cap (u_s = a_s)$ and P is equal to $(u_1 \geq a_1) \cap \cdots \cap (u_s \geq a_s)$ in a neighborhood of x. Then Q is defined by $(u = a_1 + \cdots + a_s)$ for $u = u_1 + \cdots + u_s$, and L is the minimal affine subspace containing Q. If a point $y \in P$ is in Q, then $\langle y, u_i\rangle = a_i$ for $i = 1, \ldots, s$, and hence $P - y$ and C_y are contained in the cone $C_x = (u_1 \geq 0) \cap \cdots \cap (u_s \geq 0)$. If $y \in P$ is not in Q, then it is outside L. Hence $\langle y, u_i\rangle > a_i$ for an i. Then $u_i \in C_x^\vee$ is outside $C_y^\vee$ since $\langle x - y, u_i\rangle < 0$, which implies that C_y is not contained in C_x. If $y \in \mathrm{rel.int}\, Q$ then $C_x \subset C_y$ since $x \in Q$, hence $C_x = C_y$. Conversely, if $y \in P$ satisfies $C_x = C_y$, then $y \in Q$ since $C_y \subset C_x$. Since $P - y$ is equal to $C_x = C_y$ and $P - x$ in a neighborhood of the origin, both $Q - y$ and $Q - x$ equal to $C_x \cap (u = 0)$ in a neighborhood of the origin. Hence y is also in $\mathrm{rel.int}\, Q$. Q.E.D.

For a non-degenerate strongly convex rational quasi-polyhedral set P, we denote by $\mathrm{Face}(P)$ the set of non-empty faces of P. For each $Q \in \mathrm{Face}(P)$, take an element $x \in \mathrm{rel.int}\, Q$ and define $\sigma_Q = C_x^\vee$, which does not depend on the choice of x by Lemma 1.5. This lemma also implies that the map $\mathrm{Face}(P) \to \Sigma(P)$ defined by $Q \mapsto \sigma_Q$ is a bijection.

Proposition 1.6. *Let P be a non-degenerate strongly convex rational quasi-polyhedral set. Then the map $\mathrm{Face}(P) \to \Sigma(P)$ defined by $Q \mapsto \sigma_Q$ is a bijection such that $Q \subset Q'$ if and only if $\sigma_{Q'} \prec \sigma_Q$. Furthermore, $\mathrm{rel.\,int}\,\sigma_Q$ is contained in $\mathrm{int}(\mathrm{cc}(P)^\vee)$ if and only if Q is bounded.*

Proof. The first part follows from Lemma 1.5. If $\mathrm{int}(\mathrm{cc}(P)^\vee)$ contains an element u in $\mathrm{rel.\,int}\,\sigma_Q$, as we saw in the proof of Theorem 1.3, there exists $a \in \mathbf{R}$ such that $P \subset (u \geq a)$ and $Q' = P \cap (u = a)$ is a bounded face of P. Since $u \in \mathrm{rel.\,int}\,\sigma$ defines the face Q of P, we have $Q' = Q$, and Q is bounded.

Assume that a point $u \in \mathrm{rel.\,int}\,\sigma_Q$ is on the boundary of $\mathrm{cc}(P)^\vee$. Then $\mathrm{cc}(P) \cap (u = 0) \neq \{0\}$ by Lemma 1.2. Hence there exists a non-zero $y \in \mathrm{cc}(P)$ with $\langle y, u\rangle = 0$. For $x \in Q$, the half line $x + \mathbf{R}_0 y$, which is in P by the definition of $\mathrm{cc}(P)$, is contained in Q. This implies that Q is not bounded. Q.E.D.

Example 1.7. Let $M = \mathbf{Z}^r$ with $r \geq 2$, and $(x_1, \ldots, x_r)$ the coordinates of $M_{\mathbf{R}}$. The convex hull P of the set

$$A = \{(a_1, \ldots, a_{r-1}, a_1^2 + \cdots + a_{r-1}^2)\ ;\ a_1, \ldots, a_{r-1} \in \mathbf{Z}\}$$

in $M_{\mathbf{R}}$ is a rational quasi-polyhedral set with the vertex set A, $\mathrm{cc}(P) = \mathbf{R}_0(0, \ldots, 0, 1)$ and $\mathrm{cc}\text{-}\dim(P) = 1$. Figure 1 is P for $r = 2$. The set of vertices of P is equal to A, and the faces of codimension one of P are parallelotopes defined by the hyperplane

$$\left(x_r - \sum_{i=1}^{r-1}(2a_i + 1)x_i = -\sum_{i=1}^{r-1} a_i(a_i + 1)\right)$$

for $a_1, \ldots, a_{r-1} \in \mathbf{Z}$.

For each $1 \leq i \leq r-1$, define the affine transformation $\tilde{\delta}_i(x_1, \ldots, x_r) = (x'_1, \ldots, x'_r)$ by

$$x'_j = \begin{cases} x_j & 1 \leq j \leq r-1,\ j \neq i, \\ x_i + 1 & j = i, \\ 2x_i + x_r + 1 & j = r. \end{cases}$$

Then $\tilde{\delta}_1, \ldots, \tilde{\delta}_{r-1}$ are mutually commutative and the generated free $\mathbf{Z}$-module $\widetilde{\Gamma} \simeq \mathbf{Z}^{r-1}$ acts on P. The set A is the orbit of $0 = (0, \ldots, 0)$. The dual cone $\sigma_0 = C_0^\vee$ for the origin $0 \in P$ is equal to

$$\sigma_0 = \{(u_1, \ldots, u_r)\ ;\ |u_i| \leq u_r, i = 1, \ldots, r-1\}$$

with respect to the coordinate $(u_1, \ldots, u_r)$ of $N_{\mathbf{R}} = \mathbf{R}^r$. Define the linear automorphism $\delta_i(u_1, \ldots, u_r) = (u'_1, \ldots, u'_r)$ of $N_{\mathbf{R}}$ by

$$u'_j = \begin{cases} u_j & 1 \leq j \leq r-1,\ j \neq i, \\ u_i - 2u_r & j = i, \\ u_r & j = r \end{cases}$$

for each $1 \leq i \leq r-1$. Then the free $\mathbf{Z}$-module $\Gamma \simeq \mathbf{Z}^{r-1}$ generated by $\delta_1, \ldots \delta_{r-1}$ acts on the fan $\Sigma(P)$. The actions of $\widetilde{\Gamma}$ and Γ on P and $\Sigma(P)$ are compatible with respect to the correspondences $\tilde{\delta}_i \mapsto \delta_i$ for $1 \leq i \leq r-1$. Namely, if $\sigma \in \Sigma(P)$ is equal to $C_x^\vee$ for $x \in P$, then $\delta_i(\sigma)$ is equal to $C^\vee_{\tilde{\delta}_i(x)}$ (see Figure 2).

§2. Graded rings and toric schemes

Let P be a non-degenerate strongly convex rational quasi-polyhedral set and k a field.

Lemma 2.1. *For any rational point z in P, there exist a positive integer s, a set $\{x_1, \ldots, x_s\}$ of vertices of P, non-negative rational numbers $a_1, \ldots, a_s$ with $a_1 + \cdots + a_s = 1$ and a rational point y in* $\mathrm{cc}(P)$ *such that the equality*

$$z = a_1 x_1 + \cdots + a_s x_s + y$$

holds.

Proof. We prove the lemma by induction on the rank r of $M_{\mathbf{R}}$. The assertion is trivial for $r = 0$ since then $P = \mathrm{cc}(P) = \{0\}$. Assume $r > 0$. Then any face Q of P is a non-degenerate quasi-polyhedral set in the minimal affine space containing it. By the assumption of the induction, the lemma holds for every proper face of P. Since P is strongly convex, there exists a vertex x_1. If $z = x_1$, then the lemma is satisfied for $s = 1$, $a_1 = 1$ and $y = 0$. Assume $z \neq x_1$. If $x_1 + \mathbf{R}_0(z - x_1) \subset P$, then $z - x_1 \in \mathrm{cc}(P)$ and the lemma holds for $s = 1$, $a_1 = 1$ and $y = z - x_1$. If $x_1 + \mathbf{R}_0(z - x_1) \not\subset P$, then $(x_1 + \mathbf{R}(z - x_1)) \cap P$ is a segment $\overline{x_1 w}$ for a rational point w on the boundary of P, and z is in the segment. Then w is contained in a proper face Q. By the assumption, there exist a set $\{x_2, \ldots, x_s\}$ of vertices of Q, non-negative rational numbers $b_2, \ldots, b_s$ with $b_2 + \cdots + b_s = 1$ and a rational point $y' \in \mathrm{cc}(Q) \subset \mathrm{cc}(P)$ such that the equality

$$w = b_2 x_2 + \cdots + b_s x_s + y'$$

holds. The real number t with $z = t x_1 + (1-t)w$, $0 \leq t \leq 1$, is rational since x_1, w, z are rational. Then the equality of the lemma is satisfied for $a_1 = t$, $a_i = (1-t)b_i$, $i = 2, \ldots, s$, and $y = (1-t)y'$. Q.E.D.

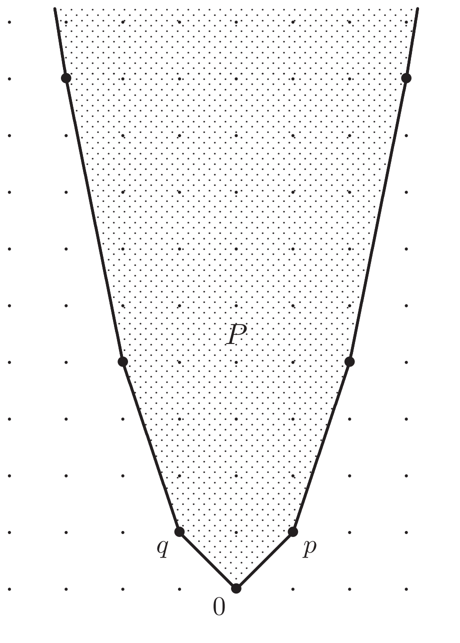

Figure 1

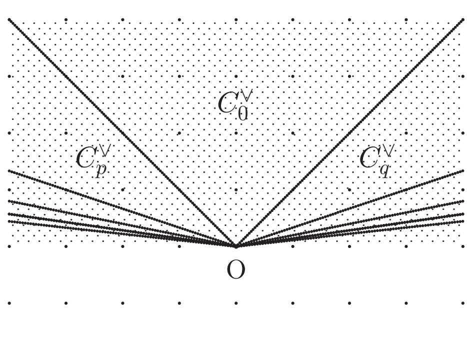

Figure 2

We define

$$\widehat{P} = \{(x,t)\ ;\ t \geq 0, x \in tP\}$$

in $M_{\mathbf{R}} \times \mathbf{R}$, where we understand $0P = \mathrm{cc}(P)$.

Lemma 2.2. *$\widehat{P}$ is a strongly convex closed cone.*

Proof. It is easy to see that $(x,s),(y,t) \in \widehat{P}$ and $\lambda, \mu \geq 0$ imply $\lambda(x,s) + \mu(y,t) \in \widehat{P}$, i.e., $\widehat{P}$ is a convex cone. Since $\widehat{P} \subset (t \geq 0)$, if $\widehat{P}$ contains a line, it is contained in $tP \times \{t\}$ for some t. This is impossible since P is strongly convex. Hence $\widehat{P}$ is strongly convex.

We will show that $\widehat{P}$ is closed. Since each tP is closed, it suffices to show that, if a sequence $\{(x_i, t_i)\}$ of points in $\widehat{P}$ converges to $(x,0)$, then x is in $\mathrm{cc}(P)$. This is clear if there exist infinitely many i with $t_i = 0$. By replacing with a subsequence, we may assume that $t_i > 0$ for all i. We take a point y in P. Then $y + (1/t_i)(x_i - t_i y) = (1/t_i)x_i$ is in P since $(t_i, x_i) \in \widehat{P}$. For any positive a, we have $1/t_i \geq a$ for sufficiently large i. By the convexity of P, we have $y + a(x_i - t_i y) \in P$ for such i. By taking the limit for i, we have $y + ax \in P$ since $x_i \to x$ and $t_i \to 0$ as $i \to \infty$. Since a is arbitrary, we have $x \in \mathrm{cc}(P)$. Q.E.D.

By this lemma, the subset

$$\mathcal{A}(P) = (M \oplus \mathbf{Z}) \cap \widehat{P} = \{(m,d)\ ;\ d \in \mathbf{Z}, d \geq 0, m \in M \cap dP\}$$

is an additive semigroup containing $0 = (0,0)$, which is not necessary finitely generated. We denote by $\mathbf{e}(m,d)$ or $\mathbf{e}(x)$ the monomial associated to $x = (m,d) \in \mathcal{A}(P)$, and consider the semigroup ring

$$A(P) = \bigoplus_{x \in \mathcal{A}(P)} k\mathbf{e}(x)$$

over k. Namely, this is a k-algebra with $\mathbf{e}(0) = 1$ and $\mathbf{e}(x)\mathbf{e}(y) = \mathbf{e}(x+y)$ for $x, y \in \mathcal{A}(P)$. The graded ring structure $A(P) = \bigoplus_{d=0}^{\infty} A(P)_d$ is defined by

$$A(P)_d = \bigoplus_{m \in M \cap dP} k\mathbf{e}(m,d)$$

for $d \geq 0$. The subring $A(P)_0$ is the semigroup ring for the semigroup $M \cap \mathrm{cc}(P)$, which is not necessarily finitely generated over k. It is equal to k if cc-dim $P = 0$ or $\mathrm{cc}(P)$ has no rational point other than the origin. Also, $A(P)$ is not necessarily finitely generated over $A(P)_0$.

Let $V(P)$ be the set of vertices of P. Each $v \in V(P)$ is a rational point by the rationality of P. Let d_v be the minimal positive integer with $d_v v \in M$. The half line $\mathbf{R}_0(v,1)$ is contained in $\widehat{P}$ and $(d_v v, d_v)$ is the primitive lattice point on the half line.

Lemma 2.3. *Let K be the ideal of $A(P)$ generated by $\{\mathbf{e}(d_v v, d_v)\ ;\ v \in V(P)\}$. Then the ideal $A(P)_+ = \bigoplus_{d=1}^{\infty} A(P)_d$ is equal to the radical of K. $A(P)$ is integral over the subring $B = A(P)_0[\mathbf{e}(d_v v, d_v)\ ;\ v \in V(P)]$.*

Proof. Let (m, d) be in $\mathcal{A}(P)$ with $d > 0$. Since $m/d \in P$, by Lemma 2.1, there exist $v_1, \ldots, v_s \in V(P)$, non-negative rational numbers $a_1, \ldots, a_s$ with $a_1 + \cdots + a_s = 1$, and a rational point $y \in \mathrm{cc}(P)$ such that the equality

$$\frac{m}{d} = a_1 v_1 + \cdots + a_s v_s + y$$

holds. We can take a positive integer c such that

$$b_1 = \frac{cda_1}{d_{v_1}}\ ,\ \ldots\ ,\ b_s = \frac{cda_s}{d_{v_s}}$$

are non-negative integers and $y' = cdy$ is in $M \cap \mathrm{cc}(P)$. By the equalities

$$cm = cda_1 v_1 + \cdots + cda_s v_s + cdy = b_1 d_{v_1} v_1 + \cdots + b_s d_{v_s} v_s + y'$$

and

$$b_1 d_{v_1} + \cdots + b_s d_{v_s} = cd(a_1 + \cdots + a_s) = cd\ ,$$

we have

$$c(m, d) = b_1(d_{v_1} v_1, d_{v_1}) + \cdots + b_s(d_{v_s} v_s, d_{v_s}) + (y', 0)$$

and

$$\mathbf{e}(m, d)^c = \mathbf{e}(y', 0)\mathbf{e}(d_{v_1} v_1, d_{v_1})^{b_1} \cdots \mathbf{e}(d_{v_s} v_s, d_{v_s})^{b_s}\ .$$

Since $cd > 0$, one of $b_1, \ldots, b_s$ is greater than or equal to 1, and $\mathbf{e}(m, d)^c$ is contained in K. Hence $\mathbf{e}(m, d)$ is in the radical of K. Since $A(P)_+$ is generated by these $\mathbf{e}(m, d)$ and has reduced quotient $A(P)/A(P)_+ \simeq A(P)_0$, $A(P)_+$ is equal to the radical of K. This equality also implies $\mathbf{e}(m, d)^c \in B$, and hence $\mathbf{e}(m, d)$ is integral over B. Since $A(P)$ is generated by these elements over $A(P)_0$, $A(P)$ is an integral extension of B. Q.E.D.

For a graded commutative ring $A = \bigoplus_{d=0}^{\infty} A_d$, $\mathrm{Proj}\, A$ is a scheme whose base space is the set of homogeneous prime ideals of A which do not contain the ideal $A_+ = \bigoplus_{d=1}^{\infty} A_d$ (cf. [EGA, II, 2]). If f is a homogeneous element of A with a positive degree, then $A[f^{-1}]$ is a graded ring which may have components of negative degrees, and $\widetilde{D}(f) = \mathrm{Spec}\, A[f^{-1}]_0$ is an affine open subscheme of $\mathrm{Proj}\, A$, where $A[f^{-1}]_0$ is the component of degree 0 of the graded ring. A homogeneous prime ideal

I is in $\widetilde{D}(f)$ if and only if I does not contain f. In this case, the prime ideal $IA[f^{-1}] \cap A[f^{-1}]_0$ of $A[f^{-1}]_0$ represents the same point in $\widetilde{D}(f)$. When A_0 contains a field k, this affine subscheme is of finite type if $A[f^{-1}]_0$ is finitely generated over k even if A is not noetherian. We use this fact for the quasi-polyhedral set P.

Recall the construction of toric varieties over the field k (cf. [O]). For a cone σ in $N_{\mathbf{R}}$, $M \cap \sigma^\vee$ is a finitely generated semigroup and the semigroup ring $k[M \cap \sigma^\vee]$ is an affine k-algebra. The affine toric variety $X(\sigma)$ is the affine variety with the coordinate ring $k[M \cap \sigma^\vee]$. If Σ is a fan of $N_{\mathbf{R}}$, then $X(\sigma)$ have the same function field for all $\sigma \in \Sigma$, and they are patched together to a toric variety $Z(\Sigma)$, which is a separated scheme over k locally of finite type. $X(\sigma)$'s are affine open sets of $Z(\Sigma)$, and $Z(\Sigma)$ is of finite type if and only if Σ is finite. $Z(\Sigma)$ contains the algebraic torus T_N with the coordinate ring $k[M]$.

Lemma 2.4. *Let $(m,d) \in \mathcal{A}(P)$ be an element with $d > 0$ and let $x = m/d$. Then $A(P)[\mathbf{e}(m,d)^{-1}]_0$ is equal to the semigroup ring $k[M \cap C_x]$ and $\widetilde{D}(\mathbf{e}(m,d)^{-1})$ is the affine toric variety associated to the cone $\sigma = C_x^\vee$.*

Proof. First, we will show the equality

$$C_x \times \{0\} + \mathbf{R}(m,d) = \widetilde{P} + \mathbf{R}(m,d) \ .$$

By $P \times \{1\} \subset \widetilde{P}$ and $-(x,1) \in \mathbf{R}(m,d)$, $P \times \{1\} - (x,1) = (P-x) \times \{0\}$ is contained in the right-hand side. Hence $C_x \times \{0\} = \mathbf{R}_0(P-x) \times \{0\}$ is also in the right-hand side. On the other hand, since $P \times \{1\} = (P-x) \times \{0\} + (x,1)$, $P \times \{1\}$ is contained in the left-hand side. Since $\widetilde{P}$ is the closure of the cone generated by $P \times \{1\}$ and the left-hand side is a polyhedral cone which is a closed cone, the right-hand side is contained in the left.

Since $(m,d) \in \widetilde{P}$, for any $(m',d') \in (M \oplus \mathbf{Z}) \cap (\widetilde{P} + \mathbf{R}(m,d))$, we have $(m'',d'') = (m',d') + c(m,d) \in (M \oplus \mathbf{Z}) \cap \widetilde{P} = \mathcal{A}(P)$ for sufficiently large integer c. Namely, $(m',d') = (m'',d'') - c(m,d) \in \mathcal{A}(P) + \mathbf{Z}(m,d)$. Hence, by this equality of cones, we have $\mathcal{A}(P) + \mathbf{Z}(m,d) = (M \oplus \mathbf{Z}) \cap (C_x \times \{0\} + \mathbf{R}(m,d))$.

Since $A(P)[\mathbf{e}(m,d)^{-1}]$ is the semigroup ring for the semigroup $\mathcal{A}(P) + \mathbf{Z}(m,d)$, the component of degree 0 is the semigroup ring

$$k[(M \times \{0\}) \cap (C_x \times \{0\})] = k[M \cap C_x] \ .$$

Q.E.D.

Proposition 2.5. *$Z(P) = \mathrm{Proj}\, A(P)$ is the toric variety associated to the fan $\Sigma(P)$ and covered by the affine toric varieties $\{\widetilde{D}(\mathbf{e}(d_v v, d_v))\ ;\ v \in V(P)\}$.*

Proof. By Lemma 2.4, these affine toric varieties are contained in $\mathrm{Proj}\, A(P)$. We will show that $\mathrm{Proj}\, A(P)$ is covered by these affine toric varieties. Let I be an arbitrary homogeneous prime ideal which does not contain $A(P)_+$. By Lemma 2.3, I does not contain $\mathbf{e}(d_v v, d_v)$ for a vertex v. Then the point I is contained in $\widetilde{D}(\mathbf{e}(d_v v, d_v))$. Q.E.D.

Now, we will call *convex toric variety* the toric variety $Z(P) = \mathrm{Proj}\, A(P)$ defined for a non-degenerate strongly convex rational quasi-polyhedral set P. If $V(P)$ is infinite, then $Z(P)$ is not of finite type. Although we do not have any further results, it might be interesting to study the general properties of convex toric varieties.

Proposition 2.6. *For any rational number a and a rational point x in $M_{\mathbf{R}}$, we have $Z(aP + x) = Z(P)$.*

Proof. For any graded ring $A = \bigoplus_{d=0}^{\infty} A_d$ and positive integer e, we have $\mathrm{Proj}\, A^{(e)} = \mathrm{Proj}\, A$ for $A^{(e)} = \bigoplus_{d=0}^{\infty} A_{de}$. Since $A(P)^{(e)} \simeq A(eP)$, we have $Z(eP) \simeq Z(P)$. These toric varieties contain the algebraic torus $T_N = \mathrm{Spec}\, k[M]$, and since the above isomorphism of graded rings induces the identity map of the coordinate ring $k[M] = k[M \times \{0\}]$, these toric varieties are identical. If we take e such that ea is an integer and $ex \in M$, then $Z(aP + x) = Z(eaP + ex) = Z(eaP) = Z(P)$. Q.E.D.

For each vertex $v \in V(P)$, $\sigma = C_v^{\vee}$ is an r-dimensional cone in $\Sigma(P)$, and the linear function $u \in \sigma$ on P has the minimum value at v. The *support function* $h_P : |\Sigma(P)| \to \mathbf{R}$ is defined by

$$h_P(u) = \min\{\langle x, u\rangle\ ;\ x \in P\}\ .$$

If $\sigma = C_v^{\vee}$, then $h_P = v$ on σ. Since every face of P has a vertex, any cone in $\Sigma(P)$ is a face of an r-dimensional cone in $\Sigma(P)$. Hence h_P is linear on every cone of the fan.

Proposition 2.7. *The h_P is a strongly convex function on $\mathrm{int}(\mathrm{cc}(P)^{\vee})$. Namely, $h(u + v) \geq h_P(u) + h_P(v)$ for any $u, v \in \mathrm{int}(\mathrm{cc}(P)^{\vee})$ and the equality holds if and only if u and v are contained in a same cone of $\Sigma(P)$.*

Proof. If u and v are contained in a cone σ, then the equality holds since h_P is linear on σ. Assume that u, v are not contained in a common cone. Since $\mathrm{cc}(P)^{\vee}$ is a convex cone, $u + v$ is also in the interior. For $a = h_P(u)$ and $b = h_P(v)$, the hyperplanes $(u = a)$ and $(v = b)$ of $M_{\mathbf{R}}$

define faces of P. If the intersection of these faces is not empty, it is also a face of P and has a vertex x. This is a contradiction, since u and v are is in the cone $C_x^\vee$ of $\Sigma(P)$ by Lemma 1.5. Hence the hyperplane $(u + v = a + b)$ does not intersect P, and

$$h_P(u+v) = \min\{\langle x, u\rangle \ ; x \in P\} > a + b = h_P(u) + h_P(v) \ .$$

Q.E.D.

§3. Quasi-polytope with group action

In this section, we study quasi-polyhedral sets with group action which define toric type cusp singularities. As in the previous sections, M is a free $\mathbf{Z}$-module of rank $r > 0$.

First, we recall the cusp singularities defined by Tsuchihashi [T], which we call *toric type cusp singularities* or simply *cusp singularities* in this paper. Assume that k is the complex number field $\mathbf{C}$. In this case, $T_N = N \otimes_{\mathbf{Z}} \mathbf{C}$.

A cusp singularity is defined for a pair (C, Γ) of an open convex cone C and a subgroup Γ of $\mathrm{GL}(N)$ satisfying the following conditions (cf. [T, Proposition 1.7]).

(1) The closure $\overline{C}$ is strongly convex, i.e., $\overline{C} \cap (-\overline{C}) = \{0\}$.
(2) C is Γ-invariant.
(3) The action of Γ on $C/\mathbf{R}_+$ is properly discontinuous and free, and the quotient $(C/\mathbf{R}_+)/\Gamma$ is compact.

For such a pair (C, Γ), there exists a Γ-invariant fan Σ of $N_{\mathbf{R}}$ such that $|\Gamma| = C \cup \{0\}$ and Σ is locally finite at each point of C.

The construction of the singularity is done as follows. Consider the toric variety $Z(\Sigma)$ associated with the fan Σ. Let $\mathrm{ord} : T_N \to N_{\mathbf{R}}$ be the surjective map defined by $1_N \otimes (-\log|\ |)$. Set $D(\Sigma) = Z(\Sigma) \setminus T_N$. Then $U(\Sigma) = D(\Sigma) \cup \mathrm{ord}^{-1}(C)$ is an open analytic subspace of $Z(\Sigma)$ on which Γ acts freely. The quotient $U(\Sigma)/\Gamma$ has divisor $D(\Sigma)/\Gamma$, and Tsuchihashi showed that this divisor is contracted to an isolated singularity.

Definition 3.1. We call a non-degenerate quasi-polyhedral set $P \subset M_{\mathbf{R}}$ a *quasi-polytope* if every proper face of P is bounded.

P in Example 1.7 is a quasi-polytope for every $r \geq 2$. If P is a quasi-polytope, then every proper face of P is the convex hull of its vertices. In particular, P is rational if and only if all vertices are rational.

Lemma 3.2 ([I1, Proposition 2.6])**.** *Let $P \subset M_{\mathbf{R}}$ be a quasi-polytope. Then $|\Sigma(P)| = \mathrm{int}(\mathrm{cc}(P)^\vee) \cup \{0\}$.*

Proof. By Proposition 1.6, every non-zero $\sigma \in \Sigma(P)\setminus\{\mathbf{0}\}$ intersects $\mathrm{int}(\mathrm{cc}(P)^\vee)$. Since this implies that every one-dimensional face of σ intersects $\mathrm{int}(\mathrm{cc}(P)^\vee)$, the set $\sigma \setminus \{0\}$ is contained in the convex open cone $\mathrm{int}(\mathrm{cc}(P)^\vee)$. Q.E.D.

We consider a pair $(P, \widetilde{\Gamma})$ of a quasi-polytope and a subgroup $\widetilde{\Gamma}$ of the affine transformation group of M with the following properties.

(1) $\text{cc-dim}\, P = r$.
(2) $\gamma(P) = P$ for every $\gamma \in \widetilde{\Gamma}$ of $M_{\mathbf{R}}$, i.e., $\widetilde{\Gamma}$ acts on P.
(3) For the induced linear group $\Gamma \subset \mathrm{Aut}(N)$, the quotient $(\Sigma(P)\setminus\{\mathbf{0}\})/\Gamma$ is finite.

Here note that, when an element $\tilde{\gamma} \in \widetilde{\Gamma}$ is expressed as $\tilde{\gamma}(x) = Ax + b$, the corresponding element $\gamma \in \Gamma$ is given by $\gamma(u) = {}^tA^{-1}u$ as in Example 1.7. In particular, we have $C^\vee_{\tilde{\gamma}(x)} = \gamma(C^\vee_x)$ for every $x \in P$.

If $(P, \widetilde{\Gamma})$ is such a pair, then $\mathrm{cc}(P)$ is strongly convex and $|\Sigma| = C \cup \{0\}$ for $C = \mathrm{int}(\mathrm{cc}(P)^\vee)$.

If the action of Γ on $\Sigma(P) \setminus \{\mathbf{0}\}$ has no fixed element, then (C, Γ) defines a toric type cusp singularity. This is equivalent to the condition that Γ has no fixed point in the open cone C. Although Γ has fixed points in general, the stabilizer Γ_x at each point x of $C/\mathbf{R}_+$ is a finite group. This follows from the fact that every proper face of P is bounded and has only finite vertices.

Proposition 3.3. *There exists a normal subgroup $\Gamma' \subset \Gamma$ of finite index which has no fixed point in $D = C/\mathbf{R}_+$. Namely, there exists a subgroup Γ' such that (C, Γ') defines a cusp singularity.*

Proof. Note that, for any element $g \in \mathrm{GL}(r, \mathbf{Z}) \setminus \{1\}$, some congruence subgroup of $\mathrm{GL}(r, \mathbf{Z})$ does not contain g. Hence, for any point $x \in D$, there exists a normal subgroup $G_x \subset \Gamma$ of finite index such that $\Gamma_x \cap G_x = \{1\}$ since Γ is a subgroup of $\mathrm{Aut}(N) \simeq \mathrm{GL}(r, \mathbf{Z})$. For the natural surjection $p : D \to D/\Gamma$, G_x has no fixed point in $p^{-1}(U_x) \subset D$ for a neighborhood U_x of $p(x)$ since G_x is a normal subgroup and the action is properly discontinuous. Since D/Γ is compact, there exist finite points $x_1, \ldots, x_s \in D$ such that $\{U_{x_i}\}$ covers D/Γ. Then $\Gamma' = G_{x_1} \cap \cdots \cap G_{x_s}$ satisfies the condition. Q.E.D.

Example 3.4 (Infinite linear Coxeter group [I2, Example 5.5])**.** Let $N = \mathbf{Z}^4$. Then the 4-dimensional cone σ generated by

$$\left\{ \begin{array}{llll} (1,0,0,1), & (1,0,1,1), & (2,1,3,2), & (2,1,6,3) \\ (1,1,0,1), & (1,1,1,1), & (1,2,3,2), & (1,2,6,3) \\ (0,1,0,1), & (0,1,1,1), & (-1,1,3,2), & (-1,1,6,3) \\ (-1,0,0,1), & (-1,0,1,1), & (-2,-1,3,2), & (-2,-1,6,3) \\ (-1,-1,0,1), & (-1,-1,1,1), & (-1,-2,3,2), & (-1,-2,6,3) \\ (0,-1,0,1), & (0,-1,1,1), & (1,-1,3,2), & (1,-1,6,3) \end{array} \right\}$$

has 14 facets. This cone is contained in the quadratic cone $V^+ = \{x \in \mathbf{R}^4 \; ; Q(x,x) \geq 0, x_4 \geq 0\}$ with respect to the quadratic form

$$Q = -4x_1^2 + 4x_1x_2 - 4x_2^2 - x_3^2 + 6x_4^2$$

defined by the symmetric matrix

$$\begin{bmatrix} -4 & 2 & 0 & 0 \\ 2 & -4 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

of index $(1,3)$. Let Γ be the group generated by the reflections R_τ defined for the 14 facets τ. Then $\{g(\sigma) \; ; g \in \Gamma\}$ and their faces form a hyperbolic fan Σ. This Γ is a right-angled orthogonal Coxeter group (cf. [PV]). Namely, if $\tau \cap \tau'$ is of dimension 2, then $(R_\tau R_{\tau'})^2 = 1$. For the definition of orthogonal Coxeter groups, see [V, Definition 4]. The open cone C is equal to $\operatorname{int} V^+$. There is a family of quasi-polytopes $P \subset M_{\mathbf{R}}$ giving this fan.

§4. Related examples

In this section, we introduce some examples of open convex cones and discrete groups acting on them.

Example 4.1 (Siegel modular cusps)**.** Let $g \geq 2$ be an integer. Then the set of positive definite symmetric matrices

$$C_g = \{X \in M_g(\mathbf{R}) \; ; {}^{\mathrm{t}}X = X, X > 0\}$$

is an open cone of dimension $g(g+1)/2$. The group $\Gamma = \mathrm{GL}_n(\mathbf{Z})/\{\pm 1\}$ acts on C_g by $(A, X) \mapsto AX\,{}^{\mathrm{t}}A$. In this case, $(C_g/\mathbf{R}_+)/\Gamma$ is not compact, and it will not give an isolated singularity. This gives a local description of the Satake compactification and toroidal compactifications of the Siegel space of degree g (cf. [N1, §8]).

Example 4.2. Let π be the cone generated by the standard basis of $\mathbf{R}^3$. The matrices

$$\tau_1 = \begin{bmatrix} -1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \quad \tau_2 = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 0 \\ 0 & 2 & 1 \end{bmatrix}, \quad \tau_3 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix}$$

define reflections with respect to the three faces of π. The infinite Coxeter group $\Gamma(\pi)$ generated by these reflections is orthogonal with respect to the quadratic form $Q = xy + yz + xz$ of index $(1, 2)$. The three edges of π are on the boundary of the closed cone

$$\overline{C} = \{(x, y, z)\ ;\ xy + yz + xz \geq 0, x + y + z \geq 0\}\ .$$

The interior C of this cone is isomorphic to C_2 of Example 4.1 by the linear isomorphism $\mathbf{R}^3 \to \{X \in M_2(\mathbf{R})\ ;\ {}^{\mathrm{t}}X = X\}$ defined by

$$(x, y, z) \mapsto \begin{bmatrix} x + y & y \\ y & z + y \end{bmatrix}.$$

By this linear isomorphism, $\Gamma(\pi)$ is mapped onto the congruence subgroup $\Gamma(2)/\{\pm 1\}$ of $\mathrm{GL}_2(\mathbf{Z})/\{\pm 1\}$. Namely, the group $\mathrm{GL}_2(\mathbf{Z})/\{\pm 1\}$ corresponds to the semi-direct product of $\Gamma(\pi)$ and the symmetric group S_3 acting on $\mathbf{R}^3$ as the permutations of the coordinates. Figure 3 is the section of the cones $\overline{C}$, π, $\tau_i(\pi)$, $i = 1, 2, 3$, by the plane $x + y + z = 1$.

Example 4.3. Let a, b be positive real numbers greater than one. In $\mathbf{R}^3$, we consider the pentagon D contained in the plane $(z = 1)$ with the vertices

$$p_1 = (0, 0, 1), \quad p_2 = (a, 0, 1), \quad p_3 = (a, b - 1, 1),$$

$$p_4 = (a - 1, b, 1), \quad p_5 = (0, b, 1)\ .$$

Let Q be the quadratic form $-bx^2 - ay^2 + ab(a + b - 1)z^2$. The cone $\pi_{a,b}$ generated by these vertices is contained in the open cone $C_{a,b} = \{(x, y, z)\ ;\ Q > 0, z > 0\}$ except the origin. The cone $\pi_{a,b}$ has five facets and the reflections with respect to these facets by the metric defined by Q generate an orthogonal right-angled Coxeter group Γ. There exists a quasi-polytope P such that $\bigcup_{\gamma \in \Gamma} \gamma(D)$ is the boundary and $\Gamma(p_3) \cup \Gamma(p_4)$ is the set of vertices. Figure 4 is the section of $\pi_{2,2}$ and $\overline{C}_{2,2}$ by the plane $(z = 1)$.

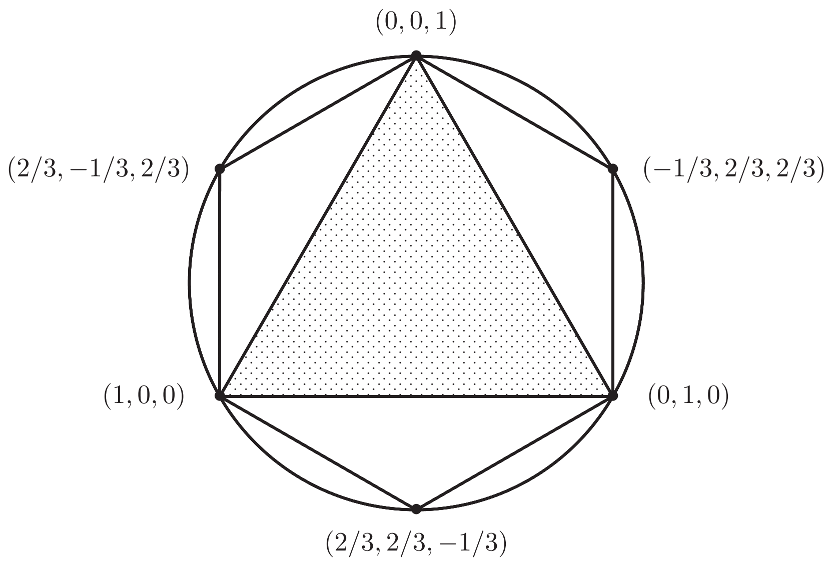

Figure 3

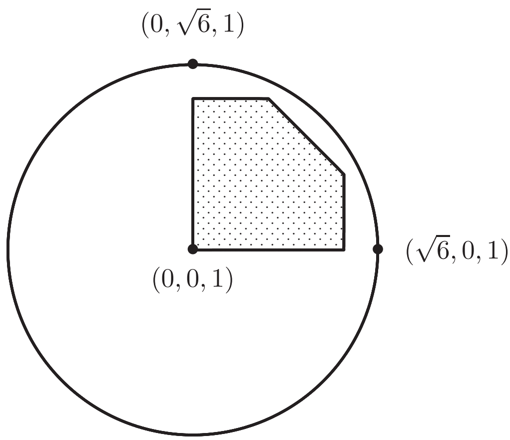

Figure 4

Any polyhedral cone C with five facets in $\mathbf{R}^3$ is equal to $\pi_{a,b}$ for some a, b for a coordinate, and hence defines an orthogonal Coxeter group. This fact is checked as follows. Let D_i, $i = 1, \dots, 5$, be the facet of C such that $D_i \cap D_{i+1}$, $i = 1, 2, 3, 4$, and $D_5 \cap D_1$ are edges. Let H_i be the plane spanned by D_i, and H_i^+ the closed half space with $C \subset H_i^+$ defined by H_i for each i. Clearly, $H_i \neq H_j$ if $i \neq j$. Then $\ell_1 = H_1 \cap H_3$ and $\ell_2 = H_2 \cap H_4$ are distinct lines. Take a basis $\{c_1 u_1, c_2 u_2, c_3 u_3\}$ of $\mathbf{R}^3$ such that $u_1 \in \ell_1 \cap H_2^+$, $u_2 \in \ell_2 \cap H_3^+$ and $u_3 \in H_2 \cap H_3 \cap C$. Then C is of the desired form for a suitable choice of the scalars $c_1, c_2, c_3 > 0$.

If we assume that the lattice N of $\mathbf{R}^3$ is $\mathbf{Z}^3$ for the standard basis, then the group is contained in $\mathrm{GL}(N)$ if and only if $(a, b) = (2, 2), (2, 3), (3, 2), (3, 3)$. In this case, P is a quasi-polytope with integral vertices. However, this assumption on N loses the generality of $\pi_{a,b}$.

For a 3-dimensional cone with more than five facets, we can also define the right-angled Coxeter group. However, that is not orthogonal in general.

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