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Deformation of morphisms onto Fano manifolds of Picard number 1 with linear varieties of minimal rational tangents

Jun-Muk Hwang

Abstract.

Let X be a Fano manifold of Picard number 1, different from projective space. We study the question whether the space $\mathrm{Hom}^s(Y, X)$ of surjective morphisms from a projective manifold Y to X is homogeneous under the automorphism group $\mathrm{Aut}_o(X)$. An affirmative answer is given in [4] under the assumption that X has a minimal dominating family $\mathcal{K}$ of rational curves whose variety of minimal rational tangents $\mathcal{C}_x$ at a general point $x \in X$ is non-linear or finite. In this paper, we study the case where $\mathcal{C}_x$ is linear of arbitrary dimension, which covers the cases unsettled in [4]. In this case, we will define a reduced divisor $B^{\mathcal{K}} \subset X$ and an irreducible subvariety $M^{\mathcal{K}} \subset \mathrm{Chow}(X)$ naturally associated to $\mathcal{K}$. We give a sufficient condition in terms of $\mathbf{B}^{\mathcal{K}}$ and $M^{\mathcal{K}}$ for the homogeneity of $\mathrm{Hom}^s(Y, X)$. This condition is satisfied if $\mathcal{C}_x$ is finite and our result generalizes [4]. A new ingredient, which is of independent interest, is a similar rigidity result for surjective morphisms to projective space in logarithmic setting.

§1. Introduction

Convention

1. We work over the complex numbers. We will use both Euclidean topology and Zariski topology. We will specify which one we are using at each occasion.
2. All manifolds are connected, but a variety may have finitely many irreducible components.

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3. For a vector space V, its projectivization $\mathbb{P}V$ is the set of 1-dimensional subspaces of V.
4. Given a morphism $f : Y' \to Y$ between nonsingular varieties, we say that f is unramified at a point $y \in Y'$ if the derivative $\mathrm{d}f_y : T_y(Y') \to T_{f(y)}(Y)$ is injective. The ramification set of f is the set of points $y \in Y'$ where f is not unramified. When f is surjective and generically finite, the branch divisor of f is the reduced divisor in Y consisting of codimension-1 components of $f(R)$ where $R \subset Y'$ is the ramification set of f.

Throughout this paper, X denotes a Fano manifold of Picard number 1. Recall that an irreducible component $\mathcal{K}$ of the space of rational curves on X is a minimal dominating component (or a minimal dominating family of rational curves) if for a general point $x \in X$, the subscheme $\mathcal{K}_x$ of $\mathcal{K}$ consisting of members passing through x is non-empty and complete. For such a choice of $\mathcal{K}$, the variety of minimal rational tangents at x is the subvariety $\mathcal{C}_x$ of the projectivized tangent space $\mathbb{P}T_x(X)$ consisting of the tangent directions at x of members of $\mathcal{K}_x$.

Let $\mathrm{Aut}_o(X)$ be the identity component of the automorphism group of X. For a projective manifold Y, denote by $\mathrm{Hom}^s(Y, X)$ the space of surjective morphisms $Y \to X$. We are mainly interested in understanding the geometry of $\mathrm{Hom}^s(Y, X)$ using $\mathcal{K}$. See Section 1 of [4] for the history of previous works and also Section 9 of [2] for the background and related results.

In Theorem 1.3 of [4], the following result was proved.

Theorem 1. *Given $(X, \mathcal{K})$, suppose the variety of minimal rational tangents associated to $\mathcal{K}$ at a general point of X is non-linear or finite. Then for any projective manifold Y and any $[f : Y \to X] \in \mathrm{Hom}^s(Y, X)$,*

$$H^0(Y, f^*T(X)) = f^*H^0(X, T(X)).$$

In particular, all deformations of a surjective morphism $Y \to X$ are unobstructed and each component of $\mathrm{Hom}^s(Y, X)$ is a reduced principal homogeneous space of the affine algebraic group $\mathrm{Aut}_o(X)$.

How restrictive is the assumption in Theorem 1 on the variety of minimal rational tangents? In Conjecture 1.2 of [4], the author naively conjectured that if this assumption is violated, then X must be projective space. An analogue of Theorem 1 is certainly not true if X is projective space. Thus if this conjecture had been verified, Theorem 1 would give a satisfactory description of $\mathrm{Hom}^s(Y, X)$ for all X different from projective space.

Recently, Munoz, Occhetta and Sola Conde (Appendix of [6]) have pointed out that the varieties $Y_a, a = 2, 4, 8$, of reductions of Severi varieties discovered by Iliev and Manivel in [5] are counterexamples to Conjecture 1.2 of [4]. In fact, the variety of minimal rational tangents associated to the family of lines on Y_a is a linear subspace of dimension $a-1$. This counterexample prompts us to question whether an analog of Theorem 1 holds when the variety of minimal rational tangents is linear and of positive dimension. Since this question is the main concern of the current paper, it will be convenient to formulate the following assumption.

Assumption 1. *X is a Fano manifold of Picard number 1 and $\mathcal{K}$ is a minimal dominating family of rational curves on X such that the associated variety of minimal rational tangents at a general point of X is linear and of dimension $k-1, k \geq 1$.*

Under Assumption 1, we will see in Definition 4.1 and Proposition 4.3 that $\mathcal{K}$ determines

1. a reduced divisor $\mathbf{B}^{\mathcal{K}} \subset X$ and
2. an irreducible subvariety $M^{\mathcal{K}} \subset \mathrm{Chow}(X)$ such that for a general point $w \in M^{\mathcal{K}}$ there exists an immersion, that is, an unramified generically injective morphism, of projective space $\nu_w : \mathbb{P}^k \to X$ whose image $\nu_w(\mathbb{P}^k) \subset X$ is the cycle corresponding to $w \in \mathrm{Chow}(X)$.

Our main result can be stated in terms of $\mathrm{B}^{\mathcal{K}}$ and $M^{\mathcal{K}}$ as follows.

Theorem 2. *Under Assumption 1, suppose that for a general $w \in M^{\mathcal{K}}$, the dual variety of the set-theoretic inverse image $\nu_w^{-1}(\mathbf{B}^{\mathcal{K}}) \subset \mathbb{P}^k$, as a hypersurface in projective space $\mathbb{P}^k$, is linearly nondegenerate. Then for any projective manifold Y and any $[f : Y \to X] \in \mathrm{Hom}^s(Y, X)$,*

$$H^0(Y, f^*T(X)) = f^*H^0(X, T(X)).$$

In particular, all deformations of a surjective morphism $Y \to X$ are unobstructed and each component of $\mathrm{Hom}^s(Y, X)$ is a reduced principal homogeneous space of the affine algebraic group $\mathrm{Aut}_o(X)$.

One can check that the condition on $\nu_w^{-1}(\mathbf{B}^{\mathcal{K}})$ in Theorem 2 holds if $k = 1$. In this sense, Theorem 2 is a generalization of Theorem 1. As such, its proof, to a certain extent, follows the line of arguments of the proof of Theorem 1. But there is one crucial new ingredient, a logarithmic version of Theorem 2 for the pair $(\mathbb{P}^k, \nu_w^{-1}(\mathbf{B}^{\mathcal{K}}))$, which will be explained in Section 2. Our Sections 3-7 generalize the arguments in Sections 3-5 of [4]. The final section, Section 8, however, follows a course different from Section 6 of [4]. A new argument incorporating the result of Section 2 is given to finish the proof of Theorem 2. We believe that this new argument is more transparent, even when $k = 1$, than the one given in Section 6 of [4].

It is natural to ask whether the condition on $\nu_w^{-1}(\mathbf{B}^{\mathcal{K}})$ in Theorem 2 is satisfied by all $(X, \mathcal{K})$ satisfying Assumption 1. In particular, it would be interesting to check whether this condition holds for the variety $Y_a, a = 2, 4, 8$, of [5]. It is also natural to ask whether this condition is removable in Theorem 2. Although this condition is essential for our argument, I guess that it should be removable. In this regard, we would like to point out that all of our arguments in Sections 3-8, up to Proposition 8.4, do not need this condition.

§2. A logarithmic version of Theorem 2 for projective space

Notation 2.1. *Let V be a complex vector space and let $\Gamma \subset \mathbb{P}V^\vee$ be a subvariety with finitely many irreducible components in the dual projective space, parametrizing a finite union of families of hyperplanes in $\mathbb{P}V$. Denote by $|\Gamma| \subset \mathbb{P}V^\vee$ the linear span of Γ, which defines a linear system of hyperplanes in $\mathbb{P}V$. Let $\mathrm{Bs}|\Gamma|$ be the base locus of $|\Gamma|$, i.e., the common intersection of members of Γ. For a point $x \in \mathbb{P}V \setminus \mathrm{Bs}|\Gamma|$, let $\mathrm{Join}(x, \mathrm{Bs}|\Gamma|)$ be the linear space spanned by x and $\mathrm{Bs}|\Gamma|$. Denote its tangent space by*

$$\mathcal{N}_x^\Gamma = T_x\left(\mathrm{Join}(x, \mathrm{Bs}|\Gamma|)\right) \subset T_x(\mathbb{P}V).$$

We will skip the proof of the next lemma, which is straight forward.

Lemma 2.1. *(i) Let $\phi_{|\Gamma|} : \mathbb{P}V \setminus \mathrm{Bs}|\Gamma| \to |\Gamma|^\vee$ be the natural morphism into the dual projective space of the linear system $|\Gamma|$. Then the closure of the fiber of $\phi_{|\Gamma|}$ through a point $x \in \mathbb{P}V \setminus \mathrm{Bs}|\Gamma|$ is $\mathrm{Join}(x, \mathrm{Bs}|\Gamma|)$.*

(ii) Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_N \subset \mathbb{P}V^\vee$ be a finite collection of subvarieties and let $\Gamma \subset \mathbb{P}V^\vee$ be their union. Then $\mathrm{Bs}|\Gamma| = \mathrm{Bs}|\Gamma_1| \cap \cdots \cap \mathrm{Bs}|\Gamma_N|$.

Lemma 2.2. *Let $\Gamma \subset \mathbb{P}V^\vee$ be the union of $d \geq 2$ distinct points corresponding to hyperplanes $G_1, \ldots, G_d \subset \mathbb{P}V$. For any point $x \in \mathbb{P}V \setminus G_1$, we have*

$$\mathcal{N}_x^\Gamma = \{v \in T_x(\mathbb{P}V),\ (\mathrm{d}\log \frac{G_i}{G_1})(v) = 0,\ \textit{for all } 2 \leq i \leq d\}$$

where G_i/G_1 denotes a choice of a rational function on $\mathbb{P}V$ with zero at G_i and pole at G_1 and $\log \frac{G_i}{G_1}$ is a choice of the logarithm of G_i/G_1 at the germ of x. It is clear that the germ of 1-form $\mathrm{d}\log \frac{G}{G_1}$ at x does not depend on the choices.

Proof. The set on the right side of the equality is equal to

$$\{v \in T_x(\mathbb{P}V), (\mathrm{d}\log \frac{G}{G_1})(v) = 0, \text{ for all } G \in |\Gamma|\}.$$

This is the set of vectors tangent to the intersection of the level sets through x of the rational functions $\frac{G}{G_1}, G \in |\Gamma|$. This intersection is exactly the fibers of $\phi_{|\Gamma|}$ whose tangent space at x is $\mathcal{N}_x^\Gamma$ by Lemma 2.1 (i). Q.E.D.

Notation 2.2. *Let V be a complex vector space and let $D \subset \mathbb{P}V$ be a reduced divisor on its projectivization. Let $\Gamma := D^\vee \subset \mathbb{P}V^\vee$ be its dual variety. We set $\mathcal{N}_x^D := \mathcal{N}_x^\Gamma$ for $x \notin \mathrm{Bs}|\Gamma|$. In particular, if $D^\vee$ is linearly nondegenerate, then $\mathrm{Bs}|\Gamma| = \emptyset$ and $\mathcal{N}_x^D = 0$ for all $x \in \mathbb{P}V$.*

Notation 2.3. *Let M be a complex manifold and $D \subset M$ be a reduced divisor. We denote by $\Omega^1_M(\log D)$ the locally free sheaf of logarithmic 1-forms defined on $M \setminus \mathrm{Sing}(D)$ and by $T_M(-\log D) \subset T(M)$ the coherent sheaf of vector fields tangent to D. On $M \setminus \mathrm{Sing}(D)$, the sheaf $T_M(-\log D)$ is locally free, dual to $\Omega^1_M(\log D)$.*

Remark 2.1. *Throughout this paper, we use $T(M)$ to denote the tangent bundle of M. The notation T_M will be used only in $T_M(-\log D)$ of Notation 2.3.*

Proposition 2.1. *Write $\mathbb{P}V = \mathbb{P}^k, \dim V = k+1$. Let $D \subset \mathbb{P}^k$ be a reduced divisor and let $\mathcal{N}_x^D$ be as in Notation 2.2. Let $J \subset \mathbb{P}V$ be a subvariety with $\dim J \leq k-2$. For a point $x \in \mathbb{P}^k \setminus (D \cup J)$ and a vector $s_x \in \Omega^1_{\mathbb{P}^k,x}(\log D) = \Omega^1_{\mathbb{P}^k,x}$, let $s_x^\perp \subset T_x(\mathbb{P}^k)$ be the annihilator of s_x. Denote by $\mathrm{Line}_x^{D,J}$ the set of all lines $\ell \subset \mathbb{P}^k \setminus J$ through x that intersect D transversally. In particular, members of $\mathrm{Line}_x^{D,J}$ are disjoint from $\mathrm{Sing}(D)$. Then*

$$\bigcap_{\substack{\text{general } \ell \in \mathrm{Line}_x^{D,J} \\ s \in H^0(\ell, \Omega^1_{\mathbb{P}^k}(\log D))}} s_x^\perp = \bigcap_{\substack{\ell \in \mathrm{Line}_x^{D,J} \\ s \in H^0(\ell, \Omega^1_{\mathbb{P}^k}(\log D))}} s_x^\perp = \mathcal{N}_x^D.$$

Proof. Given $\ell \in \mathrm{Line}_x^{D,J}$, let $x_1, \ldots, x_d$ be the intersection $\ell \cap D$ where d is the degree of the divisor D. The conormal bundle $N_\ell^\vee \subset \Omega^1_{\mathbb{P}^k}|_\ell$ of 1-forms annihilating $T(\ell)$ gives an exact sequence

$$0 \longrightarrow N_\ell^\vee \longrightarrow \Omega^1_{\mathbb{P}^k}(\log D)|_\ell \longrightarrow \Omega^1_\ell(x_1 + \cdots + x_d) \longrightarrow 0.$$

Since $N_\ell^\vee \cong \mathcal{O}(-1)^{k-1}$, we have $H^1(\ell, N_\ell^\vee) = 0$ and

$$\dim H^0(\ell, \Omega^1_{\mathbb{P}^k}(\log D)) = d - 1. \tag{1}$$

In particular, the proposition is trivially true when $d = 1$. So we will assume that $d \geq 2$.

Let $G_i \subset \mathbb{P}^k, 1 \leq i \leq d$, be the tangent hyperplane to D at x_i. Then $G_i \neq G_j$ if $i \neq j$. In fact, if $G_i = G_j$ for $i \neq j$, then the line ℓ must be contained in $G_i = G_j$ so it is not transversal to D. By the same argument, we know $x \notin G_i$ for all $1 \leq i \leq d$. Let $G[\ell]$ be the reduced divisor $G_1 + \cdots + G_d$. We have a canonical isomorphism of vector bundles on ℓ

$$\Omega^1_{\mathbb{P}^k}(\log D)|_\ell \quad = \quad \Omega^1_{\mathbb{P}^k}(\log G[\ell])|_\ell. \tag{2}$$

For each $2 \leq i \leq d$, the meromorphic form $d \log \frac{G_i}{G_1}$ on $\mathbb{P}^k$ defines an element of

$$H^0\left(\mathbb{P}^k \setminus \mathrm{Sing}(G[\ell]), \Omega^1_{\mathbb{P}^k}(\log G[\ell])\right).$$

From equations 1 and 2, the restriction of $\mathrm{d}\log \frac{G_i}{G_1}$ for $2 \leq i \leq d$ gives a basis of $H^0(\ell, \Omega^1_{\mathbb{P}^k}(\log D))$. It follows that

$$\begin{aligned} \bigcap_{s \in H^0(\ell, \Omega^1_{\mathbb{P}^k}(\log D))} s_x^\perp \quad &= \quad \{v \in T_x(\mathbb{P}^k), \mathrm{d}\log \frac{G_i}{G_1}(v) = 0 \text{ for all } 2 \leq i \leq d\} \\ &= \quad \mathcal{N}_x^{G[\ell]} \end{aligned}$$

where the last equality is from Lemma 2.2.

Let $G[\ell]^\vee$ be the finite subset of $\mathbb{P}V^\vee$, dual to $G[\ell]$. The closure of the union of $G[\ell]^\vee$ as we vary $\ell \in \mathrm{Line}_x^{D,J}$ is exactly $D^\vee$. It follows that

$$\mathrm{Bs}|D^\vee| = \bigcap_{\ell \in \mathrm{Line}_x^{D,J}} \mathrm{Bs}|G[\ell]|.$$

Thus

$$\begin{aligned} \bigcap_{\ell \in \mathrm{Line}_x^{D,J}} \mathcal{N}_x^{G[\ell]} \quad &= \quad \bigcap_{\ell \in \mathrm{Line}_x^{D,J}} T_x(\mathrm{Join}\,(x, \mathrm{Bs}|G[\ell]|)) \\ &= \quad T_x\left(\mathrm{Join}(x, \mathrm{Bs}|D^\vee|)\right) \\ &= \quad \mathcal{N}_x^D. \end{aligned}$$

This proves the proposition. Q.E.D.

Theorem 3. *Let $f : Y \to \mathbb{P}^k$ be a surjective generically finite morphism from a projective manifold Y. Let $D \subset \mathbb{P}^k$ be a reduced divisor and let $J \subset \mathbb{P}^k$ be a subvariety satisfying*

$$\dim J \leq k - 2 \textit{ and } \mathrm{Sing}(D) \subset J.$$

For each section

$$v \in H^0\left(Y \setminus f^{-1}(J), f^*T_{\mathbb{P}^k}(-\log D)\right)$$

and two points $y_1, y_2 \in f^{-1}(x)$ *over a general point* $x \in \mathbb{P}^k$*, the difference* $v_{y_1} - v_{y_2} \in T_x(\mathbb{P}^k)$ *is contained in* $\mathcal{N}_x^D$*. In particular, if the dual variety* $D^\vee$ *is linearly nondegenerate,*

$$H^0\left(Y \setminus f^{-1}(J), f^*T_{\mathbb{P}^k}(-\log D)\right) = f^*H^0\left(\mathbb{P}^k, T_{\mathbb{P}^k}(-\log D)\right).$$

Proof. For a general $\ell \in \mathrm{Line}_x^{D,J}$, the inverse image $f^{-1}(\ell)$ is a connected curve containing both y_1 and y_2 by Bertini's theorem. Choose an element $s \in H^0(\ell, \Omega^1_{\mathbb{P}^k}(\log D))$ and pull it back to $f^*s \in H^0(f^{-1}(\ell), f^*\Omega^1_{\mathbb{P}^k}(\log D))$. The pairing $\langle s, v\rangle$ is a holomorphic function on $f^{-1}(\ell)$ and must be a constant. It follows that $v_{y_1} - v_{y_2} \in s_x^\perp$. By Proposition 2.1, we see that $v_{y_1} - v_{y_2} \in \mathcal{N}_x^D$.

If $D^\vee$ is linearly nondegenerate, then $\mathcal{N}_x^D = 0$ as we mentioned in Notation 2.2. Thus

$$v \in f^*H^0\left(\mathbb{P}^k \setminus J, T_{\mathbb{P}^k}(-\log D)\right) = f^*H^0\left(\mathbb{P}^k, T_{\mathbb{P}^k}(-\log D)\right)$$

where the equality is from $\dim J \leq k-2$. Q.E.D.

§3. Effective étale family of immersed submanifolds

Definition 3.1. *Let* Y *be a projective manifold. Let* $\mathcal{P}$ *and* $\mathcal{W}$ *be two nonsingular algebraic varieties satisfying* $\dim \mathcal{P} > \dim \mathcal{W}$ *and equipped with morphisms* $\varphi : \mathcal{P} \to Y$ *and* $\psi : \mathcal{P} \to \mathcal{W}$*. The morphism* $(\varphi, \psi) : \mathcal{P} \to Y \times \mathcal{W}$ *is an* étale family of immersed submanifolds *in* Y *parametrized by* $\mathcal{W}$ *if the following conditions are satisfied.*

(i) ψ *is a smooth projective morphism with connected fibers.*
(ii) $\dim \mathcal{P} = \dim Y$ *and* φ *is unramified at every point of* $\mathcal{P}$*.*
(iii) *For each* $w \in \mathcal{W}$*, the restriction* $\varphi|_{\psi^{-1}(w)} : \psi^{-1}(w) \to Y$ *is generically injective.*

For such a pair (φ, ψ)*, write* $P_w := \varphi(\psi^{-1}(w))$ *for* $w \in \mathcal{W}$*. The* normal bundle *of* P_w *is the vector bundle* $N_{\psi^{-1}(w)/Y} := \varphi^*T(Y)/T(\psi^{-1}(w))$ *on* $\psi^{-1}(w)$*. For any* $w \in \mathcal{W}$*, the normal bundle of* P_w *is naturally isomorphic to the trivial bundle* $\psi^*T_w(\mathcal{W})$*.*

Denote by $Y^{\mathrm{mult}} \subset Y$ *the subvariety defined by the closure of the union of the singular points of* P_w *for all* $w \in \mathcal{W}$*. In other words,* Y^{mult} *is the closure of the set of points* $y \in Y$ *such that*

$$y = \varphi(p_1) = \varphi(p_2) \text{ for some } p_1 \neq p_2 \in \mathcal{P} \text{ satisfying } \psi(p_1) = \psi(p_2).$$

We say that (φ, ψ) *is an* effective *étale family of immersed submanifolds if* $P_{w_1} \neq P_{w_2}$ *if* $w_1 \neq w_2 \in \mathcal{W}$.

Lemma 3.1. *In Definition 3.1, the dimension of* Y^{mult} *is strictly smaller than* $\dim Y$. *As a consequence,* $Y \setminus Y^{\mathrm{mult}}$ *is a nonempty Zariski open subset of* Y.

Proof. Note that the singular locus of P_w has dimension strictly less than $\dim(P_w)$ for each w. Thus $\dim(Y^{\mathrm{mult}})$ is strictly smaller than $\dim(P_w) + \dim W = \dim \mathcal{P} = \dim Y$. Q.E.D.

We have the following analogue of Proposition 3.5 in [4].

Proposition 3.1. *Let* $(\varphi, \psi) : \mathcal{P} \to Y \times \mathcal{W}$ *be an étale family of immersed submanifolds as in Definition 3.1. Let* $f : \widetilde{Y} \to Y$ *be a surjective generically finite morphism from a projective manifold* $\widetilde{Y}$. *Let* $Y^f \subset Y$ *be the maximal Zariski open subset such that* $f|_{f^{-1}(Y^f)} : f^{-1}(Y^f) \to Y^f$ *is an étale covering. For* $w \in \mathcal{W}$ *with* $P_w \cap Y^f \neq \emptyset$ *and an irreducible component* $\widetilde{P}$ *of* $f^{-1}(P_w)$, *denote by* $\widetilde{\varphi} : \widehat{P} \to \widetilde{P}$ *the normalization of* $\widetilde{P}$. *For* $y \in P_w \cap Y^f \setminus Y^{\mathrm{mult}}$, *a nonsingular point of* P_w *in* Y^f, *let* y_1, y_2 *be two points in* $f^{-1}(y) \cap \widetilde{P}$ *and* $\widehat{y}_1, \widehat{y}_2$ *be two points in* $\widehat{P}$ *satisfying* $\widetilde{\varphi}(\widehat{y}_1) = y_1$ *and* $\widetilde{\varphi}(\widehat{y}_2) = y_2$. *For any element* $\sigma \in H^0(\widehat{P}, (f \circ \widetilde{\varphi})^* T(Y))$, *regarding its value* $\sigma_{\widehat{y}_1}$ *(resp.* $\sigma_{\widehat{y}_2}$*) at* $\widehat{y}_1$ *(resp.* $\widehat{y}_2$*) as a vector in* $T_y(Y)$, *we have* $\sigma_{\widehat{y}_1} - \sigma_{\widehat{y}_2} \in T_y(P_w) \subset T_y(Y)$.

Proof. We claim that $\sigma_{\widehat{y}_1} - \sigma_{\widehat{y}_2} \in T_y(Y)$ annihilates the subspace of the cotangent space $\Omega^1_{Y,y}$ spanned by the evaluation of

$$H^0(\psi^{-1}(w), \varphi^* \Omega^1_Y)$$

at y. Since $\psi^{-1}(w)$ is the normalization of P_w, we have an induced morphism $\widehat{f} : \widehat{P} \to \psi^{-1}(w)$ such that $\varphi \circ \widehat{f} = f \circ \widetilde{\varphi}$. For any $\phi \in H^0(\psi^{-1}(w), \varphi^* \Omega^1_Y)$, let $\widetilde{\phi} \in H^0(\widehat{P}, \widehat{f}^* \varphi^* \Omega^1_Y)$ be the pull-back of ϕ to $\widehat{P}$. From $\varphi \circ \widehat{f} = f \circ \widetilde{\varphi}$, we can define the pairing $\langle \widetilde{\phi}, \sigma \rangle$. This is a holomorphic function on $\widehat{P}$, hence is constant. It follows that

$$\langle \widetilde{\phi}_y, \sigma_{\widehat{y}_1} \rangle = \langle \widetilde{\phi}_y, \sigma_{\widehat{y}_2} \rangle,$$

which implies that $\sigma_{\widehat{y}_1} - \sigma_{\widehat{y}_2}$ annihilates the evaluation of ϕ at y.

Since P_w has trivial normal bundle,

$$H^0(\psi^{-1}(w), N^{\vee}_{\psi^{-1}(w)/Y}) \subset H^0(\psi^{-1}(w), \varphi^* \Omega^1_Y)$$

spans the conormal space of P_w at y. Thus $\sigma_{\widehat{y}_1} - \sigma_{\widehat{y}_2} \in T_y(P_w)$ by the claim. Q.E.D.

Definition 3.2. *Let* $(\varphi, \psi) : \mathcal{P} \to Y \times \mathcal{W}$ *be an étale family of immersed submanifolds as in Definition 3.1. Let* $B \subset Y$ *be an irreducible ample hypersurface in* Y*. We say that* $\mathcal{P}$ *is univalent on* B *if (i) there exists only one irreducible component* E *of* $\varphi^{-1}(B)$ *that is dominant over* $\mathcal{W}$*, and (ii) the morphism* $\mu|_E : E \to B$ *is birational. This is equivalent to saying that at a general point* $z \in B$*, there exists exactly one* $w \in \mathcal{W}$ *with* $z \in P_w \not\subset B$ *and this* P_w *is non-singular at* z*.*

The following is essentially the same as Proposition 3.2 in [3] and a generalization of Proposition 5.1 of [4].

Proposition 3.2. *Let* $(\varphi, \psi) : \mathcal{P} \to Y \times \mathcal{W}$ *be an effective étale family of immersed submanifolds as in Definition 3.1. Suppose that* $\mathcal{P}$ *is not univalent on an ample irreducible hypersurface* $B \subset Y$*. Then given a general point* $x \in B$*, we can find*

(1) *a Euclidean open neighborhood* $U \subset Y$ *of* x*;*
(2) *two distinct points* $x_1 \neq x_2 \in \varphi^{-1}(x)$*; and*
(3) *Euclidean open neighborhoods* $U_1 \subset \mathcal{P}$ *of* x_1 *and* $U_2 \subset \mathcal{P}$ *of* x_2 *satisfying* $U_1 \cap U_2 = \emptyset$

with the following properties.

(i) $\varphi(U_1) = U = \varphi(U_2)$.
(ii) $\varphi|_{U_1}$ *and* $\varphi|_{U_2}$ *are biholomorphic.*
(iii) *For any point* $z \in U \setminus (B \cup Y^{\mathrm{mult}})$*, set*

$$z_1 = \varphi^{-1}(z) \cap U_1,\ z_2 = \varphi^{-1}(z) \cap U_2,\ w_1 = \psi(z_1) \text{ and } w_2 = \psi(z_2).$$

Then $w_1 \neq w_2$ *and* $\varphi(\psi^{-1}(w_1) \cap U_1)$ *and* $\varphi(\psi^{-1}(w_2) \cap U_2)$ *are two distinct submanifolds of* U *each of which intersects* B *transversally.*

Proof. Let $\mathcal{E}$ be the union of irreducible components of $\varphi^{-1}(B)$ that are dominant over $\mathcal{W}$. Since $\dim \mathcal{P} > \dim \mathcal{W}$ and B is ample, $\mathcal{E}$ is non-empty. By the assumption that $\mathcal{P}$ is not univalent on B, there exist two distinct points $x_1 \neq x_2 \in \varphi^{-1}(x) \cap \mathcal{E}$ for a general point $x \in B$. Since φ is unramified, we have Euclidean open sets U, U_1 and U_2 satisfying (i) and (ii) such that $\mathcal{E} \cap U_1$ (resp. $\mathcal{E} \cap U_2$) is a nonsingular hypersurface. Furthermore, by the smoothness of ψ, we can choose U_1 and U_2 such that the fibers of ψ in U_1 (resp. U_2) intersect $\mathcal{E} \cap U_1$ (resp. $\mathcal{E} \cap U_2$) transversally. By $z \notin Y^{\mathrm{mult}}$, we have $w_1 \neq w_2$, which implies that $\varphi(\psi^{-1}(w_1) \cap U_1)$ and $\varphi(\psi^{-1}(w_2) \cap U_2)$ are distinct by the effectiveness of (φ, ψ). Q.E.D.

We will skip the proof of the following elementary lemma.

Lemma 3.2. *Let $f : \widetilde{Y} \to Y$ be a surjective generically finite morphism between projective manifolds. Let R be an irreducible component of ramification set of f such that $B = f(R)$ is a component of the branch divisor of f. At a general point $x \in B$ and $\widetilde{x} \in R \cap f^{-1}(x)$, we have Euclidean neighborhoods $x \in U$ and $\widetilde{x} \in \widetilde{U}$ such that $B \cap U$ is a nonsingular hypersurface in U and $f|_{\widetilde{U}} : \widetilde{U} \to U$ is a cyclic covering of degree > 1 branched along $B \cap U$. Consequently, if an irreducible closed submanifold $P \subset U$ intersects $B \cap U$ transversally, then $f^{-1}(P) \cap \widetilde{U}$ is irreducible.*

The following proposition is a generalization of Proposition 5.2 in [4].

Proposition 3.3. *Let $(\varphi, \psi) : \mathcal{P} \to Y \times \mathcal{W}$ be an effective étale family of immersed submanifolds as in Definition 3.1. Assume that*

(1) *there exists a nonempty Zariski open subset $Y^{\text{trans}} \subset Y \setminus Y^{\text{mult}}$ such that if $y \in Y^{\text{trans}} \cap P_{w_1} \cap P_{w_2}$ for two distinct point $w_1 \neq w_2 \in \mathcal{W}$, then $T_y(P_{w_1}) \cap T_y(P_{w_2}) = 0$ in $T_y(Y)$; and*

(2) *there exist a surjective generically finite morphism $f : \widetilde{Y} \to Y$ from a projective manifold $\widetilde{Y}$ and a section $\sigma \in H^0(\widetilde{Y}, f^*T(Y))$ such that for any $y \in Y^{\text{trans}} \cap Y^f$ in the notation of Proposition 3.1 and $y_1 \neq y_2 \in f^{-1}(y)$, the two vectors $\sigma_{y_1} \in T_y(Y)$ and $\sigma_{y_2} \in T_y(Y)$ are distinct.*

If $B \subset Y$ is an irreducible ample hypersurface contained in the branch divisor of f, then $\mathcal{P}$ is univalent on B.

Proof. Suppose that $\mathcal{P}$ is not univalent on B. Fix a general point $x \in B$ and let U, U_1, U_2 be as in Proposition 3.2. By shrinking U if necessary, we can use this U to find $\widetilde{U} \subset \widetilde{Y}$ as in Lemma 3.2. Choose $z \in U \cap Y^{\text{trans}} \cap Y^f$ with $z \notin B$ and let $w_1 \neq w_2 \in \mathcal{W}$ be the points determined by z as in Proposition 3.2 (iii).

Setting $P_1 = P_{w_1}$ (resp. $P_2 = P_{w_2}$), Lemma 3.2 shows that there exists a unique irreducible component $\widetilde{P}_1$ (resp. $\widetilde{P}_2$) of $f^{-1}(P_1)$ (resp. $f^{-1}(P_2)$) intersecting $\widetilde{U}$ such that an irreducible component of $\widetilde{P}_1 \cap \widetilde{U}$ (resp. $\widetilde{P}_2 \cap \widetilde{U}$) contains $f^{-1}(z) \cap \widetilde{U}$. In particular, $\widetilde{P}_1 \cap \widetilde{P}_2$ contains $f^{-1}(z) \cap \widetilde{U}$. Let $y_1 \neq y_2$ be two distinct points in $f^{-1}(z) \cap \widetilde{U}$. Applying Proposition 3.1 to $\widetilde{P}_1$ and $\widetilde{P}_2$,

$$\sigma_{y_1} - \sigma_{y_2} \in T_z(P_1) \text{ and } \sigma_{y_1} - \sigma_{y_2} \in T_z(P_2).$$

Since $T_z(P_1) \cap T_z(P_2) = 0$ by $y \in Y^{\text{trans}}$, we obtain $\sigma_{y_1} = \sigma_{y_2}$, a contradiction to the assumption on σ in (2). Q.E.D.

§4. Definitions of $\mathbf{B}^{\mathcal{K}}$ and $M^{\mathcal{K}}$

From now on, we will consider $(X, \mathcal{K})$ in Assumption 1. We recall the following from Proposition 2.2 in [4].

Proposition 4.1. *Let X and $\mathcal{K}$ be as above. Let Y be a projective manifold and let $f : Y \to X$ be a generically finite morphism of degree > 1. Given a general member $C \subset X$ of $\mathcal{K}$, there exists an irreducible component C' of $f^{-1}(C)$ such that the restriction $f|_{C'} : C' \to C$ is finite of degree > 1.*

The following generalizes Proposition 3.2 of [4].

Proposition 4.2. *Under Assumption 1, there exists $(\varphi, \psi) : \mathcal{P} \to X \times \mathcal{W}$, an effective étale family of immersed submanifolds parametrized by a nonsingular variety $\mathcal{W}$ with the following properties.*

(a) *The morphism $\psi : \mathcal{P} \to \mathcal{W}$ is a $\mathbb{P}^k$-bundle and the morphism φ has degree > 1.*

(b) *Each member of $\mathcal{K}_x$ for a general $x \in X$ is the image of a line in the $\mathbb{P}^k$-fiber of ψ through a point of $\varphi^{-1}(x)$. In particular, each component of $\mathcal{C}_x$ is of the form $\mathbb{P}T_x(P_w), P_w = \varphi(\psi^{-1}(w))$ for some $w \in \mathcal{W}$.*

(c) *There exists a nonempty Zariski open subset $X^{\text{trans}} \subset X \backslash X^{\text{mult}}$ such that if $x \in X^{\text{trans}} \cap P_{w_1} \cap P_{w_2}$ for two distinct points $w_1 \neq w_2 \in \mathcal{W}$, then $T_x(P_{w_1}) \cap T_x(P_{w_2}) = 0$.*

Proof. The existence of (φ, ψ) satisfying (a) and (b) follows from Theorem 3.1 of [1] and Proposition 2.1 in [3]. For (c), note that if $T_x(P_{w_1}) \cap T_x(P_{w_2}) \neq 0$, then two distinct components of $\mathcal{C}_x$ have nonempty intersection, which cannot happen for a general $x \in X \setminus X^{\text{mult}}$ by Proposition 2.2 of [3]. Q.E.D.

Definition 4.1. *Viewing $P_w, w \in \mathcal{W}$, in Proposition 4.2 as an algebraic cycle of X, we have a morphism $\mathcal{W} \to \mathrm{Chow}(X)$, which is injective by the effectiveness on (φ, ψ). Let $M^{\mathcal{K}} \subset \mathrm{Chow}(X)$ be the closure of the image of $\mathcal{W}$. From Proposition 4.2 (b), this variety $M^{\mathcal{K}}$ is uniquely determined by $\mathcal{K}$.*

Proposition 4.3. *In the setting of Proposition 4.2 and Definition 4.1, let Z be the normalization of $M^{\mathcal{K}}$. Replacing $\mathcal{P}$ and $\mathcal{W}$ by their Zariski open subsets, we can find a projective manifold X' with a generically finite morphism $\mu : X' \to X$ of degree > 1, a proper surjective morphism $\rho : X' \to Z$ and an embedding $\iota : \mathcal{P} \to X'$ as a Zariski open subset with the following properties.*

(a) *There is a natural inclusion $\mathcal{W} \subset Z$ as a Zariski open subset of the smooth locus of Z with a commuting diagram*

$$\begin{array}{ccccc} X & \stackrel{\varphi}{\longleftarrow} & \mathcal{P} & \stackrel{\psi}{\longrightarrow} & \mathcal{W} \\ \| & & \iota \downarrow & & \cap \\ X & \stackrel{\mu}{\longleftarrow} & X' & \stackrel{\rho}{\longrightarrow} & Z. \end{array}$$

(b) *For any irreducible hypersurface $H \subset X'$ satisfying* $\dim \mu(H) = \dim X - 1$, *either* $\rho(H) = Z$ *or* $\dim \rho(H) = \dim Z - 1$.

(c) *Let $\mathcal{T} \subset T(X')$ be the coherent subsheaf defined as the saturation of the relative tangent sheaf T^ρ on $\iota(\mathcal{P})$, equivalently, the saturation of* $d\iota(T^\psi)$ *in $T(X')$ where $T^\psi \subset T(\mathcal{P})$ is the relative tangent bundle of ψ. By abusing notation, we will regard $\mathcal{T}$ as a vector subbundle of $T(X')$ outside a subvariety of codimension 2. Then for any irreducible hypersurface $H \subset X'$ satisfying* $\dim \mu(H) = \dim X - 1$ *and* $\dim \rho(H) = \dim Z - 1$, *we have $\mathcal{T}_z \subset T_z(H)$ as subspaces of $T_z(X')$ at a general point $z \in H$.*

(d) *The branch divisor of μ, to be denoted by $\mathbf{B}^{\mathcal{K}} \subset X$, is uniquely determined by $\mathcal{K}$. Note that $\mathbf{B}^{\mathcal{K}}$ is nonempty because X is simply connected and μ has degree > 1.*

Proof. Let $\mathrm{Univ}_Z \to Z$ be the family of algebraic cycles of X parametrized by Z, induced by the universal family of $\mathrm{Chow}(X)$ over $M^{\mathcal{K}}$. There is a unique irreducible component Univ'_Z of Univ_Z dominant over Z. Let $\mathcal{U}$ be the normalization of Univ'_Z. Then all fibers of the natural morphism $\varrho : \mathcal{U} \to Z$ have dimension k. Note that since the normalization of a general member of $M^{\mathcal{K}}$ is $\mathbb{P}^k$, there exists a Zariski open subset in Z, over which ϱ is a $\mathbb{P}^k$-bundle. We choose X' as a desingularization $\sigma : X' \to \mathcal{U}$, which leaves the smooth locus of $\mathcal{U}$ intact. Define $\rho := \varrho \circ \sigma$ and let $\mu : X' \to X$ be the composition of σ and the natural cycle morphism $\mathcal{U} \to X$. Note that $\mathcal{U} \to X$ has degree strictly bigger than 1 because φ has degree strictly bigger than 1 from Proposition 4.2 (a). This implies that μ has degree strictly bigger than 1.

After replacing $\mathcal{W}$ by its intersection with a Zariski open subset of the smooth locus of Z over which ϱ is a $\mathbb{P}^k$-bundle, we have a natural inclusion $\iota : \mathcal{P} \to X'$ satisfying the property (a).

For (b), note that $\dim \sigma(H) = \dim \mathcal{U} - 1$ because $\dim \mu(H) = \dim X - 1 = \dim \mathcal{U} - 1$. Since all fibers of ϱ have dimension k, either $\rho(H) = \varrho(\sigma(H)) = Z$, or

$$\dim \rho(H) = \dim \varrho(\sigma(H)) = \dim \sigma(H) - k = \dim \mathcal{U} - 1 - k = \dim Z - 1.$$

For (c), since $\varrho : \mathcal{U} \to Z$ is a morphism between two normal varieties all fibers of which have dimension k, there exists a Zariski open subset $\mathcal{U}^o \subset \mathcal{U}$ such that

(1) $\mathcal{U}^o$ is in the smooth locus of $\mathcal{U}$ equipped with an isomorphism $\sigma^o : \mathcal{U}^o \cong \sigma^{-1}(\mathcal{U}^o) \subset X'$ induced by σ^{-1};
(2) $\dim(\mathcal{U} \setminus \mathcal{U}^o) \leq \dim \mathcal{U} - 2$;
(3) the underlying reduced variety of every fiber of $\varrho|_{\mathcal{U}^o}$ is nonsingular, defining a vector subbundle $T^{\varrho,\mathrm{red}} \subset T(\mathcal{U}^o)$;and
(4) the derivative

$$\mathrm{d}\sigma^o : T(\mathcal{U}^o) \cong T(\sigma^{-1}(\mathcal{U}^o))$$

sends $T^{\varrho,\mathrm{red}}$ to $\mathcal{T}|_{\sigma^{-1}(\mathcal{U}^o)}$.

From $\dim \sigma(H) = \dim \mathcal{U} - 1$ and (2), the image $\sigma(z)$ of a general point $z \in H$ is contained in $\mathcal{U}^o$. From $\dim \rho(H) = \dim Z - 1$, $\sigma(H)$ must be covered by components of k-dimensional fibers of ϱ. Thus $\mathcal{T}_z \subset T_z(H)$ from (4).

For (d), note that μ sends the ramification locus of σ to a set of codimension ≥ 2 in X. Thus the branch divisor of μ is determined by ϱ. Since ϱ is uniquely determined by $M^{\mathcal{K}}$, the branch divisor of μ is uniquely determined by $\mathcal{K}$. Q.E.D.

The following proposition is immediate from Propositions 4.2 and 4.3. It generalizes Proposition 3.3 in [4].

Proposition 4.4. *In the setting of Proposition 4.3, let $\mathcal{C} \subset \mathbb{P}T(X)$ be the closure of the union of $\mathcal{C}_x$'s for general points $x \in X$ and let $\hat{\mathcal{C}} \subset T(X)$ be the cone over $\mathcal{C}$. Denote by $0_X \subset T(X)$ the zero section and by $\pi : T(X) \to X$ the natural projection. Then there exists a Zariski open subset $X^{\mathcal{C}} \subset X^{\mathrm{trans}} \subset X$ such that*

(i) *$\mu^{-1}(X^{\mathcal{C}}) \subset \iota(\mathcal{P}) = \rho^{-1}(\mathcal{W})$;*
(ii) *$\mu|_{\mu^{-1}(X^{\mathcal{C}})} : \mu^{-1}(X^{\mathcal{C}}) \to X^{\mathcal{C}}$ is étale;*
(iii) *the restriction of π to $(\hat{\mathcal{C}} \setminus 0_X) \cap \pi^{-1}(X^{\mathcal{C}})$ is a smooth morphism;*
(iv) *for each point $x \in X^{\mathcal{C}}$ and $\mu^{-1}(x) = \{x_1, \ldots, x_j\}$, $j =$ degree of μ, the image $\rho(\mu^{-1}(x))$ consists of j distinct points in $\mathcal{W} \subset Z$ and we have a disjoint union*

$$\pi^{-1}(x) \cap (\hat{\mathcal{C}} \setminus 0_X) = \mathrm{d}\mu(T^{\rho}_{x_1} \setminus \{0\}) \cup \cdots \cup \mathrm{d}\mu(T^{\rho}_{x_j} \setminus \{0\})$$

where $T^{\rho} = \mathrm{d}\iota(T^{\psi})$ on $\mu^{-1}(X^{\mathcal{C}})$; and
(v) *we have a natural smooth morphism*

$$\chi : \ (\hat{\mathcal{C}} \setminus 0_X) \cap \pi^{-1}(X^{\mathcal{C}}) \longrightarrow \mu^{-1}(X^{\mathcal{C}})$$

defined by $\chi(T^{\rho}_{x_i} \setminus \{0\}) = x_i, 1 \leq i \leq j$, *in the notation of (iv) such that* $\pi = \mu \circ \chi$ *on* $(\hat{\mathcal{C}} \setminus 0_X) \cap \pi^{-1}(X^{\mathcal{C}})$.

§5. Decomposition of $\mu^{-1}(\mathbf{B}^{\mathcal{K}})$

Throughout this section, we work in the setting of Section 4. We will consider the following condition on X and $\mathcal{K}$, formulated in terms of Proposition 4.3.

Condition 5.1. *The étale family* $\mathcal{P}$ *constructed in Proposition 4.2 is univalent on each irreducible component of* $\mathbf{B}^{\mathcal{K}}$ *in the sense of Definition 3.2. In other words, for each irreducible component* B *of* $\mathbf{B}^{\mathcal{K}}$, *we have a unique irreducible component* B' *of* $\mu^{-1}(B)$ *that is dominant over both* Z *and* B.

The following is a generalization of Proposition 5.3 in [4].

Proposition 5.1. *In the setting of Proposition 4.3, assume that Condition 5.1 holds. Then for a general* $w \in \mathcal{W}$, *any component* P'_w *of* $\mu^{-1}(P_w)$ *which is of degree* > 1 *over* P_w *is disjoint from* B'.

Proof. Suppose not. Then for a general point x' of B', we have $w_1 \in \mathcal{W}$ and an irreducible component P'_{w_1} of $\mu^{-1}(P_{w_1})$ that contains x' and is of degree > 1 over P_{w_1}. Let $w_2 = \rho(x') \in \mathcal{W}$. Then $\rho^{-1}(w_2) = \iota(\psi^{-1}(w_2))$ is different from P'_{w_1}. So $w_1 \neq w_2$. Since $x := \mu(x') \in P_{w_1} \cap P_{w_2}$ is a general point of B, we conclude that $\mathcal{P}$ is not univalent on B from Definition 3.2, a contradiction. Q.E.D.

Proposition 5.2. *In the setting of Proposition 4.3, assume that Condition 5.1 holds. Then we can write the set-theoretical inverse image* $\mu^{-1}(\mathbf{B}^{\mathcal{K}}) \subset X'$ *as the union of three reduced divisors without common components*

$$\mu^{-1}(\mathbf{B}^{\mathcal{K}}) = \mathbf{B}^{\mathrm{exc}} \cup \mathbf{B}^{\mathrm{hor}} \cup \mathbf{B}^{\mathrm{ver}}$$

where

(a) $\dim \mu(B) \leq \dim X - 2$ *for each component* B *of* $\mathbf{B}^{\mathrm{exc}}$;

(b) $\dim \mu(B) = \dim X - 1$ *and* $\rho(B) = Z$ *for each component* B *of* $\mathbf{B}^{\mathrm{hor}}$; *and*

(c) $\dim \mu(B) = \dim X - 1$ *and* $\dim \rho(B) = \dim Z - 1$ *for each component* B *of* $\mathbf{B}^{\mathrm{ver}}$.

Furthermore, for a general $w \in \mathcal{W}$, *any component* P'_w *of* $\mu^{-1}(P_w)$ *which is of degree* > 1 *over* P_w *is disjoint from* $\mathbf{B}^{\mathrm{hor}}$.

Proof. The only nontrivial part in the decomposition of $\mu^{-1}(\mathbf{B}^{\mathcal{K}})$ into the three parts is to show that if a component B of $\mu^{-1}(\mathbf{B}^{\mathcal{K}})$ satisfies $\dim \mu(B) = \dim X - 1$, then $\dim \rho(B) \geq \dim Z - 1$. This follows from Proposition 4.3 (b). The statement about P'_w is immediate from Proposition 5.1. Q.E.D.

Proposition 5.3. *In Proposition 5.2, let z be a general point of any irreducible component of $\mathbf{B}^{\mathrm{ver}}$. Then we have $\mathcal{T}_z \subset T_z(\mathbf{B}^{\mathrm{ver}})$ where $\mathcal{T} \subset T(X')$ is the subsheaf from Proposition 4.3. In other words, $\mathcal{T} \subset T_{X'}(-\log \mathbf{B}^{\mathrm{ver}})$ as sheaves in a neighborhood of z. In particular, $\mathrm{d}\mu : T(X') \to \mu^* T(X)$ injects $\mathcal{T}$ into $\mu^* T_X(-\log \mathbf{B}^{\mathcal{K}})$ in a neighborhood of z.*

Proof. The inclusion $\mathcal{T}_z \subset T_z(\mathbf{B}^{\mathrm{ver}})$ follows from Proposition 4.3 (c) with H an irreducible component of $\mathbf{B}^{\mathrm{ver}}$. The homomorphism $\mathrm{d}\mu$ induces an isomorphism

$$T_{X'}(-\log \mathbf{B}^{\mathrm{ver}}) \to \mu^* T_X(-\log \mathbf{B}^{\mathcal{K}})$$

in a neighborhood of a general point z of $\mathbf{B}^{\mathrm{ver}}$. Thus it injects $\mathcal{T}$ into $\mu^* T_X(-\log \mathbf{B}^{\mathcal{K}})$ in that neighborhood. Q.E.D.

Proposition 5.4. *In the setting of Proposition 5.3, there exists a subvariety $E \subset X'$ with the following properties.*

(1) *$\mu(E)$ has codimension > 1 in X;*
(2) *$E = \mu^{-1}(\mu(E))$;*
(3) *$\mu|_{X' \setminus E} : X' \setminus E \to X \setminus \mu(E)$ is finite;*
(4) *$\mathbf{B}^{\mathrm{exc}} \subset E$;*
(5) *$\mathbf{B}^{\mathcal{K}} \setminus \mu(E)$ and $\mu^{-1}(\mathbf{B}^{\mathcal{K}}) \setminus E$ are nonsingular; and*
(6) *when we put $O := X' \setminus (E \cup \mathbf{B}^{\mathrm{hor}})$, the image of $\mathrm{d}\mu|_O : \mathcal{T}|_O \to \mu^* T(X)|_O$ defines a vector subbundle, to be denoted by $\mathcal{V} \subset \mu^* T(X)|_O$. Furthermore, $\mathcal{V} \subset \mu^* T_X(-\log \mathbf{B}^{\mathcal{K}})|_O$ as sheaves.*

Proof. Let $E_1 \subset X'$ be the locus where μ is not finite. By the definition of $\mathbf{B}^{\mathrm{exc}}$ in Proposition 5.2, E_1 contains $\mathbf{B}^{\mathrm{exc}}$. From Proposition 5.3, there exists a subvariety $E_2 \subset \mathbf{B}^{\mathrm{ver}}$ of codimension ≥ 2 in X' such that the $\mathrm{d}\mu$-image of $\mathcal{T}$ defines a vector subbundle of $\mu^* T(X)$ and a locally free subsheaf of $\mu^* T_X(-\log \mathbf{B})$ in a neighborhood of every point outside $\mathbf{B}^{\mathrm{hor}} \cup \mathbf{B}^{\mathrm{exc}} \cup E_2$. Set

$$E = \mu^{-1} \left(\mu(E_1) \cup \mu(E_2) \cup \mathrm{Sing}(\mathbf{B}^{\mathcal{K}}) \cup \mu(\mathrm{Sing}(\mu^{-1}(\mathbf{B}^{\mathcal{K}}))) \right).$$

It clearly satisfies (1)-(6). Q.E.D.

Proposition 5.5. *In the setting of Proposition 5.4, let $\varphi : \mathcal{P} \to X$ and $\psi : \mathcal{P} \to \mathcal{W}$ be as in Proposition 4.3. Write $\mathbf{E} := \varphi^{-1}(\mu(E)) \subset \mathcal{P}$:*

$$\begin{array}{ccccc} \mathcal{P} & \xrightarrow{\varphi} & X & \xleftarrow{\mu} & X' \\ \cup & & \cup & & \cup \\ \mathbf{E} & \longrightarrow & \mu(E) & \longleftarrow & E. \end{array}$$

Then for a general point $w \in \mathcal{W}$, the following holds.

(a) $\dim(\psi^{-1}(w) \cap \mathbf{E}) \leq k - 2$.

(b) *The intersection $\psi^{-1}(w) \cap \varphi^{-1}(\mathbf{B}^{\mathcal{K}})$ is a reduced divisor on $\psi^{-1}(w)$ and its singular loci* $\mathrm{Sing}\left(\psi^{-1}(w) \cap \varphi^{-1}(\mathbf{B}^{\mathcal{K}})\right)$ *is contained in $\psi^{-1}(w) \cap \mathbf{E}$.*

Proof. The subvariety $\mathbf{E} \subset \mathcal{P}$ has codimension > 1 because φ is unramified and $\mu(E) \subset X$ has codimension > 1 from Proposition 5.4 (1). Thus (a) holds for a general $w \in \mathcal{W}$.

Since $\mathbf{B}^{\mathcal{K}} \setminus \mu(E)$ is nonsingular from Proposition 5.4 (5), the divisor $\varphi^{-1}(\mathbf{B}^{\mathcal{K}})$ on $\mathcal{P}$ is nonsingular outside $\mathbf{E}$. Thus (b) holds for a general $w \in \mathcal{W}$. Q.E.D.

§6. Pulling back étale families

We have the following general construction.

Proposition 6.1. *Let Y be a projective manifold and let $(\varphi, \psi) : \mathcal{P} \to Y \times \mathcal{W}$ be an étale family of immersed submanifolds parametrized by $\mathcal{W}$. Let $f : Y' \to Y$ be a surjective generically finite morphism from a projective manifold Y'. Choose an irreducible component $\mathcal{P}^+$ of $Y' \times_Y \mathcal{P}$ which is dominant over Y'. Then, replacing $\mathcal{W}$ and $\mathcal{P}$ by their Zariski open subsets if necessary, we can find morphisms of nonsingular varieties (depending on the choice of $\mathcal{P}^+$)*

$$f_\flat : \mathcal{W}' \to \mathcal{W}, \ f_\sharp : \mathcal{P}' \to \mathcal{P}, \ \varphi' : \mathcal{P}' \to Y' \text{ and } \psi' : \mathcal{P}' \to \mathcal{W}'$$

with the following properties.

(1) *The following diagram commutes.*

$$\begin{array}{ccccc} \mathcal{W}' & \stackrel{\psi'}{\longleftarrow} & \mathcal{P}' & \stackrel{\varphi'}{\longrightarrow} & Y' \\ f_\flat \downarrow & & f_\sharp \downarrow & & f \downarrow \\ \mathcal{W} & \stackrel{\psi}{\longleftarrow} & \mathcal{P} & \stackrel{\varphi}{\longrightarrow} & Y. \end{array}$$

(2) *The morphism $(\varphi', \psi') : \mathcal{P}' \to Y' \times \mathcal{W}'$ defines an étale family of immersed submanifolds in Y' parametrized by $\mathcal{W}'$.*

(3) *The morphisms $f_\flat$ and $f_\sharp$ are proper, surjective and generically finite.*

(4) *$\mathcal{P}'$ is a Zariski open subset of $\mathcal{P}^+$ and the morphisms φ' and $f_\sharp$ are induced from the natural projections of $\mathcal{P}^+ \subset Y' \times_Y \mathcal{P}$ to each factor.*

Proof. Since $\varphi : \mathcal{P} \to Y$ is unramified, the natural morphism $\varphi^+ : \mathcal{P}^+ \to Y'$ is unramified by base change and $\mathcal{P}^+$ is nonsingular. Let

$$\mathcal{P}^+ \xrightarrow{\psi^+} \mathcal{W}^+ \xrightarrow{f_\flat^+} \mathcal{W}$$

be the Stein factorization of the composition $\mathcal{P}^+ \to \mathcal{P} \xrightarrow{\psi} \mathcal{W}$. Since $\mathcal{P}^+$ is nonsingular, we can choose a Zariski open subset $\mathcal{W}_o \subset \mathcal{W}$ such that ψ^+ is a smooth morphism with connected fibers over $\mathcal{W}' := (f_\flat^+)^{-1}(\mathcal{W}_o)$. Set $\mathcal{P}' = (\psi^+)^{-1}(\mathcal{W}')$ and define ψ' (resp. φ') as the restriction of ψ^+ (resp. φ^+). Setting $f_\flat$ and $f_\sharp$ as the natural morphisms induced by f and replacing $\mathcal{W}$ (resp. $\mathcal{P}$) by $\mathcal{W}_o$ (resp. $\psi^{-1}(\mathcal{W}_o)$), we have the properties (1)-(4). Q.E.D.

We will apply Proposition 6.1 to the morphism $\mu : X' \to X$ in Proposition 4.3.

Proposition 6.2. *In the setting of Proposition 4.3, replacing $\mathcal{W}$ and $\mathcal{P}$ by their Zariski open subsets if necessary, we have $(\varphi', \psi') : \mathcal{P}' \to X' \times \mathcal{W}'$, an étale family of immersed submanifolds in X' parametrized by a nonsingular variety $\mathcal{W}'$, together with a commuting diagram*

$$\begin{array}{ccccc}
\mathcal{W}' & \xleftarrow{\psi'} & \mathcal{P}' & \xrightarrow{\varphi'} & X' \\
\mu_\flat \downarrow & & \mu_\sharp \downarrow & & \mu \downarrow \\
\mathcal{W} & \xleftarrow{\psi} & \mathcal{P} & \xrightarrow{\varphi} & X
\end{array}$$

where $\mu_\sharp$ and $\mu_\flat$ are surjective proper generically finite morphisms. Furthermore, we can assume that for any $w \in \mathcal{W}$ and $w' \in \mu_\flat^{-1}(w)$, the morphism

$$(\mu_\sharp)|_{(\psi')^{-1}(w')} : (\psi')^{-1}(w') \to \psi^{-1}(w)$$

is generically finite of degree m for some integer $m \geq 2$.

Proof. Let us apply Proposition 6.1 to the generically finite morphism $\mu : X' \to X$ of Proposition 4.3 with $Y = X, Y' = X'$ and $f = \mu$. Proposition 4.1 implies that for a general $w \in \mathcal{W}$, there exists an irreducible component of $\mu^{-1}(P_w)$ that is generically finite over P_w of degree m for some integer $m \geq 2$. Thus we can find an irreducible component $\mathcal{P}^+$ of $X' \times_X \mathcal{P}$ such that, when $(\varphi', \psi') : \mathcal{P}' \to X' \times \mathcal{W}'$ is the étale family of immersed submanifolds determined by $\mathcal{P}^+$ in the sense of Proposition 6.1,

$$(\mu_\sharp)|_{(\psi')^{-1}(w')} : (\psi')^{-1}(w') \to \psi^{-1}(w)$$

is generically finite of degree m for a general $w \in \mathcal{W}$ and any $w' \in \mu_\flat^{-1}(w)$. After replacing $\mathcal{W}$ by a Zariski open subset, we can assume that this holds for any $w \in \mathcal{W}$. Q.E.D.

Proposition 6.3. *In the setting of Proposition 6.2, we have two fibration structures on $\mathcal{P}'$ given by*

$$\begin{array}{ccc} \mathcal{P}' & \xrightarrow{\varphi'} & X' \\ \psi' \downarrow & & \downarrow \rho \\ \mathcal{W}' & & Z. \end{array}$$

Then there exists a Zariski open subset $\mathcal{P}'_{\mathrm{trans}} \subset \mathcal{P}'$ such that

(i) $\varphi'|_{\mathcal{P}'_{\mathrm{trans}}} : \mathcal{P}'_{\mathrm{trans}} \to \varphi'(\mathcal{P}'_{\mathrm{trans}})$ *is étale;*

(ii) $\rho \circ \varphi'|_{\mathcal{P}'_{\mathrm{trans}}}$ *is a smooth morphism and the relative tangent bundle $T^{\rho\circ\varphi'} \subset T(\mathcal{P}'_{\mathrm{trans}})$ corresponds, via the étale morphism $\varphi'|_{\mathcal{P}'_{\mathrm{trans}}}$ in (i), to the subsheaf $\mathcal{T} \subset T(X')$ defined in Proposition 4.3; and*

(iii) $T^{\psi'}_z \cap T^{\rho\circ\varphi'}_z = 0$ *in $T_z(\mathcal{P}')$ for each $z \in \mathcal{P}'_{\mathrm{trans}}$.*

Proof. Let $X'_o \subset \iota(\mathcal{P}) \subset X'$ be a Zariski open subset such that

$$\varphi'|_{(\varphi')^{-1}(X'_o)} : (\varphi')^{-1}(X'_o) \to X'_o$$

is étale. Put $\mathcal{P}'_{\mathrm{trans}} = (\varphi')^{-1}(X'_o) \cap (\mu \circ \varphi')^{-1}(X^{\mathrm{trans}})$ where X^{trans} is from Proposition 4.2 (c). Then it satisfies (i) from the choice of X'_o and (ii) from $X'_o \subset \iota(\mathcal{P})$.

To see (iii), pick $x \in \mathcal{P}'_{\mathrm{trans}}$ and let F_1 (resp. F_2) be the fiber of ψ' (resp. $\rho\circ\varphi'$) through x. Then $\mu\circ\varphi'(F_1)$ (resp. $\mu\circ\varphi'(F_2)$)is of the form P_{w_1} (resp. P_{w_2}) for some $w_1, w_2 \in \mathcal{W}$, in the notation of Definition 3.1, such that

$$P_{w_1} \cap P_{w_2} \ni \mu\circ\varphi'(x) \in X^{\mathrm{trans}}.$$

The morphism $\varphi'(F_2) \to P_{w_2}$ is birational by the definition of ρ. On the other hand, the morphism $\varphi'(F_1) \to P_{w_1}$ has degree $m \geq 2$ by Proposition 6.2. Since $\varphi'(x) \in \varphi'(F_1) \cap \varphi'(F_2)$, we have $P_{w_1} \neq P_{w_2}$. From Proposition 4.2 (c), we see that F_1 and F_2 are transversal at x. Q.E.D.

Proposition 6.4. *In the setting of Proposition 6.2, assume that Condition 5.1 is satisfied so that we can use Proposition 5.4 and Proposition 5.5. Write $\mathbf{E}' = \mu_\sharp^{-1}(\mathbf{E})$ where $\mathbf{E} = \varphi^{-1}(\mu(E))$ is as in Proposition 5.5:*

$$\begin{array}{ccccccc} \mathcal{P}' & \xrightarrow{\mu_\sharp} & \mathcal{P} & \xrightarrow{\varphi} & X & \xleftarrow{\mu} & X' \\ \cup & & \cup & & \cup & & \cup \\ \mathbf{E}' & \longrightarrow & \mathbf{E} & \longrightarrow & \mu(E) & \longleftarrow & E. \end{array}$$

Then there exists a Zariski open subset $\mathcal{W}_\triangle \subset \mathcal{W}$ such that for every $w \in \mathcal{W}_\triangle$, (a) and (b) of Proposition 5.5 hold and, moreover, for each

$w' \in \mu_\flat^{-1}(w) \in \mathcal{W}'$,

$$(\psi')^{-1}(w') \cap (\varphi')^{-1}(O) = (\psi')^{-1}(w') \setminus \mathbf{E}'$$

where $O = X' \setminus (E \cup \mathbf{B}^{\mathrm{hor}})$ *is as in Proposition 5.4.*

Proof. We have already seen that a general $w \in \mathcal{W}$ satisfies (a) and (b) of Proposition 5.5. It remains to check

$$(\psi')^{-1}(w') \cap (\varphi')^{-1}(O) = (\psi')^{-1}(w') \setminus \mathbf{E}'$$

for a general $w \in \mathcal{W}$ and any $w' \in \mu_\flat^{-1}(w)$.

From Proposition 6.2, for any $w \in \mathcal{W}$ and $w' \in \mu_\flat^{-1}(w)$, the variety $P_{w'} = \varphi'((\psi')^{-1}(w'))$ in X' is an irreducible component of $\mu^{-1}(P_w)$ that is of degree $m > 1$ over P_w. By the last sentence in the statement of Proposition 5.2, we see that $P_{w'} \cap \mathbf{B}^{\mathrm{hor}} = \emptyset$ for a general $w \in \mathcal{W}$ and any $w' \in \mu_\flat^{-1}(w)$. Thus, for a general $w \in \mathcal{W}$ and any $w' \in \mu_\flat^{-1}(w)$, we have $\psi^{-1}(w') \cap (\varphi')^{-1}(\mathbf{B}^{\mathrm{hor}}) = \emptyset$. By $E = \mu^{-1}(\mu(E))$ of Proposition 5.4 (2), we have

$$\begin{aligned}
(\psi')^{-1}(w') \cap (\varphi')^{-1}(O) &= (\psi')^{-1}(w') \setminus (\varphi')^{-1}(E \cup \mathbf{B}^{\mathrm{hor}}) \\
&= (\psi')^{-1}(w') \setminus (\varphi')^{-1}(E) \\
&= (\psi')^{-1}(w') \setminus (\mu \circ \varphi')^{-1}(\mu(E)) \\
&= (\psi')^{-1}(w') \setminus (\varphi \circ \mu_\sharp)^{-1}(\mu(E)) \\
&= (\psi')^{-1}(w') \setminus \mu_\sharp^{-1}(\mathbf{E}).
\end{aligned}$$

This completes the proof. Q.E.D.

§7. Consequence of a hypothetical condition

Let X and $\mathcal{K}$ be as in Assumption 1. To prove Theorem 2, we may assume that X satisfies the following additional condition.

Condition 7.1. *There exist a projective manifold* Y*, a morphism* $[f : Y \to X] \in \mathrm{Hom}^s(Y, X)$ *and an element*

$$\sigma \in H^0(Y, f^*T(X)) \setminus f^*H^0(X, T(X)).$$

The goal of this section is to show that if X satisfies Condition 7.1, then we may choose $f : Y \to X$ and

$$\sigma \in H^0(Y, f^*T(X)) \setminus f^*H^0(X, T(X))$$

such that they have very special properties with respect to $\mu : X' \to X$ of Proposition 4.3 and $X^{\mathcal{C}}$ of Proposition 4.4. More precisely, we will prove the following generalization of Proposition 4.3 of [4].

Proposition 7.1. *Under Assumption 1, suppose that X satisfies Condition 7.1. Then we have a projective manifold $X^{''}$ equipped with a surjective generically finite morphism $\beta : X^{''} \to X$, an element $\theta \in H^0(X^{''}, \beta^* T(X)) \setminus \beta^* H^0(X, T(X))$ and a Zariski open subset $X^\theta \subset X$ with the following properties.*

(1) *The morphism β factors through $\mu : X' \to X$ of Proposition 4.3, i.e., there exists a surjective generically finite morphism $\gamma : X^{''} \to X$ such that $\beta = \mu \circ \gamma$:*

$$\beta : \; X^{''} \; \xrightarrow{\gamma} \; X' \; \xrightarrow{\mu} \; X.$$

(2) *In the notation of Proposition 4.4, $X^\theta \subset X^{\mathcal{C}}$ and*

$$\beta|_{\beta^{-1}(X^\theta)} : \beta^{-1}(X^\theta) \to X^\theta$$

is an étale morphism.

(3) *For any point $x \in X^\theta$, if $y_1 \neq y_2$ are two distinct points in $\beta^{-1}(x)$, then the values of θ at these points are distinct, namely, $\theta_{y_1} \neq \theta_{y_2}$ as vectors in $T_x(X)$.*

(4) *For a general point $x \in X^\theta$ and any $y \in \beta^{-1}(x)$, the value θ_y of θ at y regarded as a vector in $T_{\gamma(y)}(X')(= T_x(X))$, belongs to $T^\rho_{\gamma(y)}$ where $\rho : X' \to Z$ is as in Proposition 4.3.*

Note that $\sigma \in H^0(Y, f^*T(X))$ in Condition 7.1 defines an irreducible projective variety in $T(X)$ dominant over X given by the subset $\{\sigma_y \in T_{f(y)}, y \in Y\}$. Thus Condition 7.1 implies (in fact, equivalent to) the following.

Condition 7.2. *There exists an irreducible projective subvariety in $\Sigma \subset T(X)$ of degree > 1 over X. The natural projection $\Sigma \to X$ is necessarily finite.*

To find $\beta : X^{''} \to X$ of Proposition 7.1, we will make some intermediate constructions using Σ in Condition 7.2 in the next two propositions, which generalize Propositions 4.1 and 4.2 of [4].

Proposition 7.2. *In the setting of Proposition 4.3, assume that Condition 7.2 holds. Let $T(X) \times_X T(X)$ be the fiber product of two copies of the projection $\pi : T(X) \to X$ and let*

$$\Sigma \times_X \Sigma \subset T(X) \times_X T(X)$$

be the fiber product of two copies of $\pi|_\Sigma : \Sigma \to X$. Then there exists at least one irreducible component $\Sigma^\sharp$ of $\Sigma \times_X \Sigma$ with the following property: for a general $w \in \mathcal{W}$ and a general point $x \in P_w$, some irreducible *component of* $\pi^{-1}(P_w)\cap\Sigma$ *contains two distinct points* $x_1 \neq x_2, \pi(x_1) = \pi(x_2) = x$, *such that* $(x_1, x_2) \in \Sigma \times_X \Sigma$ *is contained in* $\Sigma^\sharp$.

Proof. For a general $w \in \mathcal{W}$, there exists an irreducible component P'_w of $\pi^{-1}(P_w)\cap\Sigma$ such that the projection $P'_w \to P_w$ is finite of degree > 1 by Proposition 4.1. Thus for a general point $x \in P_w$, we can choose two distinct points $x_1 \neq x_2 \in P'_w, \pi(x_1) = \pi(x_2) = x$. As we vary w and x, the point $(x_1, x_2) \in \Sigma \times_X \Sigma$ covers a subset of dimension $\geq \dim X$ in $\Sigma\times_X\Sigma$. Since $\dim \Sigma\times_X\Sigma = \dim X$, there exits an irreducible component $\Sigma^\sharp$ satisfying the required property. Q.E.D.

Proposition 7.3. *In the situation of Proposition 7.2, let*

$$\delta : T(X) \times_X T(X) \to T(X)$$

be the difference morphism defined by

$$\delta(v_1, v_2) := v_1 - v_2 \ \ \text{for} \ \ v_1, v_2 \in T_x(X) \ \ \text{for } x \in X.$$

Then in terms of $\hat{\mathcal{C}} \subset T(X)$ *and the morphism* $\chi : (\hat{\mathcal{C}}\setminus 0_X)\cap\pi^{-1}(X^{\mathcal{C}}) \to \mu^{-1}(X^{\mathcal{C}})$ *in Proposition 4.4,*

$$\delta(\Sigma^\sharp) \subset \hat{\mathcal{C}}, \ \ \delta(\Sigma^\sharp) \not\subset 0_X$$

and the dominant rational map $\chi^\sharp : \delta(\Sigma^\sharp) \dashrightarrow X'$ *induced by* χ *is generically finite.*

Proof. We will use the tautological section $\sigma^{\mathrm{taut}} \in H^0(\Sigma, \pi^*T(X))$ defined by

$$\sigma_a = a \in T_x(X) \text{ for each } a \in \Sigma \cap T_x(X).$$

For a general $w \in \mathcal{W}$ and a general $x \in P_w$, let $(x_1, x_2) \in \Sigma^\sharp, x_1 \neq x_2$, be as in Proposition 7.2 and let $\widetilde{P}$ be the irreducible component of $\pi^{-1}(P_w) \cap \Sigma$ containing x_1 and x_2. Applying Proposition 3.1 with the substitution of $X, \Sigma, \pi|_\Sigma$ and the pull-back of σ^{taut} to the normalization of $\widetilde{P}$ in place of $Y, \tilde{Y}, f$ and σ, we see that

$$0 \ \neq \ x_1 - x_2 \ \in \ T_x(P_w) \ \subset \ \hat{\mathcal{C}}.$$

As w varies over general points of $\mathcal{W}$, the element $x_1 - x_2$ varies over a Zariski open subset in the irreducible variety $\delta(\Sigma^\sharp)$. It follows that $\delta(\Sigma^\sharp) \subset \hat{\mathcal{C}}$ and $\delta(\Sigma^\sharp) \not\subset 0_X$. The dominant rational map $\chi^\sharp$ is generically finite because the natural projection $\delta(\Sigma^\sharp) \to X$ is finite. Q.E.D.

Now we are ready to prove Proposition 7.1.

Proof of Proposition 7.1. Choose a desingularization $\alpha : X^{''} \to \delta(\Sigma^{\sharp})$ which eliminates the indeterminacy of the generically finite rational map $\chi^{\sharp}$ in Proposition 7.3 such that $\chi^{\sharp} \circ \alpha$ defines a generically finite morphism $\gamma : X^{''} \to X'$. Denote by τ the natural projection $\delta(\Sigma^{\sharp}) \to X$ to have the commuting diagram

$$\begin{array}{ccc} X^{''} & \xrightarrow{\alpha} & \delta(\Sigma^{\sharp}) \\ \gamma \downarrow & & \downarrow \tau \\ X' & \xrightarrow{\mu} & X. \end{array}$$

From $\delta(\Sigma^{\sharp}) \subset T(X)$, there exists a tautological section

$$\theta^{\mathrm{taut}} \in H^0(\delta(\Sigma^{\sharp}), \tau^* T(X))$$

defined by $\theta^{\mathrm{taut}}(a) = a \in T_{\tau(a)}(X)$ for each $a \in \delta(\Sigma^{\sharp})$. Put $\beta = \mu \circ \gamma$ and let

$$\theta \in H^0(X^{''}, \beta^* T(X)) = H^0(X^{''}, (\tau \circ \alpha)^* T(X))$$

be the pull-back of θ^{taut} by α. We can choose a Zariski open subset $X^{\theta} \subset X$ satisfying the property (2) because α is birational and the property (3) because θ^{taut} is the tautological section. The property (4) follows from $\delta(\Sigma^{\sharp}) \subset \hat{\mathcal{C}}$ in Proposition 7.3 and the relation between $\hat{\mathcal{C}}$ and T^{ρ} in Proposition 4.4 (iv). Q.E.D.

§8. Completion of the proof of Theorem 2

In this section, we will prove Theorem 2. Let X and $\mathcal{K}$ be as in Assumption 1. As mentioned before, we may assume that X satisfies the condition 7.2, hence Proposition 7.1. We want to derive a contradiction from Proposition 7.1 and the assumptions in Theorem 2. For this, we want to descend $\theta \in H^0(X^{''}, \beta^* T(X))$ in Proposition 7.1 to some $\vartheta \in H^0(X', \mu^* T(X))$. This is obviously not possible because the morphism $\gamma : X^{''} \to X'$ may have degree ≥ 2. As we will see in Proposition 8.2 below, however, we can achieve this if we restrict to a Euclidean neighborhood of a subvariety in X', i.e., a Euclidean neighborhood of a general member of $\mathcal{P}'$ in Proposition 6.2. To make this precise, we consider the following setting.

Notation 8.1. *In the setting of Proposition 7.1, pick an irreducible component dominant over $X^{''}$ in $X^{''} \times_{X'} \mathcal{P}'$, the fiber product of $\gamma : X^{''} \to X'$ of Proposition 7.1 and $\varphi' : \mathcal{P}' \to X'$. Applying Proposition*

6.1 to this component of $X^{''} \times_{X'} \mathcal{P}'$, *we have a commuting diagram*

$$\begin{array}{ccccc}
\mathcal{W}^{''} & \xleftarrow{\psi^{''}} & \mathcal{P}^{''} & \xrightarrow{\varphi^{''}} & X^{''} \\
\gamma_\flat \downarrow & & \gamma_\sharp \downarrow & & \gamma \downarrow \\
\mathcal{W}' & \xleftarrow{\psi'} & \mathcal{P}' & \xrightarrow{\varphi'} & X' \\
\mu_\flat \downarrow & & \mu_\sharp \downarrow & & \mu \downarrow \\
\mathcal{W} & \xleftarrow{\psi} & \mathcal{P} & \xrightarrow{\varphi} & X
\end{array}$$

where the second and the third lows are from Proposition 6.2 and $(\varphi^{''}, \psi^{''})$ *is an étale family of immersed submanifolds in* $X^{''}$ *parametrized by* $\mathcal{W}^{''}$. *We may assume that* $\gamma_\flat$ *and* $\gamma_\sharp$ *are proper surjective generically finite morphisms.*

Since $\mathcal{W}^{''} \xrightarrow{\gamma_\flat} \mathcal{W}' \xrightarrow{\mu_\flat} \mathcal{W}$ *are generically finite morphisms, a general point of* $\mathcal{W}$ *has a Euclidean neighborhood* $\mathcal{M} \subset \mathcal{W}$ *such that there are Euclidean open subsets* $\mathcal{M}' \subset \mathcal{W}'$ *and* $\mathcal{M}^{''} \subset \mathcal{W}^{''}$ *which are biholomorphic to* $\mathcal{M}$ *by* $\mu_\flat$ *and* $\mu_\flat \circ \gamma_\flat$, *respectively. Write*

$$\mathcal{Q} = \psi^{-1}(\mathcal{M}),\ \mathcal{Q}' = (\psi')^{-1}(\mathcal{M}'),\ \mathcal{Q}^{''} = (\psi^{''})^{-1}(\mathcal{M}^{''})$$

and denote by $\widetilde{\mu} : \mathcal{Q}' \to \mathcal{Q}$ *and* $\widetilde{\gamma} : \mathcal{Q}^{''} \to \mathcal{Q}'$ *the natural morphisms to obtain the following diagram.*

$$\begin{array}{ccccccccc}
\mathcal{W}^{''} & \supset & \mathcal{M}^{''} & \xleftarrow{\psi^{''}} & \mathcal{Q}^{''} & \subset & \mathcal{P}^{''} & \xrightarrow{\varphi^{''}} & X^{''} \\
\gamma_\flat \downarrow & & \wr\wr & & \widetilde{\gamma} \downarrow & & \gamma_\sharp \downarrow & & \gamma \downarrow \\
\mathcal{W}' & \supset & \mathcal{M}' & \xleftarrow{\psi'} & \mathcal{Q}' & \subset & \mathcal{P}' & \xrightarrow{\varphi'} & X' \\
\mu_\flat \downarrow & & \wr\wr & & \widetilde{\mu} \downarrow & & \mu_\sharp \downarrow & & \mu \downarrow \\
\mathcal{W} & \supset & \mathcal{M} & \xleftarrow{\psi} & \mathcal{Q} & \subset & \mathcal{P} & \xrightarrow{\varphi} & X
\end{array}$$

Remark 8.1. *From our choice of* $\mathcal{P}'$ *in Proposition 6.2, the morphism* $\widetilde{\mu}$ *has degree* $m \geq 2$. *On the other hand, we have not made any special choice in the definition of* $\mathcal{P}^{''}$ *in Notation 8.1. The next proposition shows that the morphism* $\widetilde{\gamma}$ *becomes bimeromorphic, if we choose sufficiently small* $\mathcal{M}$.

Proposition 8.1. *In Notation 8.1, let* $w' \in \mathcal{W}'$ *be a general point and set*

$$P_{w'} := \varphi'((\psi')^{-1}(w')) \subset X'.$$

For each irreducible component $P^{''}$ *of* $\gamma^{-1}(P_{w'})$ *dominant over* $P_{w'}$, *the restriction* $\gamma|_{P^{''}} : P^{''} \to P_{w'}$ *is birational. In particular, we can assume that* $\widetilde{\gamma} : \mathcal{Q}^{''} \to \mathcal{Q}'$ *is bimeromorphic in Notation 8.1 by choosing* $\mathcal{M}, \mathcal{M}'$ *and* $\mathcal{M}^{''}$ *suitably.*

Proof. This follows from the special property of θ in Proposition 7.1, combined with Proposition 3.1. To be precise, we will apply Proposition 3.1 with the substitution of $X', X'', \mathcal{P}', \gamma, P''$ for $Y, \widetilde{Y}, \mathcal{P}, f, \widetilde{P}$, respectively. From the generality of $w' \in \mathcal{W}'$, we may assume that

$$P_{w'} \cap \varphi'(\mathcal{P}'_{\text{trans}}) \cap X^{\theta} \cap (X')^{\gamma} \neq \emptyset$$

where $\mathcal{P}'_{\text{trans}}$ is from Proposition 6.3, X^{θ} is from Proposition 7.1 and $(X')^{\gamma} \subset X'$ is the Zariski open subset corresponding to Y^f in Proposition 3.1.

Suppose $\gamma|_{P''}$ is not birational. For a general point

$$x \in P_{w'} \cap \varphi'(\mathcal{P}'_{\text{trans}}) \cap X^{\theta} \cap (X')^{\gamma},$$

let $x_1 \neq x_2$ be two distinct points in $\gamma^{-1}(x) \cap P''$. By the substitution of x, x_1, x_2, θ for y, y_1, y_2, σ, respectively, Proposition 3.1 says that $\theta_{x_1} - \theta_{x_2} \in T_x(P_{w'})$. But $\theta_{x_1}, \theta_{x_2} \in T^{\varrho}_x$ by Proposition 7.1 (3), while $T_x(P_{w'}) \cap T^{\varrho}_x = 0$ by Proposition 6.3 (iii). Thus $\theta_{x_1} = \theta_{x_2}$, a contradiction to Proposition 7.1 (2). Q.E.D.

Notation 8.2. *Let $f : Y' \to Y$ be a proper surjective generically finite morphism of degree m between two complex manifolds. Let V be a vector bundle on Y. We denote by* $\text{Norm}_f : H^0(Y', f^*V) \to H^0(Y, V)$ *the* norm homomorphism *of f. Recall that for a section σ of f^*V and a point $y \in Y$ where $f^{-1}(y)$ consists of m distinct points $y_1, \ldots y_m$,*

$$\text{Norm}_f(\sigma)_y = \frac{1}{m}\sum_{i=1}^{m} \sigma_{y_i} \ \in V_y.$$

In particular, if for a general $y \in Y$ and $f^{-1}(y) = \{y_1, \ldots, y_m\}$, we have $\sigma_{y_i} = \sigma_{y_j}$ for all pairs (i, j), then $\sigma = f^*\text{Norm}_f(\sigma)$.

Proposition 8.2. *Consider the setting of Notation 8.1, with the additional property that $\widetilde{\gamma} : \mathcal{Q}'' \to \mathcal{Q}'$ is bimeromorphic from Proposition 8.1. Using $\theta \in H^0(X'', \beta^*T(X))$ of Proposition 7.1, set*

$$\widehat{\theta} := (\varphi'')^*\theta \ \in H^0\left(\mathcal{P}'', (\beta \circ \varphi'')^*T(X)\right) = H^0\left(\mathcal{P}'', (\mu \circ \varphi' \circ \gamma_\sharp)^*T(X)\right)$$

and define

$$\vartheta := \text{Norm}_{\widetilde{\gamma}}(\widehat{\theta}|_{\mathcal{Q}''}) \ \in H^0(\mathcal{Q}', (\mu \circ \varphi')^*T(X)) = H^0(\mathcal{Q}', (\varphi \circ \widetilde{\mu})^*T(X)).$$

Then

(i) $\vartheta \neq 0$*;*

(ii) *ϑ takes values in the image of $(\varphi')^*\mathcal{T}$ in $(\mu \circ \varphi')^*T(X)$ where $\mathcal{T} \subset T(X')$ is the subsheaf defined in Proposition 6.2 (c) ; and*

(iii) *when we regard $T^\psi \subset T(\mathcal{P})$ as a subbundle of $\varphi^*T(X)$ via the isomorphism $\mathrm{d}\varphi : T(\mathcal{P}) \cong \varphi^*T(X)$, the difference*

$$\vartheta - \widetilde{\mu}^*\mathrm{Norm}_{\widetilde{\mu}}(\vartheta) \ \in H^0(\mathcal{Q}', (\varphi \circ \widetilde{\mu})^*T(X))$$

belongs to $H^0(\mathcal{Q}', \widetilde{\mu}^*T^\psi)$.

Proof. (i) follows from the fact that $\widehat{\theta} \neq 0$ and $\widetilde{\gamma}$ is bimeromorphic.

Since θ_y belongs to $T^\rho_{\gamma(y)}$ for general $y \in X''$ by Proposition 7.1 (3), ϑ takes values in the image of $\mathcal{T}$ in $(\mu \circ \varphi')^*T(X)$, proving (ii).

For a general point $w \in \mathcal{M}$, set $w' := \mathcal{M}' \cap \mu_\flat^{-1}(w)$. We use Proposition 3.1 with the substitution of $X, X', \mu, (\psi')^{-1}(w'), P_{w'}, \vartheta$ for $Y, \widetilde{Y}, f, \widehat{P}, \widetilde{P}, \sigma$, respectively. For a general point $x \in \psi^{-1}(w)$ and

$$\widetilde{\mu}^{-1}(x) = \{x_1, \ldots, x_m\},$$

Proposition 3.1 with $y = \varphi(x), y_1 = \varphi'(x_1)$ and $y_2 = \varphi'(x_i)$ says that

$$\vartheta_{x_i} = \vartheta_{x_1} + v_i, \ 2 \leq i \leq m, \text{ for some } v_i \in T_{\varphi(x)}(P_w) = \mathrm{d}\varphi(T^\psi_x).$$

This implies that $\mathrm{Norm}_{\widetilde{\mu}}(\vartheta)_x = \vartheta_{x_1} + v$ for some $v \in \mathrm{d}\varphi(T^\psi_x)$. Thus

$$(\vartheta - \widetilde{\mu}^*\mathrm{Norm}_{\widetilde{\mu}}(\vartheta))_{x_1} \in (\widetilde{\mu}^*T^\psi)_{x_1}.$$

By the same reasoning,

$$(\vartheta - \widetilde{\mu}^*\mathrm{Norm}_{\widetilde{\mu}}(\vartheta))_{x_i} \in (\widetilde{\mu}^*T^\psi)_{x_i}$$

for any i, which implies that

$$\vartheta - \widetilde{\mu}^*\mathrm{Norm}_{\widetilde{\mu}}(\vartheta) \ \in H^0(\mathcal{Q}', \widetilde{\mu}^*T^\psi).$$

This proves (iii). Q.E.D.

Remark 8.2. *To appreciate the geometry behind Proposition 8.2, it is worth interpreting it in the context of Proposition 6.3, namely, in terms of the two transversal fibrations $T^{\psi'}$ and $T^{\rho\circ\varphi'}$ on a Zariski open subset $\mathcal{P}'_{\mathrm{trans}} \subset \mathcal{P}'$. On the open set $\mathcal{Q}' \cap \mathcal{P}'_{\mathrm{trans}}$, (ii) says that ϑ takes values in $T^{\rho\circ\varphi'}$, while (iii) says that $\vartheta - \widetilde{\mu}^*\mathrm{Norm}_{\widetilde{\mu}}(\vartheta)$ takes values in $T^{\psi'}$.*

We need to use ϑ in Proposition 8.2 together with the results in Section 5 to prove Theorem 2. The results in Section 5 are available in our setting by the following.

Proposition 8.3. *Under Assumption 1, if X satisfies Condition 7.2, then it satisfies also Condition 5.1. In particular, all the results from Section 5 hold if X satisfies Condition 7.1.*

Proof. We will apply Proposition 3.3 with $Y = X$, $\widetilde{Y} = X''$ and $f = \beta$ using the terminology of Proposition 7.1. Setting $Y^{\rm trans} = X^{\rm trans}$ from Proposition 4.2 (c), the condition (1) of Proposition 3.3 holds. Setting $\sigma = \theta$ from Proposition 7.1, the condition (2) of Proposition 3.3 holds, too. Since any hypersurface in X is ample and $\mathbf{B}^{\mathcal{K}}$ is contained in the branch divisor of $f = \beta$, we conclude that $\mathcal{P}$ in Proposition 4.2 is univalent on each irreducible component of $\mathbf{B}^{\mathcal{K}}$. Q.E.D.

Proposition 8.4. *By Proposition 8.3, we can apply Proposition 5.4 in the setting of Notation 8.1. Let*

$$E \subset X', \ O = X' \setminus (E \cup \mathbf{B}^{\rm hor}) \text{ and } \mathcal{V} \subset \mu^* T_X|_O$$

be as in Proposition 5.4. In Notation 8.1, choose $\mathcal{M}$ such that $\mathcal{M} \subset \mathcal{W}_\triangle$ where $\mathcal{W}_\triangle$ is as in Proposition 6.4. Then

(1) $\mathcal{Q}' \cap (\varphi')^{-1}(O) = \mathcal{Q}' \setminus (\varphi')^{-1}(E)$ *and*

(2) ϑ *in Proposition 8.2 satisfies*

$$\vartheta|_{(\varphi')^{-1}(O)} \in H^0(\mathcal{Q}' \cap (\varphi')^{-1}(O), (\varphi')^* \mathcal{V}).$$

Proof. (1) is immediate from Proposition 6.4. (2) follows from Proposition 5.4 (6) and Proposition 8.2 (ii). Q.E.D.

Completion of the proof of Theorem 2. Suppose that the theorem does not hold. Then we can assume that X satisfies Condition 7.1 and use Propositions 8.2 and Proposition 8.4. The assumption on $\mathbf{B}^{\mathcal{K}}$ in Theorem 2 means that the intersection of $\varphi^{-1}(\mathbf{B}^{\mathcal{K}})$ and a general $\mathbb{P}^k$-fiber of ψ is a hypersurface in $\mathbb{P}^k$ whose dual variety is linearly nondegenerate.

Pick any $w' \in \mathcal{M}'$ and write $w = \mu_\flat(w') \in \mathcal{M} \subset \mathcal{W}_\triangle$. Using the terminology of Proposition 6.4, let

$$F := \psi^{-1}(w), \ F' := (\psi')^{-1}(w'), \ J := F \cap \mathbf{E}, \ J' := F' \cap \mathbf{E}', \ f := \widetilde{\mu}|_{F'}$$

to obtain

$$\begin{array}{ccccccc} J' = F' \cap \mathbf{E}' & \subset & F' & \subset & \mathcal{Q}' & \stackrel{\varphi'}{\to} & X' \\ \downarrow & & f\downarrow & & \widetilde{\mu}\downarrow & & \mu\downarrow \\ J = F \cap \mathbf{E} & \subset & F & \subset & \mathcal{Q} & \stackrel{\varphi}{\to} & X. \end{array}$$

By Proposition 8.4 (1),

$$F' \cap (\varphi')^{-1}(O) = F' \setminus J'.$$

Note that

$$(\varphi')^*\mathcal{V} \subset (\varphi')^*\mu^* T_X(-\log \mathbf{B}^{\mathcal{K}})$$
$$= \mu_\sharp^*\varphi^* T_X(-\log \mathbf{B}^{\mathcal{K}}) = \mu_\sharp^* T_{\mathcal{P}}(-\log \varphi^{-1}(\mathbf{B}^{\mathcal{K}})).$$

Since ϑ of Proposition 8.1 takes values in $(\varphi')^*\mathcal{V}$ on $(\varphi')^{-1}(O)$ by Proposition 8.4 (2), its restriction to $F' \cap (\varphi')^{-1}(O)$ belongs to

$$H^0\left(F' \setminus J', f^* T_{\mathcal{Q}}(-\log \varphi^{-1}(\mathbf{B}^{\mathcal{K}}))\right).$$

It follows that

$$(\vartheta - \widetilde{\mu}^* \mathrm{Norm}_{\widetilde{\mu}}(\vartheta))\,|_{F' \setminus J'} \in H^0\left(F' \setminus J', f^* T_{\mathcal{Q}}(-\log \varphi^{-1}(\mathbf{B}^{\mathcal{K}}))\right).$$

By Proposition 8.2 (iii), this takes values in $f^*T^{\psi}|_{F'} = f^*T(F)$. Thus, after setting $D := F \cap \varphi^{-1}(\mathbf{B}^{\mathcal{K}})$, we have

$$v := (\vartheta - \widetilde{\mu}^* \mathrm{Norm}_{\widetilde{\mu}}(\vartheta))\,|_{F' \setminus J'} \;\in\; H^0(F' \setminus J', f^* T_F(-\log D)).$$

From the properties of $\mathcal{W}_{\triangle}$ in Proposition 6.4, we see that J, D, v and $F \cong \mathbb{P}^k$ with F' in place of Y satisfy the conditions of Theorem 3. It follows that

$$(\vartheta - \widetilde{\mu}^* \mathrm{Norm}_{\widetilde{\mu}}(\vartheta))\,|_{F'} = f^*\lambda_F \text{ for some } \lambda_F \in H^0(F, T_F(-\log D)).$$

Since the above works for any $w' \in \mathcal{M}'$, the section

$$\vartheta - \widetilde{\mu}^* \mathrm{Norm}_{\widetilde{\mu}}(\vartheta) \in H^0(\mathcal{Q}', \xi^* T^{\psi})$$

can be written as $\widetilde{\mu}^*\lambda$ for some $\lambda \in H^0(\mathcal{Q}, T(\mathcal{Q}))$. It follows that

$$\vartheta = \widetilde{\mu}^*(\mathrm{Norm}_{\widetilde{\mu}}(\vartheta) + \lambda) \text{ where } \mathrm{Norm}_{\widetilde{\mu}}(\vartheta) + \lambda \in H^0(\mathcal{Q}, T(\mathcal{Q})).$$

Pick a general point $x \in \mathcal{Q}$ with $\varphi(x) \in X^{\mathcal{C}}$, where $X^{\mathcal{C}}$ is as in Proposition 4.4, such that $\varphi \circ \widetilde{\mu} = \mu \circ \varphi'|_{\mathcal{Q}'}$ is unramified at the points $\widetilde{\mu}^{-1}(x)$ and $(\rho \circ \varphi')(\widetilde{\mu}^{-1}(x))$ consists of m distinct points in $\mathcal{W} \subset Z$. Pick two distinct points $x_1 \neq x_2 \in \widetilde{\mu}^{-1}(x)$ such that $w_1 := \rho(\varphi'(x_1)) \neq w_2 := \rho(\varphi'(x_2))$.

Since $\varphi \circ \widetilde{\mu}$ is unramified at x_1 and x_2, we can regard the section ϑ of $(\varphi \circ \widetilde{\mu})^* T(X)$ as a vector field in a Euclidean neighborhood of x_1 (resp. x_2). Let C_1 (resp. C_2) be the local analytic curve through x_1 (resp. x_2) integrating the vector field induced by ϑ in this Euclidean neighborhood. Then $\widetilde{\mu}(C_1)$ (resp. $\widetilde{\mu}(C_2)$) is the analytic curve through x integrating the vector field induced by $\mathrm{Norm}_{\widetilde{\mu}}(\vartheta) + \lambda$ in a neighborhood of x in $\mathcal{P}$. It follows that $\widetilde{\mu}(C_1) = \widetilde{\mu}(C_2)$.

Since ϑ is a section of $(\varphi')^*\mathcal{V}$, we have

$$\varphi'(C_1) \subset \rho^{-1}(w_1) \text{ and } \varphi'(C_2) \subset \rho^{-1}(w_2),$$

and consequently,

$$\varphi(\widetilde{\mu}(C_1)) \subset P_{w_1} \text{ and } \varphi(\widetilde{\mu}(C_2)) \subset P_{w_2}.$$

This implies that P_{w_1} and P_{w_2} share a common analytic curve through $\varphi(x) \in X^{\mathcal{C}} \subset X^{\mathrm{trans}}$. Since $w_1 \neq w_2$, this is a contradiction to Proposition 4.2 (c). Q.E.D.

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Korea Institute for Advanced Study, Hoegiro 85, Seoul, 02455, Korea
E-mail address: jmhwang@kias.re.kr