Homological stability of $Aut(F_n)$ revisited

Mladen Bestvina

Abstract.

   We give another proof of a theorem of Hatcher and Vogtmann stating that the sequence $Aut(F_n)$ satisfies integral homological stability. The paper is for the most part expository, and we also explain Quillen's method for proving homological stability.

§1.   Introduction

   Let $G_1 \subset G_2 \subset G_3 \subset \cdots$ be a sequence of groups. For example, G_n could be any of the following: the permutation group S_n, or the signed permutation group $S_n^{\pm}$, braid group B_n, $SL_n(\mathbb{Z})$, $Aut(F_n)$, and many other groups, with all inclusions standard. The sequence satisfies *homological stability* if for every r there is $n(r)$ such that for $n \geq n(r)$ inclusion induced $H_r(G_n) \to H_r(G_{n+1})$ is an isomorphism. All of the above sequences satisfy homological stability.

   Homological stability of $Aut(F_n)$ over $\mathbb{Q}$ was proved by Hatcher and Vogtmann by a very elegant argument [5], as follows. First, they show that $Aut(F_n)$ acts properly on an r-connected simplicial complex $S\mathbb{A}_{n,r+1}$, and second, that for $n > 2r$ the quotient spaces $Q_{n,r+1} = S\mathbb{A}_{n,r+1}/Aut(F_n)$ and $Q_{n+1,r+1}$ are *canonically homeomorphic.* Since

$$H_r(Aut(F_n); \mathbb{Q}) = H_r(Q_{n,r+1}; \mathbb{Q})$$
$$= H_r(Q_{n+1,r+1}; \mathbb{Q}) = H_r(Aut(F_{n+1}; \mathbb{Q}))$$

stability follows.

   This is a very transparent reason for stability, and I am not aware of any other example where stability can be proved in this way.

   Question 1. *Can one prove rational homological stability for braid groups or mapping class groups in the same way?*

Integral stability of $Aut(F_n)$ is more subtle. It was first established by Hatcher and Vogtmann in [5] by studying a spectral sequence associated to the action of $Aut(F_n)$ on the complex of "split factorizations" of F_n. Further, it is known, by the work of Hatcher, Vogtmann and Wahl, that $Aut(F_n) \to Out(F_n)$ induces an isomorphism in H_r when $n \geq 2r + 4$, see [4, 5, 6, 7, 3]. The proof is based on Quillen's method and requires a rather delicate spectral sequence argument.

In this note we give a proof of integral stability in the same spirit as Hatcher-Vogtmann's proof of rational stability. We view the quotient spaces $Q_{n,r+1}$ as *orbi spaces* and observe that for $n >> r$ and canonical identification $Q_{n,r+1} = Q_{n+1,r+1}$ of underlying topological spaces, the groups $\Gamma_{n,r+1}(x), \Gamma_{n+1,r+1}(x)$ associated to a point x (i.e. stabilizers of corresponding points in $S\mathbb{A}_{n,r+1}, S\mathbb{A}_{n+1,r+1}$) themselves belong to a sequence of finite groups satisfying homological stability. More precisely, $\Gamma_{n,r}(x) = G_r(x) \times S_{n-2r}^{\pm}$ where $G_r(x)$ does not depend on n, and $S_i^{\pm}$ is the signed permutation group on i symbols. Integral stability of $Aut(F_n)$ easily follows.

We emphasize here that we will use spectral sequences only to prove homological stability for signed permutation groups. The rest of the argument is geometric, in the spirit of Hatcher-Vogtmann [5].

The price we must pay for conceptual transparency is that our stability range is far from optimal. The argument seems to require $n > 4r$ (while the best known estimate is $n \geq 2r + 2$), although it is possible that this may be improved with further effort.

We note that Galatius [2] computed stable homology groups. For a more systematic approach to homological stability of automorphism groups see [17].

Outline. In Section 2 we recall Quillen's method for proving stability. We will only need the simple form where the group acts on a highly connected space with one orbit of cells in a dimension range. We then prove homological stability for permutation groups, following an argument of Maazen, and we give a variant for signed permutation groups. The latter groups naturally appear as subgroups of $Aut(F_n)$ that act as symmetry groups of a rose. The final section elaborates on the outline given above. Instead of working with orbi spaces X/G, we use the Borel construction and consider $X \times_G EG = (X \times EG)/G$ where G acts diagonally on $X \times EG$. This is technically more expedient, but the reader should keep the orbi space picture in mind.

§2. Quillen's method

Here we describe a method, due to Quillen (unpublished), to prove homological stability. For published accounts of Quillen's method see e.g. [16, 15].

We will say that X is *r-connected* if $\tilde{H}_i(X) = 0$ for $i \le r$.

Fix a sequence $G_0 \subset G_1 \subset G_2 \subset \cdots$ of groups. Suppose that $H_i(G_{s-1}) \to H_i(G_s)$ is an isomorphism when the following hold:

- $i = r - 1$ and $s \ge n - 2$, or
- $i < r - 1$ and $s \ge n + i - r - 2$.

Also assume that the group G_n acts on a Δ-complex $X = X(n)$ of dimension $\le n - 1$ with the following properties.

(i) Action is without inversions, i.e. any element that leaves a simplex invariant fixes it pointwise.
(ii) X is *r*-connected.
(iii) in each dimension $0, 1, \cdots, r$ there is one orbit of simplices.
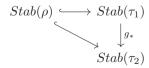(iv) There is a flag of simplices

$$\sigma^0 < \sigma^1 < \cdots < \sigma^r$$

in X such that $Stab(\sigma^i) = G_{n-i-1} \subset G_n$.
(v) If τ_1^i and τ_2^i are two *i*-simplices in X contained in ρ^{i+1} as faces $(i = 0, 1, \cdots, r - 1)$ then there exists $g \in G$ such that:
- $g(\tau_1) = \tau_2$, and
- g commutes with all elements of $Stab(\rho)$.

Remark 2. In view of (iii) and (iv), we have that the stabilizer of *every* *i*-simplex is a conjugate of G_{n-i-1}. Note that conjugation induces identity in the homology of a group. So we have a canonical isomorphism $H_*(Stab(\tau^i)) \cong H_*(Stab(\sigma^i)) = H_*(G_{n-i-1})$ for any *i*-simplex τ (by choosing any $g \in G$ with $g(\tau) = \sigma$ and passing to the isomorphism in homology induced by conjugation $g_* : Stab(\tau) \to Stab(\sigma)$, $h \mapsto ghg^{-1}$; this isomorphism is independent of the choice of g).

Property (v) guarantees that stabilizer inclusions $Stab(\rho) \hookrightarrow Stab(\tau_j)$, $j = 1, 2$, induce the *same* homomorphism $H_*(G_{n-i-2}) \to H_*(G_{n-i-1})$ in homology, after the identifications in the previous paragraph. This follows by considering the diagram

$$
\begin{array}{ccc}
Stab(\rho) & \hookrightarrow & Stab(\tau_1) \\
 & \searrow & \downarrow g_* \\
 & & Stab(\tau_2)
\end{array}
$$

which commutes (by the assumption that g_* fixes $Stab(\rho)$) and passing to homology. By (iv) this is the homomorphism induced by inclusion $G_{n-i-2} \hookrightarrow G_{n-i-1}$.

Proposition 3. *Under the above assumptions $H_r(G_{n-1}) \to H_r(G_n)$ is an isomorphism.*

Proof. Consider the "equivariant homology spectral sequence". This is the spectral sequence associated to the filtration of $Y = X \times_{G_n} EG_n$ coming from the skeleta of X: $Y_p = X^p \times_{G_n} EG_n$. Since X is r-connected we have $H_r(Y) = H_r(G_n)$. The first page is

$$E^1_{p,q} = H_{p+q}(Y_p, Y_{p-1}) = H_q(Stab(\sigma^p)) = H_q(G_{n-p-1})$$

with equalities legal since identifications are up to inner automorphisms, which induce identity in homology. When p is even the differential $E^1_{p+1,q} \to E^1_{p,q}$ is 0 by (v) (a $(p+1)$-simplex has an even number of p-faces, stabilizer inclusions are all standard, half come with positive and half with negative sign). In particular, $E^1_{0,q} = H_q(G_{n-1})$ survives to E^2. Likewise, when p is odd the differential $E^1_{p+1,q} \to E^1_{p,q}$ is the inclusion induced $H_q(G_{n-p-2}) \to H_q(G_{n-p-1})$. A portion of the first page is pictured below. The leftmost column corresponds to $p = 0$ and the top row to $q = r$.

$$H_r(G_{n-1}) \xleftarrow{0} H_r(G_{n-2}) \leftarrow H_r(G_{n-3})$$

$$H_{r-1}(G_{n-1}) \xleftarrow{0} \boxed{H_{r-1}(G_{n-2})} \xleftarrow{\cong} \boxed{H_{r-1}(G_{n-3})} \xleftarrow{0}$$

$$H_{r-2}(G_{n-1}) \xleftarrow{0} H_{r-2}(G_{n-2}) \xleftarrow{\cong} \boxed{H_{r-2}(G_{n-3})} \xleftarrow{0} \boxed{H_{r-2}(G_{n-4})} \xleftarrow{\cong}$$

$$H_{r-2}(G_{n-5})$$

The circled terms are $E^1_{p,r-p}$ and $E^1_{p,r-p+1}$ with $p > 0$ and $q < r$, and they die thanks to our assumption that the differential $d^1 : E^1_{p+1,q} \to E^1_{p,q}$ is an isomorphism when p is odd and it involves at least one of the circled terms. It now follows that the E^2 page has $E^2_{0,r} = H_r(G_{n-1})$ and all terms on the diagonals $p+q = r$ and $p+q = r+1$ with $q < r$ vanish. Thus the same holds for the E^∞ page, and since the spectral sequence converges to $H_{p+q}(Y)$ we have that $H_r(G_{n-1}) = E^\infty_{0,r} \to H_r(Y) = H_r(G_n)$ is an isomorphism, thus proving the Proposition. Q.E.D.

2.1. Stability for (signed) permutation groups

Stability for symmetric groups was established by Nakaoka [13, 14]. We will follow Maazen's proof [11]. See also [10]. We start with a couple of elementary lemmas.

Call a nonempty polyhedron X *n-spherical* if $\tilde{H}_i(X) = 0$ for $i \neq n$.

Lemma 4. *Let X be a polyhedron and $X_i \subset X$ a finite collection of m subpolyhedra covering X, and let $n \geq 0$ be an integer. Suppose that for every $k = 1, 2, \cdots, m$ any k-fold intersection $X_{i_1} \cap \cdots \cap X_{i_k}$ is $(n-k+1)$-spherical (empty for $k \geq n+2$) whenever $i_1 < i_2 < \cdots < i_k$. Then X is n-spherical.*

Proof. Induction on m. If $m = 1$ there is nothing to prove, and if $m = 2$ the statement follows from Mayer-Vietoris. Assume $m > 2$. Let $Y = X_2 \cup X_3 \cup \cdots \cup X_m$. By induction Y is n-spherical and

$$X_1 \cap Y = (X_1 \cap X_2) \cup (X_1 \cap X_3) \cup \cdots \cup (X_1 \cap X_m)$$

is $(n-1)$-spherical (again by induction). Thus the claim follows.   Q.E.D.

Lemma 5. *If X is n-spherical and F is a nonempty finite set then the join $X * F$ is $(n+1)$-spherical.*

Proof. By induction on the cardinality of F. If $|F| = 1$ then $X * F$ is contractible. If $|F| = 2$ then $X * F$ is the suspension and $\tilde{H}_{i+1}(X * F) = \tilde{H}_i(X)$ so the claim follows. When $|F| > 2$ write $X * F$ as the union of two sets whose intersection is X, with one set contractible and the other $(n+1)$-spherical by induction (join of X and the set F without one of the points). Then use Mayer-Vietoris.   Q.E.D.

Proposition 6. *The sequence of symmetric groups S_n satisfies homological stability:*

$$H_r(S_{n-1}) \to H_r(S_n)$$

is an isomorphism for $n > 2r$.

Proof. We argue by induction on r, starting with the obvious $r = 0$. We will apply Proposition 3 with $G_n = S_n$. Note that the fact that $H_i(S_{s-1}) \to H_i(S_s)$ is an isomorphism when $i = r - 1$ and $s \geq n - 2$, or when $i < r - 1$ and $s \geq n + i - r - 2$ follows by induction (for the latter case the calculation is $s \geq n + i - r - 2 > 2r + i - r - 2 = r + i - 2 \geq 2i$). Consider the Δ-complex $X = X(n)$ whose vertices are $1, 2, \cdots, n$ and there is an (ordered) simplex for every ordered subset of the vertex set. So e.g. there are $n(n-1)$ edges etc.

Claim. $X(n)$ is $(n-1)$-spherical.

Proof of Claim. Let X_i be the union of (closed) simplices in X whose first vertex is i. Thus X_i contains all simplices whose vertices don't include i (and it is the cone on this subcomplex). More generally, the k-fold intersection of the X_i's can be identified with the subcomplex (the "base") consisting of simplices that don't involve k particular vertices, with k cones attached to it. Since the base is a copy of $X(n-k)$ it is $(n-k-1)$-spherical by induction. It follows from Lemma 5 that k-fold intersections are $(n-k)$-spherical. Now Lemma 4 implies that $X = X(n)$ is $(n-1)$-spherical.

The verification of (i)-(v) is left to the reader. Thus stability follows.

<div align="right">Q.E.D.</div>

There is an identical proof for the signed permutation group $S_n^{\pm}$ (or the hyperoctahedral group), i.e. the Coxeter group of type B_n. Recall that $S_n^{\pm}$ is the semi-direct product $S_n \ltimes \mathbb{Z}_2^n$ and it can be viewed as the group of permutations π of the set $\{-n, -(n-1), \cdots, -1, 1, \cdots, n-1, n\}$ such that $\pi(-x) = -\pi(x)$ for all x.

Proposition 7 ([8]). *The signed permutation groups satisfy homological stability:*

$$H_r(S_{n-1}^{\pm}) \to H_r(S_n^{\pm})$$

is an isomorphism for $n > 2r$.

Proof. Now let $X = X(n)$ be the Δ-complex with vertex set $-n$, $-(n-1), \cdots, -1, 1, 2, \cdots, n$ and a simplex is an ordered subset with distinct absolute values. The proof that X is $(n-1)$-spherical is the same: take X_i to consist of simplices that start with i or $-i$.   Q.E.D.

Remark 8. There is one more infinite sequence of Weyl groups, namely of type D_n. This is the group of signed permutations with an even number of negative signs, and it has index 2 in $S_n^{\pm}$. One can prove homological stability as above, by considering the action on the same complex as for $S_n^{\pm}$ (there are now two orbits of simplices in the top dimension). For a generalization of this method to other sequences of Coxeter groups see [9].

§3.   Integral homological stability of $Aut(F_n)$

The following is well-known.

Proposition 9. *Let $f : X \to Y$ be a map between spaces equipped with filtrations*

$$\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_m = X$$

and

$$\emptyset = Y_{-1} \subset Y_0 \subset Y_1 \subset \cdots \subset Y_m = Y$$

such that $f(X_i) \subset Y_i$ for all i. If $f_* : H_j(X_i, X_{i-1}) \to H_j(Y_i, Y_{i-1})$ is an isomorphism for all $i = 0, \cdots, m$ and $j \leq k+1$, then $f_* : H_j(X) \to H_j(Y)$ is an isomorphism for $j \leq k$.

Proof. This can be easily proved via spectral sequences, but we will give an elementary proof. By induction on $p = 1, \cdots, m+1$ we will prove that

$$f_* : H_j(X_i, X_{i-p}) \to H_j(Y_i, Y_{i-p})$$

is an isomorphism for $p - 1 \leq i \leq m$ and $j \leq k$. For $p = 1$ this is a hypothesis. For $p > 1$ we consider the long exact sequences of triples (X_i, X_{i-1}, X_{i-p}) and (Y_i, Y_{i-1}, Y_{i-p}), and the map between them induced by f.

$$H_{j+1}(X_i, X_{i-1}) \to \ H_j(X_{i-1}, X_{i-p}) \ \to H_j(X_i, X_{i-p}) \to$$
$$\downarrow \qquad\qquad\qquad \downarrow \qquad\qquad\qquad \downarrow$$
$$H_{j+1}(Y_i, Y_{i-1}) \to \ H_j(Y_{i-1}, Y_{i-p}) \ \to H_j(Y_i, Y_{i-p}) \to$$

$$\to \ H_j(X_i, X_{i-1}) \ \to H_{j-1}(X_{i-1}, X_{i-p})$$
$$\downarrow \qquad\qquad\qquad \downarrow$$
$$\to \ H_j(Y_i, Y_{i-1}) \ \to H_{j-1}(Y_{i-1}, Y_{i-p})$$

The inductive step now follows from the 5-lemma, and the Proposition from the case $p = m + 1$.   Q.E.D.

Proposition 10. *Let (X', X) be a finite dimensional simplicial pair, G' a group and $G < G'$ a subgroup. Suppose that*

- *(i) G' acts on X' without inversions,*
- *(ii) $G < G'$ leaves X invariant,*
- *(iii) both X, X' are k-connected,*
- *(iv) every G'-orbit intersects X,*
- *(v) if two simplices of X are in the same G'-orbit, then they are in the same G-orbit,*
- *(vi) for every simplex $\sigma \in X$ the inclusion $Stab_G(\sigma) \hookrightarrow Stab_{G'}(\sigma)$ induces an isomorphism $H_j(Stab_G(\sigma)) \to H_j(Stab_{G'}(\sigma))$ for $j \leq k - \dim \sigma$.*

Then inclusion $G \hookrightarrow G'$ induces an isomorphism $H_j(G) \to H_j(G')$ for $j \leq k$.

Note that (iv) and (v) say that the induced map $X/G \to X'/G'$ is a homeomorphism, and (vi) says that groups associated to the "same" point in the orbi spaces X/G and X'/G' have the same homology in a range.

Proof. By (iii) we have $H_j(G) = H_j(X \times_G EG)$ and similarly for G', so it suffices to argue that the map

$$X \times_G EG \to X' \times_{G'} EG'$$

induced by f is an isomorphism in H_j for $j \leq k$. We will apply Proposition 9 to this map and the filtrations induced by the skeleta; thus $(X \times_G EG)_i = X^i \times_G EG$ and similarly for the target space. We have

$$H_j((X \times_G EG)_i, (X \times_G EG)_{i-1}) = H_j((X^i, X^{i-1}) \times_G EG)$$
$$= \bigoplus_{\sigma^i \in X/G} H_{j-i}(Stab_G(\sigma^i))$$

with the similar calculation for X'. By (iv) and (v) both sums are over the same set of i-simplices in $X/G = X'/G'$, and by (vi) homology groups are equal. Q.E.D.

3.1. Review of the Degree Theorem

Here we review the Hatcher-Vogtmann Degree Theorem [5]. For a simpler proof of this theorem see [12]. Let $S\mathbb{A}_n$ denote the *spine of reduced Auter space in rank n*. This is a simplicial complex whose vertices are basepointed marked graphs (Γ, v_0, ϕ) where:

- Γ is a finite connected graph (i.e. a 1-dimensional cell complex) without separating edges,
- $v_0 \in \Gamma$ is a base vertex, it has valence > 1, and all other vertices have valence > 2,
- $\phi : F_n \to \pi_1(\Gamma, v_0)$ is an isomorphism (called a *marking*).

Two triples (Γ, v_0, ϕ) and (Γ', v_0', ϕ') represent the same vertex of $S\mathbb{A}_n$ if there is a basepoint-preserving graph isomorphism $I : \Gamma \to \Gamma'$ with $\phi' = I_*\phi$.

We write $(\Gamma, v_0, \phi) > (\Gamma', v_0', \phi')$ if there is a forest (subgraph with contractible components) $F \subset \Gamma$ so that (Γ', v_0', ϕ') is equivalent to the graph obtained from Γ by collapsing all components of F, with the induced base vertex and marking.

The simplicial complex $S\mathbb{A}_n$ is the poset of this order relation, i.e. a simplex is an ordered chain. It is contractible [1]. The group $Aut(F_n)$ acts on $S\mathbb{A}_n$ by precomposing the marking and the action is proper without inversions. The stabilizer of a vertex is equal to the symmetry group of the underlying basepointed graph.

The *degree* of a basepointed graph (Γ, v_0) is the sum

$$deg(\Gamma) = \sum_{v \neq v_0} |v| - 2$$

where $|v|$ denotes the valence of v and the sum runs over all vertices of Γ distinct from v_0. Denote by $S\mathbb{A}_{n,k+1}$ the subcomplex of $S\mathbb{A}_n$ spanned by graphs with degree $\leq k+1$. This subcomplex is $Aut(F_n)$-invariant. Note also that collapsing a forest cannot increase the degree.

Theorem 11 ([5]). *$S\mathbb{A}_{n,k+1}$ is k-connected.*

Lemma 12 ([5]). *(a)   If $n > 2k+2$ then every (Γ, v_0) of degree $\leq k+1$ has a loop at v_0.*
(b)   If $n - m + 1 > 2k + 2$ then every (Γ, v_0) of degree $\leq k+1$ has m loops at v_0.
(c)   If $n > 4k$ then every (Γ, v_0) of degree $\leq k+1$ has $2k-1$ loops at v_0.

3.2.   The stability theorem

Theorem 13. *$H_k(Aut(F_n)) \to H_k(Aut(F_{n+1}))$ is an isomorphism for $n > 4k$.*

Proof. We will use Proposition 10. Let $X = S\mathbb{A}_{n,k+1}$, $X' = S\mathbb{A}_{n+1,k+1}$ with the standard actions of $G = Aut(F_n)$ and $G' = Aut(F_{n+1})$. We define a natural equivariant embedding $X \hookrightarrow X'$ as follows. Write $F_{n+1} = F_n * \langle t \rangle$ so that $Aut(F_n)$ is identified with the subgroup of $Aut(F_{n+1})$ that preserves F_n and t. A vertex of $S\mathbb{A}_{n,k+1}$ is a triple (Γ, v_0, ϕ) and we map it to the vertex of $S\mathbb{A}_{n+1,k+1}$ given by $(\Gamma \vee S^1, v_0, \phi')$. The wedge here is at the basepoint v_0, and $\phi' : F_n * \langle t \rangle \to \pi_1(\Gamma', v_0) = \pi_1(\Gamma, v_0) * \pi_1(S^1, v_0)$ is ϕ on the first factor and an isomorphism on the second, and we simply write $\phi' = \phi * id$ (there are two possible isomorphisms on the second factor, but either choice defines the same point in $S\mathbb{A}_{n+1,k+1}$).

This map on the vertices extends to a simplicial equivariant embedding

$$S\mathbb{A}_{n,k+1} \hookrightarrow S\mathbb{A}_{n+1,k+1}$$

The first three properties from Proposition 10 are clear. Property (iv) follows from Lemma 12. E.g. start with a vertex $(\Gamma', v_0, \phi') \in S\mathbb{A}_{n+1,k+1}$. Since $n + 1 > 2k + 2$ the graph Γ' has the form $\Gamma' = \Gamma \vee S^1$, and after precomposing the marking, ϕ' has the form $\phi * id$, so the $Aut(F_{n+1})$-orbit of every vertex intersects $S\mathbb{A}_{n,k+1}$. A similar argument works for any simplex in $S\mathbb{A}_{n+1,k+1}$. Say the simplex is obtained from (Γ', v_0, ϕ') by collapsing a sequence of forests. We can again write $\Gamma' = \Gamma \vee S^1$, change the marking, and observe that all the forests are contained in Γ. Property (v) is easy.

Finally, we argue (vi). We start with a vertex $(\Gamma, v_0, \phi) \in S\mathbb{A}_{n,k+1}$. Write Γ as $\Gamma = \Gamma_0 \vee R_m$ where R_m denotes the wedge of m circles,

and m is maximal possible. According to Lemma 12, $m \geq 2k - 1$. The key point is that the symmetry group of (Γ, v_0) is the direct product of symmetry groups of (Γ_0, v_0) and (R_m, v_0), and the latter one is the signed permutation group $S_m^{\pm}$. Thus we have

$$Stab_{Aut(F_n)}(\Gamma, v_0, \phi) \cong D \times S_m^{\pm}$$

and

$$Stab_{Aut(F_{n+1})}(\Gamma \vee S^1, v_0, \phi * id) \cong D \times S_{m+1}^{\pm}$$

where D is the symmetry group of (Γ_0, v_0). So we need to argue that

$$D \times S_m^{\pm} \hookrightarrow D \times S_{m+1}^{\pm}$$

induces an isomorphism in $H_{\leq k}$. This holds for $S_m^{\pm} \hookrightarrow S_{m+1}^{\pm}$ by Proposition 7, and in general by the Künneth formula.

The argument for a simplex is similar.   Q.E.D.

Question 14. *Can the stability range be improved using the same method? E.g. investigate what happens when theta graphs are wedged at the basepoint, as in* [5].

References

[1] Marc Culler and Karen Vogtmann. Moduli of graphs and automorphisms of free groups. *Invent. Math.*, Vol. 84, No. 1, pp. 91–119, 1986.

[2] Søren Galatius. Stable homology of automorphism groups of free groups. *Ann. of Math. (2)*, Vol. 173, No. 2, pp. 705–768, 2011.

[3] Allan Hatcher, Karen Vogtmann, and Natalie Wahl. Erratum to: "Homology stability for outer automorphism groups of free groups" by Hatcher and Vogtmann. *Algebr. Geom. Topol.*, Vol. 6, pp. 573–579 (electronic), 2006.

[4] Allen Hatcher. Homological stability for automorphism groups of free groups. *Comment. Math. Helv.*, Vol. 70, No. 1, pp. 39–62, 1995.

[5] Allen Hatcher and Karen Vogtmann. Cerf theory for graphs. *J. London Math. Soc. (2)*, Vol. 58, No. 3, pp. 633–655, 1998.

[6] Allen Hatcher and Karen Vogtmann. Rational homology of $Aut(F_n)$. *Math. Res. Lett.*, Vol. 5, No. 6, pp. 759–780, 1998.

[7] Allen Hatcher and Karen Vogtmann. Homology stability for outer automorphism groups of free groups. *Algebr. Geom. Topol.*, Vol. 4, pp. 1253–1272, 2004.

[8] Allen Hatcher and Nathalie Wahl. Stabilization for mapping class groups of 3-manifolds. *Duke Math. J.*, Vol. 155, No. 2, pp. 205–269, 2010.

[9] Richard Hepworth. Homological stability for families of Coxeter groups. arXiv:1502.03128.

[10] Moritz C. Kerz. The complex of words and Nakaoka stability. *Homology Homotopy Appl.*, Vol. 7, No. 1, pp. 77–85, 2005.

[11] Hendrik Maazen. *Homology Stability for the General Linear Group*. PhD thesis, Utrecht, 1979.

[12] Robert McEwen and Matthew C. B. Zaremsky. A combinatorial proof of the degree theorem in Auter space. *New York J. Math.*, Vol. 20, pp. 217–228, 2014.

[13] Minoru Nakaoka. Decomposition theorem for homology groups of symmetric groups. *Ann. of Math. (2)*, Vol. 71, pp. 16–42, 1960.

[14] Minoru Nakaoka. Homology of the infinite symmetric group. *Ann. of Math. (2)*, Vol. 73, pp. 229–257, 1961.

[15] Wilberd van der Kallen. Homology stability for linear groups. *Invent. Math.*, Vol. 60, No. 3, pp. 269–295, 1980.

[16] J. B. Wagoner. Equivalence of algebraic K-theories. *J. Pure Appl. Algebra*, Vol. 11, No. 1–3, pp. 245–269, 1977/78.

[17] Nathalie Wahl. Homological stability for automorphism groups. arXiv:1409.3541.

Department of Mathematics
University of Utah
155 S 1400 E, Rm 233
Salt Lake City, UT 84112-0090
USA
E-mail address: `bestvina@math.utah.edu`