Advanced Studies in Pure Mathematics 72, 2017
Geometry, Dynamics, and Foliations 2013
pp. 349–365

Godbillon–Vey invariants for maximal isotropic C^2 foliations

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Abstract.

For a contact manifold (M^{2m+1}, A) and an $m+1$-dimensional dA-isotropic C^2 foliation, we define *Godbillon–Vey invariants* $\{GV_i\}_{i=0}^{m+1}$ inspired by the Godbillon–Vey invariant of a codimension-one foliation, and we demonstrate the potential of this family as a tool in geometric rigidity theory. One ingredient for the latter is the Mitsumatsu formula for geodesic flows on (Finsler) surfaces.

§1. Introduction

The Godbillon–Vey invariant of a codimension-one foliation is a subtle geometric invariant of a foliation that has been of enduring interest since its inception in 1970 [9]. We show that the combination of a contact structure and a transversely orientable maximal isotropic foliation gives rise to a *sequence* of what we call Godbillon–Vey invariants that are of interest with respect to geometric rigidity. We choose the name because when the foliation is invariant under the geodesic flow of a Finsler surface, the second of these invariants coincides with the classical Godbillon–Vey invariant; we establish this by deriving a formula that reduces to the very Mitsumatsu formula for the classical Godbillon–Vey invariant in the case of the weak-unstable foliation of the geodesic flow of a negatively curved Riemannian metric [17].

Unlike in the classical case, we produce $m + 2$ invariants for an $m + 1$-dimensional maximal isotropic foliation, and our point is that as

Received January 29, 2014.
Revised March 31, 2015.
2010 *Mathematics Subject Classification.* Primary 57D30; Secondary 57R30, 53D10, 37D20.
Key words and phrases. Godbillon–Vey invariants, contact flow, Anosov flow, Mitsumatsu formula, rigidity.
Boris Hasselblatt was partially supported by the Committee on Faculty Research Awards of Tufts University.

a *sequence* they are of interest for rigidity results. Additional evidence of this interest is due to Gomes and Ruggiero, who proved a version of the Hopf conjecture using the Godbillon–Vey invariant for Finsler surfaces [10] (in a C^2 context and assuming isotropic flag curvature). Further, Paternain [19, Section 3] uses the Godbillon–Vey invariant for the invariant foliations in Anosov thermostats.

This paper presents our construction under the assumption that the foliations in question are C^2. The regularity of the invariant foliations of an Anosov flow is a subtle matter. On one hand, if the foliation is C^3 and invariant under the geodesic flow of a (Riemannian) surface, then the curvature of the metric is constant. On the other hand, for the geodesic flow of a surface with nonconstant negative Gauß curvature the weak-stable subbundle is not C^2, and in some interesting applications the desired conclusions are known consequences of the foliation in question being C^2. Therefore we will elsewhere present this construction for foliations that are $C^{1+1/2+\epsilon}$. We are nonetheless able to produce applications that illustrate the interest of this construction.

Specifically, if (M, A) is a contact manifold of dimension $2m+1$ and $\mathcal{F}$ a C^2 maximal isotropic foliation, we define *Godbillon–Vey invariants* GV_i for $i = 0, \dots, m+1$ in Definition 3.10. GV_0 is the volume of the manifold, and for a contact Anosov flow and the associated weak-stable foliation, GV_1 is the Liouville entropy (Proposition 4.2). We also show that for geodesic flows of Finsler surfaces, GV_2 is the classical Godbillon–Vey invariant by establishing the Mitsumatsu formula (Proposition 5.2).

We demonstrate the interest of these invariants for geometric rigidity as follows. Theorem 6.1 establishes that if the Margulis measure of a contact Anosov flow is absolutely continuous, then $GV_i = h^i \operatorname{vol}_A(M)$, where h denotes the topological entropy. While this result is contingent on the C^2 assumption for the foliation in question, it applies without restriction to geodesic flows of locally symmetric spaces of negative curvature. For a negatively curved Riemannian metric on a surface we then show, *with no effort*, that if $GV_0 = c$, $GV_1 = hc$, and $GV_2 = h^2c$, then the curvature is constant, c is the volume, and h is the topological entropy of the geodesic flow (Theorem 6.3).

The main rigidity results for which we present new proofs here do not mention these invariants in their statement and are of such a nature that the invariants need only be known for C^2 foliations. They are known from the work of Ghys [7] and Hurder–Katok [12], but our approach provides for an astonishingly simple main step—it consists of applying the Cauchy–Schwarz inequality.

Theorem 6.5. *Two Riemannian surfaces with topologically conjugate geodesic flows are isometric if one of them has constant curvature.*

While we invoke known results to establish that the conjugacy is smooth, the nontrivial step from there is now a simple application of Proposition 6.4, which also underlies Theorem 6.3.

Another application of this central result is:

Theorem 6.7. *Negatively curved surfaces with C^2 horospheric foliations are constantly curved.*[1]

We repeat that the common base of these applications is Proposition 6.4, and we wish to point out that we see promise in our *family* of invariants because this core result is so easy to prove—it only uses the Cauchy–Schwarz inequality and the Riemannian Mitsumatsu formula.

§2. The classical Godbillon–Vey invariant

For context we briefly present the classical Godbillon–Vey class and invariant following [21]; see also [8] for further context. If ω is a completely integrable nonsingular 1-form, then there is a 1-form η such that $d\omega = \omega \wedge \eta$ (Frobenius theorem), and $0 = d\,d\omega = d(\omega \wedge \eta) = \omega \wedge d\eta$, so there is a 1-form ξ with $d\eta = \omega \wedge \xi$, hence $\eta \wedge d\eta$ is closed, and its de Rham cohomology class is independent of such choice of η—another choice must be of the form $\eta' = \eta + u\omega$ for a function u, and then $\eta' \wedge d\eta' = \eta \wedge d\eta + d(u\,d\omega)$. Indeed, this depends only on the codimension-one foliation $\mathcal{F}$ defined by complete integrability of ω, because any ω' defining the same foliation is of the form $e^u\omega$. The cohomology class of $\eta \wedge d\eta$ is called the Godbillon–Vey class of $\mathcal{F}$, and if $\dim M = 3$, then $\int \eta \wedge d\eta$ is called the Godbillon–Vey invariant of $\mathcal{F}$; it is a characteristic class, depends only on the foliated cobordism class of $(M, \mathcal{F})$, is nontrivial, and varies continuously and nontrivially with $\mathcal{F}$.

§3. Definition of the Godbillon–Vey invariants

3.1. Contact forms

Definition 3.1. *Let M be a manifold of dimension $2m + 1$. A* contact form *on M is a 1-form A such that $A \wedge dA \wedge \cdots \wedge dA$ (with m factors of dA) is a volume. A subspace $V \subset T_xM$ is said to be* isotropic *if $dA_x\restriction_V = 0$ and* maximal isotropic *if it furthermore has dimension*

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[1] This result provides additional motivation for our forthcoming definition of these invariants when the foliations are not C^2.

$m+1$. A subbundle is said to be (maximal) isotropic if it is so at each point, and a foliation is (maximal) isotropic if its tangent bundle is so.

3.2. Normal bundle

If M is a smooth manifold and F a subbundle of TM, then we let

$$\mathcal{N}_n(F) := \left\{\text{sections } \omega \text{ of } \bigwedge^n T^*M \text{ with } \iota_\xi\omega = 0 \text{ whenever } \xi \in F\right\}.$$

Lemma 3.2. *If F is integrable, $\omega \in \mathcal{N}_k(F)$, and $Z \in F$, then $\iota_Z\, d\omega \in \mathcal{N}_k(F)$. In particular, if $\omega \in \mathcal{N}_1(F)$, then $d\omega\restriction_F = 0$.*

Proof. If $Z_0 \in F$ and $\omega \in \mathcal{N}_k(F)$, then (with "ˇ" denoting omission)

$$\begin{aligned} d\omega(Z_0,\dots,Z_k) &= \sum_{i=0}^{k}(-1)^i \mathscr{L}_{Z_i}\omega(Z_0,\dots,\check{Z}_i,\dots,Z_n) \\ &\quad + \sum_{i<j}(-1)^{i+j}\omega([Z_i,Z_j],Z_0,\dots,\check{Z}_i,\dots,\check{Z}_j,\dots,Z_n) \\ &= \mathscr{L}_{Z_0}\omega(Z_1,\dots,Z_n) \\ &\quad + \sum_{j=1}^{k}(-1)^j\omega([Z_0,Z_j],Z_1,\dots,\check{Z}_j,\dots,Z_n), \end{aligned}$$

where $\mathscr{L}$ is the Lie derivative. If $Z_l \in F$ for some $l > 0$, then $[Z_0, Z_l] \in F$ by integrablility, so each term vanishes. Q.E.D.

3.3. Transverse volume class

We define the Godbillon–Vey invariants in terms of a volume transverse to the maximal isotropic foliation $\mathcal{F}$ (with tangent bundle F) in question. Specifically, since we assume F to be transversely orientable, we henceforth fix an everywhere nonzero $\alpha \in \mathcal{N}_m(F)$.

Proposition 3.3. *If α is C^1, then $d\alpha = \beta\wedge\alpha$ for some 1-form β.*

Proof. $\mathcal{N}_m(F)$ has rank 1 and contains both α and $\iota_Z d\alpha$ for any $Z \in F$ (Lemma 3.2), so the fact that α vanishes nowhere yields a $\beta(Z)$ for which $\iota_Z\, d\alpha = \beta(Z)\alpha$, and β is a 1-form on F. Now consider an extension of β to any 1-form. Then $\beta\wedge\alpha$ and $d\alpha$ can be evaluated by decomposing each argument with respect to a local frame that contains a frame of F. On $(m+1)$-tuples of members of this frame that contain a section of F, we get $d\alpha = \beta\wedge\alpha$ from above. On $(m+1)$-tuples that do not, we get $d\alpha = 0 = \beta\wedge\alpha$ by dimension-counting. Q.E.D.

Remark 3.4. That β is uniquely defined on F means that it is well-defined modulo $\mathcal{N}_1(F)$, i.e., we defined $[\beta] := \{\beta + \omega \mid \omega \in \mathcal{N}_1(F)\}$.

Proposition 3.5. $[[\beta]] := \{\beta + df + \omega \mid f\colon M \to \mathbb{R},\ \omega \in \mathcal{N}_1(F)\}$ *is well-defined independently of the choice of* α: $\alpha' = e^f\alpha$ *with* $f\colon M \to \mathbb{R}$ *produces* $\beta' = \beta + df$.

Proof. $\beta' \wedge \alpha' = d\alpha' = d(e^f\alpha) = de^f \wedge \alpha + e^f d\alpha = e^f\, df \wedge \alpha + e^f\beta \wedge \alpha = (df + \beta) \wedge e^f\alpha = (df + \beta) \wedge \alpha'$. Q.E.D.

Lemma 3.6. $d\beta \wedge \alpha = 0$ *and* $d\beta\restriction_F = 0$.

Proof. As to the first claim,

$$0 = d\,d\alpha = d\beta \wedge \alpha - \beta \wedge d\alpha = d\beta \wedge \alpha - \beta \wedge \beta \wedge \alpha = d\beta \wedge \alpha. \tag{1}$$

If $Z_1, Z_2 \in F$, then $0 = \iota_{Z_1}\iota_{Z_2}0 = \iota_{Z_1}\iota_{Z_2}\, d\beta \wedge \alpha = d\beta(Z_1, Z_2)\alpha$ because $\alpha \in \mathcal{N}_m(F)$. Then $d\beta(Z_1, Z_2) = 0$ because α is nowhere zero. Q.E.D.

Remark 3.7. That $\alpha \in C^2$ is used for "$0 = d\,d\alpha \cdots$" in (1). Our ongoing work replaces this with a weak counterpart (in the distributional sense) because there are interesting applications outside of the C^2 context.

Remark 3.8. The content of Proposition 3.5 and Lemma 3.6 is that we have properly defined a leafwise cohomology class (represented by β). This notion is defined as follows. Let $\Omega^*(M)$ be the set of C^∞ differential forms on M and set

$$I(F) := \{\omega \in \Omega^*(M) \mid \omega\restriction_F = 0\}.$$

Then, $(\Omega^*(M)/I(F), d)$ is a well-defined cochain complex of which the cohomology is denoted by $H^*(F)$ (or $H^*(\mathcal{F})$). While $I(F)$ is different from $\mathcal{N}_n(F)$, we use here that $I(F) \cap \Omega_1(M) = \mathcal{N}_1(F)$, or, more specifically, that $d\omega \in I(F)$ if $\omega \in N_1(F)$. Since $d\beta\restriction_F = 0$ (Lemma 3.6), the differential form β represents a class in $H^1(F)$.

For extending this work to foliations of lower regularity than C^2, this leafwise cohomology will play an important role, and we will use distributional (or "weak") calculus, that is, currents (or measures).

Lemmas 3.2 and 3.6 give Proposition 3.11 via:

Lemma 3.9. $\omega \wedge (d\beta)^i \wedge d\omega^{p-i} \wedge dA^{m-p} = 0$ *for* $\omega \in \mathcal{N}_1(F)$.

Proof. Decomposing $2m+1$ linearly independent arguments with respect to a local frame that contains a frame for F gives a linear combination of expressions each of which contains at least $m+1$ sections of F. Evaluating $\omega \in \mathcal{N}_1(F)$ on a section of F gives 0, so in each such expression $m+1$ sections of F must be inserted into $(d\beta)^i \wedge d\omega^{p-i} \wedge dA^{m-p}$, and we get 0 because more than one section of F is inserted into $d\beta$ (Lemma 3.6), $d\omega$ (Lemma 3.2), or dA (isotropy). Q.E.D.

3.4. Godbillon–Vey invariants

Since $[[\beta]]$ is intrinsically defined, we can define:

Definition 3.10 (Godbillon–Vey invariants)**.** *If* (M, A) *is a closed contact manifold of dimension* $2m+1$ *and* $\mathcal{F}$ *a* C^2 *maximal isotropic foliation, define the* Godbillon–Vey invariants *by*

$$\begin{aligned}
GV_0 &= \int_M A \wedge dA^m =: \mathrm{vol}_A(M) \quad \textit{(the contact volume)},\\
GV_1 &= \int_M \beta \wedge dA^m,\\
GV_2 &= \int_M \beta \wedge d\beta \wedge dA^{m-1},\\
&\ \vdots\\
GV_{m+1} &= \int_M \beta \wedge d\beta^m.
\end{aligned}$$

Proposition 3.11. *The Godbillon–Vey invariants are well-defined.*

Proof. We check that $GV_{p+1} = \int_M \beta \wedge d\beta^p \wedge dA^{m-p}$ is unchanged when replacing β by $\beta + df + \omega$ and therefore $d\beta$ by $d\beta + d\omega$ for any $f\colon M \to \mathbb{R}$ and $\omega \in \mathcal{N}_1(F)$. First, we replace β by $\beta + df$:

$$\begin{aligned}
&\int_M (\beta + df) \wedge d(\beta + df)^p \wedge dA^{m-p} - \int_M \beta \wedge d\beta^p \wedge dA^{m-p}\\
&= \int_M df \wedge d\beta^p \wedge dA^{m-p} = \int_M d(f \cdot d\beta^p \wedge dA^{m-p}) = 0.
\end{aligned}$$

Replacing β by $\beta + \omega$ does not change GV_{p+1} either:

$$\begin{aligned}
&(\beta + \omega) \wedge d(\beta + \omega)^p \wedge dA^{m-p}\\
&\qquad = \beta \wedge d(\beta + \omega)^p \wedge dA^{m-p} + \omega \wedge d(\beta + \omega)^p \wedge dA^{m-p}\\
&[\text{Lemma 3.9}] \qquad = \beta \wedge d(\beta + \omega)^p \wedge dA^{m-p}\\
&\qquad = \beta \wedge d\beta^p \wedge dA^{m-p} + \sum_{i=1}^{p} c_i \beta \wedge d\beta^{p-i} \wedge d\omega^i \wedge dA^{m-p},
\end{aligned}$$

for suitable constants c_i, and the summands integrate to 0:

$$\begin{aligned} 0 &= \int_M d(\omega \wedge \beta \wedge d\beta^{p-i} \wedge d\omega^{i-1} \wedge dA^{m-p}) \\ &= \int_M \beta \wedge d\beta^{p-i} \wedge d\omega^i \wedge dA^{m-p} - \int_M \omega \wedge d\beta^{p-i+1} \wedge d\omega^{i-1} \wedge dA^{m-p} \\ &= \int_M \beta \wedge d\beta^{p-i} \wedge d\omega^i \wedge dA^{m-p} \end{aligned}$$

by Lemma 3.9. Q.E.D.

3.5. The Reeb field

Proposition 3.12. $GV_1 = \int \beta(X) A \wedge dA^m$, *where* X *is the* Reeb field *of* A *defined uniquely by* $\iota_X A = 1$ *and* $\iota_X\, dA = 0$.

Proof. $\iota_X\, dA = 0$ implies that

- X is tangent to $\mathcal{F}$, and $\mathcal{F}$ is therefore invariant under the flow generated by X,
- by duality there are a vector field η and a function λ with $\beta = \lambda A + \iota_\eta\, dA$,
- inserting X gives $\lambda = \beta(X)$, and
- $\iota_X(\iota_\eta\, dA \wedge dA^m) = 0$.

Thus $\beta \wedge dA^m = \beta(X) A \wedge dA^m + \iota_\eta\, dA \wedge dA^m = \beta(X) A \wedge dA^m$. Q.E.D.

§4. GV_1 for Anosov flows

Some of the interest of the Godbillon–Vey invariants lies in applications to dynamical systems, and the canonical dynamical system associated with the present context is the Reeb flow of A, which is the flow φ generated by the Reeb vector field X of A. It is said to be a *contact Anosov flow* if the tangent bundle TM splits as $TM = \mathbb{R}X \oplus E^+ \oplus E^-$ (the *flow*, *strong-unstable* and *strong-stable directions*, respectively) in such a way that there are constants $C > 0$ and $\eta > 1 > \lambda > 0$ with

$$\|D\varphi^{-t}\restriction_{E^+}\| \le C\eta^{-t} \quad \text{and} \quad \|D\varphi^{t}\restriction_{E^-}\| \le C\lambda^{t}$$

for all $t > 0$. The *weak-unstable* and *weak-stable* bundles are $\mathbb{R}X \oplus E^+$ and $F := \mathbb{R}X \oplus E^-$, respectively. F is integrable to a *continuous* foliation $\mathcal{F}$ with *smooth* leaves, called the weak-stable foliation, and our main results below assume that it is C^2, by which we mean here that its tangent bundle is C^2; this implies the existence of C^2 foliation charts.

In this case the weak-stable foliation is maximal isotropic:

Lemma 4.1. $dA\restriction_F = 0$, *i.e.*, $dA(Z_1, Z_2) = 0$ *if* $Z_1, Z_2 \in \mathbb{R}X \oplus E^-$.

Proof. $\iota_X\, dA = 0$ reduces this to the case $Z_1, Z_2 \in E^-$, where

$$dA(Z_1, Z_2) = dA(d\varphi^t(Z_1), d\varphi^t(Z_2)) \xrightarrow[t\to+\infty]{} 0$$

since A, hence dA, is φ^t-invariant and $\|d\varphi^t(Z_i)\| \xrightarrow[t\to+\infty]{} 0$. Q.E.D.

Proposition 4.2. *If the weak-stable foliation of a contact Anosov flow is C^2, then $GV_1 = h_{\mathrm{vol}} \operatorname{vol}_A(M)$, where h_{vol} is Liouville entropy.*

Proof. Choose $\beta = 0$ on E^+. Then $\mathscr{L}_X \alpha = \iota_X\, d\alpha = \beta(X)\alpha$, i.e., $\beta(X)$ is the infinitesimal relative change of the unstable volume under the flow. Rescale A so $\operatorname{vol}_A(M) = 1$. Then the time average of $\beta(X)$, hence by ergodicity of contact Anosov flows its space average GV_1, equals the sum of the positive Lyapunov exponents. By the Pesin entropy formula this is h_{vol}. Q.E.D.

Remark 4.3. Proposition 4.2 uses "Anosov" in a substantial way. While not necessary for the Pesin entropy formula, it is essential for connecting $\mathcal{F}$ to the positive exponents.

§5. The top Godbillon–Vey invariant in dimension 3

GV_2 is the classical Godbillon–Vey invariant for geodesic flows of Finsler surfaces Σ [1] whose weak-stable foliation is C^2. As mentioned, this is also in [10], and related work is in [19]. Here, $\dim M = \dim S\Sigma = 3$ gives $\operatorname{rank} E^\pm = 1$. The geodesic flow is the Reeb flow of a 1-form A whose kernel is spanned by the standard vertical vector field V and by $H := [V, X]$. Both H and X are horizontal, and

$$1 = A \wedge dA(X, V, H) = A(X)\, dA(V, H) = dA(V, H). \tag{2}$$

Then $[H, V] = X + aV + bH$, $[X, H] = KV$ (structural equations), where K is the curvature, a the *Landsberg scalar* and b the *Cartan scalar*, and $a = \dot b := \mathscr{L}_X b$ and $\mathscr{L}_X a =: \dot a = -\mathscr{L}_V K - bK$ (Bianchi identity) since

(3)
$$\begin{aligned}(\mathscr{L}_V K)V &= -[H, H] + [V, KV] = [[X, V], H] + [V, [X, H]] = [X, [V, H]]\\ &= -[X, X + aV + bH] = -\dot a V - a[X, V] - \dot b H - b[X, H]\\ &= (-\dot a - bK)V + (a - \dot b)H.\end{aligned}$$

In the Riemannian case the "Finsler defect" $\Xi := X - [H, V]$ is 0. With $\gamma := \iota_\Xi\, dA$ and $\rho := -\mathscr{L}_V a - \mathscr{L}_H b = -2\mathscr{L}_V a + \mathscr{L}_X \mathscr{L}_V b$ we get

Claim 5.1. *$d\gamma = \rho\, dA + (\mathscr{L}_V K) A \wedge \iota_V\, dA$, so $\gamma \wedge d\gamma = -b \mathscr{L}_V K A \wedge dA$.*

Proof. By (2), $\gamma(\Xi) = 0 = \gamma(X)$, $\gamma(V) = b$, $\gamma(H) = -a$, so

$$\begin{aligned}
d\gamma(X,V) &= \mathscr{L}_X\gamma(V) - \mathscr{L}_V\gamma(X) - \gamma([X,V]) = \dot{b} - a = 0\\
d\gamma(X,H) &= \mathscr{L}_X\gamma(H) - \mathscr{L}_H\gamma(X) - \gamma([X,H]) = -\dot{a} - bK = \mathscr{L}_V K\\
d\gamma(V,H) &= \mathscr{L}_V\gamma(H) - \mathscr{L}_H\gamma(V) - \gamma([V,H]) = -\mathscr{L}_V a - \mathscr{L}_H b = \rho.
\end{aligned}$$

Q.E.D.

If the invariant line bundle $F \cap \ker A$ is spanned by

$$\xi = uV + H, \tag{4}$$

then $\dot{u}+u^2+K = 0$, the Riccati equation, because comparing coefficients in $(\dot{u}+K)V - uH = [X,\xi] = f\xi = fuV + fH$ implies $f = -u$ and $-u^2 = fu = \dot{u} + K$.

Proposition 5.2 (Mitsumatsu formula)**.** *For maximal isotropic C^2 foliations[2] invariant under geodesic flows of Finsler surfaces*

$$GV_2 = \int_M (u^2 + 3(\mathscr{L}_V u)^2 + (4u\mathscr{L}_V a - 2u\mathscr{L}_X\mathscr{L}_V b - b\mathscr{L}_V K))A \wedge dA.$$

Remark 5.3. This reduces to $GV_2 = \int_M (u^2 + 3(\mathscr{L}_V u)^2)A \wedge dA$ in the Riemannian case, and the relationship can be recast via Lemma 5.6 below, which gives $\beta = \underbrace{\mathscr{L}_V u \iota_V\, dA - uA}_{=:\beta_0,\text{“Riemannian”}} + \gamma$, so

$$\begin{aligned}
\int_M \beta \wedge d\beta &= \int_M (\beta_0 + \gamma) \wedge d(\beta_0 \wedge \gamma)\\
&= \int_M \beta_0 \wedge d\beta_0 + \gamma \wedge d\beta_0 + \beta_0 \wedge d\gamma + \gamma \wedge d\gamma\\
&= \int_M \beta_0 \wedge d\beta_0 + (2\beta_0 + \gamma) \wedge d\gamma.
\end{aligned}$$

The first part of the integrand corresponds to the classical Riemannian Mitsumatsu formula, and $(2\beta_0 + \gamma) \wedge d\gamma$ is the Finsler defect.

Remark 5.4. We note that $\int_M \rho\, A \wedge dA = \int_M \mathscr{L}_X\mathscr{L}_V b + 2abA \wedge dA$ by (5) below with $(p,q) = (1,a)$. Since $\mathscr{L}_V(A \wedge dA) = bA \wedge dA$ (derived just before (5)) gives $\mathscr{L}_X(b\mathscr{L}_V(A \wedge dA)) = \mathscr{L}_X(b^2 A \wedge dA) = (\mathscr{L}_X b^2)A \wedge$

[2] see Remark 5.7

$dA = 2\dot{b}bA \wedge dA = 2abA \wedge dA$, we have

$$\begin{aligned}0 &= \int_M \mathscr{L}_X(\mathscr{L}_V(bA \wedge dA)) = \int_M \mathscr{L}_X((\mathscr{L}_V b)A \wedge dA + b\mathscr{L}_V(A \wedge dA))\\ &= \int_M \mathscr{L}_X\mathscr{L}_V b + 2ab\, A \wedge dA = \int_M \rho\, A \wedge dA.\end{aligned}$$

If $\mathscr{L}_V K \equiv 0$, then $[X, \Xi] = 0$ by (3), and hence $\mathscr{L}_X\gamma = \mathscr{L}_X(\iota_\Xi\, dA) = \iota_\Xi\mathscr{L}_X\, dA + \iota_{[X,\Xi]}\, dA = 0$, so

$$(\mathscr{L}_X\rho)\, dA = (\mathscr{L}_X\rho)\, dA + \rho\mathscr{L}_X\, dA = \mathscr{L}_X(\rho\, dA) = \mathscr{L}_X d\gamma = d\mathscr{L}_X\gamma = 0$$

by Claim 5.1, and ρ is flow-invariant. If the flow is ergodic, then $\rho \equiv \int \rho\, A \wedge dA = 0$, so $\int u\rho\, A \wedge dA = 0$, and the Mitsumatsu formula reduces to its Riemannian form. Ergodicity holds for any contact Anosov flow on a connected manifold.

The condition $\mathscr{L}_V K \equiv 0$ means that the flag curvature is isotropic; this is also referred to as the k-basic case and constitutes an intermediate generalization of the Riemannian case. Thus we just observed that the Mitsumatsu formula in the Riemannian case holds more generally in the k-basic case if the flow is ergodic, e.g., Anosov.[3]

The deviation $\int_M (3(\mathscr{L}_V u)^2 - (2u\rho + b\mathscr{L}_V K))A \wedge dA$ of GV_2 from its value for constant curvature is called the *Mitsumatsu defect.*

Lemma 5.5. *If we choose* ξ *as in* (4), $\alpha = \iota_\xi\, dA$, *then* $\alpha(H) = u$, $\alpha(V) = -1$, $\alpha(X) = 0 = \alpha(\xi)$, *and* $\alpha([H, V]) = -a + bu$.

Proof. We have $\alpha(X) = 0 = \alpha(\xi) = \alpha(uV + H) = -u + \alpha(H)$ since $\alpha(V) = dA(\xi, V) = dA(uV + H, V) = dA(H, V) = -1$. Q.E.D.

Lemma 5.6. *If we choose to take* $\beta(V) = b$, *then* $\beta(X) = -u$ *and* $\beta(H) = \mathscr{L}_V u - a$, *i.e.,* $\beta = (\mathscr{L}_V u - a)\iota_V\, dA - uA - b\iota_H\, dA$.

Proof.

$$\begin{aligned}\beta(X)\alpha(H) &= d\alpha(X, H) = \mathscr{L}_X\alpha(H) - \mathscr{L}_H\alpha(X) + \alpha([H, X])\\ &= \mathscr{L}_X u - K\alpha(V) = \dot{u} + K = -u^2\\ &= -u\alpha(H) \quad \text{(Riccati equation)}\end{aligned}$$

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[3]Therefore this is also true in the k-basic case whenever the curvature is negative, but this is vacuous because it only happens in the Riemannian situation [11].

and

$$\begin{aligned}\beta(\xi)\alpha(H) &= \beta\wedge\alpha(\xi,H) = d\alpha(\xi,H)\\ &= \mathscr{L}_\xi\alpha(H) - \mathscr{L}_H\alpha(\xi) + \alpha(\underbrace{[H,\xi]}_{=u[H,V]+(\mathscr{L}_H u)V})\\ &= \underbrace{\mathscr{L}_\xi u}_{=u\mathscr{L}_V u+\mathscr{L}_H u} + u(-a+bu) - \mathscr{L}_H u = \alpha(H)(\mathscr{L}_V u - a + bu).\end{aligned}$$

Q.E.D.

Proof of Proposition 5.2. We use that $\beta\wedge d\beta = \lambda A\wedge dA$ for some $\lambda\colon M\to\mathbb{R}$. Then

$$\begin{aligned}&\int_M (u^2 + 3(\mathscr{L}_V u)^2 + (4u\mathscr{L}_V a - 2u\mathscr{L}_X\mathscr{L}_V b - b\mathscr{L}_V K))A\wedge dA\\ &= \int_M \lambda A\wedge dA\end{aligned}$$

by (2) and by computing the terms on the right-hand side of

$$\begin{aligned}\lambda(A\wedge dA)(X,V,H) &= (\beta\wedge d\beta)(X,V,H)\\ &= \beta(X)\,d\beta(V,H) + \beta(H)\,d\beta(X,V) + \beta(V)\,d\beta(H,X).\end{aligned}$$

Lemma 5.6 implies

$$\begin{aligned}d\beta(V,H) &= \mathscr{L}_V\beta(H) - \mathscr{L}_H\beta(V) - \beta([V,H])\\ &= \mathscr{L}_V^2 u - \mathscr{L}_V a - \mathscr{L}_H b + \beta(X + aV + bH)\\ &= \mathscr{L}_V^2 u - \mathscr{L}_V a - \mathscr{L}_H b - u + b\mathscr{L}_V u,\\ d\beta(X,V) &= \mathscr{L}_X\beta(V) - \mathscr{L}_V\beta(X) - \beta([X,V]) = \mathscr{L}_X b + \mathscr{L}_V u + \mathscr{L}_V u - a\\ &= 2\mathscr{L}_V u,\\ d\beta(H,X) &= \mathscr{L}_H\beta(X) - \mathscr{L}_X\beta(H) - \beta([H,X])\\ &= -\mathscr{L}_H u - \mathscr{L}_X(\mathscr{L}_V u - a) + Kb = -\mathscr{L}_H u - \mathscr{L}_X\mathscr{L}_V u - \mathscr{L}_V K\\ &= -\mathscr{L}_V\mathscr{L}_X u - \mathscr{L}_V K\end{aligned}$$

by the Bianchi identity. This gives

$$\begin{aligned}\lambda &= \lambda(A\wedge dA)(X,V,H)\\ &= \beta(X)\,d\beta(V,H) + \beta(H)\,d\beta(X,V) + \beta(V)\,d\beta(H,X)\\ &= -u(\mathscr{L}_V^2 u - \mathscr{L}_V a - \mathscr{L}_H b - u + b\mathscr{L}_V u) + 2(\mathscr{L}_V u - a)\mathscr{L}_V u\\ &\quad - b(\mathscr{L}_V\mathscr{L}_X u + \mathscr{L}_V K)\\ &= u^2 + 2(\mathscr{L}_V u)^2 - u\mathscr{L}_V^2 u\\ &\quad + (u\mathscr{L}_V a + u\mathscr{L}_H b - ub\mathscr{L}_V u - 2a\mathscr{L}_V u) - b(\mathscr{L}_V\mathscr{L}_X u + \mathscr{L}_V K).\end{aligned}$$

To get Proposition 5.2 remove derivatives of u via integration by parts:

$$\mathscr{L}_V(A\wedge dA)=d\iota_V(A\wedge dA)=-d(A\wedge\iota_V\,dA)=A\wedge d(\iota_V\,dA)$$

and

$$\begin{aligned}(A\wedge d(\iota_V\,dA))(X,V,H)&=d(\iota_V\,dA)(V,H)=-\iota_V\,dA([V,H])\\&=dA(-V,[V,H])=dA(V,X+aV+bH)=b\end{aligned}$$

imply $\mathscr{L}_V(A\wedge dA)=bA\wedge dA$ and the integration-by-parts formula

$$\begin{aligned}0&=\int_M\mathscr{L}_V(pqA\wedge dA)\\&=\int_M(\mathscr{L}_Vp)qA\wedge dA+\int_M p(\mathscr{L}_Vq)A\wedge dA+\int_M pqbA\wedge dA.\end{aligned}\tag{5}$$

With $(p,q)=(u,\mathscr{L}_Vu)$, this gives

$$\begin{aligned}\int_M\lambda\,A\wedge dA=\int_M(u^2+3(\mathscr{L}_Vu)^2&+(u\mathscr{L}_Va+u\mathscr{L}_Hb-2a\mathscr{L}_Vu)\\&-b(\mathscr{L}_V\mathscr{L}_Xu+\mathscr{L}_VK))A\wedge dA.\end{aligned}$$

The Finsler part can be further simplified. First of all,

$$u\mathscr{L}_Hb=u(\mathscr{L}_V\mathscr{L}_Xb-\mathscr{L}_X\mathscr{L}_Vb)=u\mathscr{L}_Va-u\mathscr{L}_X\mathscr{L}_Vb.$$

Next, (5) with $(p,q)=(a,u)$ and $\mathscr{L}_X(b^2)=2\dot{b}b=2ab$ give

$$\begin{aligned}\int_M-2a\mathscr{L}_VuA\wedge dA&=\int_M2u\mathscr{L}_VaA\wedge dA+\int_M2abuA\wedge dA\\&=\int_M2u\mathscr{L}_VaA\wedge dA-\int_Mb^2\mathscr{L}_XuA\wedge dA.\end{aligned}$$

Finally, (5) with $(p,q)=(b,\mathscr{L}_Xu)$ gives

$$\begin{aligned}\int_M-b\mathscr{L}_V\mathscr{L}_XuA\wedge dA&=\int_M(\mathscr{L}_Vb)\mathscr{L}_XuA\wedge dA+\int_Mb^2\mathscr{L}_XuA\wedge dA\\&=\int_M-u\mathscr{L}_X\mathscr{L}_VbA\wedge dA+\int_Mb^2\mathscr{L}_XuA\wedge dA,\end{aligned}$$

so $\int\beta\wedge d\beta=\int\lambda A\wedge dA=\int u^2+3(\mathscr{L}_Vu)^2+(4u\mathscr{L}_Va-2u\mathscr{L}_X\mathscr{L}_Vb-b\mathscr{L}_VK)A\wedge dA$. Q.E.D.

Remark 5.7. The C^2 assumption on the foliations was used here implicitly in the definition of GV and in the integration by parts at the very end. The latter is easy to address, i.e., it is really for *defining* GV that this regularity is used.

§6. Application to geometric rigidity

Theorem 6.1. *$GV_i = h^i \operatorname{vol}_A(M)$ for contact Anosov flows with absolutely continuous Margulis measure and C^2 invariant foliations; here h is topological entropy.*

Remark 6.2. This applies to geodesic flows of negatively curved locally symmetric spaces, in particular, of surfaces with constant negative curvature.

Proof. The (un)stable conditionals of the Margulis measure are volumes and scale with h [2, Lemma 3 & §3; equation (11) concerns the Jacobian]. Therefore $h\alpha = \mathscr{L}_X\alpha = \iota_X d\alpha + d\iota_X\alpha = \beta(X)\alpha$ (since $\iota_X\alpha \equiv 0$), so $\beta(X) \equiv h$, hence $\beta = hA + \iota_\eta\, dA$ (from the proof of Proposition 3.12), and

$$\begin{aligned} h\beta \wedge \alpha = hd\alpha = dh\alpha = d\mathscr{L}_X\alpha = \mathscr{L}_X d\alpha \\ = \mathscr{L}_X\beta \wedge \alpha + \beta \wedge \mathscr{L}_X\alpha = \mathscr{L}_X\beta \wedge \alpha + \beta \wedge h\alpha. \end{aligned}$$

Thus $\mathscr{L}_X\beta \wedge \alpha = 0$, hence $\mathscr{L}_X\beta(v) = 0$ for any $v \in \mathbb{R}X \oplus E^-$. Choose $\beta = 0$ on E^+, so $\beta = fA$ for some $f\colon M \to \mathbb{R}$, hence $\beta = hA$. Q.E.D.

We now show applications of these invariants to geometric rigidity.

Theorem 6.3. *If $GV_0 = c$, $GV_1 = hc$, and $GV_2 = h^2c$ for a negatively curved Riemannian metric on a surface with C^2 invariant foliations, then the curvature is constant, c is the volume and h the topological entropy.*

This is an immediate consequence of

Proposition 6.4. *For the geodesic flow of a negatively curved Riemannian metric on a surface with C^2 invariant foliations*

$$\frac{GV_0 GV_2}{(GV_1)^2} \geq 1 \tag{6}$$

with equality if and only if the curvature is constant.

Proof. Since Lemma 5.6 and Proposition 5.2 give

$$\begin{aligned} GV_0 &= \int_M A \wedge dA, \quad GV_1 = \int_M -uA \wedge dA, \\ GV_2 &= \int_M u^2 + 3(\mathscr{L}_V u)^2 A \wedge dA, \end{aligned}$$

the Cauchy–Schwarz inequality

$$
\begin{aligned}
GV_1 &= \int_M -uA \wedge dA \\
&\le \left(\int_M u^2 A \wedge dA\right)^{\frac{1}{2}} \left(\int_M A \wedge dA\right)^{\frac{1}{2}} \le (GV_2)^{\frac{1}{2}}(GV_0)^{\frac{1}{2}}
\end{aligned}
$$

allows equality only if $u \equiv \text{const}$ (and, redundantly, $\mathscr{L}_V u \equiv 0$), hence $K = -(\dot{u} + u^2) = -u^2 \equiv \text{const}$. Q.E.D.

We recall that we assume the invariant foliations to be C^2, which by itself is known to imply constant curvature [7, 12]. We present Proposition 6.4 here to show the simplicity of the argument. The forthcoming extension of our definitions to the case of $C^{1+1/2+\epsilon}$ invariant foliations will make it substantial. While this will also add interest to Theorem 6.1, Theorem 6.7 below already does so in the present context.

It is also simple to prove that smooth conjugacy of geodesic flows implies isometry; this was a nontrivial insight in the conclusions of [7, 12].

Theorem 6.5. *If the geodesic flows φ^t and ψ^t of C^5 Riemannian surfaces M and S, respectively, are topologically conjugate and S has constant curvature -1, then M and S are isometric.*

Proof. The conjugacy h is C^2 since $\varphi^t \in C^3$ [20], [5, Corollary 4.8, Theorem 5.2], [14, p. 371], [16, Theorem 1], [15, Theorem 1.1], and a contact Anosov flow is the Reeb flow of a unique contact form. Thus h sends the contact form A for φ^t to that for ψ^t, and likewise for dA and the weak-unstable foliation—which is hence C^2. Thus, the Godbillon–Vey invariants match up, i.e., $GV_i^M = GV_i^S$ for $i = 0, 1, 2$, so $\frac{GV_0^M GV_2^M}{(GV_1^M)^2} = \frac{GV_0^S GV_2^S}{(GV_1^S)^2} = 1$ by Proposition 6.4, which implies, again by Proposition 6.4, that M has constant curvature. Constantly curved metrics are isometric if their geodesic flows are conjugate. (This follows from the work of Otal [18] and Croke [4], which does not assume constant curvature, and is proved directly and more easily by Foulon [6].) Q.E.D.

Remark 6.6. This theorem is not contingent on defining Godbillon–Vey invariants for lower regularity because the conjugacy sends the smooth maximally isotropic foliation to a C^2 maximally isotropic foliation. The same goes for the next result, which recovers a special case of a rigidity result of Hurder and Katok via a remarkably simple proof.

Theorem 6.7. *Negatively curved surfaces whose geodesic flow has C^2 horospheric foliations are constantly curved.*

We use that as in [3], the C^2 splitting yields a Bott–Kanai connection.

Proposition 6.8 ([3, Proposition 2.3 & Section 3.2, Lemma 4.1])**.** *There is a unique φ^t-invariant connection ∇ that parallelizes the geometric structure in that $\nabla A = 0$, $\nabla\, dA = 0$, $\nabla E^{\pm} \subset E^{\pm}$, and with $\nabla_{Z^{\mp}} Z^{\pm} = p^{\pm}[Z^{\mp}, Z^{\pm}]$ and $\nabla_X Z^{\pm} = [X, Z^{\pm}] \pm h_{\mathrm{vol}} Z^{\pm}$ for any sections $Z^{\pm}$ of $E^{\pm}$, where $p^{\pm}$ is the projection to $E^{\pm}$ given by the decomposition. The* (rank-1) *bundle of volume forms on a ∇-parallel subbundle of TM has a natural flat connection induced by ∇.*

Proof of Theorem 6.7. Proposition 6.8 gives a *parallel* unstable volume [3, Section 4.2], which is then holonomy-invariant and hence gives the conditionals of the Bowen–Margulis measure. Thus we can apply Theorem 6.1 and then Theorem 6.3. Q.E.D.

To summarize, the novelty of our approach is to introduce a *family* of invariants for a foliation, and to either draw conclusions about it, such as in Theorem 6.1, or to use information about the *family* to learn about the geometry, such as in Proposition 6.4 and its applications.

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