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Several problems on groups of diffeomorphisms

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Abstract.

This is a survey on several problems related to the study of groups of diffeomorphisms.

§1. Introduction

In this article we discuss several problems related to the study of groups of diffeomorphisms which the author worked on for a while with some hope to find new phenomena.

For a compact manifold M, let $\mathrm{Diff}^r(M)$ ($r = 0$, $1 \leqq r \leqq \infty$, or $r = \omega$) denote the group of C^r diffeomorphisms of M. $\mathrm{Diff}^r(M)$ is equipped with the C^r topology and let $\mathrm{Diff}^r(M)_0$ denote the identity component of it. The family of diffeomorphisms generated by a time dependent vector field is called an isotopy. A C^r diffeomorphism near the identity ($r \geqq 1$) is contained in an isotopy from the identity. $\mathrm{Diff}^r(M)$ has a manifold structure modelled on the space $\mathcal{X}^r(M)$ of C^r vector fields. The manifold structure of $\mathrm{Diff}^r(M)$ for a compact manifold M is given by using the exponential map with respect to a Riemannian metric so that a neighborhood of $0 \in \mathcal{X}^r(M)$ is homeomorphic to a neighborhood of the identity of $\mathrm{Diff}^r(M)$. It is worth noticing that the composition $(g_1, g_2) \to g_1 \circ g_2$ in $\mathrm{Diff}^r(M)$ ($1 \leqq r < \infty$) is C^∞ smooth with respect to g_1 but not smooth with respect to g_2 (see [34] §4 p. 51 for the case of $\boldsymbol{R}^n$).

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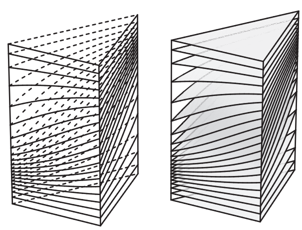
§2. Foliated products

A smooth singular simplex $\sigma\colon \Delta^m \to \mathrm{Diff}^r(M)$ corresponds to the multi dimensional isotopy which is the foliation of $\Delta^m \times M$ transverse to the fibers of the projection $\Delta^m \times M \to \Delta^m$ whose leaf passing through (t, x) is $\{\sigma(s)\sigma(t)^{-1}(x) \mid s \in \Delta\}$. These multi isotopies naturally match up along the boundary and form the universal foliated M-product over the classifying space $B\overline{\mathrm{Diff}}^r(M)$. See the following figure and [26].



Let $B\overline{\Gamma}_n^r$ be the classifying space for Haefliger's Γ_n^r structures with trivialized normal bundles ([15], [16]). Since $B\overline{\Gamma}_n^r$ classifies C^r foliations with trivialized normal bundles, for an n-dimensional parallelized manifold M^n, we obtain the map $B\overline{\mathrm{Diff}}^r(M^n) \times M^n \to B\overline{\Gamma}_n^r$, and hence the map $B\overline{\mathrm{Diff}}^r(M^n) \to \mathrm{Map}(M^n, B\overline{\Gamma}_n^r)$. The deep result by Mather–Thurston says that the last map induces an isomorphism on integral homology.

Theorem 2.1 (Mather–Thurston [24], [28]). *For an n-dimensional parallelized manifold M^n (for simplicity) and for $1 \leqq r \leqq \infty$,*

$$H_*(B\overline{\mathrm{Diff}}^r(M^n); \boldsymbol{Z}) \cong H_*(\mathrm{Map}(M^n, B\overline{\Gamma}_n^r); \boldsymbol{Z}).$$

In particular, $H_*(B\overline{\mathrm{Diff}}_c^r(\boldsymbol{R}^n); \boldsymbol{Z}) \cong H_*(\Omega^n B\overline{\Gamma}_n^r; \boldsymbol{Z})$ for the group $\mathrm{Diff}_c^r(\boldsymbol{R}^n)$ of C^r diffeomorphisms of $\boldsymbol{R}^n$ with compact support.

On the other hand, $H_1(B\overline{\mathrm{Diff}}^r(M^n); \boldsymbol{Z}) = 0$ $(1 \leqq r \leqq \infty, r \neq n+1)$ has been shown by Herman–Mather–Thurston ([18], [24], [28]). Note that $H_1(B\overline{\mathrm{Diff}}^r(M^n); \boldsymbol{Z}) \cong H_1(B\widetilde{\mathrm{Diff}}^r(M^n)_0^\delta; \boldsymbol{Z})$, where $\widetilde{\mathrm{Diff}}^r(M^n)_0$ is the universal covering group and $^\delta$ means that the group is equipped with the discrete topology when we take its classifying space. In general, the abelianization of a group G is isomorphic to $H_1(BG^\delta; \boldsymbol{Z})$ and a group is said to be perfect if its abelianization is trivial. Moreover, by the fragmentation technique, $H_1(B\overline{\mathrm{Diff}}^r(M^n); \boldsymbol{Z}) = 0$ is equivalent to $H_1(B\overline{\mathrm{Diff}}^r_c(\boldsymbol{R}^n); \boldsymbol{Z}) = 0$, and if $\widetilde{\mathrm{Diff}}^r_c(\boldsymbol{R}^n)_0$ is perfect, then $\widetilde{\mathrm{Diff}}^r(M^n)_0$ and $\mathrm{Diff}^r(M^n)_0$ are perfect (cf. [4]).

Theorem 2.2 (Herman–Mather–Thurston [18], [24], [28]). $\mathrm{Diff}^r_c(M^n)_0$ ($1 \leqq r \leqq \infty$, $r \neq n+1$) *is a perfect group. It is a simple group if* M^n *is connected.*

It is known that for $r > 2 - 1/(n+1)$, there is a characteristic cohomology class called the Godbillon–Vey class in $H^{n+1}(B\overline{\mathrm{Diff}}^r(M^n); \boldsymbol{R})$ ([13], [38]). $B\overline{\mathrm{Diff}}^r(M^n)$ is conjectured to be n-acyclic. For the higher dimensional homology, it is only known ([34], [35]) that

$$H_2(B\overline{\mathrm{Diff}}^r_c(\boldsymbol{R}^n); \boldsymbol{Z}) = 0 \quad \text{if } 1 \leqq r < [n/2],$$

$$H_m(B\overline{\mathrm{Diff}}^r_c(\boldsymbol{R}^n); \boldsymbol{Z}) = 0 \quad \text{if } 1 \leqq r < [(n+1)/m] - 1 \text{ and}$$

$$H_m(B\overline{\mathrm{Diff}}^1_c(\boldsymbol{R}^n); \boldsymbol{Z}) = 0 \quad \text{for } m \geqq 1.$$

The main technical reason of the above regularity conditions can be seen in the infinite iteration construction using $(\boldsymbol{Z}_+ * \boldsymbol{Z}_+)^n$ action on $\boldsymbol{R}^n$. As is well-known, by the homothety of ratio A, the C^r-norm of a foliated $\boldsymbol{R}^n$-product is multiplied by A^{1-r} ([25]). For the easiest case of divisible abelian m-cycle c represented by time 1 maps of commuting vector fields, we divide it into 2^m pieces $[n/m]$ times and we use $\boldsymbol{Z}_+^{2^n}$ action generated by homotheties of ratio $A = 1/(2+\varepsilon)$, then the infinite iteration construction converges in the C^r topology if $2^{-[n/m]}/(2+\varepsilon)^{1-r} < 1$, that is, if $r - [n/m] - 1 < 0$ ([34]). To treat general cycles we loose a little more regularity.

For the connectivity of $B\overline{\Gamma}^r_n$, it seems that it increases when r tends to 1. It is true that in $\mathrm{Diff}^{1+\alpha}_c(\boldsymbol{R}^n)$, we can construct a $\boldsymbol{Z}^k$ action which permutes open sets, where k tends to infinity as α tends to 0 ([36]), and we think that we can use it to construct infinite iterations of chains. The bound of the ranks of such actions has been studied by Deroin, Kleptsyn and Navas ([8]), and the study of group actions which permute open sets became a new direction of study of group of diffeomorphisms (cf. [27]).

For seeking more regular construction, it is necessary to know that abelian cycles are null homologous.

Problem 2.3. For the action $\varphi\colon \boldsymbol{R}^m \to \mathrm{Diff}^r(M^n)$, show that $B(\boldsymbol{R}^m)^\delta \to B\overline{\mathrm{Diff}}^r(M^n)$ induces the trivial homomorphism on integral homology.

Remark 2.4. It is true for $\mathrm{Diff}^\infty_c(\boldsymbol{R})$ ([29], [31]). It is probably true for $m = 1$ and $\mathrm{Diff}^\infty_c(\boldsymbol{R}^n)$. The first interesting case is $\boldsymbol{R}^2 \to \mathrm{Diff}^\infty_c(\boldsymbol{R}^2)$.

To treat non abelian cycles, we notice that the theorem of Mather–Thurston implies that any class of $H_2(B\overline{\mathrm{Diff}}^r(M^n);\boldsymbol{Z})$ $(r \neq n+1)$ can be represented by a foliated M^n product over the surface Σ_2 of genus 2 ([29]).

For the smooth codimension 1 foliations, there is the interesting problem of determining the kernel of the Godbillon–Vey class.

Problem 2.5. Determine the kernel of $GV\colon H_2(B\overline{\mathrm{Diff}}_c^r(\boldsymbol{R});\boldsymbol{Z}) \to \boldsymbol{R}$.

Remark 2.6. There is a group G which contains both $\mathrm{Diff}_c^r(\boldsymbol{R})$ $(r > 1+1/2)$ and the group $PL_c(\boldsymbol{R})$ of piecewise linear homeomorphisms of $\boldsymbol{R}$ with compact support, with a metric such that GV cocycle is continuous ([32], [33]). We know that for a G-foliated $\boldsymbol{R}$-product $\mathcal{F}$ with compact support over a compact oriented surface, $GV(\mathcal{F}) = 0$ if and only if $\mathcal{F}$ is homologous to a G-foliated $\boldsymbol{R}$-product $\mathcal{H}_0$ over a surface Σ which is the limit of G-foliated $\boldsymbol{R}$-products $\mathcal{H}_k$ with compact support over the surface Σ representing 0 in $H_2(G;\boldsymbol{Z})$ ([33]). $\mathcal{H}_k$ are in fact transversely piecewise linear foliations and the topology of the classifying space $BPL_c(\boldsymbol{R})^\delta$ has been known by the work of Peter Greenberg ([14]).

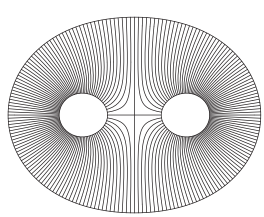
Remark 2.7. It will be nice if we can take $\mathcal{H}_k$ to be C^1 piecewise $PSL(2;\boldsymbol{R})$ foliated S^1-products. The group of C^1 piecewise $PSL(2;\boldsymbol{R})$ diffeomorphisms of S^1 contains the Thompson simple group (consisting of C^1 piecewise $PSL(2;\boldsymbol{Z})$ diffeomorphisms) which gives other interests to study this group.

Remark 2.8. It is also very interesting to know whether the amalgamate product of the k-fold covering groups $PSL(2;\boldsymbol{R})^{(k)}$ of $PSL(2;\boldsymbol{R})$ over their subgroups of rotations is dense in $\mathrm{Diff}^r(S^1)_0$ $(r \geqq 1)$. See [30].

§3. $B\overline{\Gamma}_1^\omega$

Many years ago, Haefliger showed that $B\overline{\Gamma}_1^\omega$ is a $K(\pi,1)$ space ([15]). If one understands the definition of the $\overline{\Gamma}_1^\omega$ structures, though π is a huge group, it is easy to show that $H_1(B\overline{\Gamma}_1^\omega;\boldsymbol{Z}) = 0$.

Problem 3.1. Prove or disprove that $H_2(B\overline{\Gamma}_1^\omega;\boldsymbol{Z}) = 0$.



Remark 3.2. The homology group $H_2(B\overline{\Gamma}_1^\omega; \boldsymbol{Z})$ is generated by cycles represented by surfaces Σ_2 of genus 2 with C^ω singular foliations with 2 saddles. These are obtained from the two copies of real analytic foliations of pairs of pants which looks topologically as above. Since $B\overline{\Gamma}_1^\omega$ is a $K(\pi, 1)$, a homology class represented by the map from S^2 is trivial. A homology class represented by the map from T^2 is homologous to a union of suspensions of C^ω diffeomorphisms of S^1, and these are trivial because $\mathrm{Diff}^\omega(S^1)_0$ is perfect by a result of Arnold ([2]).

As for the perfectness of the group $\mathrm{Diff}^\omega(M^n)_0$ of real-analytic diffeomorphisms of M^n, Herman showed that $\mathrm{Diff}^\omega(T^n)_0$ is simple almost 40 years ago ([18]). Rather recently, we could show that if M^n admits a nice circle action then $\mathrm{Diff}^\omega(M^n)_0$ is perfect ([39]). These are applications of Arnold's work on the small denominators ([2]). With this method, it should be at least generalized to the manifolds with non trivial circle actions. There are torus bundles which admits a flow whose orbit closures are fibers. It might be possible to apply the argument of [39].

Remark 3.3. As for the classifying space $B\overline{\Gamma}_n^{\boldsymbol{C}}$ for complex analytic Γ structures with terivialized normal bundles, there are no progress after the works by Landweber, Adachi and Haefliger–Sithananthan. In [38], the connectivity of $B\overline{\Gamma}_n^{\boldsymbol{C}}$ was wrongly stated. It should read $B\overline{\Gamma}_n^{\boldsymbol{C}}$ is n-connected in general by Landweber ([23]), Adachi ([1]) and $B\overline{\Gamma}_1^{\boldsymbol{C}}$ is 2-connected by Haefliger–Sithananthan ([17]).

§4. Uniform perfectness

For a perfect group G, every element g can be written as a product of commutators. The least number of commutators to write g is called the commutator length of g and written as $cl(g)$. A group G is uniformly perfect if cl is a bounded function. The least bound $cw(G)$ is called the commutator width. After the result by Burago–Ivanov–Polterovich ([6]), we showed that for a compact n-dimensional manifold M^n which admits a handle decomposition without handles of the middle index $n/2$, $cw(\mathrm{Diff}^r(M^n)_0) \leqq 3$ if n is even, $cw(\mathrm{Diff}^r(M^n)_0) \leqq 4$ if n is odd $(r \neq n+1)$. For a compact $2m$-dimensional manifold M^{2m} $(2m \geqq 6)$, $cw(\mathrm{Diff}^r(M^{2m})_0) < \infty$ $(r \neq 2m+1)$ ([41]).

Remark 4.1. The upper bounds 3 and 4 above are better than those in Theorem 1.1 (2) and (3) of [41] by one. This is shown by adding a simple observation in the proof in [41]. We use a self indexing Morse function and the stratification by stable manifolds and that of unstable manifolds for the gradient flow. In the proof of Theorem 1.1

(2) of [41], with respect to the stratified sets $P^{(m-1)}$ and $Q^{(m-1)}$, we write $f = g \circ h$ with $g \in \mathrm{Diff}_c(M^{2m} \setminus k(Q^{m-1}))_0$ and $h \in \mathrm{Diff}_c(M^{2m} \setminus P^{(m-1)})_0$, where the support of $k \in \mathrm{Diff}_c(M^{2m} \setminus P^{(m-1)})_0$ is contained in a small neighborhood of $Q^{(m-1)}$. Then by the argument of Lemma 2.5 of [41], there is a diffeomorphism $F \in \mathrm{Diff}_c(M^{2m} \setminus k(Q^{m-1}))$ such that $F(\mathrm{supp}(g)) \cap P^{(m-1)} = \emptyset$. Then $f = [g, F] \circ (F \circ g \circ F^{-1}) \circ h$ and $(F \circ g \circ F^{-1}) \circ h \in \mathrm{Diff}_c(M^{2m} \setminus P^{(m-1)})_0$. Since $F \circ g \circ F^{-1} \circ h$ can be written as a product of two commutators by Theorem 1.1 (1) of [41], f can be written as a product of three commutators. The proof of Theorem 1.1 (3) of [41] can be modified in a similar way.

Problem 4.2. Estimate $cw(\mathrm{Diff}^r(T^2)_0)$, $cw(\mathrm{Diff}^r(\mathbf{C}P^2)_0)$, $cw(\mathrm{Diff}^r(S^2 \times S^2)_0), \ldots$.

For the group of homeomorphisms, we managed to prove that for the spheres S^n and the Menger compact space μ^n, $cw(\mathrm{Homeo}(S^n)_0) = 1$ and $cw(\mathrm{Homeo}(\mu^n)) = 1$ ([42]). It is probably true that for the Menger-type compact space μ_k^n, $cw(\mathrm{Homeo}(\mu_k^n)_+) = 1$, where $_+$ means a certain condition concerning the orientation. The idea of proof comes from the fact that the typical homeomorphism of such a space is the one with one source and one sink and that the conjugacy class of such a homeomorphism should be unique.

Problem 4.3. Find other examples of groups of commutator width one.

In 1980, Fathi showed that for the group $\mathrm{Homeo}_\mu(M^n)_0$ of homeomorphisms preserving a good measure μ of M^n $(n \geqq 3)$, the kernel of the flux homomorphism $\mathrm{Homeo}_\mu(M^n)_0 \to H^{n-1}(M^n; \boldsymbol{R})/\Gamma$ is perfect, where Γ is the image of $\pi_1(\mathrm{Homeo}_\mu(M^n)_0)$ under the flux homomorphism defined on the universal covering group of $\mathrm{Homeo}_\mu(M^n)_0$ ([11]). It seems that he proved that the kernel is uniformly perfect (at least he proved it for the spheres). For the group $\mathrm{Diff}_{\mathrm{vol}}(M^n)_0$ of volume preserving diffeomorphisms, Thurston showed that the kernel of the flux homomorphism is perfect ([3], cf. [4] §5.1 p. 126).

Problem 4.4. Prove or disprove that $\mathrm{Diff}_{\mathrm{vol}}(S^n)_0$ $(n \geqq 3)$ is uniformly perfect.

Burago–Ivanov–Polterovich gave the notion of norms on the group and studied its properties ([6]). $\nu : G \to \boldsymbol{R}_{\geqq 0}$ is a (conjugate invariant) norm if it satisfies (i) $\nu(1) = 0$; (ii) $\nu(f) = \nu(f^{-1})$; (iii) $\nu(fg) \leqq \nu(f) + \nu(g)$; (iv) $\nu(f) = \nu(gfg^{-1})$ and (v) $\nu(f) > 0$ for $f \neq 1$. For a symmetric subset $K \subset G$ normally generating G, any $f \in G$ can be written as a product of conjugates of elements of K and the function giving the minimum number $q_K(f)$ of the conjugates is a norm. Then $cl(f) = q_K(f)$ for K being the set of single commutators.

For the groups of diffeomorphism with the fragmentation property, the perfectness implies the simplicity (cf. [4]). For a simple group G, the norm $q_{\{g,g^{-1}\}}\colon G \to \boldsymbol{Z}_{\geqq 0}$ is defined for $g \in G$. If $\{q_{\{g,g^{-1}\}}\}_{g\in G\setminus\{1\}}$ is bounded then G is said to be uniformly simple. In other words, for any $f \in G$ and $g \in G \setminus \{1\}$, f is written as a product of a bounded number of conjugates of g or g^{-1}. We have a distance function d on the set $\{C_{\{g,g^{-1}\}}\}_{g\neq 1}$ of symmetrized nontrivial conjugate classes:

$$d(C_{\{f,f^{-1}\}}, C_{\{g,g^{-1}\}}) = \log\max\{q_{\{f,f^{-1}\}}(g), q_{\{g,g^{-1}\}}(f)\}.$$

For simple groups which are not uniformly simple, for example, $\mathrm{Diff}_{\mathrm{vol},c}(\boldsymbol{R}^n)_0$ ($n \geqq 3$), A_∞, etc., it is interesting to study the metric d. For the infinite alternative group A_∞, Hiroki Kodama and Yoshifumi Matsuda showed that the set of symmetrized nontrivial conjugate classes of $A_\infty \setminus \{\mathrm{id}\}$ with the metric d is quasi-isometric to the half line ([22]). The quasi-isometry is given by the logarithm of cardinality of the support. (Added in proof: Kodama put the result on arXiv [21].)

A real valued function ϕ on a group G is a homogeneous quasimorphism if $(g_1, g_2) \mapsto \phi(g_2) - \phi(g_1g_2) + \phi(g_1)$ is bounded and $\phi(g^n) = n\phi(g)$ for $n \in \boldsymbol{Z}$. Put

$$D(\phi) = \sup\{|\phi(g_2) - \phi(g_1g_2) + \phi(g_1)| \mid (g_1, g_2) \in G \times G\}.$$

Then Bavard's duality says that

$$\mathrm{scl}(g) = \frac{1}{2} \sup_{\phi\in Q(G)/H^1(G;\boldsymbol{R})} \frac{\phi(g)}{D(\phi)},$$

where $\mathrm{scl}(g) = \lim_{n\to\infty} \frac{\mathrm{cl}(g^n)}{n}$ (stable commutator length) and $Q(G)$ is the real vector space of homogeneous quasimorphisms on G ([5], see also [7] p. 35). Of course, for groups with infinite commutator width, we need to study their stable commutator length function. If the commutator width of a group G is infinite, G is not uniformly simple, hence the distance function d is unbounded. We might have more information on the distance d by looking at relative quasimorphisms. For a symmetric subset $K \subset G$ normally generating G, let $Q(G, K)$ be the real vector space of homogeneous quasimorphisms on G which vanishes on K. If there is a nontrivial element $\phi \in Q(G, K)$ (for example, if $\dim Q(G)$ is larger than the order of K), then $\phi(f) \leqq (q_K(f) - 1)D(\phi)$ and q_K is not bounded. Entov–Polterovich ([10]), Gambaudo–Ghys ([12]), Ishida ([20]), and others have shown that $Q(\mathrm{Diff}_{\mathrm{vol}}(D^2, \mathrm{rel}\,\partial D^2))$ is infinite dimensional, and hence as they remarked, the kernel of the Calabi homomorphism $\mathrm{Diff}_{\mathrm{vol}}(D^2, \mathrm{rel}\,\partial D^2) \to \boldsymbol{R}$ is not uniformly simple.

Problem 4.5. For the kernel of the Calabi homomorphism $\mathrm{Diff}_{\mathrm{vol}}(D^2, \mathrm{rel}\,\partial D^2) \to \boldsymbol{R}$, show that $\{C_{\{g,g^{-1}\}}\}_{g\neq 1}$ with metric d is not quasi-isometric to the half line.

As for the group $\mathrm{Homeo}_{\mathrm{vol}}(D^2, \mathrm{rel}\,\partial D^2)$, despite attempts by many people, its simplicity is still an open problem. The following problem seems to be the first step to show it.

Problem 4.6. Using area preserving homeomorphisms with the Calabi invariant being infinity, show that an area preserving diffeomorphism with nontrivial Calabi invariant is a product of commutators.

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