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Billey's formula in combinatorics, geometry, and topology

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§1. Introduction

In this paper we describe a powerful combinatorial formula and its implications in geometry, topology, and algebra. This formula first appeared in the appendix of a book by Andersen, Jantzen, and Soergel [1, Appendix D]. Sara Billey discovered it independently five years later, and it played a prominent role in her work to evaluate certain polynomials closely related to Schubert polynomials [4].

To set the stage for our discussion, we review well-known foundations of Schubert calculus in Lie type A_{n-1}. Consider the group of invertible matrices $GL_n(\mathbb{C})$ with the subgroup B of upper-triangular matrices. The flag variety is the quotient $GL_n(\mathbb{C})/B$ and can be thought of geometrically as the collection of nested vector subspaces $V_1 \subseteq V_2 \subseteq \cdots V_{n-1} \subseteq \mathbb{C}^n$ where each V_i is i-dimensional. The geometry of the flag variety is interwoven with the combinatorics of the permutation group: the torus T of diagonal matrices in B acts on the flag variety, and its fixed points are the flags corresponding to permutation matrices. For each permutation w, the double coset BwB is an affine cell inside the flag variety, and the union of these double cosets forms a CW-decomposition. The closures of the cells $\overline{BwB/B}$ are the Schubert varieties, which induce a basis for the cohomology of the flag variety. Combinatorial properties

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of the permutations w determine topological properties of Schubert varieties: for instance, the number of inversions of w counts the dimension of the variety $\overline{BwB/B}$ in the flag variety.

Billey's polynomials relate all of these pieces of Schubert calculus: the geometry of Schubert varieties, the action of the torus T on the flag variety, combinatorial data about permutations, the cohomology of the flag variety and of the Schubert varieties, and the combinatorics of root systems (the generalization of inversions of a permutation).

Combinatorially, Billey's formula describes an invariant of pairs of elements of a Weyl group. On its face, this invariant is a polynomial combination of roots built from subwords of a fixed word. As we will see, it has deeper geometric and topological meaning as well:

- It tells us about the tangent spaces at each permutation flag in each Schubert variety.
- It tells us about singular points in Schubert varieties.
- It tells us about the values of Kostant polynomials.

It also reflects an aspect of GKM theory, which is a way of describing the torus-equivariant cohomology of a variety just from information about the torus-fixed points in the variety.

We will also describe some ways to use Billey's polynomials, including concrete combinatorial descriptions in special cases, and ways to bootstrap the polynomials to describe the equivariant cohomology of subvarieties of the flag variety to which GKM theory does not apply.

More precisely, Section 2 states Billey's formula, gives examples, and lists its main properties. Section 3 describes background and motivation, including the geometric viewpoint: that Billey and Andersen-Jantzen-Soergel gave a formula for the localizations of Schubert classes at torus-fixed points. Sections 4 and 5 then describe newer applications: a tool called *excited Young diagrams* with which one can compute localizations of Schubert classes in Grassmannians, and a tool called *poset pinball* with which one can compute the equivariant cohomology of various subvarieties of the flag variety. Section 5.2 even shows how poset pinball can be used to construct a complete Schubert calculus for the Peterson variety, a singular subvariety of the flag variety with important applications to quantum cohomology. Finally Section 6 concludes with some open questions and conjectures.

We used the example of Lie type A in this introduction but the theorems in this paper apply to all complex reductive linear algebraic groups. We use the following notation in the rest of this manuscript.

- G is a connected reductive complex linear algebraic group
- B is a Borel subgroup of G

- T is a maximal torus in B
- Φ is the root system corresponding to G
- $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ is the set of positive simple roots corresponding to B
- W is the Weyl group associated to G and T
- All cohomology is taken with complex coefficients

A reader interested primarily in the case of type A might find it helpful to recall that $\alpha_i = e_i - e_{i+1}$ and that each simple reflection $s_i = (i, i+1)$ in cycle notation. Thus for instance the calculation

$$s_2\alpha_1 = \alpha_1 + \alpha_2$$

is equivalent to

$$(23)(e_1 - e_2) = e_1 - e_3 = (e_1 - e_2) + (e_2 - e_3).$$

§2. Billey's Formula in Combinatorics

We begin by introducing Billey's formula from a strictly combinatorial viewpoint, from which it looks like Billey's polynomials. More formally, for each pair v, w in the Weyl group W, we will describe a polynomial $\sigma_v(w)$ in the simple roots $\alpha_1, \alpha_2, \ldots, \alpha_n$.

We need two main tools:

(1) A *reduced word* $b_1b_2b_3\cdots b_m$ is an expression for which each b_i is a simple reflection in W and the product $b_1b_2b_3\cdots b_m$ cannot be written in fewer simple reflections.

(2) To each reduced word $b_1b_2b_3\cdots b_m$ and integer j such that $1 \leq j \leq m$ we associate the root $r_j = b_1b_2b_3\cdots b_{j-1}(\alpha_j)$.

Reduced words are commonly used when considering Weyl groups, Coxeter groups, or indeed any group with prescribed generator sets.

The roots r_j deserve more comment. Surprisingly, they are always positive [12, Section 10.2]. In fact, they are exactly those positive roots negated by w^{-1} in the sense that every r_j is $-w(\alpha)$ for some positive root α. Even more important, the set of roots r_j determine w. More precisely, let $b_1b_2b_3\cdots b_m$ be a reduced word for w and denote by $N(w)$ the set $\{r_1, r_2, \ldots, r_m\}$ of all m positive roots formed according to Rule (2) in the previous paragraph. Then the set $N(w)$ uniquely specifies w in the sense that if $N(w_1) = N(w_2)$ then $w_1 = w_2$ [18]. These sets $N(w)$ satisfy many other interesting combinatorial properties.

Definition 2.1 (Billey's formula [4, 1])**.** *Let v, w be elements of W. Fix a reduced word $b_1b_2b_3\cdots b_m$ for w. Billey's formula for the*

polynomial $\sigma_v(w)$ *is*

$$\sigma_v(w) = \sum \prod_{i=1}^{k} r_{j_1} r_{j_2} \cdots r_{j_k}$$

where the sum is over all reduced subwords $b_{j_1} b_{j_2} \cdots b_{j_k}$ *for* v *in* w*, and for each* j

$$r_j = b_1 b_2 b_3 \cdots b_{j-1}(\alpha_j).$$

Andersen, Jantzen, and Soergel gave this first [1, Appendix D, Remark on p. 298]. Willems later provided a geometric proof [31].

Example 1. *Let* $v = s_1$ *and* $w = s_2 s_1 s_2$ *in the symmetric group* S_3*, namely the Weyl group of type* A_2*.*

Step 1: Find all possible subwords of w *that equal* v*:*

$$s_2 \ \textcircled{s_1} \ s_2$$

Step 2: Compute the roots in each term:

$$s_2 \ \textcircled{\alpha_1} \ \cancel{s_2} = \alpha_1 + \alpha_2$$

Step 3: Add terms:

$$\sigma_{s_1}(s_2 s_1 s_2) = \alpha_1 + \alpha_2.$$

In this case we obtain the polynomial $\alpha_1 + \alpha_2$*.*

We now consider several extreme cases of the definition.

First, suppose v is not a subword of w. In this case, the first step of the algorithm ("find all possible subwords...") fails and there are no terms in the polynomial. More formally, we have the following.

Proposition 2. *If* $v \not\leq w$ *in Bruhat order then* $\sigma_v(w) = 0$*.*

Second, suppose v is the identity $e \in W$. In this case, every element $w \in W$ contains e trivially and uniquely as the empty subword. We take the root computed in the second step of the algorithm to be 1, giving us the following.

Proposition 3. *If* $v = e$ *then* $\sigma_e(w) = 1$ *for all* $w \in W$*.*

Finally, suppose v and w are both the same element in the Weyl group. Then the polynomial $\sigma_w(w)$ is always the product of the roots in the subset $N(w)$ described above.

Proposition 4. *If* $v = w$ *is an element of the Weyl group then*

$$\sigma_w(w) = \prod_{\alpha \in N(w)} \alpha.$$

For instance if v and w are both the longest element $w_0 \in W$ then the polynomial $\sigma_{w_0}(w_0)$ is simply the product of all positive roots.

Other important properties are more subtle; before discussing them, we give one more substantive example.

Example 5. *Let $v = s_1$ and $w = s_1 s_2 s_1$ in type A.*

Step 1: Find all possible subwords of w that equal v:

$$\textcircled{s_1}\ s_2\ s_1 \qquad \textit{and} \qquad s_1\ s_2\ \textcircled{s_1}$$

Step 2: Compute the roots in each term:

$$\textcircled{α_1}\ \cancel{s_2}\ \cancel{s_1} \qquad \textit{and} \qquad s_1\ s_2\ \textcircled{α_1}$$

Step 3: Add terms:

$$\sigma_{s_1}(s_1 s_2 s_1) = \alpha_1 + \alpha_2.$$

Using a different word for w than Example 1, we still obtain $\alpha_1 + \alpha_2$.

This leads directly to the fundamental properties of $\sigma_v(w)$. Recall that the length $\ell(w)$ of a Weyl group element $w \in W$ is the number of simple reflections in a reduced word for w, namely if $w = b_1 b_2 \cdots b_m$ then $\ell(w) = m$.

Proposition 6. *Let $v, w \in W$. Billey's formula for $\sigma_v(w)$ gives:*

- *a polynomial in $\alpha_1, \alpha_2, \ldots, \alpha_n$ with nonnegative integer coefficients*
- *that has degree $\ell(v)$*
- *and is independent of the choice of reduced word for w.*

The first claim in this proposition follows from the fact that each root r_j is always positive, and that every positive root can be written in terms of the simple roots with nonnegative integer coefficients. The second claim follows from the algorithm itself: each term in $\sigma_v(w)$ is the product of roots, one for each simple reflection in a reduced word for v. Hence each term has degree $\ell(v)$. The third claim is not combinatorially trivial: Billey proved it in her original manuscript [4], and Andersen, Jantzen, and Soergel sketched a proof using the relationship between Schubert classes and the polynomials $\sigma_v(w)$ that is described in the next section [1].

§3. What Billey's formula means

We now consider the motivation behind the discovery of these polynomials and other places they first emerged.

3.1. Representations of quantum groups

Andersen, Jantzen, and Soergel wanted to answer a problem of a different flavor from the Schubert calculus described in this manuscript: identifying representations of quantum groups and of semisimple groups over fields of characteristic p. Their approach was to fit their specific questions into a larger framework. They built a more abstract endomorphism algebra and showed that its properties captured their original representation-theoretic questions.

As a small application of their results, Andersen, Jantzen, and Soergel consider a particular endomorphism ring studied earlier by Kostant and Kumar [20, 21]. They proved that classes σ_w constructed by Kostant and Kumar actually generate various sub- and quotient modules of the endomorphism ring. In order to do this, Andersen, Jantzen, and Soergel explicitly identified the polynomials $\sigma_v(w)$ [1, Chapter 19, Appendix D].

3.2. Orbit values of Kostant polynomials

Billey's original goal was to study the values of certain polynomials called *Kostant polynomials* that are also related to the classes σ_w. Let O denote a regular element of the torus $\mathfrak{t}$. Kostant polynomials are (nonhomogeneous) elements of $\mathbb{C}[\mathfrak{t}^*]$ parametrized by the Weyl group, of degree determined by the length of the Weyl group element associated to the polynomial, and defined by certain vanishing conditions on the orbits Ov for each $v \in W$. Surprisingly, these polynomials are essentially unique: Kostant showed that they are unique up to the ideal in $\mathbb{C}[\mathfrak{t}^*]$ that vanishes on the orbits OW. Moreover, the highest homogeneous component is a Schubert class in $H^*(G/B)$.

Later work expanded this deep connection between the Kostant polynomials and the cohomology of the flag variety. Carrell proved that the cohomology $H^*(G/B)$ is isomorphic as a graded ring to the coordinate ring of the variety associated to the points OW [5]. Kostant and Kumar generalized Kostant polynomials to the collections of polynomials σ_w and in a series of papers proved that the classes σ_w generate the equivariant cohomology of the flag variety G/B [20, 21].

3.3. Kumar's criterion for Schubert varieties

Of course, algebraic information about Schubert classes is closely related to geometric data of Schubert varieties. Kumar pursued this relationship, strengthening his analysis of some specific polynomials $\sigma_v(w)$ to give a criterion for when the (opposite) Schubert variety X^v is smooth at the flag wB. It turns out that smoothness is equivalent to $\sigma_v(w)$ being a specific product of distinct positive roots [23].

Theorem 7 (Kumar's Criterion)**.** *The opposite Schubert variety X^v is smooth at the flag wB if and only if*

$$\sigma_v(w) = \prod_{\substack{\alpha \in \Phi^+ s.t. \\ v \not\leq s_\alpha w}} \alpha.$$

3.4. Restriction to fixed points and GKM theory

The two previous points—(1) constructing the cohomology of the flag variety and (2) the topology of Schubert varieties—fit together in a natural framework: the family of polynomials associated to σ_v by Billey's formula actually represent the equivariant Schubert class corresponding to v and the individual polynomials $\sigma_v(w)$ encode topological information about the Schubert variety at the fixed point w. This is now viewed as a part of a larger topological construction of torus-equivariant cohomology for a wider class of varieties that is often referred to as *GKM theory*. (Andersen-Jantzen-Soergel's endomorphism algebras encapsulate key properties of GKM theory from a purely algebraic point of view.)

We describe GKM theory for flag varieties. To begin, the inclusion $(G/B)^T \hookrightarrow G/B$ induces a map

$$H_T^*(G/B) \to H_T^*((G/B)^T).$$

For the flag variety (and many other varieties, including G/P), this map is an injection:

$$H_T^*(G/B) \hookrightarrow H_T^*((G/B)^T).$$

The equivariant cohomology of a point is

$$H_T^*(pt) \cong \mathbb{C}[\alpha_1, \alpha_2, \ldots, \alpha_n].$$

The fixed points $(G/B)^T$ are the flags associated to Weyl group elements $\{wB : w \in W\}$ so

$$H_T^*((G/B)^T) \cong \bigoplus_{w \in W} \mathbb{C}[\alpha_1, \alpha_2, \ldots, \alpha_n].$$

Inclusion of fixed points induces an injection

$$H_T^*(G/B) \hookrightarrow \bigoplus_{w \in W} \mathbb{C}[\alpha_1, \alpha_2, \ldots, \alpha_n].$$

This brings us to a key point.

Fact 8. *The image of the Schubert class* $[X^v]$ *under this map is*

$$[X^v] \mapsto (\sigma_v(w))_{w \in W}$$

where $\sigma_v(w)$ *are the very same polynomials given by Billey's formula.*

In fact, Kumar showed that this fact holds for affine flag manifolds [4, Appendix] and for any G/P [24].

When the flag wB is a smooth point in the variety X^v then the polynomials $\sigma_v(w)$ describe the (torus weight on the) tangent space to the variety X^v at the fixed point w. Geometrically, then, Kumar's criterion says that a fixed point is smooth in a Schubert variety if Billey's formula gives the product of certain distinct roots.

Remark 9. *Both Billey and Andersen-Jantzen-Soergel use the same basic approach to identifying and proving the formula for the polynomials* $\sigma_v(w)$*: induction on the length of* v *together with* divided difference operators*. Divided difference operators* ∂_i *are degree-lowering operators defined on Schubert classes for* G/B*. Their key property is that*

$$\partial_i(\sigma_v) = \begin{cases} \sigma_{s_i v} & \text{if } s_i v < v \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Kostant and Kumar discovered an explicit formula for divided difference operators in the GKM setting [20, Propositions 4.20 and 4.24]*. Explicit formulas for the top-degree Schubert class* σ_{w_0} *and the lowest-degree Schubert class* σ_e *in the GKM presentation follow from the definitions. Billey uses divided difference operators to induct down from* σ_{w_0} *while Kumar (in the Appendix to Billey's paper) and Andersen-Jantzen-Soergel induct up from* σ_e*.*

We often represent equivariant Schubert classes combinatorially. The *Bruhat graph* of W is the graph with vertices given by $w \in W$ and with edges $s_\alpha w \leftrightarrow w$ between vertices $w, s_\alpha w$ that differ by left-multiplication by a reflection. We then describe the class σ_v by labeling each vertex w of the Bruhat graph with the polynomial $\sigma_v(w)$ as shown in Figure 1.

Remark 10 (GKM (Goresky-Kottwitz-MacPherson) theory)**.** *This construction of Schubert classes (and the cohomology ring) holds much more generally than just for the flag variety. In fact, we can identify the image* $H_T^*(X) \hookrightarrow H_T^*(X^T)$ *for a large family of varieties* X*. In many cases—now referred to as GKM theory—the image can be described as a subset of polynomials labeling the vertices of a graph associated to* X*. The conditions defining which polynomials are allowed are given by a straightforward combinatorial algorithm. GKM theory builds on work of*

The Bruhat graph for S_3 The Schubert class σ_{s_1}

Fig. 1. The Bruhat graph for GL_3/B and a class in $H_T^*(GL_3/B)$

Chang-Skjelbred, Kirwan, Atiyah-Bott, Guillemin-Sternberg, and many others [6, 15, 2, 9].

Figure 2 shows all of the Schubert classes for GL_3/B with large vertices indicating the support of the class.

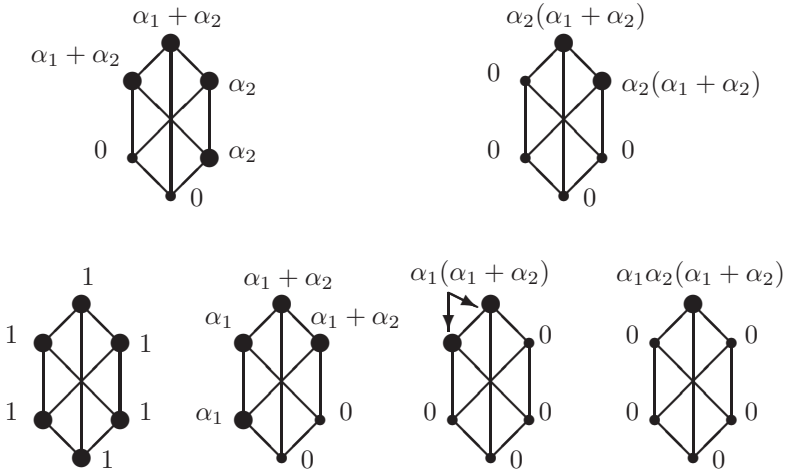


Fig. 2. Schubert classes for $H_T^*(GL_3/B)$

The central goal of Schubert calculus is to identify the products $\sigma_u\sigma_v$ in terms of Schubert classes, namely find explicit formulas for the coefficients c_{uv}^w in the expansion

$$\sigma_u\sigma_v = \sum c_{uv}^w \sigma_w.$$

With the GKM presentation of the equivariant cohomology ring, several aspects of Schubert calculus become simpler:

- The ring structure of $H_T^*(G/B)$ is streamlined: we multiply and add these classes vertex-wise, using Billey's formula to identify the polynomials associated to each vertex.
- The Schubert classes are "upper-triangular" with respect to the partial order on W in which $u \leq v$ if u can be written as a subword of v. (Figure 2 demonstrates this visually and Proposition 2 states it algebraically.)

Kostant and Kumar applied this kind of strategy directly to get a non-positive formula for the structure constants of flag varieties [20, Proposition 4.24(f)] which Willems refined [32, Theorem 4.8]. Often additional combinatorial insight is needed to give an explicitly positive formula.

For instance, Knutson and Tao used GKM theory in their paper on puzzles to give a positive formula for equivariant structure constants in $H_T^*(G(k,n))$ [17]. Their proof had three main steps: 1) proving that a small subset of structure constants determine all of the structure constants; 2) establishing certain instances of Billey's formula to identify this subset of structure constants; and 3) proving that the structure constants agreed in those instances with the polynomials given by their puzzles.

Remark 11. *An elegant closed formula for the localizations of Schubert classes exists in large part because of the strong combinatorial structures inherent in the flag variety. Nonetheless, there has been some work to generalize Billey's formula to cohomology bases for a larger family of symplectic manifolds, notably by Goldin and Tolman* [8]*. Goldin and Tolman both generalize the concept of Schubert classes to a large family of symplectic manifolds and give a (generally-non-positive) formula for the restriction of these classes to the fixed points.*

§4. Billey's formula for the Grassmannian $G(k,n)$

In brief, this formula turns calculations in geometric Schubert calculus into combinatorics. It can be restricted to subvarieties of G/P and so can be used in a wide range of applications of Schubert calculus. This section describes applications for Grassmannians. The next section shows applications when GKM theory does not hold.

In particular cases, we can make the combinatorics involved more explicit. For instance, if the ambient variety is the Grassmannian $G(k,n)$ of k-dimensional subspaces of $\mathbb{C}^n$ then the combinatorial construction for

Billey's formula is called *excited Young diagrams* by Ikeda and Naruse [13], who discovered it independently after Kreiman [22], or Knutson-Miller-Yong in the case of ordinary cohomology [16].

Theorem 12 (Kreiman, Ikeda-Naruse)**.** *In $G(k,n)$ the polynomial $\sigma_\lambda(\mu)$ is the sum of the excited Young diagrams of λ inside μ, weighted according to the position of the boxes.*

An excited Young diagram is a Young diagram in which certain boxes are marked. We draw our Young diagrams so that the partition $\mu = (\mu_1 \geq \mu_2 \geq \ldots)$ has μ_1 boxes in the first row, μ_2 boxes in the second row, and so on, with the rows aligned on the leftmost column.

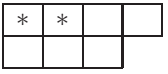
Definition 4.1. *A marked box in a Young diagram can be* excited *if it has empty boxes to the east, south, and southeast. When a box is excited, its marking moves to the box directly southeast.*

We now assign a weight (i,j) to each box in the Young diagram:

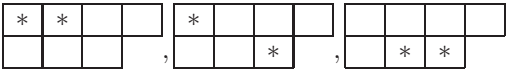
- i is the index of the column in which the box is located (read from the left).
- j is (the number of rows at or below the box) + (the index of the rightmost column in the box's row).

To construct all excited Young diagrams of λ inside μ, follow the resulting simple algorithm (which we demonstrate with an example):

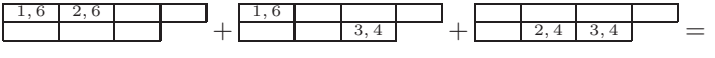
(1) Mark the boxes of λ inside μ.



(2) Excite all marked boxes in all possible ways.



(3) Sum the diagrams, weighted by position of marked boxes.



$$(t_1 - t_6)(t_2 - t_6) + (t_1 - t_6)(t_3 - t_4) + (t_2 - t_4)(t_3 - t_4)$$

We sketch the main ideas of the proof of Theorem 12, though we omit details. Young diagrams represent fixed points in $G(k,n)$ as well

as permutations. These Grassmannian permutations satisfy a strong algebraic condition: every reduced word ends in the reflection s_k. Excited Young diagrams enumerate all possible reduced subwords for λ inside the word for μ using the combinatorics of S_n.

§5. Billey's formula for subvarieties

GKM theory cannot be applied to all varieties, not even if those varieties have rich geometric and combinatorial structures. We can nonetheless construct a kind of GKM theory for suitable subvarieties Y of G/B using Billey's formula for localizations of Schubert classes in G/B. Our strategy is to bootstrap information about Schubert classes to obtain a module basis for the equivariant cohomology of Y.

The subvariety Y must satisfy two important conditions.

(1) Y must admit the action of a (one-dimensional) subtorus $S \subseteq T$ under which the fixed points $Y^S \subseteq (G/B)^T$.
(2) Y must be *equivariantly formal*.

Subvarieties of G/B that are defined by linear conditions, like Schubert varieties or Hessenberg varieties (see below) often satisfy the first condition. Any variety with no odd-dimensional ordinary cohomology satisfies the second condition.

Here are the three tools that we have developed thus far:

- a graph Γ built from G/B
- fixed points Y^S that are a subset of vertices in Γ
- basis classes σ_v of $H_T^*(G/B)$ that are indexed by the set of $v \in (G/B)^T$

The graph Γ is the Bruhat graph (see Section 3.4, or Figure 1 for an example), which is essential for GKM theory in a way that we did not need to make precise for our exposition. The basis classes σ_v are the Schubert classes, and they are specified by Billey's formula. Finally, the fixed points Y^S are guaranteed to be a subset of the vertices in Γ by our hypotheses on Y.

We will use this information to identify a subset σ_v whose images $\widetilde{\sigma}_v|_{Y^S}$ generate $H_S^*(Y)$.

Intuitively, our strategy is as follows:

• Drop one ball from each vertex $v \in Y^S$ in turn and let it roll down the edges of the graph Γ until it lands at a vertex $r(v) \in (G/B)^T$

$$v \rightsquigarrow r(v)$$

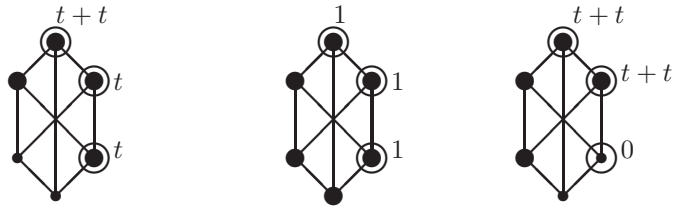


Fig. 3. Restricted Schubert classes for the Springer variety

This heuristic was originally called *poset pinball*, though others point out that *poset pachinko* may be more appropriate. The fixed point $r(v)$ is called the *roll-down* vertex of v.

We first give the main theorem and then give several examples. The theorem identifies a collection of Schubert classes, considered as functions $\sigma_{r(v)} : W \to \mathbb{C}[\alpha_1, \ldots, \alpha_n]$ for the fixed points $v \in Y^S$, and then asserts that the restrictions $\widetilde{\sigma}_{r(v)} : Y^S \to \mathbb{C}[t]$ generate the S-equivariant cohomology ring of Y.

Theorem 13 (Harada-Tymoczko)**.** *For each $u \in (G/B)^T$ let $\widetilde{\sigma}_u$ denote the image of the Schubert class σ_u under the natural restriction map $H_T^*(G/B) \twoheadrightarrow H_S^*(G/B)$. If Y is an* appropriate *subspace of G/B then the set of functions*

$$\left\{\widetilde{\sigma}_{r(w)}|_{Y^S} : w \in Y^S\right\}$$

generate $H_S^*(Y)$ as a ring and form a module basis for $H_S^*(Y)$.

Figure 3 shows this in practice. Each roll-down permutation gives a Schubert class; the vertices labeled by nonzero polynomials are emphasized, and the lowest of the marked vertices is the roll-down permutation. The circled vertices are the elements of Y^S. We record the polynomials for the Schubert class at those vertices and discard all other information. The resulting classes generate $H_S^*(Y)$ as a subring of $(\mathbb{C}[t])^3$.

Various different sets of conditions for Y can be used in this theorem [10, 11]. Our main examples are subvarieties of G/B though the results hold for a wider class of ambient varieties than even G/P.

Poset pinball applies to various important subvarieties of G/B, including both smooth and singular examples.

(1) **Schubert varieties**
Poset pinball can be used for any subvariety to which regular GKM theory applies—like Schubert varieties, including P/B.

(2) **Springer fibers of a nilpotent matrix** $X : \mathbb{C}^n \to \mathbb{C}^n$
The Springer variety of X is the subvariety

$$Spr(X) = \left\{\text{flags } gB \in GL_n(\mathbb{C})/B : g^{-1}Xg \text{ is upper-triangular}\right\}.$$

It is important in geometric representation theory: Springer showed that the symmetric group S_n acts on the cohomology of the Springer variety [29]. Moreover, the top-dimensional cohomology of $Spr(X)$ is the irreducible representation corresponding to the partition of n given by the sizes of the Jordan blocks of X. The combinatorics of Young tableaux give important information about the components and Betti numbers of Springer varieties [27, 28, 30].

(3) **Peterson varieties for a regular nilpotent** $X \in \mathfrak{g}$
The Peterson variety of a regular nilpotent $X \in \mathfrak{g}$ is

$$Pet = \left\{\text{flags } gB \in G/B : g^{-1}Xg \in \mathfrak{b} \oplus \bigoplus_{\alpha_i \in \Delta} \mathfrak{g}_{-\alpha_i}\right\}$$

Kostant showed that the quantum cohomology ring of the flag variety is isomorphic to the coordinate ring of a dense open subvariety of the Peterson variety [19]. Rietsch gave an explicit isomorphism in which the quantum parameters correspond to certain determinants, giving a beautiful collection of determinantal identities [26]. The geometry and topology of Peterson varieties is again deeply linked to combinatorics of permutations and partitions [25, 14].

(4) **Nilpotent Hessenberg varieties in type** A **for a nilpotent** $X \in \mathfrak{gl}_n$ **and appropriate subspace** $H \subseteq \mathfrak{gl}_n$
Both Springer varieties and Peterson varieties are examples of a larger class of varieties called Hessenberg varieties. Hessenberg varieties have two parameters: a nilpotent element $X \in \mathfrak{gl}_n$ and a subspace $H \subseteq \mathfrak{gl}_n$. The subspace H must satisfy two constraints: that H contains the Borel subalgebra $\mathfrak{b}$ and that H is closed under Lie bracket with $\mathfrak{b}$ in the sense that $[H, \mathfrak{b}] \subseteq H$. The Hessenberg variety for these two parameters X and H is defined as

$$Hess(X, H) = \left\{\text{flags } gB \in GL_n(\mathbb{C})/B : g^{-1}Xg \in H\right\}.$$

Hessenberg varieties have a kind of cell decomposition enumerated by Young tableaux that relates the geometry of the cells with the combinatorics of the tableaux [30].

5.1. Examples of poset pinball

We give two examples of the roll-down process.

We use the subtorus $S \subseteq T$ that consists of diagonal matrices with t^i in the i^{th} row, for each i with $1 \leq i \leq n$ and each $t \in \mathbb{C}$. This means that the map $H_T^*(G/B) \twoheadrightarrow H_S^*(G/B)$ sends each $\alpha_i \mapsto t$.

Both of our examples are for subregular Springer varieties inside the flag variety $GL_n(\mathbb{C})/B$. In other words, the nilpotent matrix X has two Jordan blocks of dimensions $n-1$ and 1.

Figure 4 shows the case when the subvariety $Y \subseteq GL_4(\mathbb{C})/B$ is the Springer variety associated to the matrix X with 1 in entries $(1,2),(2,3)$ and zero elsewhere. (We only show relevant vertices and edges of the Bruhat graph Γ.) The bold vertices in Figure 4 are the roll-down vertices; the circled vertices are the fixed points Y^S.

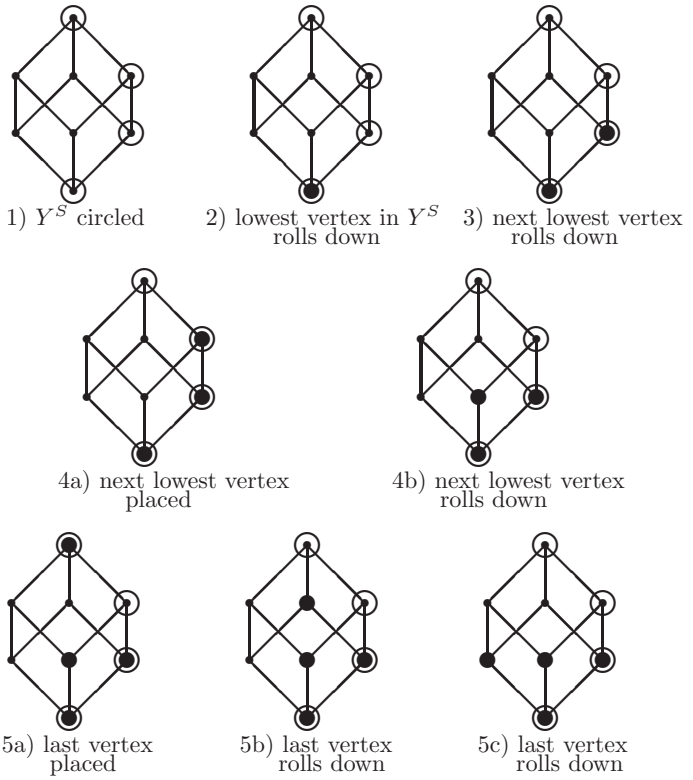


Fig. 4. Poset pinball on a Springer variety

This process is **not** deterministic: for instance, instead of rolling down in step 5b of Figure 4, the ball could instead have rolled right to give the alternate set of roll-down vertices shown in Figure 5.

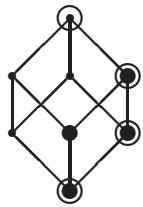


Fig. 5. Alternate roll-downs

The next example is similar but in $GL_3(\mathbb{C})/B$. The Springer variety is given by the matrix whose only nonzero entry is in position $(1, 2)$. We skip steps of the roll-down process because in this case it is essentially unique. As long as the balls drop as far down as possible first from s_2, second from $s_2 s_1$, and last from $s_1 s_2 s_1$, then the set of roll-downs must be the bottom three vertices, as in Figure 6.

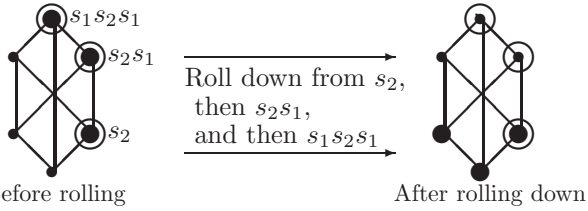


Fig. 6. Placement of balls before and after rolling down

Returning to Figure 3, we see a pictorial summary of Theorem 13. The three roll-downs from Figure 6 are vertices s_2, e, and s_1. Figure 2 shows the corresponding Schubert classes (as well as all other Schubert classes) for the flag variety; Billey's formula gives the polynomials at each vertex. Figure 6 shows the classes for vertices s_2, e, and s_1 from left-to-right; now we only show polynomials at circled vertices, namely, the fixed points associated to the subvariety.

In short, the roll-down process selects certain Schubert classes for each subvariety of the flag variety. Theorem 13 then says that restricting those Schubert classes just to the fixed points in the subvariety—and discarding all additional information—generates the equivariant cohomology of the subvariety. Finally, Billey's formula for these polynomials gives an explicit description of each basis class.

5.2. Schubert calculus and poset pinball

We use these results to actually compute Schubert calculus for suitable subvarieties of the flag variety, in this case for the Peterson variety

Y of type A_{n-1}. First we identify roll-down classes for the Peterson variety. Then we give two key formulas that together describe the entire cohomology ring of Peterson varieties: a Chevalley-Monk formula, which shows how to multiply any roll-down class by one of a specific family of generators; and a Giambelli formula, which expresses an arbitrary roll-down class in terms of the generators.

For each subset $\mathcal{A} \subseteq \{1, 2, \ldots, n-1\}$ define $v_{\mathcal{A}} = \prod_{i \in \mathcal{A}} s_i$.

Theorem 14. [10, 11] *Let Y be the Peterson variety of type A_{n-1} and S be the subtorus of diagonal matrices with t^i in the i^{th} row for each i with $1 \leq i \leq n$ and each $t \in \mathbb{C}$.*

The equivariant cohomology $H_S^*(Y)$ is generated by

$$H_S^*(Y) = \left\langle \widetilde{\sigma}_{v_{\mathcal{A}}}(w) : \mathcal{A} \subseteq \{1, 2, \ldots, n-1\}, wB \in Y^S \right\rangle.$$

This result is true in type A_{n-1} [10] and in general Lie type [11].

We now give the Chevalley-Monk and Giambelli formulas for Peterson varieties. The integers in these formulas are straightforward, but their precise descriptions require distracting notation.

Theorem 15. *Let Y be the Peterson variety of type A_{n-1} and S be the subtorus of diagonal matrices with t^i in the i^{th} row for each i with $1 \leq i \leq n$ and each $t \in \mathbb{C}$.*

(1) **Chevalley-Monk formula for Peterson varieties** [10]

$$\widetilde{\sigma}_{s_i} \widetilde{\sigma}_{v_{\mathcal{A}}} = t c_{i,\mathcal{A}}^{\mathcal{A}} \widetilde{\sigma}_{v_{\mathcal{A}}} + \sum c_{i,\mathcal{A}}^{\mathcal{B}} \widetilde{\sigma}_{v_{\mathcal{B}}}$$

where the sum is over $\mathcal{B} \supseteq \mathcal{A}$ with $|\mathcal{B}| = |\mathcal{A}| + 1$.

(2) **Giambelli's formula for Peterson varieties** [3]

$$\widetilde{\sigma}_{v_{\mathcal{A}}} = c \prod_{i \in \mathcal{A}} \widetilde{\sigma}_{s_i}.$$

All coefficients are explicit, easily-computed, positive integers.

Drellich recently extended this to Peterson varieties of all Lie types [7]. She proves her results in several cases; however, the uniformity across type suggests an underlying topological cause.

§6. Conjectures and open questions

First we ask for something like Billey's formula for varieties other than G/P. We want a *positive* formula that *explicitly* describes each polynomial in the GKM presentation of $H_T^*(X)$.

Question 16. *We ask for explicit, positive formulas for localizations of cohomology classes for varieties other than G/P.*

Using the methods of Section 5, we can identify a Schubert variety X_H for each regular nilpotent Hessenberg variety Y_H so that

$$\{\widetilde{\sigma}_v|_{X_H} : vB \in X_H\} \text{ generates } H_S^*(X_H)$$

and we conjecture

$$\{\widetilde{\sigma}_v|_{Y_H} : vB \in X_H\} \text{ generates } H_S^*(Y_H)$$

Analogous conjectures have been posed for Springer and all other nilpotent Hessenberg varieties.

Question 17. *Is there a geometric reason for the similarity between the S-equivariant cohomology of X_H and Y_H?*

For instance, is there a deformation of Hessenberg varieties to unions of Schubert varieties? More generally, are there structural (possibly non-geometric) explanations for the similarity between the cohomology of these two families of varieties?

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