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Tableaux and Eulerian properties of the symmetric group

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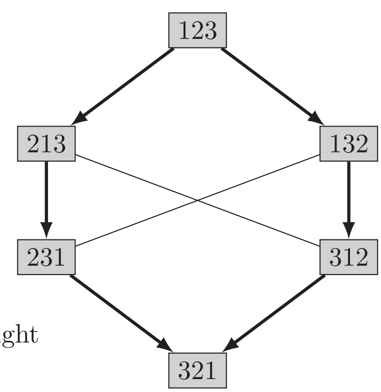
Abstract.

The Ehresmann-Bruhat order on the symmetric group satisfies a symmetry property that we generalize to a directed graph with permutations as vertices, labeling edges with tableaux of a given shape.

§1. Eulerian structures

Ehresmann defined an order (called Bruhat order) on the symmetric group $\mathfrak{S}_n$ which plays a fundamental role in geometry, algebra and representation theory:
two permutations are *consecutive* with respect to the Ehresmann-Bruhat order if they have consecutive *lengths* and differ by multiplication by a transposition.
For example, the hexagon with its diagonals encodes the order on $\mathfrak{S}_3$, the thick arrows corresponding to its generation by multiplication on the right by simple transpositions.



The Ehresmann-Bruhat order possesses many symmetry properties. For example, in every interval, there are as many permutations of even length than of odd length. These kinds of properties are accounted by the notion of Eulerian structure. Given a graded poset X, its incidence matrix E (i.e. $E[x,y] = 1 \Leftrightarrow x \leq y$) is graded: $E = E_0 + E_1 + E_2 + \ldots$

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Departing slightly from the classics [14, p. 135], let us call the poset *Eulerian* if $E^{-1} = E_0 - E_1 + E_2 - E_3 + \dots$.

Instead of ranked posets, it is more fruitful to use labeled directed graphs with a rank. Edges are labeled by elements of an algebra with an involution $\clubsuit$. The incidence matrix, still graded, is the matrix whose entries are the labels. Writing an incidence matrix amounts considering the graph to be a subset of the complete directed graph on the same vertices, some edges being labeled 0. The graph is called *Eulerian* if

$$(E_0 + E_1 + E_2 + \cdots)\left(E_0^\clubsuit - E_1^\clubsuit + E_2^\clubsuit - E_3^\clubsuit + \cdots\right) = 1\,.$$

In the next sections, we shall take the algebra of Laurent polynomials in $x_1, \dots, x_n$, the involution $\clubsuit$ being the inversion $x_1 \to x_1^{-1}, \dots, x_n \to x_n^{-1}$.

Given a second Eulerian structure, with incidence matrix F, on the same underlying complete graph, then EFE and FEF are Eulerian, but not EF nor FE if E and F do not commute. This remark will allow us to combine the Ehresmann-Bruhat order with another Eulerian structure.

The main results have already been exposed in a text with M.P. Schützenberger [11]. We use here a different algebraic approach.

§2. Keys

Many properties of symmetric functions are better understood by using words rather than monomials, and realizing, thanks to Schensted construction, the algebra of symmetric functions $\mathfrak{Sym}$ as a subalgebra of a non-commutative algebra, .

There are several ways to pass from the ring of polynomials $\mathfrak{Pol}(\mathbf{x})$ to the free algebra. In the present text, we shall use Demazure characters for type A (also called *key polynomials*) [10].

Recall the definition of the *isobaric divided differences* π_i and $\widehat{\pi}_i$ (denoted on the right):

$$f \to f\pi_i = \frac{x_i f - x_{i+1} f^{s_i}}{x_i - x_{i+1}} \quad \& \quad f \to f\widehat{\pi}_i = \frac{f - f^{s_i}}{x_i x_{i+1}^{-1} - 1}\,,$$

where s_i is the simple transposition exchanging x_i, x_{i+1}.

The two related families of Demazure characters $\{K_v\}$, $\{\widehat{K}_v\}$, $v \in \mathbb{N}^n$, are two linear bases of the ring of $\mathfrak{Pol}(\mathbf{x})$ which can be defined recursively as follows. If v is *dominant*, i.e. if v is equal to a partition λ, then $K_\lambda = \widehat{K}_\lambda = x^\lambda$. Otherwise, when v and i are such that $v_i > v_{i+1}$, one has the recursion

$$K_{vs_i} = K_v \pi_i \quad \& \quad \widehat{K}_{vs_i} = \widehat{K}_v \widehat{\pi}_i \,.$$

One extends the Ehresmann-Bruhat order to permutations of the same partition. The number of inversions $\ell(v)$ of v is the number of pairs $i < j$ such that $v_i > v_j$. Two elements are consecutive if they differ by a transposition, and their number of inversions differ by 1.

The two families $\{K_v\}$, $\{\widehat{K}_v\}$ are related by summations over intervals [10]:

$$K_v = \sum_{u \le v} \widehat{K}_u \quad \& \quad \widehat{K}_v = \sum_{u \le v} (-1)^{\ell(v)-\ell(u)} K_u \,.$$

The elements of $\mathfrak{Free}$ are words in the letters $1, 2, 3, \ldots$. *Tableaux* are the words obtained by reading the planar objects called *Young tableaux.*

We shall make no difference between $\begin{array}{ccc} 3 & & \\ 2 & 3 & \\ 1 & 1 & 2 \end{array}$ and the word 323112.

Simple transpositions s_i, as well as the operators $\pi_i, \widehat{\pi}_i$ can be lifted to operators on $\mathfrak{Free}$, that we shall denote by the same letters [9]. Given i, one decomposes the set of words into i-strings [5, 6, 8]. Then s_i preserves each i-string and is the symmetry with respect to its middle. The image under π_i of an element t in the left part of an i-string is the sum of all the elements between itself and its image under s_i. Imposing that $t + t^{s_i}$ be invariant under π_i completes the description of the action of π_i [10]. Moreover, $\widehat{\pi}_i = \pi_i - 1$.

For example,

$$\begin{array}{ccccc} \mathbf{3} & 4 & & & \\ 2 & 2 & 3 & & \\ 1 & 1 & 2 & \mathbf{3} & \mathbf{3} \end{array} \; \widehat{\pi}_3 = \begin{array}{ccccc} \mathbf{3} & 4 & & & \\ 2 & 2 & 3 & & \\ 1 & 1 & 2 & \mathbf{3} & \mathbf{4} \end{array} + \begin{array}{ccccc} \mathbf{3} & 4 & & & \\ 2 & 2 & 3 & & \\ 1 & 1 & 2 & \mathbf{4} & \mathbf{4} \end{array} + \begin{array}{ccccc} \mathbf{4} & 4 & & & \\ 2 & 2 & 3 & & \\ 1 & 1 & 2 & \mathbf{4} & \mathbf{4} \end{array} ,$$

the 3-string being

$$\begin{array}{ccccc} \mathbf{3} & 4 & & & \\ 2 & 2 & 3 & & \\ 1 & 1 & 2 & \mathbf{3} & \mathbf{3} \end{array} \to \begin{array}{ccccc} \mathbf{3} & 4 & & & \\ 2 & 2 & 3 & & \\ 1 & 1 & 2 & \mathbf{3} & \mathbf{4} \end{array} \to \begin{array}{ccccc} \mathbf{3} & 4 & & & \\ 2 & 2 & 3 & & \\ 1 & 1 & 2 & \mathbf{4} & \mathbf{4} \end{array} \to \begin{array}{ccccc} \mathbf{4} & 4 & & & \\ 2 & 2 & 3 & & \\ 1 & 1 & 2 & \mathbf{4} & \mathbf{4} \end{array} ,$$

which is isomorphic, after suppression of the letters $\neq 3, 4$ and the paired 43, to the 3-string

$$\boxed{\mathbf{3}\,\mathbf{3}\,\mathbf{3}} \to \boxed{\mathbf{3}\,\mathbf{3}\,\mathbf{4}} \to \boxed{\mathbf{3}\,\mathbf{4}\,\mathbf{4}} \to \boxed{\mathbf{4}\,\mathbf{4}\,\mathbf{4}} \,.$$

For a partition λ, one defines $t_\lambda = K^{\mathcal{F}}_\lambda = \widehat{K}^{\mathcal{F}}_\lambda = \cdots 3^{\lambda_3} 2^{\lambda_2} 1^{\lambda_1}$ (this is the *Yamanouchi* tableau of shape λ). According to [10], the elements $\{K^{\mathcal{F}}_v\}$, $\{\widehat{K}^{\mathcal{F}}_v\}$ of the free algebra $\mathfrak{Free}$ satisfy the same recursion as $\{K_v\}$, $\{\widehat{K}_v\}$:

$$K^{\mathcal{F}}_{vs_i} = K^{\mathcal{F}}_v \pi_i \quad \& \quad \widehat{K}^{\mathcal{F}}_{vs_i} = \widehat{K}^{\mathcal{F}}_v \widehat{\pi}_i \text{ when } v_i > v_{i+1}\,. \tag{1}$$

For example, for $\lambda = [2, 1, 0]$, one has

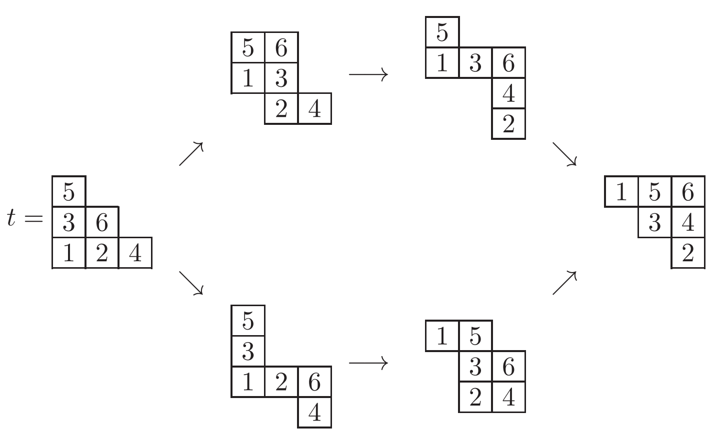
$$\widehat{K}^{\mathcal{F}}_{210} = \begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline\end{array}$$

$$\widehat{K}^{\mathcal{F}}_{120} = \begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array} \qquad \widehat{K}^{\mathcal{F}}_{201} = \begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline\end{array}$$

$$\widehat{K}^{\mathcal{F}}_{102} = \begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline\end{array} + \begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline\end{array} \qquad \widehat{K}^{\mathcal{F}}_{021} = \begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array} + \begin{array}{|c|c}\hline 3 & \\ \hline 2 & \multicolumn{1}{c|}{2} \\ \hline\end{array}$$

$$\widehat{K}^{\mathcal{F}}_{012} = \begin{array}{|c|c}\hline 3 & \\ \hline 2 & \multicolumn{1}{c|}{3} \\ \hline\end{array}$$

The elements $K^{\mathcal{F}}_v$, $\widehat{K}^{\mathcal{F}}_v$, are sums of tableaux without multiplicities. Given a tableau t of shape λ, then it belongs to one and only one $\widehat{K}^{\mathcal{F}}_v$, with v a permutation of λ. We call v the *right key* of t and denote it $\mathcal{C}_+(t)$. Keys can be determined by using the jeu de taquin on consecutive columns [10]:



The recipe is the following: with the rightmost columns of the skew tableaux generated by the jeu de taquin from t, build a tableau and write its commutative evaluation in exponential form x^v. Then v is the right key $\mathcal{C}_+(t)$ of t.

$$\left\{ \boxed{4},\ \begin{array}{|c|}\hline 6\\\hline 4\\\hline\end{array},\ \begin{array}{|c|}\hline 6\\\hline 4\\\hline 2\\\hline\end{array} \right\} \Leftrightarrow \begin{array}{|c|c|c|}\cline{1-1} 6 & \multicolumn{2}{c}{}\\\cline{1-2} 4 & 6 & \multicolumn{1}{c}{}\\\hline 2 & 4 & 4\\\hline\end{array} \to x^{010302} \ \Leftrightarrow\ \mathcal{C}_+(t) = [0,1,0,3,0,2]\,.$$

Using the leftmost columns, one defines similarly the *left key* $\mathcal{C}_-(t)$.

$$\left\{ \boxed{1},\ \begin{array}{|c|}\hline 5\\\hline 1\\\hline\end{array},\ \begin{array}{|c|}\hline 5\\\hline 3\\\hline 1\\\hline\end{array} \right\} \Leftrightarrow \begin{array}{|c|c|c|}\cline{1-1} 5 & \multicolumn{2}{c}{}\\\cline{1-2} 3 & 5 & \multicolumn{1}{c}{}\\\hline 1 & 1 & 1\\\hline\end{array} \to x^{30102} \ \Leftrightarrow\ \mathcal{C}_-(t) = [3,0,1,0,2]\,.$$

The computation of $\widehat{K}_v^{\mathcal{F}}$ is no more complicated than that of $\widehat{K}_v$. Indeed, when v and i are such that $v_i > v_{i+1}$, then $\widehat{K}_v^{\mathcal{F}}$ decomposes into a sum of single elements which are heads of their i-strings (and sent to the full string minus its head under $\widehat{\pi}_i$) and full i-strings (which are sent to 0 under $\widehat{\pi}_i$).

Instead of a Yamanouchi tableau t_λ, one can start from a power t_λ^r. The element $\widehat{K}_v^{\mathcal{F}} = t_\lambda \widehat{\pi}_i \cdots \widehat{\pi}_j$ will then be replaced by the sum [10]

$$t_\lambda^r \widehat{\pi}_i \cdots \widehat{\pi}_j = \sum t_1 t_2 \cdots t_r\,, \tag{2}$$

sum over r-tuples of tableaux of shape λ such that $\mathcal{C}_+(t_1) \le \mathcal{C}_-(t_2)$, $\mathcal{C}_+(t_2) \le \mathcal{C}_-(t_3), \ldots, \mathcal{C}_+(t_{r-1}) \le \mathcal{C}_-(t_r)$, $\mathcal{C}_+(t_r) = v$. We shall call *chain* such r-tuple of tableaux.

§3. Tensor product

The operator $\widehat{\pi}_i$ satisfies a Leibniz formula when acting on $\mathfrak{Pol}(\mathbf{x})$:

$$fg\,\widehat{\pi}_i = f\,(g\widehat{\pi}_i) + f\widehat{\pi}_i(gs_i)\,.$$

This is no more true when acting on $\mathfrak{Free}$. However, the action of a product $\widehat{\pi}_i \cdots \widehat{\pi}_j$ on a product $t_\lambda t_\lambda$ does satisfy Leibniz rule, as well as the braid relations [10]. This is due to the fact that one does not act on general elements of $\mathfrak{Free}$, but on heads of strings, or on full strings, or on full strings minus their head, as we have seen in the preceding section.

Instead of $t_\lambda t_\lambda$, let us write $t_\lambda \otimes t_\lambda$, and write $\mathbb{U}_i$ for the action of $\widehat{\pi}_i$ on $\mathfrak{Free} \otimes \mathfrak{Free}$: the image of $w \otimes w'$ is obtained from $ww'\widehat{\pi}_i$ by cutting words into two factors of the same lengths as w and w'.

When starting from $t_\lambda \otimes t_\lambda$, the operator $\mathbb{U}_i$ coincides with

$$\begin{aligned} \mathbb{U}_i &= \widehat{\pi}_i \otimes s_i + 1 \otimes \widehat{\pi}_i \\ &= \pi_i \otimes s_i + 1 \otimes \theta_i \,, \end{aligned}$$

with $\theta_i = \widehat{\pi}_i - s_i = \pi_i - 1 - s_i$.

Adapting notations, let us define $\widehat{\mathbb{K}}_\lambda = t_\lambda \otimes t_\lambda$ for partitions λ, and define the other $\widehat{\mathbb{K}}_v$, for $v \in \mathbb{N}^n$, by the recursions

$$\widehat{\mathbb{K}}_{vs_i} = \widehat{\mathbb{K}}_v \, \mathbb{U}_i \,,$$

for $v_i > v_{i+1}$.

The tensor notation renders more evident the following property of the action of the operators $\widehat{\pi}_i$ on $\mathfrak{Free}$.

Theorem 3.1. *Given a partition λ, v a permutation of it, then*

$$\widehat{\mathbb{K}}_v = \sum_{t_2} \sum_{t_1 \colon \mathcal{C}_+(t_1) \le u} t_1 \otimes t_2 = \sum_{t_2} K_u^{\mathcal{F}} \otimes t_2 \,, \tag{3}$$

sum over all tableaux t_2 of shape λ, of right key v, and left key u.

Proof. The first sum was already given for chains of tableaux of length r. Summing over t_1, one obtains the second sum. QED

Corollary 3.2. *Let λ be a strict partition, $\sigma \ge \nu$ be two permutations, $s_i \cdots s_j$ be a reduced decomposition of σ of length ℓ. Then the sum*

$$t_\lambda \sum_{\epsilon \in \{0,1\}} \left(s_i^{\epsilon_1} \theta_i^{1-\epsilon_1}\right) \cdots \left(s_j^{\epsilon_\ell} \theta_j^{1-\epsilon_\ell}\right) \,,$$

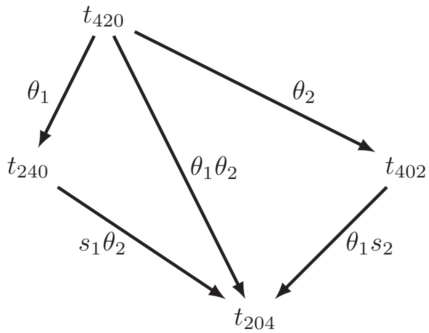
sum over $\epsilon_1, \ldots, \epsilon_\ell$ such that $\pi_i^{\epsilon_1} \cdots \pi_j^{\epsilon_\ell} = \pi_\nu$, is equal to the sum of all tableaux of left key $\lambda\nu$ and right key $\lambda\sigma$.

For example, for $\lambda = [4,2,0]$, $\sigma = [3,2,1]$, $\nu = [2,1,3]$, the choice $\sigma = s_2 s_1 s_2$ gives that the sum of tableaux of left key $[2,4,0]$, right key $[0,2,4]$, which is $\begin{array}{|c|c|cc}\hline 2 & 3 \\ \hline 1 & 2 & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{3} \\ \hline\end{array} + \begin{array}{|c|c|cc}\hline 2 & 3 \\ \hline 1 & 2 & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{3} \\ \hline\end{array}$, is equal to $t_{420}\theta_2 s_1 \theta_2$. The reduced decomposition $s_1 s_2 s_1$ expresses the same sum of two tableaux as $t_{420} s_1 \theta_2 \theta_1 + t_{420} \theta_1 \theta_2 s_1 + t_{420} s_1 \theta_2 s_1$, with cancelations occurring.

§4. Tableauhedron

Let λ be a strict partition: $\lambda = [\lambda_1, \ldots, \lambda_n]$, $\lambda_1 > \lambda_2 > \cdots > \lambda_n \ge 0$, ζ be a permutation in $\mathfrak{S}_n$, $v = \lambda\zeta$. The *v-tableauhedron* Γ_v is the directed graph with vertices the tableaux $t_{\lambda\sigma} = t_\lambda \sigma$, for $\sigma \in \mathfrak{S}_n$ such that $\sigma \le \zeta$, an edge $[t_{\lambda\nu}, t_{\lambda\sigma}]$ being labelled by the sum of all tableaux in $\mathfrak{Tab}(\lambda\nu, \lambda\sigma)$, that is all tableaux of left key $t_{\lambda\nu}$, right key $t_{\lambda\sigma}$.

For example, for $v = [4,2,0]s_1s_2 = [2,0,4]$, the edges and vertices of Γ_{204} are given by the expansion of $t_{420}\,(s_1{+}\theta_1{+}1)(s_2{+}\theta_2{+}1)$:



$$t_{420}\,\theta_1 = \begin{array}{|c|c|cc}\hline 2&2\\ \hline 1&1&\multicolumn{1}{c|}{1}&\multicolumn{1}{c|}{2}\\ \hline\end{array}$$

$$t_{420}\,\theta_2 = \begin{array}{|c|c|cc}\hline 2&3\\ \hline 1&1&\multicolumn{1}{c|}{1}&\multicolumn{1}{c|}{1}\\ \hline\end{array}$$

$$t_{420}\,\theta_1 s_2 = \begin{array}{|c|c|cc}\hline 3&3\\ \hline 1&1&\multicolumn{1}{c|}{1}&\multicolumn{1}{c|}{3}\\ \hline\end{array}$$

$$t_{420}\,\theta_1\theta_2 = \begin{array}{|c|c|cc}\hline 2&2\\ \hline 1&1&\multicolumn{1}{c|}{1}&\multicolumn{1}{c|}{3}\\ \hline\end{array} + \begin{array}{|c|c|cc}\hline 2&3\\ \hline 1&1&\multicolumn{1}{c|}{1}&\multicolumn{1}{c|}{3}\\ \hline\end{array}$$

and $t_{420}s_1\theta_2 = \begin{array}{|c|c|cc}\hline 2&3\\ \hline 1&1&\multicolumn{1}{c|}{3}&\multicolumn{1}{c|}{3}\\ \hline\end{array} + \begin{array}{|c|c|cc}\hline 2&2\\ \hline 1&1&\multicolumn{1}{c|}{3}&\multicolumn{1}{c|}{3}\\ \hline\end{array} + \begin{array}{|c|c|cc}\hline 2&2\\ \hline 1&1&\multicolumn{1}{c|}{2}&\multicolumn{1}{c|}{3}\\ \hline\end{array}$.

The *polynomial incidence matrix* of Γ_v is the matrix E with polynomial entries $E_{\lambda\nu,\lambda\sigma} = \mho\left(\sum_{t\in\mathfrak{Tab}(\lambda\nu,\lambda\sigma)} t\right)$ (and thus $E_{\lambda\sigma,\lambda\sigma} = \mho(t_{\lambda\sigma})$). This matrix is graded by the length of permutations: $E = E_0 + E_1 + \cdots + E_{\binom{n}{2}}$. Recall that $\clubsuit$ is the morphism $x_1 \to x_1^{-1}, \ldots, x_n \to x_n^{-1}$, and, accordingly, that $E_i^{\clubsuit}$ is the image of E_i under $\clubsuit$.

This section is devoted to show the Euler relation

$$(E_0 + E_1 + E_2 + \cdots)\left(E_0^{\clubsuit} - E_1^{\clubsuit} + E_2^{\clubsuit} - E_3^{\clubsuit} + \cdots\right) = 1\,.$$

Notice that the divided difference $\pi_{x_i^{-1},x_{i+1}^{-1}}$ is equal to $-\theta_i$, and therefore, the image of $\mathbb{U}_i$, as an operator on $\mathfrak{Pol}(\mathbf{x}^{\pm}) \otimes \mathfrak{Pol}(\mathbf{x}^{\pm})$, under the inversion of variables of the first component, is

$$\widetilde{\mathbb{U}}_i = -\theta_i \otimes s_i + 1 \otimes \theta_i\,.$$

With $\widetilde{\mathbb{U}}_i$ instead of $\mathbb{U}_i$, Corollary 3.2 entails

Corollary 4.1. *Let λ be a strict partition, $\zeta \ge \nu$ be two permutations, $s_i \cdots s_j = \zeta$ be a reduced decomposition of ζ. Then*

$$(4)\qquad \sum_{\nu\le\sigma\le\zeta} (-1)^{\ell(\zeta)-\ell(\sigma)} \sum_{\substack{t_1\in\mathfrak{Tab}(\lambda\nu,\lambda\sigma)\\ t_2\in\mathfrak{Tab}(\lambda\sigma,\lambda\zeta)}} t_1 \otimes t_2$$

$$= \sum_{\epsilon_i\in\{0,1\}} t_\lambda \otimes t_\lambda (s_i\otimes s_i)^{\epsilon_i}\widetilde{\mathbb{U}}_i^{1-\epsilon_i} \cdots (s_j\otimes s_j)^{\epsilon_j}\widetilde{\mathbb{U}}_j^{1-\epsilon_j}\,,$$

sum over all ϵ such that $\pi_i^{\epsilon_i} \cdots \pi_j^{\epsilon_j} = \pi_\nu$.

For example, let $\zeta = [3,2,1]$, $\nu = [2,1,3]$. For the choice of the reduced decomposition $\zeta = s_2 s_1 s_2$, the right hand side is $t_\lambda \otimes t_\lambda \widetilde{\mathbb{U}}_2 (s_1 \otimes s_1) \widetilde{\mathbb{U}}_2 = t_\lambda \otimes t_\lambda \big(s_1 \otimes \theta_2 s_1 \theta_2 - s_1 \theta_2 \otimes \theta_2 s_1 s_2 - \theta_2 s_1 \otimes s_2 s_1 \theta_2 + \theta_2 s_1 \theta_2 \otimes s_2 s_1 s_2 \big)$.

The reduced decomposition $s_1 s_2 s_1$ involves more terms, the right hand side being now $t_\lambda \otimes t_\lambda (s_1 \otimes s_1) \widetilde{\mathbb{U}}_2 \widetilde{\mathbb{U}}_1 + t_\lambda \otimes t_\lambda \widetilde{\mathbb{U}}_1 \widetilde{\mathbb{U}}_2 (s_1 \otimes s_1) + t_\lambda \otimes t_\lambda (s_1 \otimes s_1) \widetilde{\mathbb{U}}_2 (s_1 \otimes s_1)$.

Let $\widetilde{\mho}$ be the algebra morphism $\mathfrak{Free} \otimes \mathfrak{Free} \to \mathfrak{Pol}(\mathbf{x}^{\pm})$ defined by $\widetilde{\mho}(i \otimes j) = x_i^{-1} x_j$.

Lemma 4.2. *One has the commutation*

$$\begin{tikzcd}
\mathfrak{Free} \otimes \mathfrak{Free} \arrow[r, "\widetilde{\mathbb{U}}_i"] \arrow[d, "\widetilde{\mho}"'] & \mathfrak{Free} \otimes \mathfrak{Free} \arrow[d, "\widetilde{\mho}"] \\
\mathfrak{Pol}(\mathbf{x}^{\pm}) \arrow[r, "\widehat{\pi}_i"] & \mathfrak{Pol}(\mathbf{x}^{\pm})
\end{tikzcd}$$

Proof. The statement is an assertion about words in an alphabet in two letters, say $\{1,2\}$. Pairing letters as much as possible, and ignoring them afterwards, one is reduced to products of the type $1^\alpha \otimes 1^\beta$, $1^\alpha \otimes 2^\beta$, $2^\alpha \otimes 1^\beta$, $2^\alpha \otimes 2^\beta$. Let us check only the first case, with $\alpha, \beta \geq 1$, the other cases being similar. Then

$$1^\alpha \otimes 1^\beta \widetilde{\mathbb{U}}_1 = - \left(\sum_{i=0}^{\alpha-1} 1^{\alpha-1-i} 2^{i+1} \right) \otimes 2^\beta + 1^\alpha \otimes \left(\sum_{j=0}^{\beta-1} 1^{\beta-1-j} 2^{j+1} \right) .$$

On the other hand, the image of $x_1^{\beta-\alpha} = \widetilde{\mho}(1^\alpha \otimes 1^\beta)$ under $\widehat{\pi}_1$ is $(x_1^{\beta-\alpha} - x_1^{\alpha-\beta})\,(x_1 - x_2)^{-1} x_2 = \widetilde{\mho}\left(1^\alpha \otimes 1^\beta \widetilde{\mathbb{U}}_1\right)$. QED

Notice that for any tableau t, and any i, then $\widetilde{\mho}\left(t \otimes t \widetilde{\mathbb{U}}_i\right) = 0$, since it is equal to $1 \widehat{\pi}_i$.

We are now in position to check that the tableauhedron is Eulerian.

Theorem 4.3. *Let $\lambda \in \mathbb{N}^n$ be a strict partition, η be a permutation in $\mathfrak{S}_n$, $v = \lambda\eta$. Then the tableauhedron Γ_v is Eulerian, that is for any* *pair of permutations $\nu < \zeta \leq \eta$ one has the nullity*

$$\widetilde{\mho}\left(\sum_{\substack{t_1\in\mathfrak{Tab}(\lambda\nu,\lambda\sigma)\\ t_2\in\mathfrak{Tab}(\lambda\sigma,\lambda\zeta)}} (-1)^{\ell(\zeta)-\ell(\sigma)} t_1\otimes t_2\right) = 0\,.$$

Proof. According to Corollary 4.1, the sum $\sum t_1 \otimes t_2$ is equal, when $\nu \neq \zeta$, to a sum of terms of the type $t_\lambda \otimes t_\lambda (s_i \otimes s_i)\cdots(s_j\otimes s_j)\widetilde{\mathbb{U}}_k\cdots = t\otimes t\,\widetilde{\mathbb{U}}_k\cdots$. Therefore, it vanishes under $\widetilde{\mho}$. QED

Continuing with the same example Γ_{204}, the sum in the theorem is

$$t_{420}\otimes t_{420}\,\widetilde{\mathbb{U}}_1\widetilde{\mathbb{U}}_2 = \sum\nolimits_{t_5} t_{420}\otimes t_5 - \sum\nolimits_{t_3} t_1\otimes t_3 - t_2\otimes t_4 + \sum\nolimits_{t_5} t_5\otimes t_{204}\,.$$

$$\begin{tikzcd}
 & t_{420} \arrow[dl, "t_1"'] \arrow[dr, "t_2"] \arrow[dd, "\sum t_5"] & \\
t_{240} \arrow[dr, "\sum t_3"'] & & t_{402} \arrow[dl, "t_4"] \\
 & t_{204} &
\end{tikzcd}$$

The Eulerian property of the tableauhedron translates into the fact that the following two matrices are inverse of each other (dots replacing 0's).

$$\begin{bmatrix}
{x_2}^2{x_1}^4 & x_2x_3{x_1}^4 & {x_2}^3{x_1}^3 & x_2{x_3}^2{x_1}^3 + {x_2}^2{x_1}^3x_3 \\
\cdot & {x_3}^2{x_1}^4 & \cdot & {x_3}^3{x_1}^3 \\
\cdot & \cdot & {x_2}^4{x_1}^2 & x_2{x_3}^3{x_1}^2 + {x_2}^2{x_1}^2{x_3}^2 + {x_2}^3{x_1}^2x_3 \\
\cdot & \cdot & \cdot & {x_3}^4{x_1}^2
\end{bmatrix},$$

$$\begin{bmatrix}
\frac{1}{{x_2}^2{x_1}^4} & -\frac{1}{x_2x_3{x_1}^4} & -\frac{1}{{x_2}^3{x_1}^3} & \frac{1}{x_2{x_3}^2{x_1}^3} + \frac{1}{{x_2}^2{x_1}^3x_3} \\
\cdot & \frac{1}{{x_3}^2{x_1}^4} & \cdot & -\frac{1}{{x_3}^3{x_1}^3} \\
\cdot & \cdot & \frac{1}{{x_2}^4{x_1}^2} & -\frac{1}{x_2{x_3}^3{x_1}^2} - \frac{1}{{x_2}^2{x_1}^2{x_3}^2} - \frac{1}{{x_2}^3{x_1}^2x_3} \\
\cdot & \cdot & \cdot & \frac{1}{{x_3}^4{x_1}^2}
\end{bmatrix}.$$

§5. Postulation

A flag variety $\mathcal{F}(\mathbb{C}^n)$ is equipped with tautological line bundles $L_1, \dots, L_n$. For each partition λ, let $L^\lambda = L_1^{\otimes \lambda_1} \otimes \cdots \otimes L_n^{\otimes \lambda_n}$. Schubert subvarieties of the flag variety are indexed by permutations. Geometers need to determine the dimensions of the space of sections of the line bundles L^λ over Schubert subvarieties.

Combinatorially, thanks to [2], these dimensions (*postulation*) are equal to the specialization $x_1 = 1 = \cdots = x_n$ of the polynomials $x^\lambda \pi_\sigma$, i.e. of the polynomials K_v.

Instead of considering a single L^λ, one prefers to take all its powers at the same time, and thus compute the specialization $\mathbf{x} = \mathbf{1}$ of the generating series

$$1 + zK_v + z^2 K_{2v} + z^3 K_{3v} + \dots$$

where $v = \lambda\sigma$, and kv denotes $[kv_1, \dots, kv_n]$. In terms of the preceding sections, one has to count the number of chains of tableaux $t_1 \cdots t_r$ with $\mathcal{C}_+(t_r) \le v$.

In the case of Graßmannians, this problem was solved by Hodge [4] (with the help of Littlewood [12] for what concerns the determinantal formula giving the postulation number). One has in that case to enumerate chains of partitions (with respect to inclusion of diagrams), or, equivalently, to enumerate plane partitions.

Instead of computing only dimensions, I determined with Fulton [3] the class of $L^\lambda \mathcal{O}_\sigma$, where $\mathcal{O}_\sigma$ is the structure sheaf of a Schubert variety, in the Grothendieck ring $K^0(\mathcal{F})$ of classes of vector bundles over the flag variety. Dimensions now occur as the number of terms in the expansion of $[L^\lambda \mathcal{O}_\sigma]$ as a sum of classes $[\mathcal{O}_\nu]$. This involves a combinatorics of Grothendieck polynomials still relying on the notion of keys. Pittie and Ram [13] generalized this work to the flag varieties associated to any semisimple group.

Chains may be obtained using powers of matrices. Indeed, if $M^{\lambda,\sigma}$ is the matrix with entries $M^{\lambda,\sigma}_{\nu,\zeta} = \sum_{t \in \mathfrak{Tab}(\lambda\nu, \lambda\zeta)} t$, $\nu \le \zeta \le \sigma$, and if B^σ is the restriction of the incidence matrix of the Ehresmann-Bruhat order to the permutations $\le \sigma$, then the N-E entry of $\left(B^\sigma M^{\lambda,\sigma}\right)^r$ is precisely $\widehat{K}^{\mathcal{F}}_{rv}$.

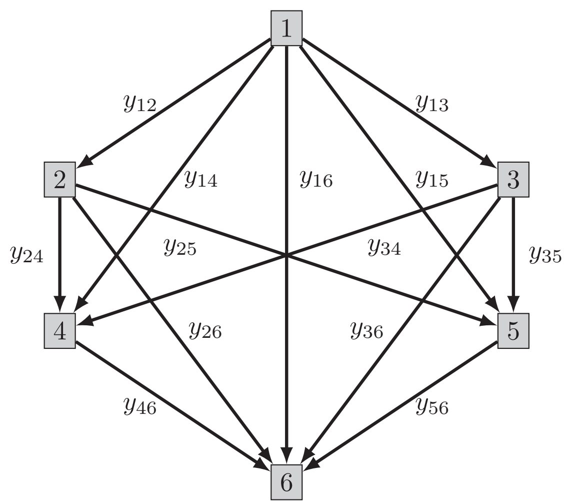
Let us go back to generating series. Given a strict partition λ and a permutation σ, the function $F_{\lambda,\sigma} = (1 - zx^\lambda)^{-1} \widehat{\pi}_\sigma \Big|_{\mathbf{x}=\mathbf{1}}$ is rational with denominator $(1-z)^{\ell(\sigma)+1}$. Its numerator, that we shall denote $\widehat{\mathcal{E}}_{\lambda,\sigma}(z)$, is of degree $\ell(\sigma)-1$. The function $F_{\lambda,\sigma}$ is the specialization $\mathbf{x} = \mathbf{1}$ of the generating function $1+z\widehat{K}_v^{\mathcal{F}}+z^2\widehat{K}_{2v}^{\mathcal{F}}+z^3\widehat{K}_{3v}^{\mathcal{F}}+\ldots$ of chains of tableaux $t_1\cdots t_r$ with $\mathcal{C}_+(t_r)=v=\lambda\sigma$.

The denominator of $(1-zx^\lambda)^{-1}\widehat{\pi}_\sigma$ is of degree the number of permutations $\leq\sigma$. To obtain instead a rational function with denominator of the right degree $\ell(\sigma)+1$, one can use Leibniz formula. For example, for $\sigma=s_1s_2$, $\lambda=[6,3,0]$, one has

$$\frac{1}{1-zx^{630}}\widehat{\pi}_1=(1-zx^{360})\widehat{\pi}_1\frac{1}{(1-zx^{630})(1-zx^{360})}=\frac{1}{1-zx^{630}}\frac{\widehat{K}_{360}}{1-zx^{360}}$$

$$\frac{1}{1-zx^{630}}\widehat{\pi}_1\widehat{\pi}_2=\frac{1}{(1-zx^{630})(1-zx^{360})}\frac{(\widehat{K}_{360}(1-zx^{306})\widehat{\pi}_2)}{1-zx^{306}} + \frac{1}{1-zx^{630}}\frac{\widehat{K}_{603}}{1-zx^{603}}\frac{(\widehat{K}_{360}s_2)}{1-zx^{306}}\ .$$

For $\sigma=s_1s_2s_1$, and λ a strict arbitrary partition, writing y_{ij} for the multiplicities of the edges,



one obtains

$$\begin{aligned}\widehat{\mathcal{E}}_{\lambda,s_1s_2s_1}(z) &= z\left(1+\sum\nolimits_{i<6}y_{i6}\right)\\ &+z^2\left(2+\sum w_1+\frac{1}{2}\sum w_2+\sum \overset{\bullet}{w}_3+\sum w_3-2y_{16}\right)\\ &+z^3\left(1+y_{12}+y_{13}+\frac{1}{2}(y_{12}y_{24}+y_{12}y_{25}+y_{13}y_{34}+y_{13}y_{35})+y_{16}\right)\ ,\end{aligned}$$

where w_k, $k = 1, 2, 3$, is a path consisting of k consecutive length 1-edges, and $\overset{\bullet}{w}_3$ is the image of w_3 under suppression of the middle edge.

There are relations among the multiplicities y_{ij}, apart from the Euler relations, and we leave it as an open problem to obtain a satisfactory expression of the numerators $\widehat{\mathcal{E}}_{\lambda,\sigma}(z)$ and of the rational functions $(1-zx^\lambda)\pi_\sigma$ and $(1-zx^\lambda)\widehat{\pi}_\sigma$.

The enumeration of paths in the Ehresmann-Bruhat order resort to the theory of *shellability* [1]. In our case, we have replaced enumeration of paths or chains by the computation of rational functions.

§6. Eulerian polynomials

The choice $\lambda = \rho = [n-1, \dots, 0]$ corresponds to the Plücker embedding of the flag variety. Let $\mathcal{E}_\sigma(z)$ be the numerator of the function $(1 - zx^\rho)^{-1}\pi_\sigma\big|_{\mathbf{x}=\mathbf{1}}$.

We shall show in this section that the family $\{\mathcal{E}_\sigma(z)\}$ is a natural generalization of the family of Eulerian polynomials $\mathbb{E}_2(z) = 1+z$, $\mathbb{E}_3(z) = 1+4z+z^2$, $\mathbb{E}_4(z) = 1+11z+11z^2+z^3, \dots$. In fact the polynomial $\mathcal{E}_\sigma(z)$, when σ is a maximal element of Young subgroup, is equal to some Eulerian polynomial.

The relation

$$\omega x^{-\rho}\pi_i x^\rho \omega = -\widehat{\pi}_{n-i}$$

allows to relate the functions $(1-zx^\rho)^{-1}\widehat{\pi}_\sigma$ and $(1-z^{-1}x^\rho)\pi_\sigma$. According to [7, Prop. 5.1], one has

$$(-1)^{\ell(\sigma)+1}(1 - zx^\rho)^{-1}\,\widehat{\pi}_\sigma\, x^\rho = z^{-1}(1 - z^{-1}x^\rho)\,\pi_\sigma \clubsuit\,. \tag{5}$$

Hence the polynomials $\widehat{\mathcal{E}}_{\rho,\sigma}$ and $\mathcal{E}_\sigma(z)$ are reciprocal of each other, that is $z^{\ell(\sigma)}\mathcal{E}_\sigma(z^{-1}) = \widehat{\mathcal{E}}_\sigma(z)$, and one can choose to use either π_σ or $\widehat{\pi}_\sigma$ to compute them.

Let D be the isobaric derivative on functions of a variable z,

$$D:\ f(z) \to D(f) := \frac{d}{dz}\big(zf(z)\big)\,.$$

The *Eulerian polynomial* $\mathbb{E}_n(z)$ is defined as

$$\mathbb{E}_n(z) = (1-z)^{n-1}\, D^n\big((1-z)^{-1}\big)\,.$$

The operator D is the limit, for $x_1 = x_2 = z$ of the *isobaric divided difference* π_1 acting on functions of x_1. Indeed, for any positive integer k, $x_1^k\,\pi_1 = x_1^k + x_1^{k-1}x_2 + \cdots + x_2^k$, and this last function specializes to $z^k(k+1)$.

Equivalently, one can view D as the following composite operator :

$$f(z) \to f(zx_1) \to f(zx_1)\,\pi_1\big|_{x_1=x_2=1}\,.$$

This has the following generalisation.

Definition 6.1. *Let w be an arbitrary permutation of $\mathfrak{S}_n$, and let $\rho = [n-1, \ldots, 1, 0]$. Then D_w is the following operator on functions of z :*

$$f(z) \to f(zx^\rho)\,\pi_w\big|_{x_1=\cdots=x_n=1}\,.$$

For example, testing the operators on $1/(1-z)$, one finds that

$$D_{123} = 1,\ D_{213} = D_{132} = D,\ D_{231} = D_{312} = \frac{3}{2}D^3 - \frac{1}{2}D,\ D_{321} = D^3\,.$$

Indeed, $(1-z)^{-1}D_{213} = (1-z)^{-1}D_{132} = (1-z)^{-2}$, $(1-z)^{-1}D_{231} = (1-z)^{-1}D_{312} = (1+2z)(1-z)^{-3}$, $(1-z)^{-1}D_{321} = (1+4z+z^2)(1-z)^{-4}$.

Let us notice that if w belongs to a Young subgroup $\mathfrak{S}_k \times \mathfrak{S}_{n-k}$, i.e. if w factorizes into $w'w''$, with w' fixing $k+1, \ldots n$, and w'' fixing $1, \ldots, k$, then the operator D_w factorizes into $D_{w'}D_{w''}$.

The reader may be willing to show that, when w is the maximal cycle $[n, 1, \ldots, n-1]$, then

$$D_w = \frac{1}{n!}(nD-0)(nD-1)\cdots(nD-n+2)\,.$$

For example,

$$D_{21} = \frac{1}{2!}(2D-0),\, D_{312} = \frac{1}{3!}(3D-0)(3D-1),$$
$$D_{4123} = \frac{1}{4!}(4D-0)(4D-1)(4D-2)\,.$$

We give below the polynomials $\mathcal{E}_{n,1,\ldots,n-1}(z)$ for $n = 2, \ldots, 9$ (read by rows, increasing powers of z) :

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 10 & 5 & 0 & 0 & 0 & 0 & 0 \\
1 & 37 & 73 & 14 & 0 & 0 & 0 & 0 \\
1 & 126 & 651 & 476 & 42 & 0 & 0 & 0 \\
1 & 422 & 4770 & 8530 & 2952 & 132 & 0 & 0 \\
1 & 1422 & 31851 & 114612 & 95943 & 17886 & 429 & 0 \\
1 & 4853 & 202953 & 1317133 & 2162033 & 987261 & 107305 & 1430
\end{bmatrix}$$

The following property implies that Eulerian polynomials are associated to maximal permutations of Young subgroups.

Proposition 6.2. *Given n, then for the maximal permutation $\omega = [n, \ldots, 1]$, one has $D_\omega = D^{\binom{n}{2}}$.*

Proof. For any k, writing $k\rho$ for the partition $[k(n-1), k(n-2), \ldots, k, 0]$, then the image of $(x^\rho)^k$ under $\pi_{n\ldots1}$ is the Schur function $S_{k\rho}(x_1 + \cdots + x_n)$, which specializes into

$$S_{k\rho}(n) = \prod_{\square \in diagram(k\rho)} \frac{n + c_\square}{h_\square} = (k+1)^{\binom{n}{2}} \ .$$

But this last constant is precisely the eigenvalue of $D^{\binom{n}{2}}$ when acting on z^k. QED

Corollary 6.3. *Given a Young subgroup $\mathfrak{S}_a \times \mathfrak{S}_b \times \cdots \times \mathfrak{S}_d$, let $\omega_{a\ldots d}$ be its element of maximal length $\ell = \binom{a}{2} + \cdots + \binom{d}{2}$. Then $D_{\omega_{a\ldots d}}$ coincides with D^ℓ, and sends $(1-z)^{-1}$ onto $\mathbb{E}_\ell(z)\,(1-z)^{-\ell-1}$.*

For example, $D_{321654} = D^6 = D_{4321}$ sends $(1-z)^{-1}$ onto

$$(z^5 + 57\,z^4 + 302\,z^3 + 302\,z^2 + 57\,z + 1)\,(1-z)^{-7}$$

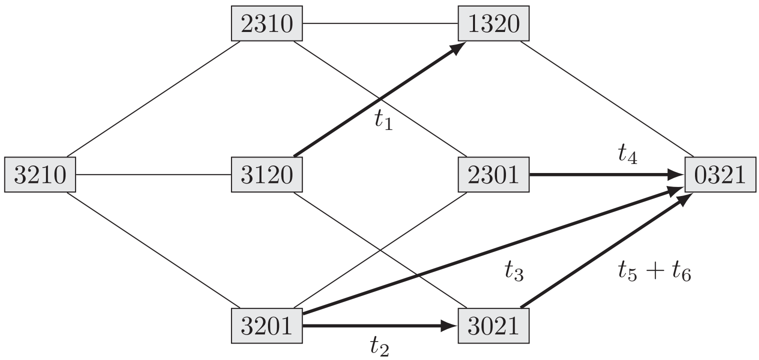
One may use the tableauhedron to compute the polynomials $\mathcal{E}_\sigma(z)$ and $\widehat{\mathcal{E}}_\sigma(z)$. For example, for the cycle $\sigma = [4, 1, 2, 3]$, one has

$$(1 - zt_{3210})^{-1}\, \widehat{\pi}_3 \widehat{\pi}_2 \widehat{\pi}_1 = z\widehat{K}^{\mathcal{F}}_{0321} + z^2 \widehat{K}^{\mathcal{F}}_{0642} + z^3 \widehat{K}^{\mathcal{F}}_{0963} + \cdots$$

The tableauhedron Γ_{0321} has eight vertices, and edges are labelled by 6 tableaux:

$$t_1 = \begin{array}{|c|c|c|} \hline 3 \\ \cline{1-1} 2 & 3 \\ \cline{1-3} 1 & 1 & 2 \\ \hline \end{array}\ , \ t_2 = \begin{array}{|c|c|c|} \hline 4 \\ \cline{1-1} 2 & 3 \\ \cline{1-3} 1 & 1 & 1 \\ \hline \end{array}\ , \ t_3 = \begin{array}{|c|c|c|} \hline 4 \\ \cline{1-1} 2 & 3 \\ \cline{1-3} 1 & 1 & 2 \\ \hline \end{array}\ , \ t_4 = \begin{array}{|c|c|c|} \hline 4 \\ \cline{1-1} 2 & 3 \\ \cline{1-3} 1 & 2 & 2 \\ \hline \end{array}\ , \ t_5 = \begin{array}{|c|c|c|} \hline 4 \\ \cline{1-1} 3 & 3 \\ \cline{1-3} 1 & 1 & 2 \\ \hline \end{array}\ ,$$

$$t_6 = \begin{array}{|c|c|c|} \hline 4 \\ \cline{1-1} 3 & 3 \\ \cline{1-3} 1 & 2 & 2 \\ \hline \end{array}\ .$$



Keeping $t_1, \ldots, t_6$ as indeterminates, and specializing $t_{3210}, \ldots, t_{0321}$ to 1, one finds

$$
\begin{aligned}
&(z\widehat{K}^{\mathcal{F}}_{0321} + z^2\widehat{K}^{\mathcal{F}}_{0642} + \cdots)(1-z)^{-4} \\
&= z(1+t_3 + t_4 + t_5 + t_6)+z^2(4+t_1 + t_2 - t_3 + t_4 + t_5 + t_6+t_2t_5+t_2t_6)+z^3 \,.
\end{aligned}
$$

One could keep t_1, t_2, t_4, t_5, t_6 as arbitrary parameter, t_3 being determined by the Euler relation $2t_3 = t_2(t_5 + t_6)$.

Specializing all t_i to 1, and reversing the polynomial (we have used $\widehat{\pi}_3\widehat{\pi}_2\widehat{\pi}_1$ instead of $\pi_3\pi_2\pi_1$), one finds that $\mathcal{E}_{4123} = 1 + 10z + 5z^2$.

A systematic study of the polynomials $\mathcal{E}_\sigma(z)$ would be welcome. We list below the polynomials for $\sigma \in \mathfrak{S}_4$, correcting an error in [7].

$\mathcal{E}_{1234}(z) = 1 = \mathcal{E}_{1243}(z) = \mathcal{E}_{1324}(z) = \mathcal{E}_{2134}(z)$, $\mathcal{E}_{1342}(z) = \mathcal{E}_{3124}(z) = \mathcal{E}_{2314}(z) = \mathcal{E}_{1423}(z) = 1 + 2z, \mathcal{E}_{2143}(z) = 1 + z$, $\mathcal{E}_{3142}(z) = 1 + 9z + 4z^2, \mathcal{E}_{2413}(z) = 1 + 8z + 3z^2$, $\mathcal{E}_{2341}(z) = \mathcal{E}_{4123}(z) = 1 + 10z + 5z^2$, $\mathcal{E}_{3214}(z) = \mathcal{E}_{1432}(z) = 1+4z+z^2$, $\mathcal{E}_{2413}(z) = \mathcal{E}_{4213}(z) = 1+18z+24z^2 + 3z^3$, $\mathcal{E}_{4132}(z) = \mathcal{E}_{3241}(z) = 1 + 19z + 25z^2 + 3z^3$, $\mathcal{E}_{3412}(z) = 1 + 25z + 44z^2+8z^3$, $\mathcal{E}_{4312}(z) = \mathcal{E}_{3421}(z) = 1+38z+120z^2+58z^3+3z^4$, $\mathcal{E}_{4231}(z) = 1+43z+150z^2+81z^3+5z^4$, $\mathcal{E}_{4321}(z) = 1+57z+302z^2+302z^3+57z^4+z^5$.

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