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Stable quasimaps to holomorphic symplectic quotients

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Abstract.

We study the moduli space of twisted quasimaps from a fixed smooth projective curve to a Nakajima's quiver variety and the moduli space of δ-stable framed twisted quiver bundles with moment map relations. We show that they carry symmetric obstruction theories and when δ is large enough, they exactly coincide. These results generalize works by D.E. Diaconescu [12] about the ADHM quiver, in the framework of the quasimap theory of I. Ciocan-Fontanine, D. Maulik and the author [8, 9].

§1. Introduction

There are, so far, two classes of moduli examples which naturally carry symmetric obstruction theories:

- Moduli of stable objects in the abelian category of coherent sheaves on a Calabi-Yau threefold ([29, 23]).
- Moduli of stable objects in the abelian category of representations of a quiver with relations given by a superpotential ([28]).

In this paper, we add one more such class:

- Moduli of stable objects in the abelian category of coherent $\mathbf{M}$-twisted quiver sheaves on a projective smooth curve C associated to double quivers with moment map relations. Here $\mathbf{M}$ is a certain collection of pairs of line bundles on C such that the product of each pair is isomorphic to the canonical sheaf ω_C.

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It arises as a curve counting on a Nakajima's quiver variety. If C is an elliptic curve and Y is a holomorphic symplectic quasi-projective variety, then the canonical perfect obstruction theory on the Hom scheme $\mathrm{Hom}(C, Y)$ of morphisms from C to Y is symmetric since for a map $f : C \to Y$,

$$\begin{aligned} \mathrm{ob}(f)^* := H^1(C, f^*\mathcal{T}_Y)^* &\cong H^0(C, f^*\Omega_Y \otimes \omega_C) \\ &\cong H^0(C, f^*\mathcal{T}_Y) = \mathrm{def}(f), \end{aligned}$$

where $\mathrm{ob}(f)$ is the obstruction space at f and $\mathrm{def}(f)$ is the 1st order deformation space at f.

When Y is given by a holomorphic symplectic quotient of an affine algebraic variety X by a reductive complex algebraic group G action, we can apply the quasimap construction of [8, 9] in order to compactify $\mathrm{Hom}(C, Y)$ relatively over the affine quotient of X. The advantage of the construction compared to the stable map construction is that we can keep the fixed domain curve C so that the natural extension of the obstruction theory still remains symmetric (Proposition 3.5). When the affine variety X carries a torus T action commuting with the G action such that the fixed locus $(X /\!\!/\!\!/ G)^T$ is proper over $\mathbb{C}$, then the induced T-fixed locus on the moduli space of stable quasimaps is also proper over $\mathbb{C}$. Therefore, in this case one can obtain well-defined localization residue "invariants" by the virtual localization [15].

If Y is a Nakajima's quiver variety, a quasimap is nothing but a quiver bundle on C. Using the idea of twisted quiver bundles, we obtain the corresponding notion of twisted quasimap and allow any genus of C. We show that the moduli space of stable $\mathbf{M}$-twisted quasimaps carries a symmetric obstruction theory (Theorem 4.3) and the stability as quasimaps coincides with the δ-slope stability of quiver bundles for $\delta \gg 0$ (Proposition 6.10). In the case of ADHM quiver, these facts have been shown by Diaconescu in [11, 12], which is, together with the quasimap construction [8, 9], the main source of inspiration for this work. We also show that with respect to any δ-slope stability, the moduli space of stable twisted quiver bundles carries a canonical symmetric obstruction theory (Theorem 6.6), following [11, 12].

The typical examples for double quivers in our framework are ADHM quiver ([11, 12, 7, 6]) and the framed ADE quivers. The wall-crossing phenomena are studied in [12, 7, 6, 17] which show that the conditions corresponding to (a) (the moduli stack is analytic-locally the critical locus of a holomorphic function on a smooth complex domain) and (b) (the Euler-like form is numerical) in §1.5 [16] hold in our setting.

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§2. Holomorphic symplectic quotients

2.1. Symplectic quotients

We set up holomorphic symplectic quotients suitable for our purpose.

Let X be a smooth affine algebraic variety over $\mathbb{C}$ equipped with a holomorphic (algebraic) symplectic form

$$\omega : \mathcal{T}_X \otimes \mathcal{T}_X \to \mathcal{O}_X.$$

Suppose a connected reductive complex algebraic group G acts on X as a hamiltonian action, i.e., the action preserves ω and there is a so-called *complex* moment map $\mu : X \to \mathfrak{g}^*$ where $\mathfrak{g}$ denotes the Lie algebra of G. This complex moment map μ is, by definition, a G-equivariant algebraic morphism such that for every tangent vector ξ of X and every $g \in \mathfrak{g}$, the equation

$$\langle d\mu(\xi), g\rangle = \omega(d\alpha(g), \xi) \tag{2.1}$$

holds where $d\alpha$ is the derivative of the map $\alpha : G \to \mathrm{Aut}(X)$ induced from the G-action on X. Here G-action on $\mathfrak{g}^*$ is the coadjoint representation Ad^*.

Choose an Ad^*-invariant element λ in $\mathfrak{g}^*$. Denote $\mu^{-1}(\lambda)$ by W. Suppose $W \hookrightarrow X$ is a regular imbedding of codimension $\dim \mathfrak{g}$. For a character $\chi \in \mathrm{Hom}(G, \mathbb{C}^\times)$, let $\mathbb{C}_\chi$ be the G-representation space $\mathbb{C}$ associated to χ and let L be the linearization $W \times \mathbb{C}_\chi$. The *holomorphic symplectic quotient* $X /\!\!/\!\!/_{\lambda,\chi} G$ is defined to be a GIT quotient

$$W /\!\!/_\chi G := \mathbf{Proj}(\bigoplus_{k\geq 0} H^0(W, L^k)^G)$$

(see [13, §4.5] and references therein for Nakajima's quiver variety case).

Definition 2.1. A point p of W is called *semistable* with respect to χ if there is $s \in H^0(W, L^k)^G$ for some $k > 0$ such that $s(p) \neq 0$. The semistable point p is called *stable* with respect to χ if the stabilizer G_p is finite and the action of G on $\{q : s(q) \neq 0\}$ is closed (i.e., every orbit is closed in $\{q : s(q) \neq 0\}$).

The stable (resp. semistable) locus is the open subset of W consisting of all stable (resp. semistable) points. According to the proof of Theorem 1.10 of [21], the GIT quotient is the categorical quotient of the semistable locus W^{ss} and it contains as an open subset the geometrical quotient of the stable locus W^s (see also [13, Definition 2.2.3, Theorem 2.2.4]).

The equation (2.1) shows that the stabilizer of a point $p \in X$ is a finite group if and only if $d\mu_{|_{T_pX}} : T_pX \to \mathfrak{g}^*$ is surjective if and only if p is a regular point of the moment map μ. Hence the stack quotient $[W^s/G]$ of the stable locus W^s is a holomorphic symplectic smooth DM stack.

For a 1-parameter subgroup, i.e., a homomorphism $\kappa : \mathbb{C}^\times \to G$, denote by $\langle \chi, \kappa \rangle$ the exponent m of $t^m = \chi(\kappa(t))$, $t \in \mathbb{C}^\times$. There is a numerical criterion for semistable and stable points due to King [18]. This will be used in Section 6.1.

Proposition 2.2. ([18]) *A point p is semistable (resp. stable) if*

$$\langle \chi, \kappa \rangle \geq 0, \ \textit{(resp. } \langle \chi, \kappa \rangle > 0\textit{)}$$

for any nontrivial 1-parameter subgroup κ for which the limit of $\kappa(t) \cdot p$ as $t \to 0$ exists.

Proof. The G-action on X is linear in the sense that there is a G-equivariant closed embedding of X into a G-linear space V (see e.g. [19], page 94). Since G is geometrically reductive, the semistability of p is equivalent to the fact that the closure of the orbit $G\cdot(p,1)$ in $W \times \mathbb{C}_{-\chi}$ for $(p,1) \in W \times \mathbb{C}_{-\chi}$ is disjoint from the zero section $W \times \{0\}$ as in [18, Lemma 2.2]. This translation shows that a point p of W is (semi)stable if and only if it is (semi)stable as a point of the G-space V with respect χ. Now the proof immediately follows from [18, Proposition 2.5]. Q.E.D.

Remark 2.3. In the above, the sign convention is correct since $(g \cdot s)(p) = \chi(g)^l s(g^{-1} \cdot p)$ for $s \in H^0(W, L^l)$.

2.2. Symmetry

The derivatives of α and μ together give rise to a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_X \otimes \mathfrak{g} & \xrightarrow{d\alpha} & \mathcal{T}_X & \xrightarrow{d\mu} & \mathcal{O}_X \otimes \mathfrak{g}^* & \longrightarrow & 0 \\
\downarrow & & \mathrm{id}\downarrow & & \omega\downarrow{=-\omega^\vee} & & \downarrow -\mathrm{id} & & \downarrow \\
0 & \longrightarrow & \mathcal{O}_X \otimes \mathfrak{g} & \xrightarrow[d\mu^\vee]{} & \Omega_X & \xrightarrow[d\alpha^\vee]{} & \mathcal{O}_X \otimes \mathfrak{g}^* & \longrightarrow & 0
\end{array}$$

of G-sheaves due to equation (2.1). Here the G-action on $\mathfrak{g}$ is the adjoint representation Ad. The first horizontal line of the diagram is a G-equivariant complex after the restriction to W. Note that the 3-term perfect G-equivariant complex

$$\mathbb{T}_{[W/G]} := \left[\mathcal{O}_X \otimes \mathfrak{g} \xrightarrow{d\alpha} \mathcal{T}_X \xrightarrow{d\mu} \mathcal{O}_X \otimes \mathfrak{g}^*\right]_{|W}$$

concentrated in degrees $-1, 0, 1$ is nothing but (the pullback of) the tangent complex $\mathbb{T}_{[W/G]}$ of the quotient stack $[W/G]$. Note that the commutative diagram shows that

$$\mathbb{T}^{\vee}_{[W/G]} \cong \mathbb{T}_{[W/G]} \tag{2.2}$$

as complexes of G-sheaves.

§3. Stable Quasimaps

The notion of stable quasimaps appeared for a compactification of maps from a smooth projective curve C to a suitable GIT quotient with no strictly semistable points. We first begin with conventions and definitions.

Definition 3.1. By a *principal G-bundle* $\pi : P \to Y$ on a scheme Y, we mean a scheme P with a free *left* G-action which is étale locally trivial, i.e, there is an étale surjective morphism from a scheme Y' to Y making the pullback $P \times_Y Y'$ isomorphic to $G \times Y'$ as a G-space over Y'. By a *morphism between two principal bundles* on Y, we mean a G-equivariant morphism over Y.

If P is a principal bundle with right action as usual, then we will consider it as a principal bundle with left action through $G \to G$, $g \mapsto g^{-1}$.

Definition 3.2. Define a *degree class* β of P as the homomorphism

$$\beta : \mathrm{Hom}(G, \mathbb{Z}) \to \mathbb{Z}, \quad \theta \mapsto \deg(P \times_G \mathbb{C}_\theta).$$

3.1. Quasimaps

We are interested in 1-morphisms $[u]$ from C to the quotient stack $[W/G]$. Such morphisms correspond to pairs (P, u) of principal G-bundles P on C and sections u of fiber bundle $P \times_G W$ on C. For such a pair (P, u), $[u]$ is the composite of

$$C \xrightarrow{u} P \times_G W = [(P \times W)/G] \xrightarrow{\mathrm{pr}_2} [W/G] \ ,$$

where pr_2 is the projection to the second factor.

Definition 3.3. ([8, 9]) In this paper, a pair

$$(P, u), \text{ i.e., } [u] : C \to [W/G]$$

is called a *quasimap* to $[W^s/G]$. The pair is called *stable* with a *rigid* domain curve C if the preimage $u^{-1}(P \times_G W^s)$ is nonempty.

In this paper, we always fix C. Often, we will consider u also as a G-equivariant map from P to W. Two quasimaps (P, u) and (P', u') are said isomorphic if there is an isomorphism

$$\begin{array}{ccc} P & \xrightarrow{\phi} & P' \\ & \searrow \quad \swarrow & \\ & C & \end{array} \tag{3.1}$$

of principal G-bundles preserving sections, that is, $\tilde{\phi} \circ u = u'$ where $\tilde{\phi} : P \times_G W \to P' \times_G W$ is the map induced from ϕ.

The *degree class* of (P, u) is defined to be the degree class of P. The *moduli stack*

$$QG_\beta$$

of stable quasimaps with a rigid domain C and a fixed degree class β is an open substack of Hom stack $\mathrm{Hom}(C, [W/G])$ which is an Artin stack locally of finite type over $\mathbb{C}$ (see [20, Proposition 2.11]).

3.2. Some results from [9]

The space QG_β is a finite type DM stack by the boundedness theorem [9, Theorem 3.2.5]. In the paper [9], QG_β is denoted by

$$Qmap_{g(C),0}([W^s/G], \beta; C)$$

and called the *quasimap graph space*. (Here we do not need the condition that $W^s = W^{ss}$.)

There is a natural morphism from QG_β to the *affine quotient* $W/\!\!/_0 G := \mathrm{Spec}\mathbb{C}[W]^G$ obtained by assignment:

$$(P, u) \mapsto \mathrm{Im}(C \stackrel{[u]}{\to} [W/G] \to W/\!\!/_0 G),$$

where the composite is a constant map since C is projective and the affine quotient is affine.

It is shown in [9] that QG_β is proper over $W/\!\!/_0 G$ if $W^s = W^{ss}$ and G acts on W^s freely (for the proof see also §4.4).

3.3. Symmetric obstruction theory

When C is an elliptic curve, we shall show that QG_β carries a symmetric obstruction theory. We begin with recalling the definition.

Definition 3.4. ([4]) A perfect obstruction theory $E \to \mathbb{L}_{\mathfrak{M}}$ for a finite type DM-stack $\mathfrak{M}$ is called *symmetric* if it is endowed with an isomorphism $\sigma : E \to E^\vee[1]$ in the derived category of coherent sheaves on $\mathfrak{M}$ such that $\sigma^\vee[1] = \sigma$ holds, where $\mathbb{L}_{\mathfrak{M}}$ is the cotangent complex of $\mathfrak{M}$ relative to $\mathbb{C}$.

Denote the natural maps as in diagram

$$\begin{array}{ccc} C \times QG_\beta & \xrightarrow{[u]} & [W/G] \\ \downarrow{\scriptstyle \pi} & & \\ QG_\beta & & \end{array} \tag{3.2}$$

We use the universal maps (e.g. u) by the same letters. Now we can obtain an obstruction theory

$$\mathbb{E} := (R^\bullet \pi_* [u]^* \mathbb{T}_{[W/G]})^\vee \to \mathbb{L}_{QG_\beta} \tag{3.3}$$

as in [3, Proposition 6.3].

Let $\mathcal{P}$ denote the universal G-bundle on $C \times QG_\beta$ and let

$$\mathcal{P}(\mathbb{T}_{[W/G]})$$

denote the complex on $\mathcal{P} \times_G W$ associated to the G-equivariant perfect complex $\mathbb{T}_{[W/G]}$. It is useful to note that

$$[u]^* \mathbb{T}_{[W/G]} = u^* \mathcal{P}(\mathbb{T}_{[W/G]}).$$

Proposition 3.5. *If C is an elliptic curve, then the obstruction theory* (3.3) *for QG_β is a symmetric obstruction theory.*

Proof. By the Grothendieck duality and (2.2),

$$\begin{aligned} (R^\bullet \pi_*([u]^* \mathbb{T}_{[W/G]}))^\vee &\cong R^\bullet \pi_*(([u]^* \mathbb{T}^\vee_{[W/G]}) \otimes \omega_\pi[1]) \\ &\cong R^\bullet \pi_*([u]^* \mathbb{T}_{[W/G]})[1]. \end{aligned}$$

Denote by σ the composite of the isomorphisms. We need to check that $\sigma^\vee = \sigma[-1]$. This can be seen by the equality $\mathrm{tr}(i(t_{0,0}) \otimes t_{0,1}) = \mathrm{tr}(i(t_{0,1}) \otimes t_{0,0})$ for $t_{0,0} \in H^0(C, [u]^* \mathbb{T}_{[W/G]}), t_{0,1} \in H^1(C, [u]^* \mathbb{T}_{[W/G]})$, where tr is the trace map and i is the isomorphism $[u]^*(\mathbb{T}_{[W/G]}) \to [u]^*(\mathbb{T}_{[W/G]})^\vee \otimes \omega_C$.

The symmetry together with the fact that there are no nontrivial infinitesimal automorphisms implies that $\mathbb{E}$ is a two term perfect complex in the derived category of coherent sheaves. Q.E.D.

§4. Twisted Quasimaps

By twisting quasimaps, we shall generalize the examples of symmetric obstruction theory above for any smooth projective curve C. The ideas of the twisting in our context and Theorem 4.3 are originated from [11, 12].

4.1. Set-up

Let A be a finite index set and for $a \in A$ let (M_{a^+}, M_{a^-}) be a pair of line bundles on C equipped with an isomorphism $M_{a^+} \otimes M_{a^-} \cong \omega_C$. For $a \in A$, consider a G-representation space V_{a^+} and denote by V_{a^-} the G-space $V_{a^+}^*$, the dual space of V_{a^+}. Let X be the canonical symplectic space

$$T^*V_+ = V_+ \oplus V_-, \text{ where } V_+ := \bigoplus_{a \in A} V_{a^+}.$$

Suppose that the moment map μ is the sum of each moment map μ_a, *bilinear* on $X_a := T^*V_{a^+}$.

4.2. Twisted quasimaps

Consider the total space of a G-equivariant vector bundle on C:

$$\mathbf{M} \otimes X := \mathrm{Tot}(\bigoplus_{a \in A, i=\pm} M_{a^i} \otimes V_{a^i})$$

and define the pullback of μ_a:

$$\begin{array}{cccc} \tilde{\mu}_a : & \mathrm{Tot}(\sum_i M_{a^i} \otimes V_{a^i}) & \to & \mathrm{Tot}(\omega_C \otimes \mathfrak{g}^*), \\ & m_{a^+} \otimes v_{a^+} + m_{a^-} \otimes v_{a^-} & \mapsto & m_{a^+} m_{a^-} \otimes \mu_a(v_{a^+}, v_{a^-}), \end{array}$$

which is well-defined by the bilinearity of μ_a. This in turn defines

$$\mathbf{M} \otimes W := \tilde{\mu}^{-1}(0), \text{ where } \tilde{\mu} := \sum_{a \in A} \tilde{\mu}_a.$$

Note that $\mathbf{M} \otimes W$ is a G-invariant subvariety of $\mathbf{M} \otimes X$ (i.e., $G \cdot \mathbf{M} \otimes W \subset \mathbf{M} \otimes W$). We denote by $\tilde{\alpha}$ the induced homomorphism $G \to \mathrm{Aut}(\mathbf{M} \otimes V)$. Locally over C, $\mathbf{M} \otimes X$ (resp. $\mathbf{M} \otimes W$) can be G-equivariantly trivialized with fiber X (resp. W).

Let P be a principal G-bundle and write

$$P_{\mathbf{M}}(X) := (P \times_C (\mathbf{M} \otimes X))/G,$$

which is (the total space of) a vector bundle on C. Let u be a section of $P_{\mathbf{M}}(X)$.

Definition 4.1. A pair (P, u) is called a $\mathbf{M}$-*twisted quasimap* to $X/\!\!/\!\!/G$ if it is a map over C from C to the stack quotient $[\mathbf{M} \otimes W/G]$.

Let $\Gamma(C, [\mathbf{M} \otimes W/G])$ be the stack of $\mathbf{M}$-twisted quasimaps (P, u):

$$\begin{array}{ccc} C & \xrightarrow{(P,u)} & [\mathbf{M} \otimes W/G] \\ & \underset{=}{\searrow} & \downarrow{\scriptstyle \rho} \\ & & C \end{array} \quad , \text{ with } \rho \circ (P, u) = \mathrm{id}$$

for the canonical projection ρ. The stack is a closed substack of the Artin stack $\mathrm{Hom}(C, [\mathbf{M} \otimes W/G])$.

With respect to the character χ, we may consider the stable locus $(\mathbf{M} \otimes W)^s$ and its quotient stack $[(\mathbf{M} \otimes W)^s/G]$.

Definition 4.2. We say that a $\mathbf{M}$-twisted quasimap $[u] := (P, u)$ to $X/\!\!/\!\!/G$ is *stable* if $[u]^{-1}([(\mathbf{M} \otimes W)^s/G])$ is nonempty.

Define

$$QG_{\mathbf{M},\beta}$$

to be the stack of stable $\mathbf{M}$-twisted quasimaps with the degree class β.

Theorem 4.3. (1) *The stack $QG_{\mathbf{M},\beta}$ is a finite type DM stack over $\mathbb{C}$ with a canonical symmetric perfect obstruction theory.*

(2) *Assume that $W^s = W^{ss}$ and G acts on W^s freely. Suppose that for all a^i there is a torus T action on V_{a^i} which is linear and commutes with the G action such that the induced T actions on X and $\mathbf{M} \otimes X$ preserve subvarieties W and $\mathbf{M} \otimes W$. If $(W/\!\!/_0 G)^T$ is isolated, then the T-fixed part $QG^T_{\mathbf{M},\beta}$ of $QG_{\mathbf{M},\beta}$ is proper over $\mathbb{C}$.*

Proof. 1) Since a stable $\mathbf{M}$-twisted quasimap is a rational map to a DM stack $[(\mathbf{M} \otimes W)^s/G]$, it has no infinitesimal automorphisms. Thus the Artin stack $QG_{\mathbf{M},\beta}$ is DM. It is of finite type over $\mathbb{C}$ by [9, Theorem 3.2.5]. The obstruction part will be proven in §4.3.

2) This will be proven in §4.4. Q.E.D.

4.3. Obstruction Theory

Denote universal morphisms as in the following commuting diagram:

$$\begin{array}{ccc} C \times QG_{\mathbf{M},\beta} & \xrightarrow{[u]} & [\mathbf{M} \otimes W/G] \\ {\scriptstyle\pi}\downarrow & \searrow^{\pi_C} & \downarrow{\scriptstyle\rho} \\ QG_{\mathbf{M},\beta} & & C. \end{array}$$

From this, immediately we obtain a natural obstruction theory for $QG_{\mathbf{M},\beta}$:

$$\mathbb{E}_{\mathbf{M}} := (R\pi_*[u]^*\mathbb{T}_\rho)^\vee \to \mathbb{L}_{QG_{\mathbf{M},\beta}}. \tag{4.1}$$

Here the relative tangent complex $\mathbb{T}_\rho$ of ρ is explicitly

$$\begin{array}{cccccc} \mathcal{O}_{\mathbf{M}\otimes W} \otimes \mathfrak{g} & \xrightarrow{d\tilde{\alpha}} & \mathcal{T}_{\mathbf{M}\otimes X}|_{\mathbf{M}\otimes W} & \xrightarrow{d\tilde{\mu}} & \rho^*\omega_C \otimes \mathfrak{g}^* & \\ \text{at } \sum m_{a^i} \otimes w_{a^i} \overset{g}{} & \mapsto & \sum \begin{matrix} m_{a^i} \\ \otimes d\alpha(g)_{w_{a^i}} \end{matrix} & & & \\ & & \sum m_{a^i} \otimes v_{a^i} & \mapsto & \sum \begin{matrix} m_{a^+} m_{a^-} \\ \otimes d\mu_a(v_{a^+}, v_{a^-}) \end{matrix} & \end{array}$$

concentrated at $[-1, 1]$.

Parallel to (2.2),

$$\mathbb{T}_\rho^\vee \otimes \rho^*\omega_C \cong \mathbb{T}_\rho \tag{4.2}$$

by the symplectic form and the isomorphisms $M_{a^+} \otimes M_{a^-} \cong \omega_C$. Therefore, (4.1) is a symmetric obstruction theory for $QG_{\mathbf{M},\beta}$ as in the proof of Proposition 3.5.

4.4. T-action

We verify Theorem 4.3 part (2) by the valuative criterion for properness, following [9]. First we note that by the assumption $(W/\!\!/_0 G)^T =$ isolated, $(\mathbf{M} \otimes W/\!\!/_0 G)^T$ is isomorphic to a projective scheme $(W/\!\!/_0 G)^T \times C$; and by the assumption $W^s = W^{ss}$ with the free G-action, $[(\mathbf{M} \otimes W)^s/G]^T = [(\mathbf{M} \otimes W)^{ss}/G]^T = (\mathbf{M} \otimes W/\!\!/G)^T$ which is a projective scheme over $(\mathbf{M} \otimes W/\!\!/_0 G)^T$.

Let $(\Delta, 0)$ be a pointed smooth curve and consider a family of T-fixed stable quasimaps $(P, u) : C \times (\Delta \setminus \{0\}) \to [\mathbf{M} \otimes W/G]$. Since $[(\mathbf{M} \otimes W)^s/G]^T$ is a projective scheme, the stable quasimap (P, u) extends uniquely to a stable quasimap $(\bar{P}, \bar{u}) : C \times \Delta \to [\mathbf{M} \otimes W/G]$ possibly except finitely many points on the central fiber of $C \times \Delta$. We may extend $(\bar{P}, \bar{u})$ to a quasimap defined everywhere on $C \times \Delta$ by extending the principal bundle P and the section u. The latter extensions are possible due to [9, Lemma 4.3.2] and Hartogs' theorem, respectively. The uniqueness of these extensions are clear.

Remark 4.4. If the isomorphism (4.2) is T-equivariant, then the T-fixed part of $\mathbb{E}_{\mathbf{M}}$ (see [15, Proposition 1]) is a symmetric obstruction theory for $QG^{T}_{\mathbf{M},\beta}$.

§5. The Quiver Example

5.1. Nakajima's quiver varieties

The typical examples of $X/\!\!/\!\!/G$, to which Theorem 4.3 can be applied, are Nakajima's quiver varieties. We set up quiver varieties, basically following [13, 22].

Let Q be a finite quiver, which means that it is equipped with two finite sets Q_0, Q_1 and two maps $t, h : Q_1 \to Q_0$. We call Q_0 (resp. Q_1) the vertex (resp. arrow) set of Q and ta (resp. ha) the tail (resp. head) of arrow $a \in Q_1$. Let $\overline{Q}$ be the double quiver of Q. It is defined as follows. The vertex set $\overline{Q}_0$ of $\overline{Q}$ is exactly Q_0. For each arrow a in Q_1, create exactly two associated arrows a^+, a^- of $\overline{Q}$, by making the head (resp. tail) of a^+ (resp. a^-) = the head (resp. tail) of a.

Given a *dimension vector* $v = (v_i) \in \mathbb{N}^{Q_0}$, let

$$\mathrm{Rep}(\overline{Q}, v) := \bigoplus_{a \in Q_1} \left(\mathrm{Hom}(\mathbb{C}^{v_{ta^+}}, \mathbb{C}^{v_{ha^+}}) \oplus \mathrm{Hom}(\mathbb{C}^{v_{ta^-}}, \mathbb{C}^{v_{ha^-}})\right).$$

After $\mathrm{Rep}(\overline{Q}, v)$ being canonically identified with the total space of the cotangent bundle of

$$\bigoplus_{a \in Q_1} \mathrm{Hom}(\mathbb{C}^{v_{ta^+}}, \mathbb{C}^{v_{ha^+}}),$$

the space

$$X := \mathrm{Rep}(\overline{Q}, v) \tag{5.1}$$

can be regarded as a holomorphic symplectic manifold with the canonical holomorphic symplectic form ω. The linear symplectic form is defined by

$$\omega(A, B) = \sum_{a} \mathrm{tr}(A_{a^+} B_{a^-}) - \mathrm{tr}(A_{a^-} B_{a^+})$$

for tangent vectors $A, B \in T_x\mathrm{Rep}(\overline{Q}, v) = \mathrm{Rep}(\overline{Q}, v)$ at $x \in X$.

Fix a subset Q_0' of Q_0 and let $Q_0'' := Q_0 \setminus Q_0'$. Let

$$G := \prod_{i \in Q_0'} GL_{v_i}(\mathbb{C}).$$

There is a natural hamiltonian G-action on X with a moment map

$$\mu(x) = \bigoplus_{i \in Q_0'} \Big(\sum_{a \in \overline{Q}_1 : t\overline{a} = i} (-1)^{|a|} x_a x_{\overline{a}} \Big) \text{ for } x = (x_a)_{a \in \overline{Q}_1} \in X \tag{5.2}$$

where $\overline{a}$ denotes the opposite arrow of a and $|b^+| = 0$, $|b^-| = 1$ for $b \in Q_1$. Here we identified $\mathfrak{g}$ with its dual by trace. The equation (2.1) can be checked easily since for $g \in \mathfrak{g}$, $d\alpha(g)$ is the linear vector field $(g_{ha}x_a - x_a g_{ta})_a \in \mathrm{Rep}(\overline{Q}, v)$.

Let

$$\lambda = \sum_{i \in Q_0'} \lambda_i \mathrm{Id}_{\mathrm{End}_{v_i}(\mathbb{C})} \in \bigoplus_{i \in Q_0'} \mathrm{End}_{v_i}(\mathbb{C}),\ \lambda_i \in \mathbb{C}$$

and choose a character $\chi = (\theta_i) \in \mathbb{Z}^{Q_0'}$ of G by sending $g \in G$ to $\prod_{i \in Q_0'} (\det g_i)^{\theta_i} \in \mathbb{C}^\times$. This defines a linearization $L = W \times \mathbb{C}_\chi$ where $W := \mu^{-1}(\lambda)$. Now we may consider the holomorphic symplectic quotient $X/\!\!/\!\!/G := W/\!\!/_\chi G$. We call the quotient a *quiver variety.*

5.2. Twisted quasimap to a Nakajima quiver variety

In this subsection let $\lambda = 0$. For each arrow $a \in Q_1$, fix two line bundles $M_{a^\pm}$ on C with a fixed isomorphism $M_{a^+} \otimes M_{a^-} \to \omega_C$. Below we identify $M_{a^+} \otimes M_{a^-} \cong \omega_C$ and for $M_{a^-} \otimes M_{a^+} \cong \omega_C$ we will use the natural isomorphism followed by the given one: $M_{a^-} \otimes M_{a^+} \to M_{a^+} \otimes M_{a^-} \to \omega_C$.

We rewrite the notion of $\mathbf{M}$-twisted quasimaps to the quiver variety $X/\!\!/\!\!/G$ in terms of twisted quiver bundles.

Definition 5.1. A pair (P, u) is called a $\mathbf{M}$-*twisted quasimap* to $X/\!\!/\!\!/G$ with a rigid domain curve C if:

- P is a principal G-bundle on C.
- $u = (u_a)_{a \in \overline{Q}_1}$ where

$$u_a \in \Gamma(C, P \times_G \mathrm{Hom}(\mathbb{C}^{v_{ta}}, \mathbb{C}^{v_{ha}}) \otimes M_a).$$

- The section u satisfies the "moment map" equation (for $\lambda = 0$): for all $i \in Q_0'$,

$$\sum_{a \in \overline{Q}_1 : t\overline{a} = i} (-1)^{|a|} (u_a \otimes \mathrm{Id}_{M_{\overline{a}}}) \circ u_{\overline{a}}) = 0.$$

An $\mathbf{M}$-twisted quasimap is called *stable* if u hits unstable locus only at finitely many points of C.

This $\mathbf{M}$-twisted quasimap is a generalization of a stable ADHM sheaf of Diaconescu [11] from the point of view of quasimaps [8, 9].

5.3. Obstruction theory

We note that $[u]^*\mathbb{T}_\rho$ becomes

$$\begin{aligned} 0 \to \mathcal{P} \times_G \mathfrak{g} \quad &\to \quad \bigoplus\nolimits_{a \in Q_1} (\mathcal{H}om(\mathcal{V}_{ta^+}, \mathcal{V}_{ha^+}) \otimes \pi_C^* M_{a^+} \\ &\qquad\qquad \oplus \mathcal{H}om(\mathcal{V}_{ta^-}, \mathcal{V}_{ha^-}) \otimes \pi_C^* M_{a^-}) \\ &\to \quad (\mathcal{P} \times_G \mathfrak{g}^*) \otimes \pi_C^* \omega_C \to 0, \end{aligned}$$

where $\mathcal{V}_i = \mathcal{P} \times_G \mathbb{C}^{v_i}$. This is a generalization of the complex (4.3) in [11, Definition 4.3].

§6. Stabilities on Quiver bundles

6.1. King's stability

The stability with respect to χ we used in the previous section can be rephrased as a Rudakov's stability condition on a suitable abelian category of representations of the path algebra $\mathbb{C}\overline{Q}$ with relations (see [24], [13, §2.3]).

The path algebra is a $\mathbb{C}$-algebra spanned by, as a $\mathbb{C}$-vector space, all finite paths $a_n ... a_1$ of consecutive arrows and an extra arrow e_i, for each $i \in Q_0$, where $a_l \in \overline{Q}_1$ and $ha_l = ta_{l+1}$ for all l. The product is given by a sort of compositions. Namely, $(a_{j_m} ... a_{j_1}) \cdot (a_{k_n} .... a_{k_1})$ is $a_{j_n} ... a_{j_1} a_{k_n} .... a_{k_1}$ if $ha_{k_n} = ta_{j_1}$, 0 otherwise. The generators are subject to relations: $e_i^2 = e_i$; $e_i a$ is a if $ha = i$, 0 otherwise; and ae_i is a if $ta = i$, 0 otherwise. Impose one more relation coming from the moment map together with an element $\lambda \in \mathbb{C}^{Q_0'}$:

$$\sum_{a \in \overline{Q}_1 : t\overline{a} \in Q_1'} (-1)^{|a|} a\overline{a} = \sum_{i \in Q_0'} \lambda_i e_i$$

which will be denoted symbolically by $\mu - \lambda = 0$.

Denote by $(\mu - \lambda)$ the two-sided ideal generated by $\mu - \lambda$. Note that a $\mathbb{C}\overline{Q}/(\mu - \lambda)$-module V amounts to data $(V_i, \phi_a)_{i \in Q_0, a \in \overline{Q}_1}$ where V_i is a $\mathbb{C}$-vector space and ϕ_a is a homomorphism from V_{ta} to V_{ha} subject to the condition coming from $\mu - \lambda = 0$. A homomorphism from a module (V_i, ϕ_a) to another (V_i', ϕ_a') is nothing but a collection $(\varphi_i)_{i \in Q_0}$ of linear maps $\varphi_i : V_i \to V_i'$ making $\phi_a' \circ \varphi_{ta} = \varphi_{ha} \circ \phi_a$ for every $a \in \overline{Q}_1$.

Now, following [18], we are ready to reformulate the χ-stability of $V = (V_i, \phi_a) \in \mathrm{Rep}(\overline{Q}, v)$ satisfying the moment map relation $\mu - \lambda = 0$. Suppose $v_0 := \sum_{i \in Q_0''} \dim V_i \neq 0$. Let $\theta_0 = -(\sum_{i \in Q_0'} \theta_i v_i)/v_0$ and for a $\mathbb{C}\overline{Q}/(\mu - \lambda)$-module S write

$$\theta(S) = \theta_0 \dim S_0 + \sum_{i \in Q_0'} \theta_i \dim S_i,$$

where $S_0 := \oplus_{i \in Q_0''} \dim S_i$. Note that $\theta(V) = 0$.

Theorem 6.1. *The followings are equivalent.*

(1) *V is semistable (resp. stable) with respect to $\chi = (\theta_i)$ as a point in $\mu^{-1}(\lambda) \subset \mathrm{Rep}(\overline{Q}, v)$.*

(2) *$\theta(S) \geq 0$ (resp. $\theta(S) > 0$) for every nonzero, proper, submodule S of V with $S_0 = V_0$ or $S_0 = 0$.*

Proof. A slightly modified argument of [18, §3], which uses Proposition 2.2, works with group G, too. We provide its details. For a 1-parameter subgroup $\kappa : \mathbb{C}^\times \to G$, we may consider a linear action on the vector space $V = \oplus_{i \in Q_0} \mathbb{C}^{v_i}$ via the standard action of $G = \prod_{i \in Q_0'} GL_{v_i}(\mathbb{C})$. If $V^n := \{p \in V : \kappa(t) \cdot p = t^m p, m \geq n, \forall t \in \mathbb{C}^\times\}$, we obtain a $\mathbb{Z}$-filtration $\cdots \supset V^{n-1} \supset V^n \supset \cdots$.

For (2) $\Rightarrow$ (1), suppose $\lim_{t \to 0} \kappa(t) \cdot (V_i, \phi_a)$ exists for a nontrivial κ. The existence means that for every n, V^n is a submodule of V. Since $(V^n)_0 = V_0$ or 0, we see that $\theta(V^n) \geq 0$. Since the κ-action is nontrivial, some V^n is a proper, nonzero submodule of V. Now the proof follows from the identity $\langle \chi, \kappa \rangle = \sum_{n \in \mathbb{Z}} n\theta(V^n/V^{n+1}) = \sum_{n \in \mathbb{Z}} \theta(V^n)$.

For (1) $\Rightarrow$ (2), let S be a nonzero, proper submodule of V. If $S_0 = V_0$, then there is a nontrivial κ such that $V^{-1} = V$, $V^0 = S$ and $V^1 = 0$. If $S_0 = \{0\}$, then there is a nontrivial κ such that $V^0 = V$, $V^1 = S$ and $V^2 = \{0\}$. In either case, $\lim_{t \to 0} \kappa(t) \cdot (V_i, \phi_a)$ exists and $\langle \chi, \kappa \rangle = \sum_{n \in \mathbb{Z}} n\theta(V^n/V^{n+1}) = \sum_{n \in \mathbb{Z}} \theta(V^n) = \theta(S)$. Therefore, we conclude the proof by Proposition 2.2. Q.E.D.

This motivates the following. First, *from now on we assume that Q_0'' has exactly one vertex* 0. Fix a finite dimensional vector space K.

Definition 6.2. Denote by $\mathrm{Rep}_\lambda(\overline{Q}, K)$ the category whose objects are finite dimensional $\mathbb{C}\overline{Q}/(\mu - \lambda)$-modules V with an identification $V_0 = \bar{V}_0 \otimes_{\mathbb{C}} K$ for some vector space $\bar{V}_0$ and whose morphisms, say from V to V' are module homomorphisms $(\varphi_i)_{i \in Q_0}$ satisfying the condition: $\varphi_0 = \bar{\varphi}_0 \otimes \mathrm{id}_K : V_0 = \bar{V}_0 \otimes K \to \bar{V}_0{}' \otimes K = V_0'$ where $\bar{\varphi}_0$ is a $\mathbb{C}$-linear map from $\bar{V}_0$ to $\bar{V}_0{}'$. It is clear that $\mathrm{Rep}_\lambda(\overline{Q}, K)$ is an abelian category.

Define a homomorphism

$$Z : \mathbb{Z}^2 \to \mathbb{C}, \quad (x, y) \mapsto y + \sqrt{-1}x.$$

For $V \in \mathrm{Rep}_\lambda(\overline{Q}, K)$, let $Z(V) = Z(v_0, v_1)$ where $v_0 = \dim V_0$, $v_1 = \dim V_1$ and $V_1 := \oplus_{i \in Q_0'} V_i$. Note that for nonzero V, $Z(V) \neq 0$ and $0 \leq \mathrm{Arg}(Z(V)) \leq \pi/2$.

For a $V \in \mathrm{Rep}_\lambda(\overline{Q}, K)$ with $v_0 \neq 0$, if we take

$$\theta_0 := -v_1/v_0, \ \theta_i := 1, \forall i \in Q_0'$$

then $\mathrm{Arg}Z(S) \leq \mathrm{Arg}Z(V)$ if and only if $\theta(S) \geq 0$ (resp. $\mathrm{Arg}Z(S) < \mathrm{Arg}Z(V)$ if and only if $\theta(S) > 0$) for every nonzero proper subobject S of V in $\mathrm{Rep}_\lambda(\overline{Q}, K)$. Therefore, by Theorem 6.1, Rudakov's stability ([24]) defined via the stability function Z coincides with King's θ-stability.

6.2. Quiver sheaves

In this subsection, for every $i \in Q_0'$ fix $\lambda_i \in \Gamma(C, \omega_C)$. For a systematic study of stabilities on quasimaps we interpret quasimaps as linear objects. Fix a finite dimensional vector space K as before.

Definition 6.3. A data $(E_i, \phi_a)_{i \in Q_1, a \in \overline{Q}_1}$ is called $\mathbf{M}$-*twisted quiver (coherent) sheaf on* C with respect to $(\overline{Q}, \lambda)$ if E_i is a (coherent) sheaf of $\mathcal{O}_C$-modules; E_0 is $\bar{V}_0 \otimes_{\mathbb{C}} (K \otimes_{\mathbb{C}} \mathcal{O}_C)$ for some finite dimensional vector space $\bar{V}_0$; and ϕ_a is a $\mathcal{O}_C$-module homomorphism from E_{ta} to $E_{ha} \otimes M_a$. The homomorphisms are subject to relation (which will be denoted also by $\mu - \lambda = 0$):

$$\sum_{a \in \overline{Q}_1 : t\overline{a} = i} (-1)^{|a|} (\phi_a \otimes \mathrm{Id}_{M_{\overline{a}}}) \circ \phi_{\overline{a}} - \mathrm{Id}_{E_i} \otimes \lambda_i = 0, \quad \forall i \in Q_0' \tag{6.1}$$

unless stated otherwise.

Remark 6.4. For the history of the studies of (the moduli spaces of) twisted quiver sheaves usually without the relation, see [1, 2, 10, 14, 25] and references therein. See also [27].

Like a quiver representation and a $\mathcal{O}_C$-sheaf, a quiver sheaf can be considered as a module of the $\mathbf{M}$-twisted path algebra $\mathbf{M}\overline{Q}$ over $\mathcal{O}_C$ ([14, 2]). For each path $p = a_m ... a_1$, let $M_p^\vee = M_{a_m}^\vee \otimes_{\mathcal{O}_C} M_{a_{m-1}}^\vee \otimes_{\mathcal{O}_C} ... \otimes_{\mathcal{O}_C} M_{a_1}^\vee$ and for e_i, let $M_{e_i}^\vee = \mathcal{O}_C$. Let

$$\mathbf{M}\overline{Q}/(\mu - \lambda) = (\bigoplus_{\text{all paths } p} M_p^\vee)/(\mu - \lambda)$$

which has a $\mathcal{O}_C$-algebra structure similar to the path algebra $\mathbb{C}\overline{Q}$. Here $(\mu - \lambda)$ is the two-sided ideal generated by the relations (6.1) for "abstract" ϕ_a: For every local section $\xi \in \omega_C^\vee$, consider

$$(\mu - \lambda)_i(\xi) := \sum_{ha=i} (-1)^{|a|} \xi_a \otimes \xi_{\bar{a}} - \langle \xi, \lambda_i \rangle e_i$$

where $\xi_a \otimes \xi_{\bar{a}}$ is an element in $M_a^\vee \otimes M_{\bar{a}}^\vee$ corresponding to ξ under the given isomorphism $M_a^\vee \otimes M_{\bar{a}}^\vee \cong \omega_C^\vee$. The ideal sheaf $\mu - \lambda$ is defined to be the ideal sheaf generated by $(\mu - \lambda)_i(\xi)$ for all $i \in Q_0'$, $\xi \in \omega_C^\vee$.

Given a $\mathbf{M}\overline{Q}/(\mu - \lambda)$-module structure on E, a $\mathbf{M}$-twisted quiver sheaf can be associated by letting $E_i = M_{e_i}^\vee E$ and $(\phi_a)_{|U}(m_a \otimes s) = m_a s$ for an open set $U \subset C$, $m_a \in M_a^\vee(U)$, $s \in E_{ta}(U)$. Here we regard ϕ_a as a homomorphism from $M_a^\vee \otimes E_{ta}$ to E_{ha}. Conversely, a quiver sheaf defines a module structure on $\oplus E_i$. For full details, see [10, Lemma 2.1, §4], [17, Proposition 2.3], [2, Proposition 5.1].

Denote by $\mathrm{Rep}_C(\overline{Q}, \mathbf{M}, K)$ the abelian category of $\mathbf{M}$-twisted quiver coherent sheaves E with framing $E_0 = \bar{V}_0 \otimes_{\mathbb{C}} (K \otimes_{\mathbb{C}} \mathcal{O}_C)$ for some finite dimensional vector space $\bar{V}_0$. A morphism from (E_i, ϕ_a) to (E_i', ϕ_a') is, by definition, a collection $(\varphi_i)_{i \in Q_0}$ of $\mathcal{O}_C$-homomorphism $\varphi_i : E_i \to E_i'$ making $\phi_a' \circ \varphi_{ta} = (\varphi_{ha} \otimes 1_{M_a}) \circ \phi_a$ for every $a \in \overline{Q}_1$ and satisfying $\varphi_0 = \bar{\varphi}_0 \otimes \mathrm{id}_{K \otimes_{\mathbb{C}} \mathcal{O}_C}$ for some $\mathbb{C}$-linear map $\bar{\varphi}_0$. Denoted by

$$\mathrm{Rep}_{\lambda, C}(\overline{Q}, \mathbf{M}, K)$$

to be the full subcategory of $\mathrm{Rep}_C(\overline{Q}, \mathbf{M}, K)$ whose objects satisfy the moment map relation (6.1).

For $\delta > 0$, define a homomorphism $Z_\delta : \mathbb{Z}^3 \to \mathbb{C}$ by assignments

$$\begin{aligned} Z(v_0, v_1, d) &= Z_{(1)}(v_1, d) + Z_{(2)}(v_0, v_1), \\ Z_{(1)}(v_1, d) &= \tfrac{v_1}{2} + \sqrt{-1} d, \\ Z_{(2)}(v_0, v_1) &= \tfrac{v_1}{2} + \sqrt{-1} \delta v_0. \end{aligned}$$

Also define $Z_\delta(E) \in \mathbb{C}$ by the rank-degree map $\mathrm{Rep}_C(\overline{Q}, \mathbf{M}, K) \to \mathbb{Z}^3$, $E \mapsto (\mathrm{rank} E_0, \mathrm{rank} E_1, \deg E_1)$ followed by Z, where $E_1 := \oplus_{i \in Q_0'} E_i$.

We remark that the homomorphism (followed by $\pi/2$-rotation) is a stability function on $\mathbb{Z}^3$ with Harder-Narasimhan property in the sense of Bridgeland (see [5, Proposition 2.4]). In more elementary way, this gives rise to a Rudakov's stability [24] in an abelian category.

Let $\mu_\delta(E) \in (-\infty, \infty]$ be the slope of $Z_\delta(E)$ for a nonzero quiver sheaf E. We abuse notation by letting μ stand for slopes as well as moment maps. This shouldn't cause any confusion.

Following the δ-stability introduced in [12, Definition 2.1] for ADHM case, we come to this.

Definition 6.5. An $\mathbf{M}$-twisted quiver coherent sheaf

$$E \in \mathrm{Rep}_{\lambda,C}(\overline{Q}, \mathbf{M}, K), \text{ with } \mathrm{rank} E_0 \neq 0$$

is called *δ-semistable* (resp. *δ-stable*) if $\mu_\delta(E') \leq \mu_\delta(E)$ (resp. $\mu_\delta(E') < \mu_\delta(E)$) for every nonzero proper subobject E' of E in $\mathrm{Rep}_{\lambda,C}(\overline{Q}, \mathbf{M}, K)$.

It is necessary that a δ-semistable quiver sheaf E is locally free (i.e., a quiver sheaf with E_i being locally free sheaf for every i). We call a locally free quiver sheaf a *quiver bundle.*

From now on assume that $\lambda = 0$ *and by a* $\mathbf{M}$*-twisted quiver sheaf* E *we mean* $E \in \mathrm{Rep}_{\lambda=0,C}(\overline{Q}, \mathbf{M}, K)$ *so that* E *satisfies the relation* (6.1).

If $X := \mathrm{Rep}(\overline{Q}, v)$ as in (5.1), a $\mathbf{M}$-twisted quasimap to $X/\!\!/\!\!/G$ with degree β amounts to a $\mathbf{M}$-twisted quiver bundle with

$$\mathrm{rank} E_i = v_i, \ \deg E_i = \beta(\det_i),$$

where $\det_i$ is the character of G given by the determinant of i-th general linear group. More precisely speaking, if $W := \mu^{-1}(0)$ (see (5.2)), there is a natural 1-morphism from $\Gamma(C, [\mathbf{M} \otimes W/G])$ to the moduli stack of $\mathbf{M}$-twisted quiver bundles on C with $E_i = P \times_G \mathbb{C}^{v_i}$.

Let $\mathfrak{M}_{\delta,\mathbf{M},\{v_i\}}$ be the stack of δ-stable objects in $\mathrm{Rep}_{\lambda=0,C}(\overline{Q}, \mathbf{M}, K)$ with ranks v_i and let $QG_{\delta,\mathbf{M}}$ be its inverse image under the above natural 1-morphism.

Theorem 6.6. *The moduli space* $QG_{\delta,\mathbf{M},\beta}$ *of* δ*-stable* $\mathbf{M}$*-twisted quiver bundles of degree* β *on* C *is an algebraic space of finite type over* $\mathbb{C}$*, equipped with a natural symmetric obstruction theory.*

Proof. There are no nontrivial automorphisms of stable objects except overall multiplications. The nontrivial overall multiplications of stable objects are not allowed as objects in $QG_{\delta,\mathbf{M}}$ because the framing with the trivial G-action and the δ-stability remove such automorphims in (3.1). Thus $QG_{\delta,\mathbf{M}} \times B\mathbb{C}^* \cong \mathfrak{M}_{\delta,\mathbf{M},\{v_i\}}$ as stacks. The section §4.3 shows the complex corresponding to (4.1) is a symmetric obstruction theory for $QG_{\delta,\mathbf{M},\beta}$ since the δ-stability is an open condition as usual.

We prove the boundedness using Harder-Narasimhan filtration with respect to the standard slope μ_{st}. Let (E, ϕ^a) be μ_δ*-semistable* quiver sheaf. Considering E as a $\mathcal{O}_C$-module $\oplus_i E_i$ on C, take the Harder-Narasimhan (HN) filtration

$$0 = E^0 \subset E^1 \subset ... \subset E^l = E$$

of E for μ_{st} in the category of $\mathcal{O}_C$-modules. Since the HN filtration of a direct sum is a certain sum of each HN filtration, we see $E^i = \oplus_{j\in Q_0} E^i_j$ for $E^i_j := E^i \cap E_j$. Also, note that $E_0 = E^i_0/E^{i-1}_0$ for some i.

Claim: The following inequality holds:

$$\mu_{st}(E^i/E^{i-1}) \leq N_i(\mu_\delta(E), l, \deg \mathbf{M}) \\ := \max\{0, \mu_\delta(E)\} + (l-i)\max\{0, \deg M_a \mid a \in \overline{Q}_1\}.$$

Proof of Claim. We prove Claim by induction on $l-i$. When $i=l$, the claim is true since $\mu_{st}(E^l/E^{l-1}) \leq \mu_{st}(E^l)$. We define a composite

$$\psi^a_i : E^i \to E^i_{ta} \to E_{ha} \otimes M_a \to E_{ha} \otimes M_a / E^i_{ha} \otimes M_a \to E \otimes M_a / E^i \otimes M_a,$$

where the first map is the canonical epimorphism and the second map is the restriction of ϕ^a to E^i_{ta}, the third map is the projection, and the last map is the canonical monomorphism. Define

$$\psi_i = \oplus_{a\in\overline{Q}_1} \psi^a_i : \oplus_a E^i \to \oplus_a (E \otimes M_a / E^i \otimes M_a),$$

where $\oplus_a E^i$ is the sum of $\overline{Q}_1$ - many copies of E^i.

If $\psi_i = 0$, then E^i is a subobject of E in $\mathrm{Rep}_{\lambda=0,C}(\overline{Q}, \mathbf{M}, K)$ since $E^j/E^{j-1} = E_0$ for some j. Hence,

$$\mu'_{st}(E^i) := \frac{\deg E^i}{\mathrm{rank} E^i_1} \leq \mu_\delta(E^i) \leq \mu_\delta(E)$$

which, combined with

$$\mu_{st}(E^i/E^{i-1}) \leq \mu_{st}(E^i) = \mu'_{st}(E^i) - \frac{\mathrm{rank} E^i_0 \deg E^i}{\mathrm{rank} E^i \mathrm{rank} E^i_1},$$

implies

$$\mu_{st}(E^i/E^{i-1}) \leq \begin{cases} 0 & \text{if } \deg E^i \leq 0 \\ \mu_\delta(E) & \text{if } \deg E^i > 0. \end{cases}$$

Thus, $\mu_{st}(E^i/E^{i-1}) \leq \max\{0, \mu_\delta(E)\}$.

If $\psi_i \neq 0$, then $\mu_{st}(E^{i'}/E^{i'-1}) \leq \mu_{st}(E^{i''}/E^{i''-1} \otimes M_a)$ for some a and $i' \leq i \leq i''-1$. By induction hypothesis,

$$\mu_{st}(E^{i'}/E^{i'-1}) \leq N_i(\mu_\delta(E), l, \deg \mathbf{M}).$$

□

Now we are ready to prove the boundedness. Let F be a nonzero subsheaf of E. If

$$0 = F^0 \subset F^1 \subset ... \subset F^m = F$$

is the HN filtration of F for μ_{st}, for every $1 \leq k \leq m$ the natural map

$$F^k/F^{k-1} \to E^i/E^{i-1}$$

is nonzero for some i. Hence,

$$\mu_{st}(F^k/F^{k-1}) \leq \mu_{st}(E^i/E^{i-1})$$

for some i, which implies, combined with Claim, that

$$\mu_{st}(F) \leq N_1(\mu_\delta(E), \mathrm{rank}(E), \deg \mathbf{M}).$$

Now the boundedness follows from [26, Theorem 1.1]. Q.E.D.

Remark 6.7. Note that the above proof of the boundedness holds also for δ-*semistable* $\mathbf{M}$-twisted quiver bundles of degree β on C.

The following Lemma will be used in the proof of Proposition 6.10 below.

Lemma 6.8. *Fix the curve C and a branched covering map $\phi : C \to \mathbb{P}^1$. Let E be a vector bundle on C.*

(1) *If $H^1(C, E \otimes \phi^*\mathcal{O}(m_0)) = 0$ for some $m_0 \geq 0$, then there is a number n_0 depending only on* $\deg E$, rank$E$, *and m_0 satisfying $H^1(C, E^\vee \otimes \phi^*\mathcal{O}(m)) = 0$ for all $m \geq n_0$.*

(2) *If $H^1(C, E^\vee \otimes L) = 0$ for a line bundle L on C, then* $\deg F \leq (|\deg L| + |1-g|)\mathrm{rank}E$ *whenever F is a subsheaf of E.*

Proof. Let b be the degree of the covering map and let $r = \mathrm{rank}(E)$.

For (1): Since $0 = H^1(C, E \otimes \phi^*\mathcal{O}(m_0)) = H^1(\mathbb{P}^1, (\phi_*E)(m_0))$, we have $\phi_*E = \oplus \mathcal{O}(a_i^E)$ with $a_i^E + m_0 \geq -1$. Hence

$$\sum_i a_i^E \geq a_j^E - (m_0+1)(br-1), \text{ for any } j.$$

On the other hand, by Riemann-Roch theorem,

$$\deg \phi_*E + br = \deg E + r(1-g).$$

Therefore, $a_j^E \leq r(1-g) + m_0 br + \deg E$. If we set

$$n_0 = |r(1-g) + m_0 br + \deg E| + l + 1$$

for some positive l with a nonzero homomorphism $\omega_C \to \phi^*\mathcal{O}(l)$, the proof follows by the Serre duality.

For (2): Note that

$$H^0(C, F\otimes\omega_C\otimes L^\vee) \subset H^0(C, E\otimes\omega_C\otimes L^\vee) = H^1(C, E^\vee\otimes L)^\vee = 0,$$

which implies that

$$0 \geq \chi(C, F\otimes\omega_C\otimes L^\vee) = \deg F + \mathrm{rank}F(2g-2-\deg L) + \mathrm{rank}F(1-g).$$

Hence, we conclude the proof. Q.E.D.

Definition 6.9. Denote by $\langle E_0\rangle$ the smallest quiver *saturated* subsheaf of E containing E_0 (which can be obtained by the intersection of all submodules F satisfying $E_0 \subset F$ and E/F is torsion free). Here the saturation means that the sheaf $E_i/\langle E\rangle_i$ is torsion free for every i.

Proposition 6.10. *Fix $v = (v_i) \in \mathbb{N}_{\geq 0}^{Q_0}$ with $v_0 = \dim K$ and let $d = (d_i) \in \mathbb{Z}^{Q_0'}$. There is a number $\delta_0 > 0$ such that for all $\delta \geq \delta_0$, the following conditions are equivalent for $\mathbf{M}$-twisted quiver bundles E with numerical data (v, d) in the category $\mathrm{Rep}_{\lambda=0,C}(\overline{Q}, \mathbf{M}, K)$.*

(1) *δ-semistability.*
(2) *δ-stability.*
(3) *the stability as a $\mathbf{M}$-twisted quasimap to $X/\!/\!/_\theta G$ where $\theta = (1, ..., 1)$.*
(4) *$\langle E_0\rangle = E$.*

Proof. Let $v_1 := \sum_{i\in Q_0'} v_i$. If $v_1 = 0$, then the proof is obvious since in this case any $\mathbf{M}$-twisted sheaf is both δ-stable and quasimap-stable. Thus, we assume that $v_1 \neq 0$.

(3) $\Leftrightarrow$ (4): This follows from that θ-stability of $E|_p$ for general $p \in C$ if and only if $\langle E_0\rangle|_p = E|_p$ for general $p \in C$ if and only if $\langle E_0\rangle = E$.

For a δ-semistable $\mathbf{M}$-twisted quiver coherent sheaf $\{E_i\}$ with the numerical data or for a $\mathbf{M}$-twisted quasimap (P, u) to $X/\!/\!/G$ with the numerical data, let $E := \oplus_i E_i$ or $E := \oplus_i P\times_G \mathbb{C}^{v_i}$ as a coherent sheaf of $\mathcal{O}_C$-modules. By the boundedness in Theorem 6.6 and Theorem 4.3(1), such E's are bounded. Therefore by Lemma 6.8, there is a number m_0 for which the condition $H^1(C, E\otimes\phi^*\mathcal{O}(m_0)) = 0$ in Lemma 6.8 holds. (In fact, such $E^\vee$'s are also bounded.) Let $N = 2(|\deg L| + |1-g|)\mathrm{rank}E$ in Lemma 6.8 with $L = \phi^*\mathcal{O}(n_0)$. Let Δ be a bound of $2|\deg\langle E_0\rangle|$ which is finite since E is bounded and the construction of $\langle E_0\rangle$ from E can be lifted in the level of family.

Denote $(-\infty, +\infty)$-valued slope functions $\mu_{(1)}$ and $\mu_{(2)}$ by

$$\mu_{(1)}(v', d') = 2d'/v_1' \quad \text{and} \quad \mu_{(2)}(v', d') = 2v_0'/v_1' \quad \text{for } v_1' > 0.$$

Note that $2\mu_\delta(E) = \delta\mu_{(2)}(E) + \mu_{(1)}(E)$. Take any number δ_0 such that

$$\delta_0 \mathrm{Min}_{0\neq v'<v}|\mu_{(2)}(v) - \mu_{(2)}(v')| > N + |\mu_{(1)}(E)| + \Delta, \tag{6.2}$$

for any pair $v' = (v'_0, v'_1)$ of integers satisfying $v'_0 = v_0$ or 0, $0 < v'_1 \leq v_1$, and $v' \neq v$.

(1) $\Rightarrow$ (4): If $\langle E_0\rangle \neq E$, then $\mu_{(2)}(\langle E_0\rangle) > \mu_{(2)}(E)$, which implies that $\delta(\mu_{(2)}(E) - \mu_{(2)}(\langle E_0\rangle)) < -|\mu_{(1)}(E)| - \Delta$ by (6.2). Hence $\mu_\delta(E) < \mu_\delta(\langle E_0\rangle)$.

(3) $\Rightarrow$ (2): First, recall that for a point $p \in W \subset X$ the θ-stabilities as a point in W and in X coincide as seen in the proof of Proposition 2.2. Let E' be a nonzero quiver subsheaf of E and let E be stable as a **M**-twisted quasimap. Since $\mu_{(2)}(E') \leq \mu_{(2)}(E)$, which implies that $\mu_\delta(E') < \mu_\delta(E)$ by (6.2). Q.E.D.

Remark 6.11. Theorem 6.6 and Proposition 6.10 generalize the corresponding Diaconescu's works for ADHM case in [11, 12].

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