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Lectures on equivariant Schubert polynomials

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Abstract.

The notes are aimed at a reasonably self-contained introduction to the theory of Schubert polynomials (in a wider sense) for all the classical Lie types in the setting of torus equivariant cohomology. As a powerful combinatorial device, we use the restriction maps to the set of torus fixed points throughout the lectures.

Contents

§0. Introduction

The article is based on a series of lectures delivered in 5th MSJ Seasonal Institute, Schubert calculus 2012, Osaka, where I tried to give a survey on the theory of Schubert polynomials, with particular emphasis on a combinatorial approach which is available in the torus equivariant cohomology.

The theory of Schubert polynomials aims to produce explicit representatives for Schubert classes in the cohomology ring of the flag variety. In the classical literature, cohomology ring of a (generalized) flag variety is naturally presented as a quotient of some polynomial ring. Many attempts to find "canonical" representatives of the Scubert classes in the

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quotient ring have been done. As far as the group G is classical type, a principle of such a choice is given by considering infinite rank setting. In fact, if G_n be the classical group of types A_n, B_n, C_n, and D_n, we can consider the Weyl group of the infinite rank which contains all finite rank Weyl groups of any fixed type. Then we can introduce a Schubert class for each element in the infinite Weyl group. In fact, for the full flag variety of type A, *Les Polynômes de Schubert* introduced by Lascoux and Schützenberger in [37] can be characterized by this principle (*cf.* [11]). For other classical types, it was pointed out by Fomin-Kirillov [10] that the Borel type presentation of cohomology ring is not suited for this purpose. However, as Billey and Haiman [3] showed, a presentation using Schur's Q-functions (or P-functions) introduced in [46] gives a satisfactory theory of Schubert polynomials in the infinite rank setting. This is our model to develop the theory of Schubert polynomials in the setting of torus equivariant cohomology.

The purpose of this whole lectures is to show some fundamental ideas of introducing the double version of the Billey-Haiman polynomials [20]. By the *equivariant Schubert polynomials*, I mean the Schubert polynomials in these context. Although the original papers were published in [18], [20], [21]. I will try to put them in a simple frame so that the basic ideas become manifest.

I intended to include an introduction to the theory of Schubert polynomials relevant for the torus equivariant K-theory also. In fact, the last lecture of this series was an account of the results in [22] which provides a natural K-theoretic analogue of the contents in §2. I had tried to make the results in [22] more comprehensive, however, I finally gave up doing that. This is because, although the formalism goes parallel as in the case of cohomology, we can not avoid some of complicated calculations, up to my knowledge now. Here I just expect the reader to consult the original article [22], and more fundamental papers by Kostant and Kumar [29] and Buch [6].

In Section 1, we consider the torus equivariant cohomology of the Grassmannian. The goal is to identify the equivariant Schubert classes with the so-called *factorial Schur functions*. Our approach is purely combinatorial and uses the GKM ([14]) description of the equivariant cohomology and its Schubert basis. I also emphasize the role of the "left" divided difference operators, which play a fundamental role when we consider the extension of the story to the equivariant K-theory. I just briefly discuss how the role of the double Schubert polynomial in this context. The last subsection on the Kempf-Laksov formula is added so that we can reinterpret the results in the context of degeneracy loci formulas.

In Section 2, we discuss the Lagrangian Grassmannian (the maximal isotropic Grassmannian of orthogonal type is treated similarly, however we mainly consider The Lagrangian case) which is the key geometric object related to the construction of double Schubert polynomials. The first important result is to relate the equivariant Schubert class with the factorial Q-function, which is a deformation of the Q-function introduced by V. N. Ivanov in [23]. Next we extend this to the full flag variety and introduce the double Schubert polynomials. Also in the last subsection, I included a review on a result by Kazarian in [25] which corresponds to the Kempf-Laksov formula in the Lagrangian case.

Section 3 is an application of the previous section. We study the singularity of the Schubert variety by using the equivariant Schubert polynomials. If the generalized flag variety is *cominuscule* type, we can calculate the Hilbert-Samuel multiplicity of the singular points by using the equivariant Schubert polynomials.

Open questions and related works.

Recently, Anderson and Fulton reconstructed in [2] the double Schubert polynomials of the classical types by a geometric method in the context of the degeneracy loci of vector bundles. They also obtained explicit formulas for the double Schubert polynomials for a class of Weyl group element which they call Vexillary signed permutations. These formula extends Theorem 2.2 and Theorem 2.5. This result makes up for the present article from geometric point of view. For more on the degeneracy loci approach, the recent paper [49] by Tamvakis will be an excellent guide. See also the references therein.

We do not discuss the problem to determine the multiplicative structure constants for the various Schubert basis. Of course, this is a main question in Schubert calculus. So I briefly mention about what is known. The results are classical for the Schur functions and known for the factorial Schur functions (Morev-Sagan [42], Knutson-Tao [27], Kreiman [30]) and also for the Schur Q-functions (Stembridge [47], Cho [8]). The case of the factorial Q-function is open.

Acknowledgements. I am particularly grateful to my collaborators H. Naruse and L. Mihalcea for the wonderful conversation and the work over a period of years. Since the early stage of the works presented in the lecture, I have been very much inspired by [23], [27], and [34]. I am deeply indebted to their authors. V. N. Ivanov, A. Knutson, V. Lakshmibai, K. N. Raghavan, P. Sankaran, T. Tao. Without their works, I could not even start the research on the Schubert calculus.

Moreover, I learned the crucial meaning of cominuscule property in Section 3 by A. Knutson. On this occasion, I would like to express my thanks to K. N. Raghavan and his collaborators for sharing some results on the multiplicities of the singular points on the Schubert variety. Impressively enough, M. Kazarian [25] had proved essentially the same result in my paper [18] by using completely different languages. This experience gives me a great pleasure. I would like to also thank him for deep understanding on my work. I also thank Tomoo Matsumura and Takashi Sato for valuable comments on the draft.

§1. Grassmannians and Factorial Schur functions

We review a combinatorial description of equivariant Schubert classes in the ordinary Grassmannian. We put much emphasis on combinatorial properties of special polynomials called the factorial Schur functions.

1.1. Schubert varieties in Grassmannians

We start with the ordinary Grassmannian $\mathbb{G}_{d,n}$ of linear d-spaces in $\mathbb{C}^n$. We fix d throughout in this section, while we take a limit of n tends to infinity later. An element in $\mathbb{G}_{d,n}$ has *unique* basis $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_d$ such that the matrix $(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_d)$ is a *column echelon form* like (Cell) in the following

$$\text{(Cell)} \begin{pmatrix} 1 & 0 & 0 \\ * & 0 & 0 \\ 0 & 1 & 0 \\ * & * & 0 \\ * & * & 0 \\ 0 & 0 & 1 \\ * & * & * \end{pmatrix}.$$

Pivot ones in this example have the row indices $1, 3, 6$. We denote by ${}_n\mathcal{C}_d$ the set of all subsets in $\{1, \ldots, n\}$ of cardinality d. The elements of the Grassmannian $\mathbb{G}_{d,n}$ are divided into the parts labeled by their set of pivot ones in ${}_n\mathcal{C}_d$. Collecting the elements in $\mathbb{G}_{d,n}$ of a fixed label in ${}_n\mathcal{C}_d$, we define a *Schubert cell*.

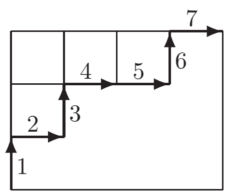
It is convenient to consider ${}_n\mathcal{C}_d$ as the coset space $S_n/(S_d \times S_{n-d})$ in the following way, where S_n is the symmetric group of degree n. In fact, S_n acts transitively on ${}_n\mathcal{C}_d$ and the stabilizer of $\{1, \ldots, d\}$ is $S_d \times S_{n-d}$. Let $J \in {}_n\mathcal{C}_d$ (we consider J as the set of row indices of pivot ones). We can choose a permutation $w \in S_n$ so that

$$(1.1) \quad 1 \leq w(1) < \ldots < w(d) \leq n, \quad 1 \leq w(d+1) < \ldots < w(n) \leq n.$$

and $J = \{w(1), \dots, w(d)\}$. Then we call w the *Grassmannian permutation* corresponding to J. For the above example, the corresponding Grassmannian permutation is $w = 1362457$ in one line notation.

One more interpretation of the index set is given by Young diagrams (or *partition*). Let $\mathcal{P}_{d,n}$ be the set of partitions $\lambda = (\lambda_1 \geq \cdots \geq \lambda_d)$ such that $\lambda_1 \leq n - d$. From a Grassmannian permutation w we form $\lambda \in \mathcal{P}_{d,n}$ in a way illustrated by the following example:

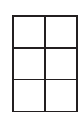
Figure 1.



Let $J = \{w(1), \dots, w(d)\}$. Consider $d \times (n - d)$ rectangular. Starting from SW corner, we walk to NE corner as follows: If $i \in J$, then the i-th step is vertical (upper direction), if not, horizontal (right direction). Then the corresponding Young diagram is the set of boxes sitting upper left side of the route. Thus in the above example we have $\lambda = (3, 1)$.

Let $\lambda \in \mathcal{P}_{d,n}$ and J be the corresponding element in ${}_n\mathcal{C}_d$. Consider the set of matrices of the form (Nbd) below. The rows which are not labeled by the elements of J are arbitrary. This set can be considered to be an open set in $\mathbb{G}_{d,n}$ isomorphic to the affine space $\mathbb{C}^{d(n-d)}$. We denote this open set by $\mathcal{U}_\lambda$. The Schubert cell labeled by λ is denoted by X_λ°, which form a "coordinate subspace" in $\mathcal{U}_\lambda$. The coordinate functions vanishing on the cell can be naturally identified with the Young diagram rotated by $90°$ clockwise as follows:

$$\text{(Nbd)} \begin{pmatrix} 1 & 0 & 0 \\ * & * & * \\ 0 & 1 & 0 \\ * & * & * \\ * & * & * \\ 0 & 0 & 1 \\ * & * & * \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 0 \\ * & \boxed{0} & \boxed{0} \\ 0 & 1 & 0 \\ * & * & \boxed{0} \\ * & * & \boxed{0} \\ 0 & 0 & 1 \\ * & * & * \end{pmatrix}$$



This in particular implies the codimension of X_λ° is $|\lambda| = \sum_{i=1}^d \lambda_i$. The set $\mathcal{P}_{d,n}$ is partially ordered by $\lambda \leq \mu \iff \lambda \subset \mu$ (the inclusion as Young diagrams). If we set $X_\lambda = \sqcup_{\mu \geq \lambda} X_\mu^\circ$, then $X_\lambda = \overline{X_\lambda^\circ}$ (the Zariski closure) which is the Schubert variety. The condition for V to be in X_λ is equivalent to the following:

$$\dim(V \cap F_{n-d-\lambda_i+i}) \geq i \quad (1 \leq i \leq d), \tag{1.2}$$

where $F_i = \langle \boldsymbol{e}_{n-i+1}, \dots, \boldsymbol{e}_n \rangle$ $(1 \le i \le n)$.

The (rational) cohomology of $\mathbb{G}_{d,n}$ is given as a $\mathbb{Q}$-vector space by

$$H^*(\mathbb{G}_{d,n}) = \bigoplus_{\lambda \in \mathcal{P}_{d,n}} \mathbb{Q}\,\sigma_\lambda,$$

where $\sigma = [X_\lambda] \in H^{2|\lambda|}(\mathbb{G}_{d,n})$ is the fundamental class of X_λ called the Schubert class.

1.2. Weights of coordinate functions

We denote the standard coordinate functions z_{ij} on $\mathcal{U}_\mu$ as follows.

$$\begin{array}{cc}
 & \begin{array}{ccc}\boxed{1} & \boxed{3} & \boxed{6}\end{array} \\
\begin{array}{c}1\\ \boxed{2}\\ 3\\ \boxed{4}\\ \boxed{5}\\ 6\\ 7\end{array} & \begin{pmatrix}1 & 0 & 0\\ z_{21} & \boxed{z_{23}} & \boxed{z_{26}}\\ 0 & 1 & 0\\ z_{41} & z_{43} & \boxed{z_{46}}\\ z_{51} & z_{53} & \boxed{z_{56}}\\ 0 & 0 & 1\\ z_{71} & z_{73} & z_{76}\end{pmatrix}
\end{array}$$

Note that the column index j of z_{ij} is the row index of pivot one in the column, *i.e.*, the indices corresponding to the elements of J. We assign z_{ij} the *weight* $t_j - t_i$. This records the action of $T = (\mathbb{C}^\times)^n$ given by the left multiplication.

In the affine space $\mathcal{U}_\lambda$, the variety $X_\lambda \cap \mathcal{U}_\lambda$ is a coordinate subspace defined by $z_{ij} = 0$ with $i < j$. Let us fill in the boxes of the Young diagram with the corresponding weights as follows. We call this the *weighted Young diagram*.

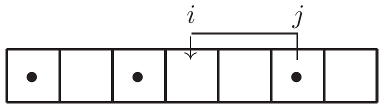
Figure 2.

$$\begin{array}{|c|c|c|}\hline t_6 - t_2 & t_6 - t_4 & t_6 - t_5 \\ \hline t_3 - t_2 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}$$

Let $J \in {}_d\mathcal{C}_n$. Then the *inversion set* of J is defined by

$$\mathrm{Inv}(J) = \{(i,j) \mid j \in J \text{ (pivot)},\ i \notin J \text{ (non-pivot)},\ i < j\}.$$

We depict $J = \{1,3,6\} \in {}_3\mathcal{C}_7$ by a diagram with dots, as follows. These diagrams are called *Maya diagrams*. We draw n boxes in one line such that the dots correspond to the elements of J. Then an element in $\mathrm{Inv}(J)$ is represented a move on the dots in a pivot diagram as follows:

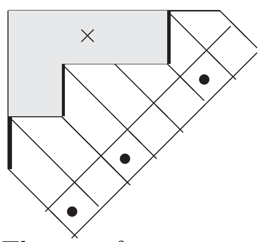
Figure 3. 

Pick out a dot at j and move it down (left) to a vacant box of position i. We equip the element $(i, j) \in \mathrm{Inv}(J)$ a weight $t_j - t_i$.

Proposition 1.1. Let $J \in {}_d\mathcal{C}_n$ and let λ be the corresponding Young diagram in $\mathcal{P}_{d,n}$. There is a weight preserving bijection

$$\mathrm{Inv}(J) \cong \{\text{the boxes of the Young diagram } \lambda\}.$$

We explain this bijection by the following example. The element $(i, j) \in \mathrm{Inv}(J)$ in Figure 3 corresponds to the box indicated by $\times$ in the Young diagram of $\lambda = (3, 1)$. Note the the weight of this box is $t_6 - t_4$ (see Figure 2).

Figure 4. 

Let $P = \oplus_{i=1}^n \mathbb{Q}t_i$. The set of *positive roots* (type A_{n-1}) is $\Delta_+ = \{t_j - t_i \mid 1 \leq i < j \leq n\} \subset P$. For $\alpha = t_j - t_i \in \Delta_+$ denote $s_\alpha = (i, j) \in S_n$, the transposition of i and j. Recall that S_n acts on $\mathcal{P}_{d,n} \cong S_n/(S_d \times S_{n-d})$.

Proposition 1.2. The set of weights associated with the boxes of the weighted Young diagram of $\lambda \in \mathcal{P}_{d,n}$ is given by

$$\{\alpha \in \Delta_+ \mid s_\alpha \lambda < \lambda\}. \tag{1.3}$$

Proof. Let J be the element in ${}_d\mathcal{C}_n$ corresponding to λ. We see that $s_\alpha\lambda < \lambda$ if and only if the transposition $s_\alpha = (i, j)$ is in $\mathrm{Inv}(J)$. Then the corresponding box of the Young diagram of λ has the weight $\alpha = t_j - t_i$. Thus the proposition follows from Proposition 1.1. Q.E.D.

1.3. GKM graph and equivariant Schubert classes

Let T be the subgroup of $GL_n(\mathbb{C})$ consisting of the diagonal matrices. Then T is isomorphic to the algebraic torus $(\mathbb{C}^\times)^n$. Let T act on $\mathbb{G}_{d,n}$ through the natural action of $GL_n(\mathbb{C})$ on $\mathbb{G}_{d,n}$. We consider the T-equivariant (rational) cohomology ring $H_T^*(\mathbb{G}_{d,n})$ of the Grassmannian variety $\mathbb{G}_{d,n}$. It is known that this cohomology ring can be calculated purely combinatorial way. There are many excellent expositions (*e.g.* [1]) and related papers ([15]).

1.3.1. *Geometric background* Here we briefly explain the geometric idea underlying the following combinatorial description. The set of T-fixed points in $\mathbb{G}_{d,n}$ is naturally labeled by the set $\mathcal{P}_{d,n}$. If $w \in S_n$ represents the element $\lambda \in \mathcal{P}_{d,n}$, we define $e_\lambda = \mathrm{span}\{\boldsymbol{e}_{w(1)}, \ldots, \boldsymbol{e}_{w(d)}\}$, which is the *origin* of the cell $\mathcal{U}_\lambda$. Then the set $\mathbb{G}_{d,n}^T$ of T-fixed points $\mathcal{P}_{d,n}$ is given by $\{e_\lambda \mid \lambda \in \mathcal{P}_{d,n}\}$. The embedding map $i : \mathbb{G}_{d,n}^T \hookrightarrow \mathbb{G}_{d,n}$ induces the pull-back morphism $i^* : H_T^*(\mathbb{G}_{d,n}) \to H_T^*(\mathbb{G}_{d,n}^T)$. Since $\mathbb{G}_{d,n}^T$ is a discrete finite set, we have $H_T^*(\mathbb{G}_{d,n}^T) = \prod_{e_\lambda \in \mathcal{P}_{d,n}} H_T^*(e_\lambda)$. It is known that i^* is injective. Each $H_T^*(e_\lambda)$ is isomorphic to the polynomial ring $\mathbb{Q}[t_1, \ldots, t_n]$, where t_i corresponds to a character of T, *i.e.*, a group homomorphism from T to $\mathbb{C}^\times$. Thus we can identify $H_T^*(\mathbb{G}_{d,n}^T)$ with $\mathrm{Map}(\mathcal{P}_{d,n}, \mathbb{Q}[t_1, \ldots, t_n])$ with pointwise multiplication.

Thus we have an embedding of rings:

$$i^* : H_T^*(\mathbb{G}_{d,n}) \hookrightarrow \mathrm{Map}(\mathcal{P}_{d,n}, \mathbb{Q}[t_1, \ldots, t_n]).$$

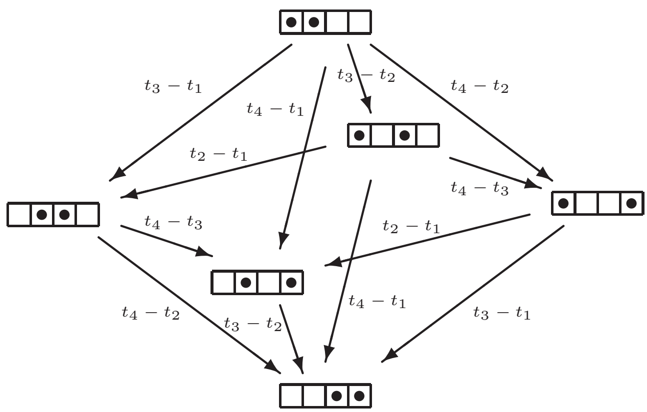
Let us denote the inclusion map $\{e_\lambda\} \hookrightarrow \mathbb{G}_{d,n}$ by i_λ, which is T-equivariant. The induced map $i_\lambda^* : H_T^*(\mathbb{G}_{d,n}) \to H_T^*(e_\lambda) \cong \mathbb{Q}[t_1, \ldots, t_n]$ is called the restriction (localization) map. For $\phi \in H_T^*(\mathbb{G}_{d,n})$, we denote by $\phi|_\lambda$ for $\lambda \in \mathcal{P}_{d,n}$ the polynomial $i_\lambda^*(\phi)$. There is a remarkable result due to Chang and Skjelbred [7], and in more general context, by Goresky-Kottwitz-MacPherson [14] that describes the image of the localization map i^* in a purely combinatorial manner. In this article, we adopt this description as the definition of the equivariant cohomology ring.

1.3.2. *Combinatorial description of* $H_T^*(\mathbb{G}_{d,n})$ Here we set up the combinatorial framework to discuss the equivariant cohomology of the Grassmannian.

Definition 1.1 (The graph $\mathcal{G}_{d,n}$)**.** Let us consider the following weighted oriented graph:

- vertices: $\mathcal{P}_{d,n} \cong S_n/(S_d \times S_{n-d}) \cong {}_d\mathcal{C}_n$,
- oriented edges : if there is a positive root α such that $\mu = s_\alpha\lambda > \lambda$, we draw an oriented edge $\lambda \overset{\alpha}{\to} \mu$,
- weight of $\lambda \overset{\alpha}{\to} \mu$ is α.

If $n = 4$ and $d = 2$ the graph looks as follows. Here we depict each element of ${}_4\mathcal{C}_2$ as the corresponding Maya diagram.



Definition 1.2 (Combinatorial definition of equivariant cohomology ring)**.** An element $\phi \in \mathrm{Map}(\mathcal{P}_{d,n}, \mathbb{Q}[t_1, \ldots, t_n])$ is in $H_T^*(\mathbb{G}_{d,n})$ if for all edge $\lambda \overset{\alpha}{\to} \mu$ the difference $\phi|_\lambda - \phi|_\mu$ is divisible by α.

Clearly $H_T^*(\mathbb{G}_{d,n})$ is a $\mathbb{Q}[t_1, \ldots, t_n]$-subalgebra of the product ring. Next is a fundamental property of the ring $H_T^*(\mathbb{G}_{d,n})$. If $\lambda \in \mathcal{P}_{d,n}$ corresponds to $J \in {}_n\mathcal{C}_n$, we also denote $\mathrm{Inv}(J)$ by $\mathrm{Inv}(\lambda)$.

Proposition 1.3. Let $\phi \in H_T^*(\mathbb{G}_{d,n})$. If $\nu \in \mathcal{P}_{d,n}$ is a minimal element in $\mathrm{Supp}(\phi) := \{\mu \in \mathcal{P}_{d,n} \mid \phi|_\mu \neq 0\}$, then $\phi|_\nu$ is divisible by

$$d_\nu := \prod_{\alpha \in \mathrm{Inv}(\nu)} \alpha.$$

Proof. Consider the set of all arrows $\mu \overset{\alpha}{\to} \nu$, which is naturally in bijection with $\mathrm{Inv}(\nu)$ by (1.1). Since $\mu < \nu$, we have $\phi|_\mu = 0$ by the assumption minimality of ν. Then the GKM condition implies that $\phi|_\nu$ is divisible by all α in $\mathrm{Inv}(\nu)$. Note that $\mathbb{Q}[t_1, \ldots, t_n]$ is a unique factorization domain and the elements in $\mathrm{Inv}(\nu)$ are pairwise relatively prime elements. Hence $\phi|_\nu$ is divisible by their product $\prod_{\alpha \in \mathrm{Inv}(\nu)} \alpha$. Q.E.D.

Definition 1.3. Let $\lambda \in \mathcal{P}_{d,n}$. An element σ_λ in $H_T^*(\mathbb{G}_{d,n})$ is called a Schubert class indexed by λ if it satisfies the following properties:

(i) $\sigma_\lambda|_\mu$ is homogeneous of degree $|\lambda|$ with $\deg(t_i) = 1$,
(ii) $\sigma_\lambda|_\mu = 0$ unless $\lambda \leq \mu$,
(iii) $\sigma_\lambda|_\lambda = d_\lambda$.

Existence of the Schubert classes follows from general results by Kostant-Kumar [28] (see also Kumar [33]). We give a constructive proof later.

Proposition 1.4 ([27], Lemma 1)**.** Let $\lambda \in \mathcal{P}_{d,n}$. There is at most one Schubert class corresponding to λ.

Proof. Suppose both σ_λ and σ'_λ satisfy the defining properties (i),(ii),(iii) in Definition 1.3. Put $\tau = \sigma_\lambda - \sigma'_\lambda$ and assume $\tau \neq 0$. Let ν be any minimal element of $\mathrm{Supp}(\tau) \neq \emptyset$. Then by (ii) we have $\nu \geq \lambda$. By (iii) we have $\tau|_\lambda = 0$. So $\lambda \notin \mathrm{Supp}(\tau)$. Hence we have $\nu \gneqq \lambda$. In particular, we have $|\nu| > |\lambda|$. By Proposition 1.3, $\tau|_\nu$ is divisible by d_ν. This means $\deg(\tau|_\nu) \geq \deg(d_\nu) = |\nu|$. However from (i) we have $|\lambda| = \deg(\tau|_\nu)$. This is a contradiction. Q.E.D.

Proposition 1.5 ([27])**.** $H_T^*(\mathbb{G}_{d,n})$ is a free $\mathbb{Q}[t_1, \ldots, t_n]$-module with basis $\{\sigma_\lambda\}_{\lambda \in \mathcal{P}_{d,n}}$.

Proof. We first show that the set $\{\sigma_\lambda | \ \lambda \in \mathcal{P}_{d,n}\}$ spans $H_T^*(\mathbb{G}_{d,n})$ as a $\mathbb{Q}[t_1, \ldots, t_n]$-module. Let ϕ be an arbitrary nonzero element in $H_T^*(\mathbb{G}_{d,n})$. We may suppose that ϕ is homogeneous. Let $\nu \in \mathcal{P}_{d,n}$ be a minimal element in $\mathrm{Supp}(\phi)$ then write $\phi|_\nu = c_\nu d_\nu$ ($c_\nu \in \mathbb{Q}[t_1, \ldots, t_n]$) by Proposition 1.3. So $\phi' = \phi - c_\nu \sigma_\nu$ vanishes at ν. By the minimality of ν, ϕ' vanishes on $D(\nu) := \{\kappa \mid \kappa \leq \nu\}$. If $\phi' = 0$ then we have $\phi = c_\nu \sigma_\nu$ so we are done. Suppose $\phi' \neq 0$. Let ν' be a minimal element in $\mathrm{Supp}(\phi')$. Then $\nu' \notin D(\nu)$ *i.e.*, $\nu' \not\leq \nu$. By the same procedure, we set $\phi'' = \phi' - c_{\nu'} \sigma_{\nu'}$, which vanishes on $D(\nu')$. Moreover, ϕ'' vanishes on $D(\nu)$. In fact, if $\kappa \leq \nu$, since $\nu' \not\leq \nu$ as above we have $\nu' \not\leq \kappa$, and so $\sigma_{\nu'}|_\kappa = 0$ by (ii). Since $D(\nu) \subsetneqq D(\nu) \cup D(\nu')$, we can deduce the support successively to make it empty by subtracting a suitable linear combination of the Schubert classes.

Now we prove the linear independence. Suppose we have a nontrivial linear relation $\sum_\lambda c_\lambda \sigma_\lambda = 0$ over $\mathbb{Q}[t_1, \ldots, t_n]$. Among λ such that $c_\lambda \neq 0$ we choose a minimal one. Then by (ii), we have $c_\lambda \sigma_\lambda|_\lambda = 0$. But we have $\sigma_\lambda|_\lambda = d_\lambda \neq 0$, and that $c_\lambda = 0$. This is a contradiction. Q.E.D.

Example 1.1. We first consider the projective space $\mathbb{G}_{1,n} = \mathbb{P}^{n-1}$. For each codimension k, there is a unique Schubert class $\sigma_{(k)}$ represented by $s_k \cdots s_1$ (a row of k boxes). If we define the *generalized factorial*

$$(z|t)^k = (z - t_1) \cdots (z - t_k), \tag{1.4}$$

the class $\sigma_{(k)}$ evaluated at the element $s_i \cdots s_2 s_1$ is given by

$$(t_{i+1}|t)^k = (t_{i+1} - t_1) \cdots (t_{i+1} - t_k).$$

Example 1.2. It is strongly recommended to work out the table for equivariant Schubert classes in $\mathbb{G}_{2,4}$, by any convincing way. Here we write $t_{ij} = t_i - t_j$.

$\lambda\backslash\mu$	$(0,0)$	$(1,0)$	$(1,1)$	(2)	$(2,1)$	$(2,2)$
σ_0	1	1	1	1	1	1
σ_1	0	t_{32}	t_{31}	t_{42}	t_{41}	$t_{41}+t_{32}$
$\sigma_{1,1}$	0	0	$t_{31}t_{21}$	0	$t_{41}t_{21}$	$t_{41}t_{31}$
σ_2	0	0	0	$t_{42}t_{43}$	$t_{41}t_{43}$	$t_{41}t_{42}$
$\sigma_{2,1}$	0	0	0	0	$t_{41}t_{43}t_{21}$	$t_{41}t_{42}t_{31}$
$\sigma_{2,2}$	0	0	0	0	0	$t_{41}t_{42}t_{31}t_{32}$

1.3.3. *Digression on Gröbner degeneration* So far we have set up the algebraic and combinatorial framework for the computation of the equivariant Schubert classes. Now we want to turn ourselves to the geometric aspect that underlies the construction. The following results in this subsection are not used in the rest of the paper, but they will help us to understand the whole picture.

Proposition 1.6. If $X_\lambda \cap \mathcal{U}_\mu$ is a coordinate subspace defined by $z_\alpha = 0$ $(\alpha \in I)$ for some subset $I \subset \Delta_+$, then $\sigma_\lambda|_\mu = \prod_{\alpha\in I} \alpha$.

For the proof, see Theorem 3, [34] for more general results and the idea of Gröbner degeneration. See also Remark 3.2.

By using Proposition 1.6, we can calculate some $\sigma_\lambda|_\mu$.

Example 1.3. We consider the equivariant Schubert classes of $\mathbb{G}_{2,4}$. Let $\lambda = (1,1)$ and $\mu = (2,1)$. The condition that $V \in \mathbb{G}_{2,4}$ is in X_λ is given by $V \subset F_3$. Now $\mathcal{U}_\mu$ and F_3 is described as follows:

$$\mathcal{U}_\mu = \left\{ \begin{pmatrix} z_{12} & z_{14} \\ 1 & 0 \\ z_{32} & z_{34} \\ 0 & 1 \end{pmatrix} \right\}, \quad F_3 = \left\{ \begin{pmatrix} 0 \\ * \\ * \\ * \end{pmatrix} \right\}.$$

We see that the affine patch $X_\lambda \cap \mathcal{U}_\mu$ is given by the equations

$$z_{12} = z_{14} = 0.$$

So by Proposion 1.6, we have $\sigma_\lambda|_\mu = (t_4 - t_1)(t_2 - t_1)$.

Example 1.4. The only exception that we can calculate $\sigma_\lambda|_\mu$ for $\mathbb{G}_{2,4}$ using Proposition 1.6 is the case $\lambda = (1,0)$, $\mu = (2,2)$. The Schubert condition is

$$\dim(V \cap F_2) \geq 1,$$

where $V \in \mathbb{G}_{2,4}$. Now $\mathcal{U}_\mu$ and F_2 is described as follows:

$$\mathcal{U}_\mu = \left\{ \begin{pmatrix} z_{13} & z_{14} \\ z_{23} & z_{24} \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = (\boldsymbol{v}_1, \boldsymbol{v}_2) \right\}, \quad F_2 = \left\{ \begin{pmatrix} 0 \\ 0 \\ * \\ * \end{pmatrix} \right\}.$$

In order that $V = \langle \boldsymbol{v}_1, \boldsymbol{v}_2 \rangle \in \mathcal{U}_\mu$ has non-zero intersection with F_2, we must have

$$\begin{vmatrix} z_{13} & z_{14} \\ z_{23} & z_{24} \end{vmatrix} = 0. \tag{1.5}$$

Thus $X_\lambda \cap \mathcal{U}_\mu$ is defined by the equation (1.5). In section §3.2 below, we see how the ideal determine the polynomial $\sigma_\lambda|_\mu$ in general. Here we give a rough idea of *Gröbner degeneration* employed in [34]. The equation is *deformed* into $z_{14}z_{23} = 0$ in a certain sense, and thus the corresponding variety is deformed into the union of hyperplanes defined by $z_{14} = 0$ and $z_{23} = 0$. Each components (hyperplanes) contribute to t_{41} and t_{32} as a summand of $\sigma_\lambda|_\mu$.

This naive idea in the last example also shows how the polynomial $\sigma_\lambda|_\mu$ is related to the singularity of X_λ at e_μ. Later in §3, we discuss the Hilbert-Samuel multiplicity of the point e_μ in X_λ. In fact, in the above example, the multiplicity of the local ring $\mathcal{O}_{X_\lambda, e_\mu}$ at e_μ is two.

1.4. Factorial Schur functions

Let us introduce the functions, called the *factorial Schur functions*, which play the leading part in this lecture.

1.4.1. *Definition and basic properties of factorial Schur functions* For any partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_d \geq 0)$ we set

$$s_\lambda(z_1, \ldots, z_d|t) = \frac{\det((z_i|t)^{\lambda_j + d - j})_{d\times d}}{\prod_{1\leq i<j\leq d}(z_i - z_j)},$$

where we used the generalized factorial defined by (1.4). Here t is the infinite sequence of $(t_1, t_2, \ldots)$. Since the numerator is an anti-symmetric polynomial in z's with coefficients in t's, it is divisible by the difference product in the denominator. Thus this rational function is indeed a polynomial in z's and t's which is symmetric in $z_1, \ldots, z_d$. One sees that it is homogeneous of degree $|\lambda|$ with $\deg(z_i) = \deg(t_i) = 1$.

Remark 1.1. Note that $s_\lambda(z_1, \ldots, z_d|t)$ and $s_\lambda(z_1, \ldots, z_d, z_{d+1}|t)|_{z_{d+1}=0}$ are different.

Let $\mu \in \mathcal{P}_{d,n}$, and $f = f(z_1, \ldots, z_d)$ be any symmetric polynomial in $\mathbb{Q}[z_1, \ldots, z_d]$. Take any permutation v in S_n representing μ. Then the following polynomial in $\mathbb{Q}[t_1, \ldots, t_n]$ given by substitution

$$f(t_{v(1)}, \ldots, t_{v(d)})$$

does not depend on v but only on μ. We denote the resulting polynomial by $f(t_\mu)$. If you want to be more explicit, take $v \in S_n$ to be the Grassmannian permutation corresponding to μ. Then we have $z_i \mapsto t_{\mu_{d-i+1}+i}$ for $1 \leq i \leq d$.

First important result about this function is the vanishing property.

Lemma 1.1. We have $s_\lambda(t_\mu|t) = 0$ unless $\lambda \leq \mu$.

Proof. Since the polynomial is symmetric in $z_1, \ldots, z_d$, we can substitute $z_i \mapsto t_{\mu_i+d-i+1}$ instead (in the reverse order). Then the (i,j) component of the matrix in the numerator is $(t_{\mu_i+d-i+1}|t)^{\lambda_j+d-j}$. If $\lambda_k > \mu_k$ for some k, then all (i,j) components such that $i \geq k,\ j \leq k$ are vanish. The numerator vanishes while the denominator does not. Q.E.D.

Lemma 1.2. We have $s_\lambda(t_\lambda|t) = \prod_{\alpha \in \mathrm{Inv}(\lambda)} \alpha$.

Proof. We substitute $z_i \mapsto t_{\lambda_i+d-i+1}$. Then the matrix in the numerator is upper triangular, so we have

$$s_\lambda(t_\lambda|t) = \prod_{i=1}^{d} \frac{(t_{\lambda_i+d-i+1}|t)^{\lambda_i+d-i}}{\prod_{i<s\leq d}(t_{\lambda_i+d-i+1} - t_{\lambda_s+d-s+1})}.$$

For each i, after some cancelation, this corresponds to the ith row of the corresponding weighted Young diagram (*cf.* (1.3)). Q.E.D.

1.4.2. *Factorial Schur function represents the Schubert class*

Lemma 1.3. Let f be any symmetric polynomial in $\mathbb{Q}[z_1, \ldots, z_d]$. Then the family of polynomials $(f(t_\mu))_{\mu \in \mathcal{P}_{d,n}})$, considered as an element of $\mathrm{Map}(\mathcal{P}_{d,n}, \mathbb{Q}[t_1, \ldots, t_n])$, belongs to $H_T^*(\mathbb{G}_{d,n})$.

Proof. Let $\mu \to s_\alpha \mu$ be an arbitrary edge of the graph $\mathcal{G}_{d,n}$. Let $v \in S_n$ be any permutation representing μ. We have to show that

$$f(t_{v(1)}, \ldots, t_{v(d)}) - f(t_{s_\alpha v(1)}, \ldots, t_{s_\alpha v(d)})$$

is divisible by α. Let $\alpha = t_j - t_i$ $(1 \leq i < j \leq n)$. Then $s_\alpha = (i,j)$. Since $s_\alpha \mu < \mu$, we have $j \in J_\mu,\ i \notin J_\mu$, where $J_\mu = \{v(1), \ldots, v(d)\}$. Then the divisibility is obvious. Q.E.D.

Theorem 1.1 ([27]). Let $\lambda \in \mathcal{P}_{d,n}$. Then the family of polynomials $(s_\lambda(t_\mu|t))_{\mu\in\mathcal{P}_{d,n}}$ is the Schubert class associated to λ. Therefore the Schubert class σ_λ exists.

Proof. Note first that $s_\lambda(t_\mu|t) \in \mathbb{Q}[t_1,\ldots,t_n]$ for $\lambda,\mu \in \mathcal{P}_{d,n}$. By lemma 1.3, we know that this family of polynomials satisfy the GKM condition. The first conditions in Definition 1.1 is obviously satisfied. The second and the third hold by Lemmas 1.1 and 1.2. Q.E.D.

1.5. Localization map

Let $\mathcal{P}_d = \bigcup_{n\geq d}\mathcal{P}_{d,n}$, and let $\mathbb{Q}[t] = \mathbb{Q}[t_1,t_2,\ldots]$. A homomorphism of $\mathbb{Q}[t]$-algebras defined by

$$\Phi : \mathbb{Q}[t][z_1,\ldots,z_d]^{S_d} \to \mathrm{Map}(\mathcal{P}_d,\mathbb{Q}[t]),$$

$$f(z_1,\ldots,z_d|t) \mapsto (\mu \mapsto f(t_\mu|t)) \tag{1.6}$$

is called the *localization map.*

We can naturally consider the graph $\mathcal{G}_{d,\infty}$ as the union $\bigcup_{n\geq d}\mathcal{G}_{d,n}$. Let $H_T^*(\mathbb{G}_{d,\infty})$ be the set of $\phi \in \mathrm{Map}(\mathcal{P}_d,\mathbb{Q}[t])$ satisfying the following conditions:

- the GKM condition, *i.e.* the condition in Definition 1.2
- $\deg(\phi_\mu)$ $(\mu \in \mathcal{P}_d)$ is bounded above.

We define the notion of the *Schubert classes* $\sigma_\lambda^{(\infty)}(\lambda \in \mathcal{P}_d)$ for $H_T^*(\mathbb{G}_{d,\infty})$ in the same way as in Definition 1.3.

Proposition 1.7. There exists a unique family of Schubert classes $\{\sigma_\lambda^{(\infty)}\}_{\lambda\in\mathcal{P}_d}$.

Proof. We have established the existence and uniqueness of the Schubert classes of $\mathbb{G}_{d,n}$ (Proposition 1.4 , Theorem 1.1). Let $\sigma_\lambda^{(n)}$ denote the Schubert class for $\mathbb{G}_{d,n}$ associated to $\lambda \in \mathcal{P}_{d,n}$. If $n \leq m$, we claim that

$$\sigma_\lambda^{(m)}|_\mu = \sigma_\lambda^{(n)}|_\mu \ (\mu \in \mathcal{P}_{d,n}). \tag{1.7}$$

In fact, the polynomials $\sigma_\lambda^{(m)}|_\mu, \mu \in \mathcal{P}_{d,n}(\subset \mathcal{P}_{d,m})$ satisfy the defining property for $\sigma_\lambda^{(n)}$, because the weighted graph $\mathcal{G}_{d,n}$ is embedded in $\mathcal{G}_{d,m}$, so by the uniqueness of Schubert class, we have (1.7).

We can define $\sigma_\lambda^{(\infty)}$ for $\lambda \in \mathcal{P}_d$ as follows. Choose n so that $\lambda \in \mathcal{P}_{d,n}$. For any $\mu \in \mathcal{P}_d$, we choose m such that $n \leq m$ and $\mu \in \mathcal{P}_{d,m}$. Then set $\sigma_\lambda^{(\infty)}|_\mu = \sigma_\lambda^{(m)}|_\mu$. This does not depend on the choices of n and m since we have (1.7). From the construction, $\sigma_\lambda^{(\infty)}$ clearly satisfies the defining property of Schubert class for $\mathbb{G}_{d,\infty}$ associated with $\lambda \in \mathcal{P}_d$. Q.E.D.

Proposition 1.8. The set $\{\sigma_\lambda^{(\infty)}\}_{\lambda \in \mathcal{P}_d}$ form a $\mathbb{Q}[t]$-basis of $H_T^*(\mathbb{G}_{d,\infty})$.

Proof. Let ϕ be a homogeneous element in $H_T^*(\mathbb{G}_{d,\infty})$. The same procedure in the proof for Proposition 1.5 works also for ϕ. We can successively extract $\mathbb{Q}[t]$-linear combinations of $\sigma_\lambda^{(\infty)}$'s so that the differences of them to ϕ have smaller and smaller supports. By the degree reason, only a finite number of $\sigma_\lambda^{(\infty)}$'s can appear in the linear combination. Hence ϕ is a finite $\mathbb{Q}[t]$-linear combination of the Schubert classes. The linear independence is proved by the same argument. Q.E.D.

The next is the main result of this section.

Theorem 1.2. The map Φ gives an isomorphism of $\mathbb{Q}[t]$-algebras

$$\mathbb{Q}[t][z_1, \ldots, z_d]^{S_d} \overset{\cong}{\to} H_T^*(\mathbb{G}_{d,\infty}),$$

sending $s_\lambda(z_1, \ldots, z_d|t)$ to $\sigma_\lambda^{(\infty)}$ $(\lambda \in \mathcal{P}_d)$.

Proof. By Theorem 1.1 and the proof of Proposition 1.7, we see that Φ sends $s_\lambda(z_1, \ldots, z_d|t)$ to $\sigma_\lambda^{(\infty)}$. By a standard argument, one can show that the polynomials $s_\lambda(z_1, \ldots, z_d|t)$, $\lambda \in \mathcal{P}_{d,n}$ form a $\mathbb{Q}[t]$-basis. Hence the theorem follows from Proposition 1.8. Q.E.D.

1.6. Left divided difference operators

This subsection is supplementary in the sense that we do not need it in order to prove Theorem 1.2 which is the main result of this section, however, I would like to show how naturally the left divided difference operators behave in our framework.

Consider the action of $S_\infty = \bigcup_{n \geq 1} S_n$ on $\mathbb{Q}[t][z_1, \ldots, z_d]$ by permutation of the variables $t_1, t_2, \ldots$. There is an action of S_∞ on $\mathrm{Map}(\mathcal{P}_d, \mathbb{Q}[t])$ such that Φ is a S_∞-module homomorphism. Let $\phi \in \mathrm{Map}(\mathcal{P}_d, \mathbb{Q}[t])$ and $w \in S_\infty$. We define

$$(w \cdot \phi)|_\mu = w(\phi|_{w^{-1}\mu}) \quad (\mu \in \mathcal{P}_d).$$

We will show that $H_T^*(\mathbb{G}_{d,\infty})$ is a S_∞-submodule of $\mathrm{Map}(\mathcal{P}_d, \mathbb{Q}[t])$.

Definition 1.4 (Local form of the left divided difference operator)**.** We can define an operator δ_i on $H_T^*(\mathbb{G}_{d,\infty})$ by

$$\delta_i \phi = \frac{\phi - s_i \cdot \phi}{\alpha_i}. \tag{1.8}$$

As for the well-definedness of δ_i on $H_T^*(\mathbb{G}_{d,\infty})$, we refer the reader to Appendix of [27]. The next result is important because it can be considered as a definition of the Schubert classes $\{\sigma_\lambda\}$ (*cf.* [27], see also Exercise 1.1).

Proposition 1.9 ([27])**.** We have $\delta_i\sigma_\lambda^{(\infty)} = \begin{cases} \sigma_{s_i\lambda}^{(\infty)} & \text{if } s_i\lambda < \lambda, \\ 0 & \text{if } s_i\lambda \geq \lambda. \end{cases}$

Proof. See Lemma 6 in [27]. Q.E.D.

Remark 1.2. There is a nice combinatorial description of the action of S_∞ on $\mathcal{P}_d$. See for example [21], [27].

Definition 1.5 (Global form of the left divided difference operators)**.** Let $f \in \mathbb{Q}[t][z_1, \ldots, z_d]^{S_d}$. For $i \geq 1$, we define

$$\delta_i f = \frac{f - s_i \cdot f}{\alpha_i}, \quad \alpha_i = t_{i+1} - t_i,$$

where $s_i \cdot f$ is obtained from f by exchanging t_i and t_{i+1}.

Since Φ is $\mathbb{Q}[t]$-linear and commutes with the action of S_∞, we have $\Phi \circ \delta_i = \delta_i \circ \Phi$. From this commutativity, Theorem 1.2, and Proposition 1.9, we have the following results. However, the analogous results play crucial roles in [22], where we developed K-theory analogue of the factorial Q-functions. Furthermore, the left divided difference operators turns out to be quite useful when we study the non-maximal isotropic Grassmannian ([19]). So we give a direct proof of this fact.

Proposition 1.10. We have $\delta_i s_\lambda(z|t) = \begin{cases} s_{s_i\lambda}(x|t) & \text{if } s_i\lambda < \lambda, \\ 0 & \text{if } s_i\lambda \geq \lambda. \end{cases}$

Proof. We only have to calculate the numerator $\det((z_i|t)^{\lambda_j+d-j})$. We first remark that $(z|t)^k$ for $k \neq i$ is invariant the exchange of variables t_i and t_{i+1}. One easily see that

$$\delta_i(z|t)^i = (z|t)^{i-1}. \tag{1.9}$$

If $s_i\lambda < \lambda$ then there is an index j such that the jth columns are all of the form $(z|t)^i$. Then by the above remark and (1.9) we easily obtain the result. If $s_i\lambda \geq \lambda$ then by the above remark all the entries $(z_i|t)^{\lambda_j+d-j}$ are symmetric under s_i^t. In fact, there is no j such that $\lambda_j + d$. Hence the result follows. Q.E.D.

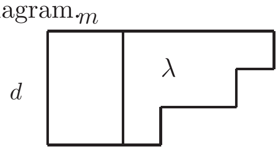
Exercise 1.1 (*cf.* [36])**.** Suppose $\mu \neq \emptyset$. We can choose some i such that $s_i\mu < \mu$. Then the restrictions to μ of all the Schubert classes

σ_λ can be calculated by restrictions to $s_i\mu$ by the following recurrence equation

$$\sigma_\lambda|_\mu = \begin{cases} s_i(\sigma_\lambda|_{s_i\mu}) + \alpha_i \cdot s_i(\sigma_{s_i\lambda}|_{s_i\mu}) & \text{if } s_i\lambda < \lambda, \\ s_i(\sigma_\lambda|_{s_i\mu}) & \text{if } s_i\lambda \geq \lambda. \end{cases}$$

Derive the equation from the divided difference equation in Proposition 1.9.

Exercise 1.2. Let $m \geq 1$ and $\lambda \in \mathcal{P}_d$. Let $m^d \cup \lambda$ denote the following Young diagram



Then

$$s_{m^d\cup\lambda}(z_1, \ldots, z_d|t) = s_{m^d}(z_1, \ldots, z_d|t) \cdot s_\lambda(z_1, \ldots, z_d|t_{m+1}, t_{m+2}, \ldots),$$

$$\text{where} \quad s_{m^d}(z_1, \ldots, z_d|t) = \prod_{1\leq i\leq d,\ 1\leq j\leq m} (z_i - t_j).$$

1.7. Double Schubert polynomials

Now we discuss briefly how the story is extended to the full flag variety $\mathcal{F}l_n = GL_n(\mathbb{C})/B$, where B is the Borel subgroup of $GL_n(\mathbb{C})$ consisting of the upper triangular matrices in $GL_n(\mathbb{C})$. The proofs for the results in this section are omitted.

Let f be a polynomial in $z = (z_1, z_2, \ldots)$. For $i \geq 1$, define

$$\partial_i f = \frac{f - s_i^z f}{z_i - z_{i+1}},$$

where s_i^z exchanges z_i and z_{i+1}.

Let $\mathbb{Q}[z, t]$ denote the polynomial ring $\mathbb{Q}[z_1, z_2, \ldots, t_1, t_2, \ldots]$ in two set of infinite variables $z = (z_1, z_2, \ldots)$ and $t = (t_1, t_2, \ldots)$.

Proposition 1.11 ([37], *cf.* [20])**.** There is a unique family of polynomials $\{\mathfrak{S}_w(z,t)\}_{w\in S_\infty}$ in $\mathbb{Q}[z, t]$ satisfying the following conditions:

- $\mathfrak{S}_e(z,t) = 1$,
- $\delta_i\mathfrak{S}_w(z,t) = \begin{cases} \mathfrak{S}_{s_iw}(z,t) & \text{if } \ell(s_iw) = \ell(w) - 1, \\ 0 & \text{if } \ell(s_iw) = \ell(w) + 1, \end{cases}$
- $\partial_i\mathfrak{S}_w(z,t) = \begin{cases} \mathfrak{S}_{ws_i}(z,t) & \text{if } \ell(ws_i) = \ell(w) - 1, \\ 0 & \text{if } \ell(ws_i) = \ell(w) + 1. \end{cases}$

Remark 1.3. For the longest element $w_0^{(n)} = (n, \dots, 2, 1)$ of S_n, the following explicit formula is known:

$$\mathfrak{S}_{w_0^{(n)}}(z,t) = \prod_{i+j\le n} (z_i - t_j). \tag{1.10}$$

Let $w \in S_\infty$. We can choose $n \ge 1$ such that $w \in S_n$. Then we can calculate $\mathfrak{S}_w(z,t)$ by successive applications of ∂_i's (or δ_i's also) to the "top" polynomial (1.10) for S_n. One can prove that the resulting polynomial does not depend on the choice of n. This is the way the double Schubert polynomials are originally introduced in [37]. Here we adopt the above definition in order to emphasize the infinite rank situation and the symmetry of left and right divided difference operators.

Define $V = \bigoplus_{i\ge1} \mathbb{Q}t_i$, $\Delta_+ = \{t_j - t_i \mid 1 \le i < j\} \subset V$, and $\Delta_- = -\Delta_+$. For $\alpha = (i,j) \in \Delta_+$, denote by s_α the transposition $(i,j) \in S_\infty$. The Bruhat order on S_∞ is denoted by $\le$. We define the GKM graph for $\mathcal{F}l_\infty = \bigcup_{n\ge1} \mathcal{F}l_n$. The vertex set is S_∞. The oriented edges are given by the reflections with respect to all the positive roots, the elements in Δ_+. If α is a positive root such that $v = s_\alpha w,\ v > w$ in the Bruhat order, then we draw an oriented edge $v \overset{\alpha}{\to} w$ with the weight α.

Definition 1.6. An element $\phi \in \mathrm{Map}(S_\infty, \mathbb{Q}[t])$ is in $H_T^*(\mathcal{F}l_\infty)$ if for all edge $v \overset{\alpha}{\to} w$ the difference $\phi|_w - \phi|_v$ is divisible by α, and $\deg(\phi|_w)$ $(w \in S_\infty)$ is bounded.

The Schubert class $\sigma_w^{(\infty)} \in H_T^*(\mathcal{F}l_\infty)$ $(w \in S_\infty)$ is defined by the following conditions :

(1) $\sigma_w^{(\infty)}|_v$ is homogeneous of degree $\ell(w)$ for each $v \ge w$,
(2) $\sigma_w^{(\infty)}|_w = \prod_{\alpha\in\Delta_+\cap w(\Delta_-)} \alpha$,
(3) $\sigma_w^{(\infty)}|_v$ vanishes unless $v \ge w$.

For $v \in S_\infty$, let $\Phi_v : \mathbb{Q}[z,t] \to \mathbb{Q}[t]$ be the $\mathbb{Q}[t]$-algebra homomorphim defined by $z_i \mapsto t_{v(i)}$ $(i \ge 1)$. Define the $\mathbb{Q}[t]$-algebra homomorphism $\Phi : \mathbb{Q}[z,t] \to \mathrm{Map}(S_\infty, \mathbb{Q}[t])$ by $f \mapsto (\Phi_v(f))_{v\in S_\infty}$.

Proposition 1.12 ([20])**.** The map Φ is an isomorphism of $\mathbb{Q}[t]$-algebras

$$\Phi : \mathbb{Q}[z,t] \xrightarrow{\cong} H_T^*(\mathcal{F}l_\infty)$$

such that $\mathfrak{S}_w(z,t) \mapsto \sigma_w^{(\infty)}$.

In the rest of this section, we discuss the relation between the double Schubert polynomials and the equivariant cohomology of the Grassmannian.

Let $S_{\infty,(d)}$ be the subgroups of S_∞ formed by the permutations which preserves both $\{1,\ldots,d\}$ and $\{d+1,d+2,\ldots\}$. One sees that $S_{\infty,(d)}$ is the stabilizer of S_∞ action on $\mathcal{P}_d$, and there is a natural bijection $\mathcal{P}_d \cong S_\infty/S_{\infty,(d)}$. Let us denote by $S_\infty^{(d)}$ be the set of all Grassmannian permutations in S_∞, which is the representatives for the coset space $S_\infty/S_{\infty,(d)}$.

Remark 1.4. The left multiplication induces an action of S_∞ on $S_\infty^{(d)} \cong \mathcal{P}_d$, whereas the right multiplication does not. This is the reason we introduced only δ_i's in the Grassmannian case.

Let S_∞ act on $\mathrm{Map}(S_\infty,\mathbb{Q}[t])$ by the right multiplication, *i.e.*, $(\phi_v)_{v\in S_\infty} \mapsto (\phi_{vw})_{v\in S_\infty}$, for $w\in S_\infty$. If we denote by $\mathrm{Map}(S_\infty,\mathbb{Q}[t])^{S_{\infty,(d)}}$ the subring of $\mathrm{Map}(S_\infty,\mathbb{Q}[t])$ invariant under the subgroup $S_{\infty,(d)}$ of S_∞. Via the bijection $\mathcal{P}_d \cong S_\infty/S_{\infty,(d)}$, we have the following isomorphisms

$$\mathrm{Map}(\mathcal{P}_d,\mathbb{Q}[t]) \cong \mathrm{Map}(S_\infty/S_{\infty,(d)},\mathbb{Q}[t]) \cong \mathrm{Map}(S_\infty,\mathbb{Q}[t])^{S_{\infty,(d)}}. \tag{1.11}$$

Then we have the following commutative diagram:

$$\begin{array}{ccccc}
\mathbb{Q}[t][z_1,\ldots,z_d]^{S_d} & \xrightarrow{\Phi} & \mathrm{Map}(S_\infty,\mathbb{Q}[t])^{S_{\infty,(d)}} & \xleftarrow[i^*]{} & H_T^*(\mathbb{G}_{d,\infty}) \\
{\scriptstyle j}\big\downarrow & & {\scriptstyle k}\big\downarrow & & {\scriptstyle l}\big\downarrow \\
\mathbb{Q}[z,t] & \xrightarrow{\Phi} & \mathrm{Map}(S_\infty,\mathbb{Q}[t]) & \xleftarrow[i^*]{} & H_T^*(\mathcal{F}l_\infty),
\end{array}$$

where j and k are the natural injections, and we use the identification (1.11) in the upper row. The map l above is induced by k. Note that l can be thought of the "pull-back" π^* of the "projection" $\pi:\mathcal{F}l_\infty \to \mathbb{G}_{d,\infty}$, although we do not discuss such geometric constructions here.

Note that $\mathbb{Q}[t][z_1,\ldots,z_d]^{S_d}$ is the invariant subalgebra of $\mathbb{Q}[z,t]$ with respect to the action of $S_{\infty,(d)}$ given by the permutations of the variables z's. Furthermore, we have

$$\mathfrak{S}_{{}_d w_\lambda}(z,t) = s_\lambda(z_1,\ldots,z_d|t),$$

where $\lambda\in\mathcal{P}_d$ and ${}_d w_\lambda$ is the corresponding Grassmannian permutations in $S_\infty^{(d)}$. This last fact can be directly proved (for example, we can use similar argument in §2.6.4, [39]).

1.8. Appendix to §1— Kempf-Laksov formula

Here we give a supplementary discussion on how the factorial Schur functions had been appeared in the study of degeneracy loci formulas of the vector bundles. See [12] for a more thorough account on the theory of the degeneracy loci formula up to the 1990's.

Let E be a vector bundle of rank n on a variety X. We denote by $G_d(E)$ the Grassmann bundle over X parametrizing vector bundles C of rank d with $C \subset E$. Suppose we are given a flag of vector bundles on X such that

$$\underline{A} : 0 \subsetneqq A_1 \subsetneqq A_2 \subsetneqq \cdots \subsetneqq A_d \subset E$$

Let $a(i) = \mathrm{rank}(A_i)$. So we have

$$1 \leq a(1) < \cdots < a(d) \leq n.$$

We denote by $\Omega(\underline{A})$ the subscheme of $G_d(E)$ parameterzing $C \subset E$ of rank d such that

$$\mathrm{rank}(A_i \cap C) \geq i \quad (1 \leq i \leq d).$$

The goal of this section is to express $[\Omega(\underline{A})]$ in $H^*(G_d(E))$ as a polynomial of the Chern classes of the vector bundles involved in the present situation. It is useful to introduce the following sequence which we consider as a Young diagram

$$\lambda_i = n - d - a(i) + i \quad (1 \leq i \leq d).$$

Then we have

$$(n-d) \geq \lambda_1 \geq \cdots \geq \lambda_d \geq 0,$$

so our Young diagram is included in the rectangular shape $(n-d)^d$. Let S denote the tautological subbundle of $\varphi^* E$ on $G_d(E)$, where $\varphi : G_d(E) \to X$ is the structure morphism. The following exact sequence of vector bundles on $G_d(E)$ is called the universal sequence:

$$0 \longrightarrow S \longrightarrow \varphi^* E \longrightarrow Q \longrightarrow 0,$$

where Q is the quotient bundle $\varphi^* E/S$.

For vector bundles E, F on a variety, set $c_i(E-F)$ be the term of degree i in $c(E-F) = c(E)c(F)^{-1}$, where $c(F)^{-1}$ is the inverse of $c(F)$

$$1 - c_1(F) + (c_1(F)^2 - c_2(F)) - (c_1(F)^3 - 2c_1(F)c_2(F) + c_3(F)) + \cdots .$$

For example, we have

$$\begin{aligned} c_1(E-F) &= c_1(E) - c_1(F), \\ c_2(E-F) &= c_2(E) - c_1(E)c_1(F) + c_1(F)^2 - c_2(F), \ldots . \end{aligned}$$

Let $[\Omega(\underline{A})]$ denote the fundamental class of $\Omega(\underline{A})$ in $H^*(G_d(E))$.

Theorem 1.3 (Kempf-Laksov [26])**.** We have

$$[\Omega(\underline{A})] = \det\left(c_{\lambda_i+j-i}(Q - \varphi^* A_i)\right)_{d\times d}.$$

Note that this expression is universal in the sense that it does not depend on X apparently. Another important feature of this formula is that it does not depend on n. We will show the more precise meaning of this claim in the next section.

Now we discuss the relation between the Kempf-Laksov formula to the factorial Schur polynomials. Consider a full flag of vector bundles over X

$$0 \subset E_1 \subset E_2 \subset \cdots \subset E_n = E, \quad \mathrm{rank}(E_i) = i$$

such that

$$A_i = E_{n-d-\lambda_i+i} \quad (1 \le i \le d).$$

We introduce variables $t_1, t_2, \ldots, t_n$ by

$$t_i = -c_1(E_{n-i+1}/E_{n-i}).$$

Let $z_1, \ldots, z_d$ be the Chern roots of S^*, the dual bundle of the tautological subbundle S. Define the polynomials $h_r^{(k)}(z_1, \ldots, z_d)$ by the following generating function:

$$\sum_{r=0}^{\infty} h_r^{(k)}(z_1, \ldots, z_d) = \prod_{i=1}^{d} \frac{1}{1-z_i} \prod_{i=1}^{k} (1-t_i).$$

Using these notation we can rewrite the Kempf-Laksov formula in the following way.

Proposition 1.13. $[\Omega(\underline{A})] = \det(h^{(d+\lambda_i-i)}_{\lambda_i+j-i}(z_1, \ldots, z_d))_{d\times d}$.

Remark 1.5. $d+\lambda_i-i$ is the co-rank of A_i. Note that the polynomial in the right hand side does not deepend on n.

Proof. It suffices to show

$$c_r(Q - \varphi^* A_i) = h_r^{(d+\lambda_i-i)}(z_1, \ldots, z_d).$$

By the universal exact sequence we have

$$\prod_{i=1}^{d} (1-z_i) \times c(Q) = \prod_{j=1}^{n} (1-t_j).$$

Then we have

$$c(Q-\varphi^*A_i) = \frac{\prod_{j=1}^n(1-t_j)}{\prod_{j=1}^d(1-z_j)}\frac{1}{\prod_{j>d+\lambda_i-i}^n(1-t_j)} = \frac{\prod_{j=1}^{d+\lambda_i-i}(1-t_j)}{\prod_{j=1}^d(1-z_j)}.$$

Q.E.D.

Exercise 1.3. Prove that the polynomial in the right hand side of Proposition 1.13 is $s_\lambda(z_1,\dots,z_d|t)$.

Remark 1.6. There is a general principle that the study of the T-equivariant Schubert class is equivalent to the problem of degeneracy loci of vector bundles. This is repeatedly pointed out in the literature (*e*.g. [1], [2], [41]).

§2. Schur's Q-functions and the Lagrangian Grassmannian

We generalize the results in the previous section to the Lagrangian Grassmannian, which is a homogeneous space of the symplectic group. Main results in this section were proved in [18], [20], [21].

2.1. Lagrangian Grassmannian

Let V be an even finite dimensional complex vector space. Suppose there is a non degenerate skew symmetric bilinear form $\langle\cdot,\cdot\rangle$ on V. A linear subspace U of V is *isotropic* if $\langle u,u'\rangle = 0$ for all $u,u'\in U$. A maximal isotropic subspace is called a *Lagrangian* subspace. An isotropic subspace L is Lagrangian if and only if $\dim L = n$, when $\dim V = 2n$. The bilinear form gives a natural isomorphism $\phi : V\to V^*$ of vector spaces such that

$$\phi(v)(w) = \langle v,w\rangle \quad (v,w\in V).$$

Proposition 2.1. Let L be a Lagrangian subspace in the symplectic vector space $V\cong\mathbb{C}^{2n}$. There is a natural isomorphism of vector spaces:

$$V/L \longrightarrow L^*.$$

Proof. For $v\in V$, the restriction $\phi(v)|_L$ of $\phi(v)$ to L is an element of L^*. Let $\alpha : V\to L^*$ denote the map sending $v\in L$ to $\phi(v)|_L\in L^*$. We will prove that $\mathrm{Ker}(\alpha) = L$. Let $v\in\mathrm{Ker}(\alpha)$, *i.e.*, $\phi(v)(l) = \langle v,l\rangle = 0$ for all $l\in L$. Then $L+\mathbb{C}v$ is an isotropic subspace containing L, because $\langle v,v\rangle = 0$ for any $v\in V$. Since L is a maximal isotropic subspace, we have $v\in L$. Hence we have $\mathrm{Ker}(\alpha)\subset L$. Since L is isotropic we have

$\mathrm{Ker}(\alpha) \supset L$. Thus we have $\mathrm{Ker}(\alpha) = L$. Now we know that the image of α, which is isomorphic to $V/\mathrm{Ker}(\alpha) = V/L$, has dimension n. So the map α is surjective, and it induces the isomorphism $V/L \to L^*$. Q.E.D.

Let $LG(n)$ denote the set of all Lagrangian subspaces of $V \cong \mathbb{C}^{2n}$. This is a closed subvariety of the Grassmannian of n-dimensional subspaces in V. Let E be the trivial vector bundle on $LG(n)$ with fiber V. Let S be the tautological subbundle of E, whose the fiber over each point $L \in LG(n)$ is given by L. By using Proposition 2.1, we identify the quotient bundle E/S with S^*. Thus we have the following exact sequence

$$0 \longrightarrow S \longrightarrow E \longrightarrow S^* \longrightarrow 0.$$

By the Whitney formula, we have $c(S)c(S^*) = 1$. Actually we have the following presentation of cohomology ring:

$$H^*(LG(n)) = \mathbb{Q}[c_1(S), \ldots, c_n(S)]/\langle c(S)c(S^*) = 1\rangle. \tag{2.1}$$

Since $c_i(S^*) = (-1)^i c_i(S)$ we have

$$c_i(S)^2 + 2\sum_{j=1}^{i}(-1)^j c_{i+j}(S)c_{i-j}(S) = 0 \quad (i \geq 1). \tag{2.2}$$

Remark 2.1. If we use the Chern roots $z_1, \ldots, z_n$ of S, then $c_i(S) = e_i(z_1, \ldots, z_n)$, where e_i is the ith elementary symmetric function in the corresponding variables. The relation $c(S)c(S^*) = 1$ is written in the following from:

$$\prod_{i=1}^{n}(1 - z_i)(1 + z_i) = 1,$$

which is also equivalent to the following:

$$e_i(z_1^2, \ldots, z_n^2) = 0 \quad (1 \leq i \leq n). \tag{2.3}$$

Then the presentation (2.1) reads

$$H^*(LG(n)) = \mathbb{Q}[z_1, \ldots, z_n]^{S_n}/\langle e_i(z_1^2, \ldots, z_n^2)\ (1 \leq i \leq n)\rangle. \tag{2.4}$$

2.2. Schubert varieties in $LG(n)$

For the Lagrangian Grassmannian $LG(n)$, we develop the analogous argument to §1.2.

Fix a basis $\{\boldsymbol{e}_i, \boldsymbol{e}_i^* \,|1 \leq i \leq n\}$ of V on which we define a skew symmetric bilinear form by $\langle \boldsymbol{e}_i, \boldsymbol{e}_j\rangle = \langle \boldsymbol{e}_i^*, \boldsymbol{e}_j^*\rangle = 0$ and $\langle \boldsymbol{e}_i^*, \boldsymbol{e}_j\rangle = \delta_{i,j}$. Let $G = Sp(V)$ be the associated symplectic group *i.e.* the group of linear automorphisms of V preserving the skew symmetric form. Fix a flag $F_\bullet : F_1 \subset \cdots \subset F_n$ defined by

$$F_i = \langle \boldsymbol{e}_{n-i+1}, \ldots, \boldsymbol{e}_n \rangle \quad (1 \le i \ \le n), \tag{2.5}$$

which is isotropic in the sense that each F_i is isotropic, so in particular, F_n is a Lagrangian subspace. The stabilizer B of $F_\bullet$ is a Borel subgroup of G. Let T denote the torus of the symplectic group $G = Sp(V)$ diagonalized by the above basis.

Let W_n be the Weyl group of (G, T). We can identify W_n with the group of signed permutations of $\bar{1}, \ldots, \bar{n}, 1, \ldots, n$; that is, an element in W_n is a permutation w of $\{\bar{1}, \ldots, \bar{n}, 1, \ldots, n\}$ such that $w(\bar{i}) = \overline{w(i)}$ for $1 \le i \le n$. Here we use $\bar{i}$ to denote the negative element $-i$, and so we understand that $\bar{\bar{i}} = i$.

Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be a strict partition such that $\lambda_1 \le n$. We denote the set of such λ's by $\mathcal{SP}(n)$. For each $\lambda \in \mathcal{SP}(n)$, there is a unique element w_λ of W_n such that

$$w_\lambda(i) = \overline{\lambda_i} \ (1 \le i \le r), \quad w_\lambda(r+1) < \cdots < w_\lambda(n), \tag{2.6}$$

where $\bar{n} < \cdots < \bar{1} < 1 < \cdots < n$. the Grassmannian element corresponding to λ. Let e_λ be the point $\langle \boldsymbol{e}^*_{w_\lambda(1)}, \ldots, \boldsymbol{e}^*_{e_{w_\lambda}(n)} \rangle$ in $LG(n)$. If B^- is the opposite Borel subgroup of B, the Schubert variety X_λ is the Zariski closure of B^--orbit of e_λ. The Schubert variety for $\lambda \in \mathcal{SP}(n)$ can be also defined as

$$X_\lambda = \{L \in LG(n) \mid \dim(L \cap F_{n+1-\lambda_i}) \ge i \quad (1 \le i \le r)\},$$

where $F_1 \subset \cdots \subset F_n \subset V$ is the isotropic flag defined by (2.5).

Then we have

$$H^*(LG(n)) = \bigoplus_\lambda \mathbb{Q}[X_\lambda],$$

where λ runs for all strict partition such that $\lambda_1 \le n$.

There is a T-stable affine neighbourhood $\mathcal{U}_\lambda$ of e_λ such that $\mathcal{U}_\lambda$ is naturally identified with the space of symmetric $n \times n$ matrices with respect to the anti-diagonal. For example, let $\lambda = (3, 1)$, $w_\lambda = \bar{3}\bar{1}2$. The Schubert cell and the canonical neighborhood $\mathcal{U}_\lambda$ are represented by the following matrices:

$$(\text{Cell}) \begin{pmatrix} 0 & 0 & \boxed{0} \\ 1 & 0 & 0 \\ \bullet & \boxed{0} & \boxed{0} \\ 0 & 1 & 0 \\ * & * & \boxed{0} \\ 0 & 0 & 1 \end{pmatrix}, \quad (\text{Nbd}) \begin{pmatrix} \bullet & \bullet & * \\ 1 & 0 & 0 \\ \bullet & * & * \\ 0 & 1 & 0 \\ * & * & * \\ 0 & 0 & 1 \end{pmatrix},$$

where $*$ are arbitrary element in $\mathbb{C}$ and $\bullet$ are determined by $*$ by the isotropic condition. In fact, we can introduce the coordinate functions on $\mathcal{U}_\lambda$ as in the following example:

$$\begin{array}{c} \bar{3} \\ \boxed{\bar{2}} \\ \bar{1} \\ \boxed{1} \\ 2 \\ \boxed{3} \end{array} \begin{pmatrix} z_{\bar{3}\bar{2}} & z_{\bar{3}1} & \boxed{z_{\bar{3}3}} \\ 1 & 0 & 0 \\ z_{\bar{1}\bar{2}} & \boxed{z_{\bar{1}1}} & \boxed{z_{\bar{1}3}} \\ 0 & 1 & 0 \\ z_{2\bar{2}} & z_{21} & \boxed{z_{23}} \\ 0 & 0 & 1 \end{pmatrix}$$

$2t_3$	t_3+t_1	t_3-t_2
	$2t_1$	

We have

$$z_{ij} = z_{\bar{j}\bar{i}}.$$

The diagram illustrates the corresponding weighted *shifted* Young diagram.

Exercise 2.1. The defining equation of $X_{(2)} \cap \mathcal{U}_{(3,1)}$ is given by

$$z_{\bar{3}3} = z_{\bar{1}3} = 0.$$

Deduce that $\sigma_{(2)}|_{(3,1)} = 2t_3(t_3+t_1)$.

Example 2.1. The defining equation of $X_{(1)} \cap \mathcal{U}_{(3,1)}$ is given by

$$\begin{vmatrix} z_{\bar{3}1} & z_{\bar{3}3} \\ z_{\bar{1}1} & z_{\bar{1}3} \end{vmatrix} = 0.$$

As was illustrated in the Grassmannian case in Example 1.4, the defining equation can be deformed into $z_{\bar{1}1}z_{\bar{3}3} = 0$. Thus we have $\sigma_{(1)}|_{(3,1)} = 2t_1 + 2t_3$. See [13] by Ghorpde and Raghavan for the Gröbner degeneration of the Schubert varieties in the Lagrangian Grassmannian.

Here we briefly comment on the general case. For example, let $\lambda = (2)$ and $\mu = (3,2)$. Then the weighted shifted Young diagram of μ is the following:

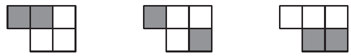
$2t_3$	t_3+t_2	t_3-t_1
	$2t_1$	t_2-t_1

Figure 5

The equivariant Schubert class σ_λ restricted to e_μ is given by

$$\sigma_\lambda|_\mu = 2t_3(t_3+t_2) + 2t_3(t_2-t_1) + 2t_1(t_2-t_1).$$

By a combinatorial description given in [21], each term of the above expression is interpreted as the contribution of the following diagram respectively:



We called these kind of combinatorial object as *Excited Young diagrams*, EYD for short. See the original article for the precise definition of the EYD.

Remark 2.2. The same notion of EYD was independently discovered by V. Kreiman [31], [32].

Remark 2.3. The number of EYDs is shown to be the Hilbert-Samuel multiplicity of X_λ at e_μ. See Theorem 3.1 and Remark 3.2.

2.3. Schur's Q-functions

In this subsection, we collect some definitions and results of the Schur Q-functions ([46]) which will be used in the rest of this section. Detailed proofs can be found in [38] Chap. III, §8, and [17].

2.3.1. *Definition and basic properties of the Schur Q-functions* Let $x_1, x_2, \ldots$ be countably many variables. Define the formal power series $q_k(x)$ $(k \geq 0)$ by the following generating function:

$$q(u) = \prod_{i=1}^{\infty} \frac{1 + x_i u}{1 - x_i u} = \sum_{k=0}^{\infty} q_k(x) u^k, \tag{2.7}$$

where

$$\frac{1}{1 - x_i u} = \sum_{k=0}^{\infty} x_i^k u^k.$$

For example, we have

$$q_0(x) = 1,\ q_1(x) = 2\sum_{i\geq 1} x_i,\ q_2(x) = 2\sum_{i\geq 1} x_i^2 + 4\sum_{i<j} x_i x_j, \ldots.$$

We have $q(u)q(-u) = 1$, or equivalently,

$$q_i(x)^2 + 2\sum_{j=1}^{i}(-1)^j q_{i+j}(x) q_{i-j}(x) = 0 \quad (i \geq 1). \tag{2.8}$$

Let k, l be positive integers. Then we set

$$Q_{k,l}(x) = q_k(x) q_l(x) + 2\sum_{i=1}^{l}(-1)^i q_{k+i}(x) q_{l-i}(x). \tag{2.9}$$

We have $Q_{k,l}(x) = -Q_{l,k}(x)$. In fact, if $k \neq l$, then it obviously from the definition that we have $Q_{k,l}(x) = -Q_{l,k}(x)$, and $Q_{k,k}(x) = 0$ by (2.9).

For arbitrary strict partition $\lambda = (\lambda_1 > \cdots > \lambda_r > 0)$, we define $Q_\lambda(x)$ as follows. If r is even, $Q_\lambda(x)$ is the Pfaffian of the skew-symmetric matrix M_λ of size r with k, l entries $Q_{\lambda_i,\lambda_j}(x)$. If r is odd, we set $\lambda_{r+1} = 0$ and consider the skew symmetric matrix $M_\lambda = (Q_{\lambda_i,\lambda_j}(x))_{1\leq i,j\leq r+1}$ of size $(r + 1)$, with $Q_{k,0}(x) = q_k(x)$ $(k \geq 1)$. Then $Q_\lambda(x)$ is also the Pfaffian of M_λ. Let $\epsilon = 0$ if r is even and $\epsilon = 1$ if r is odd. With the convention $\lambda_{r+1} = 0$ as above, we have

$$Q_\lambda(x) = \mathrm{Pf}(Q_{\lambda_i,\lambda_j}(x))_{1\leq i,j\leq r+\epsilon}. \tag{2.10}$$

The functions $Q_{\lambda_1,\ldots,\lambda_r}(x)$ can be also defined recursively as follows. If r is even

$$Q_{\lambda_1,\ldots,\lambda_r}(x) = \sum_{i=2}^{r}(-1)^{i-1}Q_{\lambda_1,\lambda_i}(x)Q_{\lambda_2,\ldots,\hat{\lambda_i},\ldots\lambda_r}(x),$$

and if r is odd

$$Q_{\lambda_1,\ldots,\lambda_r}(x) = \sum_{i=1}^{r}(-1)^{i-1}q_{\lambda_i}(x)Q_{\lambda_1,\ldots,\hat{\lambda_i},\ldots\lambda_r}(x).$$

Now we define the ring Γ to be

$$\mathbb{Q}[q_1(x), q_2(x), \ldots],$$

which is naturally $\mathbb{Q}$-graded so that the degree of $q_k(x)$ is k.

Proposition 2.2 ([38], (8.9), Chap. III)**.** The set of $Q_\lambda(x)$'s, $\lambda \in \mathcal{SP}_\infty$, form a $\mathbb{Q}$-basis of Γ.

Proposition 2.3 ([17], Corollary 7.6, (ii))**.** Γ is isomorphic to the quotient ring of the polynomial ring $\mathbb{Q}[q_1, q_2, \ldots]$ by the ideal generated by $q_i^2 + 2\sum_{j=1}^{i}(-1)^j q_{i+j}q_{i-j} (i \geq 1)$.

2.3.2. *Schur Q-function represents the Schubert class of the Lagrangian Grassmannian* The following fact is a key to the geometric applications of the Schur Q-functions.

Proposition 2.4. There is a surjective homomorphism of graded rings:

$$\pi_n : \Gamma \longrightarrow H^*(LG(n))$$

sending $q_i(x)$ to $c_i(S)$ for $1 \leq i \leq n$ and $q_i(x)$ to zero for $i > n$.

Proof. The proposition follows from the presentation of $H^*(LG(n))$ given by (2.1), (2.2), or equivalently (2.2), if we use Proposition 2.3. Q.E.D.

Theorem 2.1 (Pragacz [43])**.** Let λ be a strict partition such that $\lambda_1 \leq n$. Then $Q_\lambda(x)$ is sent by π_n to the Schubert class $[X_\lambda] \in H^*(LG(n))$.

The first proof in [43] was given by comparing the Pieri rules for the Schubert classes by Hiller-Boe [16] and for the Q-functions. We later prove the equivariant version of this result (Theorem 2.2).

2.4. GKM graph and Schubert classes for $LG(\infty)$

2.4.1. *W_∞ and its quotient $\mathcal{SP}_\infty$* We consider the Weyl group W_∞ of type C_∞. This is defined by generators $s_0, s_1, s_2, \ldots$ and relations

$$s_i^2 = 1, \quad s_0s_1s_0s_1 = s_1s_0s_1s_0, \quad s_is_{i+1}s_i = s_{i+1}s_is_{i+1}\ (i \geq 1),$$

$$s_is_j = s_js_i\ (|i-j| \geq 2).$$

The subgroup $S_\infty = \langle s_1, s_2, \ldots \rangle$ is isomorphic to the symmetric group of infinite order.

Proposition 2.5. There is a natural bijection between the left coset space W_∞/S_∞ and the set of all strict partitions $\mathcal{SP}_\infty$.

Proof. The correspondence is given by (2.6). Q.E.D.

2.4.2. *GKM graph and the equivariant cohomology of $LG(\infty)$* We have a natural action of W_∞ on the vector space $\bigoplus_{i=1}^\infty \mathbb{Q}t_i$. The element s_0 exchanges t_1 to $-t_1$ and $t_i \mapsto t_i$ for $i \geq 1$, while s_i $(i \geq 1)$ exchanges t_i and t_{i+1} and $t_j \mapsto t_j$ $(j \neq i, i+1)$. We introduce the *simple roots*

$$\alpha_0 = 2t_1, \quad \alpha_i = t_{i+1} - t_i\ (i \geq 1)$$

and the set of roots Δ as the orbit of W_∞ of the simple roots. In fact, if we set

$$\Delta_+ = \{2t_i \mid i \geq 1\} \cup \{t_i - t_j \mid i > j\},$$

then $\Delta = \Delta_+ \cup (-\Delta_+)$. For $\alpha \in \Delta_+$, define $s_\alpha = ws_iw^{-1}$ with $\alpha = w(s_i)$ $(w \in W_\infty)$. Note that W_∞ acts naturally on $\mathcal{SP}_\infty \cong W_\infty/S_\infty$.

Definition 2.1 (GKM graph for $LG(\infty)$)**.** Define the following graph with weights

- vertices: $\mathcal{SP}_\infty \cong W_\infty/S_\infty$,
- oriented edges: if there is a positive root α such that $\mu = s_\alpha\lambda > \lambda$, we draw an oriented edge $\lambda \stackrel{\alpha}{\to} \mu$,
- weight of $\lambda \stackrel{\alpha}{\to} \mu$ is α.

For $\lambda \in \mathcal{SP}_\infty$, define

$$\mathrm{Inv}(\lambda) = \{\alpha \in \Delta_+ \mid s_\alpha\lambda < \lambda\}.$$

We denote $d_\mu = \prod_{\alpha \in \mathrm{Inv}(\lambda)} \alpha$ as in the type A case.

Example 2.2. If $\lambda = (3,1)$, then $\mathrm{Inv}(\lambda) = \{2t_3, t_3 + t_2, t_3 - t_1, 2t_1, t_2 - t_1\}$. These are the weights in the shifted weighted Young diagram of λ (see Figure 5 in §2.2). It shows that an analogous result to Proposition 1.2 holds. For the precise statement and proof, we refer to [21], Proposition 7.2.

Definition 2.2. Define $H_T^*(LG(\infty))$ to be the set of all $\phi \in \mathrm{Map}(\mathcal{SP}_\infty, \mathbb{Q}[t])$ satisfying the following properties:

(1) the GKM condition, *i.e.* for all edge $\lambda \xrightarrow{\alpha} \mu$ the difference $\phi|_\lambda - \phi|_\mu$ is divisible by α,

(2) $\deg(\phi_\mu)$ $(\mu \in \mathcal{SP}_\infty)$ is bounded from above,

2.4.3. *Schubert classes for* $LG(\infty)$ We define the Schubert classes $\sigma_\lambda^{(\infty)}, \lambda \in \mathcal{SP}_\infty$, by the similar conditions as in Definition 1.3.

Definition 2.3. Let $\lambda \in \mathcal{SP}_\infty$. An element σ_λ in $H_T^*(LG(\infty))$ is called a Schubert class indexed by λ if it satisfies the following properties:

(i) $\sigma_\lambda|_\mu$ is homogeneous of degree $|\lambda|$ with $\deg(t_i) = 1$,

(ii) $\sigma_\lambda|_\mu = 0$ unless $\lambda \le \mu$,

(iii) $\sigma_\lambda|_\lambda = d_\lambda$.

We will give an algebraic proof of the existence of the Schubert classes in §2.5 using the factorial Q-functions.

Proposition 2.6. For $\lambda \in \mathcal{SP}_\infty$, the Schubert class associated with λ, if exists, is unique.

Proof. Now we are assuming, as (3) of Definition 2.2, the analogue of the consequence of Proposition 1.3. Hence the uniqueness follows from the similar proof for Proposition 1.4. Q.E.D.

Proposition 2.7. The set $\{\sigma_\lambda^{(\infty)}\}_{\lambda \in \mathcal{SP}_\infty}$ of Schubert classes, if exists, form a $\mathbb{Q}[t]$-basis of $H_T^*(LG(\infty))$.

Proof. The proof is similar to the proofs for Proposition 1.5 and Proposition 1.8. Q.E.D.

2.5. Factorial Q-functions

We introduce a deformed version of Schur Q-functions called the *factorial Schur Q-function* $Q_\lambda(x|t)$ introduced by Ivanov in [23].

Let t be the infinite sequence of the variables $t_1, t_2, \ldots$. For $l \ge 1$, let $q_k^{(l)}(x|t)$ $(k \ge 0)$ be defined by the generating function

$$\sum_{k=0}^{\infty} q_k^{(l)}(x|t) z^k = \prod_{i=1}^{\infty} \frac{1 + x_i z}{1 - x_i z} \prod_{j=1}^{l-1} (1 - t_j z).$$

Note that $q_k^{(1)}(x|t) = q_k(x)$. Let k, l be integers such that $k > l \geq 0$. Then we set

$$(2.11) \qquad Q_{k,l}(x|t) = q_k^{(k)}(x|t)q_l^{(l)}(x|t) + 2\sum_{i=1}^{l}(-1)^i q_{k+i}^{(k)}(x|t)q_{l-i}^{(l)}(x|t).$$

Note that $Q_{k,0}(x|t) = q_k^{(k)}(x|t)$. For arbitrary strict partition λ, we define

$$(2.12) \qquad Q_\lambda(x|t) = \mathrm{Pf}(Q_{\lambda_i,\lambda_j}(x|t))_{1\leq i<j\leq r+\epsilon}$$

as in (2.12), or the recursive definition also works[1].

Remark 2.4. The expression (2.11) is from [20], but essentially the formula was in Kazarian's work [25].

Proposition 2.8 ([23])**.** The set of $Q_\lambda(x|t)$'s with $\lambda \in \mathcal{SP}_\infty$ form a $\mathbb{Q}[t]$-basis of the ring $\Gamma[t]$.

2.5.1. *Localization map* Let $\mu \in \mathcal{SP}_\infty$ and $v \in W_\infty$ be a representative of μ. We can regard v as a signed permutation v which acts on the set $\{\pm 1, \pm 2, \ldots\}$. Then consider the following substitution

$$x_i \mapsto \begin{cases} t_{-v(i)} & \text{if } v(i) < 0, \\ 0 & \text{if } v(i) > 0, \end{cases}$$

into $Q_\lambda(x|t)$. This makes sense as an element in $\mathbb{Q}[t]$. We denote it by $Q_\lambda(t_\mu|t)$ since it does not depend on the representatives.

The most important property of these functions is the following:

Lemma 2.1 ([23])**.** Let $\lambda \in \mathcal{SP}_\infty$.
(1) $Q_\lambda(t_\mu|t) \neq 0$ if and only if $\lambda \leq \mu$,
(2) $Q_\lambda(t_\lambda|t) = d_\lambda$.

A proof of this lemma is given by using an alternative expression similar to Hall-Littlewood symmetric function, which is valid for finite variable version, *i.e.* $Q_\lambda(x_1, \ldots, x_n|t)$ is equal to

$$(2.13) \qquad \frac{1}{(n-r)!}\sum_{w\in S_n} w\left(\prod_{i=1}^{r} 2x_i(x_i|t)^{\lambda_i - 1}\prod_{i=1}^{r}\prod_{j=r+1}^{n}\frac{x_i + x_j}{x_i - x_j}\right),$$

---

[1] Here we only defined $Q_{k,l}(x|t)$ for $k > l$ in (2.11). We can also define $Q_{l,k}(x|t)$ just as $-Q_{k,l}(x|t)$, or as $q_l^{(k)}(x|t)q_k^{(l)}(x|t) + 2\sum_{j=1}^{k}(-1)^j q_{l+j}^{(k)}(x|t)q_{k-j}^{(l)}(x|t)$, so that $Q_{l,k}(x|t) = -Q_{k,l}(x|t)$ holds and the Pfaffian is defined. Note also that the recursive definition makes sense, when we only have $Q_{k,l}(x|t)$ with $k > l$.

where we assume the number r of nonzero parts in λ is less than or equal to n. We have $Q_\lambda(x_1,\ldots,x_n,0|t) = Q_\lambda(x_1,\ldots,x_n|t)$. Then $Q_\lambda(x|t)$ is defined as the projective limit of $Q_\lambda(x_1,\ldots,x_n|t)$.

Let Φ be the map defined as follows:

$$\Gamma[t] \to \mathrm{Map}(\mathcal{SP}_\infty, \mathbb{Q}[t]), \quad f(x|t) \mapsto (\mu \mapsto f(t_\mu|t))_{\mu\in\mathcal{SP}_\infty}.$$

Lemma 2.2. We have $\Phi(Q_\lambda(x|t)) \in H_T^*(LG(\infty))$.

Proof. The condition (2) in Definition 2.2 is obvious. For the GKM condition, it suffices to consider $\Phi(q_k(x))$. Let $\mu \to s_\alpha\mu$ be an arrow in the graph for $LG(\infty)$. We have to show $q_k(t_\mu) - q_k(t_{s_\alpha\mu})$ is divisible by α. If $\alpha = 2t_i$, then it is easy to see that $q_k(t_\mu) - q_k(t_{s_\alpha\mu})$ is divisible by t_i. On the other hand, one sees that the coefficients of q_k $(k \geq 1)$ is divisible by 2. The case of $\alpha = t_j - t_i$ $(i < j)$ is similar to type A case and is left to the reader. Q.E.D.

Lemma 2.3. Let $\lambda \in \mathcal{SP}_\infty$. Then the family of polynomials $(Q_\lambda(t_\mu|t))_{\mu\in\mathcal{SP}_\infty}$ is the Schubert class associated to λ. In particular, the Schubert class σ_λ exists.

Proof. By Lemma 2.2, $\Phi(Q_\lambda(x|t))$ is in $H_T^*(LG(\infty))$. Lemma 2.1 shows it satisfies the defining condition for $\sigma_\lambda^{(\infty)}$. Q.E.D.

Theorem 2.2 ([18])**.** There is an isomorphism of $\mathbb{Q}[t]$-algebras

$$\Phi : \Gamma[t] \xrightarrow{\cong} H_T^*(LG(\infty))$$

sending $Q_\lambda(x|t)$ to $\sigma_\lambda^{(\infty)}$ for all strict partitions λ.

Proof. The theorem follows from Proposition 2.7, Proposition 2.8, and Lemma 2.3. Q.E.D.

Remark 2.5. Original proof [18] is based on the equivariant Chevalley formula.

2.6. Double Schubert polynomials for the symplectic flag variety

Let me briefly show you how the story is extended to the full flag variety.

Now we discuss an action of W on $\mathbb{Q}[t] \otimes \Gamma$. The natural action of S_∞ on $\mathbb{Q}[t]$ can be extended to $\mathbb{Q}[t] \otimes \Gamma$ by letting it act on Γ trivially.

The element s_0 acts in a funny way. But this is the heart of our theory. One may describe this action by

$$\phi(x_1, x_2, \ldots; t_1, t_2, t_3, \ldots) \mapsto \phi(-t_1, x_1, x_2, \ldots; -t_1, t_2, t_3, \ldots).$$

More explicitly, on the algebra generators, s_0 operates

$$s_0^t : q_k(x) \mapsto q_k(x) + 2\sum_{i=1}^{k} (-t_1)^i q_{k-i}(x), \quad t_1 \mapsto -t_1, \quad t_j \mapsto t_j \ (j \geq 2).$$

Or more essential way of description is

$$\prod_{i=1}^{\infty} \frac{1+x_i}{1-x_i} \mapsto \frac{1-t_1}{1+t_1} \prod_{i=1}^{\infty} \frac{1+x_i}{1-x_i}.$$

From this, one easily sees that $(s_0^t)^2 = \mathrm{id}$. Define s_i^t $(i \geq 1)$ on $\Gamma[t]$ by $s_i^t(t_i) = t_{i+1},\ s_i^t(t_{i+1}) = t_i$ and $s_i^t(t_j) = t_j$ $(j \neq i, i+1)$. Then $s_i \mapsto s_i^t$ gives an action of W_∞ on $\Gamma[z,t]$.

Let us define the left divided difference oparators on $\Gamma[t]$:

$$\delta_i f = \frac{f - s_i^t f}{\alpha_i}.$$

Theorem 2.3 ([20]). We have $\delta_i Q_\lambda(x|t) = \begin{cases} Q_{s_i\lambda}(x|t) & \text{if } s_i\lambda < \lambda, \\ 0 & \text{if } s_i\lambda \geq \lambda. \end{cases}$

This is a part of results in [20]. A proof can be carried out by a direct calculation using expression (2.13). See [22] for the similar calculation in the case of equivariant K-theory.

Let $\Gamma[z,t]$ denote the polynomial ring of the variables $t = (t_1, t_2, \ldots)$ and $z = (z_1, z_2, \ldots)$ with coefficients in Γ. There are operators s_i^z $(i \geq 0)$ on $\Gamma[z,t]$ such that $s_i \mapsto s_i^z$ $(i \geq 0)$ gives an action of W_∞ on $\Gamma[z,t]$ as $\mathbb{Q}[z]$-algebra automorphisms. In order to define this second action, we can use the following ring automorphism:

$$\omega(t_i) = -z_i, \quad \omega(z_i) = -t_i, \quad \omega(Q_i(x)) = Q_i(x).$$

Then $s_i^z = \omega s_i^t \omega$ $(i \geq 0)$. Define the right divided difference operators by

$$\partial_i f = \frac{f - s_i^z f}{\omega(\alpha_i)} \quad (i \geq 0).$$

We define $H_T^*(\mathcal{F}l^{C_\infty})$ as the $\mathbb{Q}[t]$-subalgebra of $\mathrm{Map}(W_\infty, \mathbb{Q}[t])$.

Theorem 2.4 ([20])**.** There is an isomorphism of $\mathbb{Q}[t]$-algebras

$$\Gamma[z,t] \overset{\cong}{\to} H_T^*(\mathcal{F}l^{C_\infty}).$$

Explicitly, the map is given by specialization $z_i \mapsto t_{w(i)}$, $x_i \mapsto t_\mu$ for all $w \in W$, where $\mu \in \mathcal{SP}_\infty$ corresponds to the coset wS_∞.

Let $\mathfrak{C}_w(x;z,t)$ denote the element in $\Gamma[z,t]$ corresponding to σ_w ($w \in W_\infty$). We call this *double Schubert polynomials* rather than "triple" Schubert polynomials. This is a natural extension of functions $\mathfrak{C}_w(x;z)$ introduced by Billey and Haiman, which is the "single" Schubert polynomial that correspond to the non-equivariant cohomology $H^*(\mathcal{F}l^{C_\infty})$.

Proposition 2.9. $\{\mathfrak{C}_w(x;z,t)\}_{w\in W_\infty}$ are characterized by the following conditions:

- $\mathfrak{C}_e(x;z,t) = 1$,
- $\delta_i\mathfrak{C}_w(x;z,t) = \begin{cases} \mathfrak{C}_{s_iw}(x;z,t) & \ell(s_iw) = \ell(w) - 1, \\ 0 & \ell(s_iw) = \ell(w) + 1, \end{cases}$
- $\partial_i\mathfrak{C}_w(x;z,t) = \begin{cases} \mathfrak{C}_{ws_i}(x;z,t) & \ell(ws_i) = \ell(w) - 1, \\ 0 & \ell(ws_i) = \ell(w) + 1. \end{cases}$

We have the following explicit formula.

Theorem 2.5 ([20])**.** Let $w_0^{(n)}$ be the longest element of $W_n = \langle s_0, s_1, \ldots, s_{n-1}\rangle$. Then we have

$$\mathfrak{C}_{w_0^{(n)}}(x;z,t) = Q_{2n-1,2n-3,\cdots,3,1}(x|t_1, -z_1, t_2, -z_2, \ldots).$$

This is a remarkable expression, since the right hand side is a single Pfaffian. Even more remarkably, Anderson and Fulton [2] proved Pfaffian formula for a wider class of Weyl group elements called "Vexillary signed permutations" introduced there.

2.7. Factorial Q-functions and Kazarian's formula

Recall the universal sequence:

$$0 \longrightarrow S \longrightarrow E \longrightarrow S^* \longrightarrow 0.$$

Now we consider T-equivariant cohomology of $LG(n)$. Note that E is a trivial vector bundle but it is not equivariantly trivial. Let $\pm t_1, \ldots, \pm t_n$ be the equivariant Chern roots of E.

Let $E = E_0 \oplus E_0^\perp$ ($E_0 = \bigoplus_{i=1}^n L_i$ is a standard Lagrangian subspace of E and $E_0^\perp$ is the orthocomplement). For $1 \le k \le n$, let $U_k = \bigoplus_{i=k}^n L_i$.

Proposition 2.10 (*cf.* [20]). There is a natural homomorphism $\Gamma[t] \to H_T^*(LG(n))$ such that

$$q_i^{(k)}(x|t) \mapsto c_i^T(E - S - U_k).$$

Moreover, the map equals the composition of the map Φ of Theorem 2.2 and the natural projection $H_T^*(LG(\infty)) \to H_T^*(LG(n))$.

Proof. By the Whitney relation $c^T(E_0)c^T(E_0^\perp) = c^T(S)c^T(S^*)$ we have

$$c_i^T(S^* - E_0) = c_i^T(E_0^\perp - S).$$

We denote the class by β_i. Then β_i's satisfy the quadratic relations

$$\beta_i^2 + 2\sum_{j=1}^{i-1}(-1)^j \beta_{i+j}\beta_{i-j} = 0 \quad (i \geq 1).$$

as a consequence of the Whitney relation. So we can define $\Gamma[t] \to H_T^*(LG(n))$ by sending $q_i(x)$ to β_i. The second statement is directly checked (see [20]). Q.E.D.

Kazarian [25] proved the Lagrangian degeneracy loci formula in terms of multi-Schur Pfaffian with the entry of the right hand the above proposition. This means that Kazarian's formula is nothing but the factorial Q-function!

Remark 2.6. Proposition 2.10 explains *ad hoc* introduction of the ring $\Gamma[z,t]$ of the double Schubert polynomials. In the recent survey paper [49] by Tamvakis, he call the construction the *geometrization map*. Also this shows the image of Q-functions in $H_T^*(\mathcal{F}l^{C_n})$ are written as polynomials of the equivariant Chern roots $z_1, \ldots, z_n$ of S with coefficients in $H_T^*(pt) = \mathbb{Q}[t_1, \ldots, t_n]$. This justifies the formula of the action of s_0^t and s_0^z on the Q-functions. Note that these polynomial expressions in the variables z_i, t_i are not at all unique but the expression using Q-functions is unique in $\Gamma[z,t]$.

§3. Equivariant multiplicity

So far we have used the map of restriction to a torus fixed point as a convenient algebraic tool to describe the equivariant Schubert classes. In this section we will describe an image of the restriction map as a special kind of more general notion called *equivariant multiplicity* ([45]). This is a multi-variable polynomial while the classical multiplicity of Hilbert-Samuel is a natural number. The notion has been called in many different ways, Joseph's characteristic polynomials ([24]), multidegrees ([48]), equivariant Hilbert polynomials ([9]).

Let T be an algebraic torus, and V be a finite dimensional vector space with algebraic T-action such that all weights of V in T are in one side of a hyperplane in the weight space of T. The equivariant multiplicity can be defined for any T-equivariant coherent sheaf $\mathcal{F}$ on such U. In this lecture, however, we restrict ourselves to the case of our interest, namely $U = \mathcal{U}_\mu$ is the standard T-stable neighbourhood of a torus fixed point e_μ, and $\mathcal{F}$ is the structure sheaf of the Schubert variety X_λ. Note that most of the arguments are applicable to more general setting mentioned above.

Combined with some combinatorial results in the previous sections, we deduce some combinatorial results for the classical multiplicity of the Schubert variety at a T-fixed point.

3.1. Notation

For some general geometric background related to this section, we refer to [4]. Let G be a complex semisimple, connected and simply connected, linear algebraic group. Let B be a Borel subgroup of G. There is a unique maximal torus T contained in B. Any subgroup P of G containing B is called a standard parabolic subgroup. Let W and W_P be Weyl groups of G and P respectively. Each left coset of W/W_P has a unique element of the minimum length. We denote by W^P the set of all such elements. For $\lambda \in W^P$, set $X_\lambda^\circ = B_- e_\lambda$, with $e_\lambda = w_\lambda P \in G/P$, where $w_\lambda \in W$ is a representative of λ. The Schubert variety is defined to be the closure $X_\lambda = \overline{X_\lambda^\circ}$.

Let $R_u(P)$ denote the unipotent radical of P, and let $\Delta_P^+ = \{\beta \in \Delta^+ \mid U_\beta \subset R_u(P)$, where U_β is the root subgroup associated to β. For a given $\mu \in W^P$, let U_μ^- be the subgroup of G generated by the root subgroups $U_{-\beta},\ \beta \in w_\mu(\Delta^+ - \Delta_P^+)$. Under the map $G \to G/P,\ g \mapsto ge_\mu$, U_μ^- is mapped isomorphically onto its image $\mathcal{U}_\mu = U_\mu^- e_\mu$. Thus we obtain a canonical T-stable affine neighbourhood of e_μ.

3.2. Formal character

The *characters* $\chi : T \to \mathbb{C}^\times$ of T form a free abelian group $X^*(T)$. The differential of $\chi \in X^*(T)$ is a linear form $d\chi : \mathfrak{t} \to \mathbb{C}$ on the Lie algebra $\mathfrak{t}$ of T, called an *integral weight*. Let $\hat{T}$ denote the lattice in $\mathfrak{t}^*$ of all integral weights.

Let M be a T-module. For each character χ with differential $d\chi = \lambda$, we denote by M_λ the corresponding weight space

$$M_\lambda = \{v \in M \mid tv = \chi(t)v \ (t \in T)\}$$

If $M = \bigoplus_{\lambda\in\hat{T}} M_\lambda$ is a direct sum of weight spaces such $\dim M_\lambda$ are all finite, we define the *formal character* of M as the formal sum

$$\mathrm{ch}(M) := \sum_{\lambda\in\hat{T}} (\dim M_\lambda) e^\lambda.$$

For example, for $\mu \in W^P$, the coordinate ring $\mathbb{C}[\mathcal{U}_\mu]$ of the T-stable affine neighbourhood of e_μ is a T-module with the formal character D_μ^{-1} with

$$D_\mu = \prod_{\gamma\in\Delta_\mu} (1 - e^{-\gamma}),$$

where $\Delta_\mu = w_\mu(\Delta^+ - \Delta_P^+)$. More generally we have the following result.

Proposition 3.1. Let $\lambda, \mu \in W^P$. There exists $\psi_\lambda|_\mu \in R(T)$ such that

$$\mathrm{ch}_T\, \mathbb{C}[X_\lambda \cap \mathcal{U}_\mu] = \frac{\psi_\lambda|_\mu}{D_\mu}.$$

Proof. For a proof, we refer to [5], Cor. 3.6. Q.E.D.

Remark 3.1. The most natural interpretation for $\psi_\lambda|_\mu$ is given in T-equivariant K-theory. Namely, if $\mathcal{O}_{X_\lambda}$ is the structure sheaf of X_λ, the restriction of its class $[\mathcal{O}_{X_\lambda}]$ at e_μ is $\psi_\lambda|_\mu$. For the proof of this fact, see for example [9, Claim 6.6.8].

For each $\gamma \in \hat{T}$, we define the formal power series

$$e^\gamma = \sum_{i=0}^{\infty} \frac{\gamma^i}{i!}$$

considered as element of $\hat{S}(\mathfrak{t}^*)$, where $\hat{S}(\mathfrak{t}^*)$ is the completion of the symmetric algebra $S(\mathfrak{t}^*)$ by the ideal $\mathfrak{t}^* S(\mathfrak{t}^*)$.

Proposition 3.2. Suppose $\lambda \leq \mu$. The lowest non-vanishing degree of $\psi_\lambda|_\mu$ considered as an element of $\hat{S}(\mathfrak{t}^*)$ equals $\mathrm{codim}(X_\lambda)$. Moreover we have

$$\psi_\lambda|_\mu = \sigma_\lambda|_\mu + \text{higher degree terms},$$

where $\sigma_\lambda|_\mu$ is the image of the equivariant Schubert class σ_λ by the restriction map $\iota_\mu^* : H_T^*(G/P) \to H_T^*(e_\mu) \cong S$.

Proof. A proof of this fact in general framework is given in [5], Theorem 3.10. See also [9], Theorem 6.6.12 for a more geometric proof. Q.E.D.

3.3. Restriction to one-parameter subgroups

Now we will describe a passage from the equivariant multiplicity to the classical multiplicity.

Let $X_*(T)$ denote the set of *one-parameter subgroups* of T, *i.e.* the set of all homomorphisms $\phi : \mathbb{C}^\times \to T$ as algebraic groups. Let $\phi \in X_*(T)$. Let $R(T)$ denote the representation ring of the algebraic torus T. We can consider the natural restriction map

$$\phi^* : R(T) \to \mathbb{Z}[z, z^{-1}] = R(\mathbb{C}^\times), \quad e^\gamma \mapsto z^{\langle \gamma, \phi \rangle},$$

where $\langle \cdot, \cdot \rangle : \hat{T} \times X_*(T) \to \mathbb{Z}$ is the natural pairing. We also denote by ϕ^* the map from $S(\mathfrak{t}^*)$ to $\mathbb{Z}$ given by $\gamma \mapsto \langle \gamma, \phi \rangle$ ($\gamma \in \mathfrak{t}^*$).

By Proposition 3.2, we have that the Taylor expansion of $\phi^*(\psi_\lambda|_\mu) \in \mathbb{Q}[z, z^{-1}]$ at $z = 1$ has the following form:

$$\phi^*(\psi_\lambda|_\mu) = \phi^*(\sigma_\lambda|_\mu) \cdot (1 - z)^{\operatorname{codim}(X_\lambda)} + \text{higher order terms}. \tag{3.1}$$

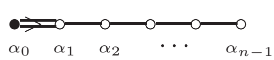
In particular, if $\phi^*(\sigma_\lambda|_\mu) \neq 0$, then the order of zero of $\phi^*(\psi_\lambda|_\mu)$ at $z = 1$ is $\operatorname{codim}(X_\lambda)$.

3.4. G/P of cominuscule type

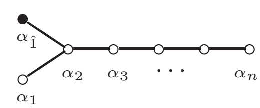
A maximal (standard) parabolic subgroup P is of *cominuscule* type if the corresponding simple root α occurs with coefficient 1 in the expression of the highest root as a linear combination of the simple roots. Let $\varpi_\alpha^\vee$ be the fundamental coweight corresponding to α. Note that $\varpi_\alpha^\vee \in X_*(T)$.

The following pairs of Dynkin diagrams and simple roots corresponds to cominuscule G/P:

(C_n, α_0)



$(D_{n+1}, \alpha_{\hat{1}})$



The pair (C_n, α_0) corresponds to $LG(n)$ and $(D_{n+1}, \alpha_{\hat{1}})$ corresponds to the maximal isotropic Grassmannian $OG(n)$ for orthogonal $(2n+2)$-dimensional linear space. In both cases, there is a natural bijection $W^P \cong \mathcal{SP}(n)$.

Lemma 3.1. Let P be a maximal parabolic subgroup of cominuscule type, and α be the corresponding simple root. For $\mu \in W^P$, let $\phi_\mu = w_\mu(\varpi_\alpha^\vee)$. Then

$$\Delta_\mu = \{\beta \in \Delta^+ \mid \langle \beta, \phi_\mu \rangle = 1\}.$$

Proof. It suffices to consider the identity coset id. By assumption on α, we have $0 \leq \langle \beta, \varpi_\alpha^\vee \rangle \leq 1$ for any positive root β. Since $\beta \notin \Delta_P^+$ is equivalent to $\langle \beta, \varpi_\alpha^\vee \rangle = 0$, this is also equivalent to the condition $\langle \beta, \varpi_\alpha^\vee \rangle = 1$. Q.E.D.

Let $T_\mu = \phi_\mu(\mathbb{C}^\times) \cong \mathbb{C}^\times$ be the image of ϕ_μ. Then $T_\mu \subset T$ acts on $\mathcal{U}_\mu$ by scaler multiplication. This implies that $X_\lambda \cap \mathcal{U}_\mu$ is a *cone*, *i.e.* a subvariety of $\mathcal{U}_\mu$ which is stable under the scaler multiplication. This also implies that $\mathbb{C}[X_\lambda \cap \mathcal{U}_\mu]$ is naturally a $\mathbb{Z}$-graded ring isomorphic to $\mathrm{gr}_{\mathfrak{m}} \mathcal{O}_{X_\lambda, e_\mu}$, the associated graded ring of the local ring $\mathcal{O}_{X_\lambda, e_\mu}$ with respect to the maximal ideal $\mathfrak{m}$.

Note for $\gamma \in \Delta_\mu$ we have $\phi_\mu^*(e^{-\gamma}) = z$. Denote by $\mathrm{mult}_{e_\mu}(X_\lambda)$ the multiplicity of X_λ at e_μ.

Proposition 3.3. Let P and ϕ_μ be as in Lemma 3.1. Then for $\lambda \in W^P$, we have

$$\phi_\mu^*(\sigma_\lambda|_\mu) = \mathrm{mult}_{e_\mu}(X_\lambda).$$

Proof. By Proposition 3.1 and $\phi_\mu^*(e^{-\gamma}) = z$ for $\gamma \in \Delta_\mu$, we see that the Poincare series of $\mathbb{C}[X_\lambda \cap \mathcal{U}_\mu] \cong \mathrm{gr}_{\mathfrak{m}} \mathcal{O}_{X_\lambda, e_\mu}$ is given by

$$\begin{aligned} \phi_\mu^* \mathrm{ch}_T\, \mathbb{C}[X_\lambda \cap \mathcal{U}_\mu] &= \frac{\phi_\mu^*(\psi_\lambda|_\mu)}{(1-z)^{\dim G/P}} \\ &= \frac{\phi_\mu^*(\sigma_\lambda|_\mu) + (\text{term vanishes at } z=1)}{(1-z)^{\dim \mathbb{C}[X_\lambda \cap \mathcal{U}_\mu]}}, \end{aligned}$$

where we used (3.1) and $\dim \mathbb{C}[X_\lambda \cap \mathcal{U}_\mu] = \dim G/P - \mathrm{codim}(X_\lambda)$. Then by the definition of the Hilbert-Samuel multiplicity (*e*.g. [40, Chap. 5]) we have the result. Q.E.D.

Theorem 3.1 ([21], *cf.* [18])**.** Let $X = G/P$ be the generalized flag veriety of type $(D_{n+1}, \alpha_{\hat{1}})$ or (C_n, α_0). Let λ, μ be elements in $\mathcal{SP}(n) \cong W^P$. Then

$$\mathrm{Pf}(\mathrm{mult}_{e_\mu}(X_{\lambda_i, \lambda_j}))_{1 \leq i < j \leq r} = \mathrm{mult}_{e_\mu}(X_\lambda).$$

Proof. Since G/P is cominuscule, for the case (C_n, α_0), the result follows immediately from Theorem 2.2 and Proposition 3.3 and (2.12). The case $(D_{n+1}, \alpha_{\hat{1}})$ is similar (see [21]). Q.E.D.

Exercise 3.1. Lakshmibai and Weyman [35] derived a recurrence equation for $\mathrm{mult}_{e_\mu}(X_\lambda)$. By comparing this with Pieri formula for $Q_\lambda(x|t)$ due to Ivanov [23], prove Theorem 3.1. (See [21, Remark after Proposition 9.1].)

Remark 3.2. By using Gröbner degeneration, Ghorpade and Raghavan [13] and Raghavan and Upadhyay [44] proved combinatorial descriptions of the multiplicity of the local rings of the Schubert variety for G/P corresponding to (C_n, α_0) and $(D_{n+1}, \alpha_{\hat{1}})$ respectively.

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