

## Restricted Lazarsfeld–Mukai bundles and canonical curves

Marian Aprodu, Gavril Farkas and Angela Ortega

*Dedicated to Professor Shigeru Mukai on his sixtieth birthday, with admiration*

For a  $K3$  surface  $S$ , a smooth curve  $C \subset S$  and a globally generated linear series  $A \in W_d^r(C)$  with  $h^0(C, A) = r + 1$ , the *Lazarsfeld–Mukai* vector bundle  $E_{C,A}$  is defined via the following elementary modification on  $S$

$$(1) \quad 0 \longrightarrow E_{C,A}^\vee \longrightarrow H^0(C, A) \otimes \mathcal{O}_S \longrightarrow A \longrightarrow 0.$$

The bundles  $E_{C,A}$  have been introduced more or less simultaneously in the 80's by Lazarsfeld [L1] and Mukai [M1] and have acquired quite some prominence in algebraic geometry. On one hand, they have been used to show that curves on general  $K3$  surfaces verify the Brill–Noether theorem [L1], and this is still the only class of smooth curves known to be general in the sense of Brill–Noether theory in every genus. When  $\rho(g, r, d) = 0$ , the vector bundle  $E_{C,A}$  is rigid and plays a key role in the classification of Fano varieties of coindex 3. For  $g = 7, 8, 9$ , the corresponding Lazarsfeld–Mukai bundle has been used to coordinatize the moduli space of curves of genus  $g$ , thus giving rise to a new and more concrete model of  $\mathcal{M}_g$ , see [M2], [M3], [M4]. Furthermore, Lazarsfeld–Mukai bundles of rank two have led to a characterization of the locus in  $\mathcal{M}_g$  of curves lying on  $K3$  surfaces in terms of existence of linear series with unexpected syzygies [F], [V]. For a recent survey on this circle of ideas, see [A].

Recently, Lazarsfeld–Mukai bundles have proven to be effective in shedding some light on an interesting conjecture of Mercat in Brill–Noether theory, see [FO1], [FO2], [LMN]. Recall that the Clifford index

---

Received April 2, 2014.

Revised October 3, 2014.

2010 *Mathematics Subject Classification.* 14D20, 14H10, 14J28.

of a semistable vector bundle  $E \in \mathcal{U}_C(n, d)$  on a smooth curve  $C$  of genus  $g$  is defined in [LN1] as

$$\gamma(E) := \mu(E) - \frac{2}{n}h^0(C, E) + 2 \geq 0.$$

Then the *higher Clifford indices* of the curve  $C$  are defined as the quantities

$$\text{Cliff}_n(C) := \min \left\{ \gamma(E) : E \in \mathcal{U}_C(n, d), \quad d \leq n(g-1), \quad h^0(C, E) \geq 2n \right\}.$$

For any line bundle  $L$  on  $C$  such that  $h^i(C, L) \geq 2$  for  $i = 0, 1$ , that is, contributing to the classical Clifford index  $\text{Cliff}(C)$ , by computing the invariants of the strictly semistable vector bundle  $E := L^{\oplus n}$ , one finds that  $\text{Cliff}_n(C) \leq \text{Cliff}(C)$ . Mercat [Me1] predicted that for any smooth curve  $C$  of genus  $g$ , the following equality

$$(M_n) : \quad \text{Cliff}_n(C) = \text{Cliff}(C).$$

should hold. Counterexamples to  $(M_2)$  have been found on curves lying on  $K3$  surfaces that are special in Noether-Lefschetz sense, see [FO1], [FO2] and [LN2]. However,  $(M_2)$  is expected to hold for a general curve of genus  $g$ , and in fact even for a curve  $C$  lying on a  $K3$  surface  $S$  such that  $\text{Pic}(S) = \mathbb{Z} \cdot C$ . For instance, it is known that statement  $(M_2)$  holds on  $\mathcal{M}_{11}$  outside a certain Koszul divisor (which also admits a Noether-Lefschetz realization), see [FO2] Theorem 1.3. It has also been shown that  $(M_2)$  holds generically on  $\mathcal{M}_g$  for  $g \leq 16$ , see [FO1].

It has been proved in [LMN] that rank three restricted Lazarsfeld-Mukai bundles invalidate statement  $(M_3)$  in genus 9 and 11 respectively, that is, Mercat's conjecture in rank three fails generically on  $\mathcal{M}_9$  and  $\mathcal{M}_{11}$  respectively. This was then extended in [FO2] Theorem 1.4, to show that on a  $K3$  surface  $S$  with  $\text{Pic}(S) = \mathbb{Z} \cdot C$ , where  $C^2 = 2g - 2$ , if  $A \in W_d^2(C)$  is a linear system where  $d := \lfloor \frac{2g+8}{3} \rfloor$ , the restriction to  $C$  of the Lazarsfeld-Mukai bundle  $E_{C,A}$  is stable and has Clifford index strictly less than  $\lfloor \frac{g-1}{2} \rfloor$ , in particular, statement  $(M_3)$  fails for the curve  $C$ . For further background on this problem, we refer to the papers [Me1], [LN1] and [GMN].

The restricted Lazarsfeld-Mukai bundle  $E|_C := E_{C,A} \otimes \mathcal{O}_C$  sits in the following exact sequence on the curve  $C \subset S$

$$(2) \quad 0 \longrightarrow Q_A \longrightarrow E|_C \longrightarrow K_C \otimes A^\vee \longrightarrow 0,$$

where  $Q_A = M_A^\vee$  is the dual of the kernel bundle defined by the sequence

$$0 \longrightarrow M_A \longrightarrow H^0(C, A) \otimes \mathcal{O}_C \longrightarrow A \longrightarrow 0.$$

One then easily shows [V], [FO2] that the sequence (2) is exact on global sections, that is,

$$h^0(C, E|_C) = h^0(C, K_C \otimes A^\vee) + h^0(C, Q_A) = g - d + 2r + 1.$$

By choosing the degree  $d$  minimal such that  $W_d^r(C) \neq \emptyset$ , precisely when  $d = r + \lfloor \frac{r(g+1)}{r+1} \rfloor$ , it becomes clear that, for sufficiently high  $g$ , one has

$$\gamma(E|_C) < \text{Cliff}(C),$$

that is,  $E|_C$ , when semistable, provides a counterexample to Mercat’s conjecture ( $M_{r+1}$ ). We prove the following result, extending to rank 4 a picture studied in smaller ranks in the papers [M1], [V], respectively [FO2].

**Theorem 0.1.** *Let  $S$  be a K3 surface with  $\text{Pic}(S) = \mathbb{Z} \cdot L$ , where  $L^2 = 2g - 2$  and write*

$$g = 4i - 4 + \rho \quad \text{and} \quad d = 3i + \rho,$$

*with  $\rho \geq 0$  and  $i \geq 6$ . Then for a general curve  $C \in |L|$  and a globally generated linear series  $A \in W_d^3(C)$  with  $h^0(C, A) = 4$ , the restriction to  $C$  of the Lazarsfeld–Mukai bundle  $E_{C,A}$  is stable.*

Note that in Theorem 0.1,  $\dim W_d^3(C) = \rho$ . The rank 3 version of this result was proved in [FO2]. We record the following consequence of Theorem 0.1:

**Corollary 0.2.** *For  $C \subset S$  with  $g \geq 20$  and  $\text{Pic}(S) = \mathbb{Z} \cdot C$ , we set  $d := \lfloor \frac{4g+14}{3} \rfloor$  and  $A \in W_d^3(C)$  with  $h^0(C, A) = 4$ . Then  $E|_C$  is a stable rank 4 bundle with  $\gamma(E|_C) < \lfloor \frac{g-1}{2} \rfloor$ . It follows that the statement ( $M_4$ ) fails for  $C$ .*

The curves  $C$  appearing in Corollary 0.2 are Brill–Noether general, that is, they satisfy  $\text{Cliff}(C) = \lfloor \frac{g-1}{2} \rfloor$ , see [L1].

Theorem 0.1 and Corollary 0.2 fit into a more general set of results that are independent from the structure of  $\text{Pic}(S)$ . For example, we show that under mild restrictions, on a very general K3 surface, the extension (2) is non-trivial and the restricted Lazarsfeld–Mukai bundle  $E|_C$  is simple (see Theorem 1.3). We expect that the bundle  $E|_C$  remains stable also for higher rank  $r+1 = h^0(C, A)$ , at least when  $\text{Pic}(S) = \mathbb{Z} \cdot C$ . However, our method of proof based on the Bogomolov inequality, seems not to extend easily for  $r \geq 4$ .

The second topic we discuss in this paper concerns a connection between normal bundles of canonical curves and Mercat’s conjecture. The

question we pose is however fundamental and interesting irrespective of Mercat’s conjecture.

For a smooth non-hyperelliptic canonical curve  $C \subset \mathbf{P}^{g-1}$  of genus  $g$ , we consider the normal bundle  $N_C := N_{C/\mathbf{P}^{g-1}}$ ; we then define the twist of the conormal bundle  $E := N_C^\vee \otimes K_C^{\otimes 2}$ . By direct calculation

$$\det(E) = K_C^{\otimes (g-5)} \quad \text{and} \quad \text{rk}(E) = g - 2.$$

In particular, the vector bundle  $E$  contributes to  $\text{Cliff}_{g-2}(C)$  if and only if  $g \leq 8$ . Since  $M_{K_C}(-1) = \Omega_{\mathbf{P}^{g-1}|C}$ , the bundle  $E$  sits in the following exact sequence

$$(3) \quad 0 \longrightarrow E \longrightarrow M_{K_C} \otimes K_C \xrightarrow{\gamma_{K_C}} K_C^{\otimes 3} \longrightarrow 0,$$

where  $\gamma_{K_C} : H^0(C, M_{K_C} \otimes K_C) \rightarrow H^0(C, K_C^{\otimes 3})$  is the Gaussian map of  $C$ , see [W]. The map  $\gamma_{K_C}$  vanishes on symmetric tensors, hence  $\text{Ker}(\gamma_{K_C}) = I_2(K_C) \oplus \text{Ker}(\psi_{K_C})$ , where

$$\psi_{K_C} := \gamma_{K_C}|_{\wedge^2 H^0(C, K_C)} : \bigwedge^2 H^0(C, K_C) \rightarrow H^0(C, K_C^{\otimes 3}),$$

and  $I_2(K_C) = K_{1,1}(C, K_C)$  is the space of quadrics containing the canonical curve  $C$ . The map  $\psi_{K_C}$  has been studied intensely in the context of deformations in  $\mathbf{P}^g$  of the cone over the canonical curve  $C \subset \mathbf{P}^{g-1}$ , see [W]. It is in particular known [CHM], [V] that  $\psi_{K_C}$  is surjective for a general curve  $C$  of genus  $g \geq 12$ .

We now specialize to the case  $g = 7$ , when  $E$  contributes to  $\text{Cliff}_5(C)$ . Then

$$\text{rk}(E) = 5 \quad \text{and} \quad \det(E) = K_C^{\otimes 2},$$

therefore  $\mu(E) = \frac{24}{5}$ . It is easy to show that the Gaussian map  $\psi_{K_C}$  is injective for every smooth curve  $C$  of genus 7 having maximal Clifford index  $\text{Cliff}(C) = 3$ . In particular, the space

$$H^0(C, E) = I_2(K_C)$$

is 10-dimensional and  $\gamma(E) = 2 + \frac{4}{5} < \text{Cliff}(C)$ . We establish the following result:

**Theorem 0.3.** *The normal bundle  $N_{C/\mathbf{P}^6}$  of every canonical curve  $C$  of genus 7 with maximal Clifford index is stable. In particular, the Mercat conjecture ( $M_5$ ) fails for a general curve of genus 7.*

The proof of Theorem 0.3 uses in an essential way Mukai’s realisation [M3] of a canonical curve  $C$  of genus 7 with  $\text{Cliff}(C) = 3$  as a linear

section of the 10-dimensional spinorial variety  $OG(5, 10) \subset \mathbf{P}^{15}$ . In particular, the vector bundle  $E$  is the restriction to  $C$  of the rank 5 spinorial bundle on  $OG(5, 10)$ , which endows  $E$  with an extra structure that only exists in genus 7. Note that the normal bundle of every canonical curve of genus at most 6 is unstable, and more generally, the normal bundle of a tetragonal canonical curve of any genus is unstable (see also Section 3). In particular, we have the following identification of cycles on  $\mathcal{M}_7$

$$(4) \quad \left\{ [C] \in \mathcal{M}_7 : N_C \text{ is unstable} \right\} = \mathcal{M}_{7,4}^1,$$

where the right hand side denotes the divisor of tetragonal curves of genus 7. We make the following conjecture:

**Conjecture 0.4.** *The normal bundle  $N_{C/\mathbf{P}^{g-1}}$  of a general canonical curve  $C$  of genus  $g \geq 7$  is stable.*

Note that the stability of the normal bundle  $N_{C/\mathbf{P}^r}$  of a curve of genus  $g$  is not known even in the case of a non-special embedding  $C \hookrightarrow \mathbf{P}^r$  given by a line bundle  $L \in \text{Pic}(C)$  of large degree. This is in stark contrast with the case the kernel bundle  $M_L = \Omega_{\mathbf{P}^r|C}(1)$ , whose stability easily follows by a filtration argument due to Lazarsfeld [L2]. For some very partial results in this direction, see [EL]. In general, one can show by degenerating a canonical curve  $C \subset \mathbf{P}^{g-1}$  to the transversal union of two rational normal curves in  $\mathbf{P}^{g-1}$  meeting transversally in  $g + 1$  points, that  $N_{C/\mathbf{P}^{g-1}}$  is not too unstable. Due to the fact that the slope  $\mu(N_{C/\mathbf{P}^{g-1}})$  is not an integer, this simple minded technique does not seem to lead to a full solution, because one cannot expect to find a specialization in which the corresponding limit of the normal bundle is a direct sum of line bundles of the same degree. It is of course, natural to ask whether a generalization of the equality (4) exists for higher genus.

*Acknowledgments.* The second author is grateful to Shigeyuki Kondo for organizing a wonderful conference "Developments in moduli theory" in Kyoto in June 2013 to celebrate Mukai's 60th birthday. The first author was partly supported by a grant of the Romanian Ministry of National Education, CNCS-UEFISCDI project number PN-II-ID-PCE-2012-4-0156, and by a Humboldt fellowship. He thanks Humboldt Universität zu Berlin and Max Planck Institut für Mathematik Bonn for hospitality during the preparation of this work. The second and third authors are partly financed by the Sonderforschungsbereich 647 "Raum-Zeit-Materie".

**§1. Simplicity of restricted Lazarsfeld-Mukai bundles**

We fix a K3 surface  $S$ , a smooth curve  $C \subset S$  of genus  $g$  and a globally generated linear series  $A \in W_d^r(C)$ , with  $h^0(C, A) = r + 1$ . Using the evaluation sequence (1), we form the vector bundle  $F = F_{C,A}$ ; by dualizing, we obtain an exact sequence for the dual vector bundle  $E = E_{C,A} := F_{C,A}^\vee$ :

$$(5) \quad 0 \longrightarrow H^0(C, A)^\vee \otimes \mathcal{O}_S \longrightarrow E_{C,A} \longrightarrow K_C \otimes A^\vee \longrightarrow 0.$$

It is well-known [M1], [L1] that  $c_1(E) = [C]$  and  $c_2(E) = d$ ; moreover  $h^0(S, F) = 0$  and  $h^1(S, E) = h^1(S, F) = 0$ . Finally, one also has that

$$\chi(S, E \otimes F) = 2 - 2\rho(g, r, d);$$

in particular, if  $E$  is a simple bundle, then  $\rho(g, r, d) \geq 0$ . Assuming furthermore that  $\text{Pic}(S) = \mathbb{Z} \cdot C$ , it is also well-known that both  $E$  and  $F$  are  $C$ -stable bundles on  $S$ .

**1.1. The rank 2 case**

We begin by showing that in rank 2, irrespective of the structure of  $\text{Pic}(S)$ , a splitting of the restriction  $E|_C$  can only be induced by an elliptic pencil on the K3 surface.

**Theorem 1.1.** *Let  $C \subset S$  be as above and  $A \in W_d^1(C)$  a base point free pencil of degree  $2 < d < g - 1$  with  $K_C \otimes A^\vee$  globally generated. The following conditions are equivalent:*

- (i)  $E|_C \cong A \oplus (K_C \otimes A^\vee)$ ;
- (ii) *There exists an elliptic pencil  $N \in \text{Pic}(S)$  such that  $N|_C = A$ .*
- (iii)  $h^0(S, E \otimes F) < h^0(C, E \otimes F|_C)$ .

**Corollary 1.2.** *With notation as above, if  $g \leq 2d - 2$  and  $A$  is not induced by an elliptic pencil on  $S$ , then  $E|_C$  is simple if and only if  $E$  is simple.*

Note that it is easy to see that if  $E|_C$  is simple, then  $E$  is also simple. It is also known that if  $E$  is simple, then automatically  $g \leq 2d - 2$ .

*Proof.* (of Theorem 1.1)  $(ii) \Rightarrow (i)$ . Let  $N$  be an elliptic pencil with  $N|_C = A$ . Consider the exact sequence

$$0 \longrightarrow N^\vee \longrightarrow F \longrightarrow N(-C) \longrightarrow 0.$$

Its restriction to  $C$  gives a splitting of the dual of the sequence (2) characterizing  $E|_C$ . Observe that since  $d < g - 1$ , there is no morphism from  $A^\vee$  to  $K_C^\vee \otimes A$ .

(i)⇒(ii). Conversely, suppose that  $E|_C = A \oplus (K_C \otimes A^\vee)$ . Applying  $\text{Hom}(K_C \otimes A^\vee, -)$  to the sequence (1), we obtain an exact sequence

$$0 \longrightarrow \text{Ext}^1(K_C \otimes A^\vee, F) \longrightarrow \text{Ext}^1(K_C \otimes A^\vee, H^0(C, A) \otimes \mathcal{O}_S) \longrightarrow \text{Ext}^1(K_C \otimes A^\vee, A).$$

Since the extension class  $[E] \in \text{Ext}^1(K_C \otimes A^\vee, H^0(C, A) \otimes \mathcal{O}_S)$  maps to the trivial extension in  $\text{Ext}^1(K_C \otimes A^\vee, A)$ , it follows that there exists a rank 2 bundle  $G$  on  $S$  which fits into a commutative diagram:

$$(6) \quad \begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F & \longrightarrow & H^0(A) \otimes \mathcal{O}_S & \longrightarrow & A \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & G & \longrightarrow & E & \longrightarrow & A \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & K_C \otimes A^\vee & \xlongequal{\quad} & K_C \otimes A^\vee & & \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

Using that  $H^0(S, F) = H^1(S, F) = 0$ , we obtain  $H^0(S, G) \cong H^0(C, K_C \otimes A^\vee)$ . Since  $h^0(S, E) = h^0(C, A) + h^1(C, A) = h^0(C, A) + h^0(S, G)$ , and  $h^1(S, E) = 0$ , it follows that  $H^1(S, G) = 0$ . From the second row of (6), we find that  $H^0(S, G(-C)) = 0$ .

Furthermore, we compute  $c_1(G) = 0$  and  $c_2(G) = 2d - 2g + 2$ . So  $c_2(G) < 0 = c_1^2(G)$ , that is,  $G$  violates Bogomolov’s inequality, and then it sits in an extension

$$(7) \quad 0 \longrightarrow M \longrightarrow G \longrightarrow M^\vee \otimes \mathcal{I}_{\Gamma/S} \longrightarrow 0,$$

where  $\Gamma$  is a zero-dimensional subscheme of  $S$ , and  $M \in \text{Pic}(S)$  is such that  $M^2 > 0$  and  $M \cdot H > 0$  for any ample line bundle  $H$  on  $S$ . In particular,  $H^0(S, M^\vee) = 0$ , and hence  $H^0(S, M) \cong H^0(S, G) \cong H^0(C, K_C \otimes A^\vee) \neq 0$ . Moreover, since

$$h^0(S, M^\vee \otimes \mathcal{I}_{\Gamma/S}) = h^1(S, G) = 0,$$

it also follows that  $H^1(S, M) = 0$ .

On the other hand  $H^0(S, F) = 0$ , which implies that the composed map

$$M \longrightarrow G \longrightarrow K_C \otimes A^\vee$$

is non-zero; in fact, we claim it is surjective, hence  $M|_C = K_C \otimes A^\vee$ . Suppose that  $M|_C = K_C \otimes A^\vee(-D')$ , with  $D' \neq 0$  an effective divisor on  $C$ . Since  $h^0(S, G(-C)) = 0$ , we have  $h^0(S, M(-C)) = 0$ , which implies  $h^0(S, M) \leq h^0(C, M|_C)$ . Since we assumed  $K_C \otimes A^\vee$  to be globally generated, we have that

$$h^0(S, M) \leq h^0(C, K_C \otimes A^\vee(-D')) < h^0(C, K_C \otimes A^\vee) = h^0(S, M),$$

a contradiction.

Setting  $N := M^\vee(C)$ , we have shown that  $N|_C = A$  and there is an exact sequence

$$0 \longrightarrow M^\vee \longrightarrow N \longrightarrow A \longrightarrow 0.$$

Since  $h^0(S, M^\vee) = h^1(S, M^\vee) = 0$ , it follows that  $h^0(S, N) = h^0(C, A) = 2$  and hence  $N$  defines an elliptic pencil.

(iii) $\Rightarrow$ (i). From the sequence (1) twisted by  $E(-C) \cong F$ , we obtain that

$$H^0(S, E \otimes F(-C)) \subset H^0(C, A) \otimes H^0(S, E(-C)),$$

and, since  $F$  has no sections, it follows that  $H^0(S, E \otimes F(-C)) = 0$ . We have an exact sequence

$$0 \longrightarrow H^0(S, E \otimes F) \longrightarrow H^0(S, E \otimes F|_C) \longrightarrow H^1(S, E \otimes F(-C)).$$

The hypothesis implies that  $H^1(S, E \otimes F(-C)) \neq 0$ . From (1) twisted by  $E(-C) \cong F$ , we obtain the exact sequence in cohomology

$$\begin{aligned} 0 \longrightarrow H^0(C, E|_C \otimes K_C^\vee \otimes A) &\longrightarrow H^1(S, E \otimes F(-C)) \\ &\longrightarrow H^0(C, A) \otimes H^1(S, E(-C)) = 0, \end{aligned}$$

therefore  $h^0(C, E|_C \otimes K_C^\vee \otimes A) \neq 0$ . The sequence (2) yields to an exact sequence

$$\begin{aligned} 0 = H^0(C, K_C^\vee \otimes A^{\otimes 2}) &\longrightarrow H^0(C, E|_C \otimes K_C^\vee \otimes A) \\ &\longrightarrow H^0(C, \mathcal{O}_C) \rightarrow H^1(C, K_C^\vee \otimes A^{\otimes 2}). \end{aligned}$$

Then  $H^0(C, E|_C \otimes K_C^\vee \otimes A) \rightarrow H^0(C, \mathcal{O}_C)$  is an isomorphism and, under the coboundary map

$$H^0(C, \mathcal{O}_C) \ni 1 \mapsto 0 \in H^1(C, K_C^\vee \otimes A^{\otimes 2}),$$



that is, the sequence (2) is split.

Note that we also have  $h^1(S, E \otimes F(-C)) = 1$  and  $h^0(C, E \otimes F|_C) = h^0(S, E \otimes F) + 1$ .

(i)⇒(iii). From the hypothesis and from the sequence (2), we find

$$h^0(C, E|_C \otimes A^\vee) = h^0(C, K_C \otimes A^{\otimes(-2)}) + 1.$$

Furthermore,  $h^0(S, E \otimes F) = h^0(C, E|_C \otimes A^\vee)$ ; twist (5) by  $F$  and use the vanishing of  $h^0(F)$  and that of  $h^1(F)$ .

On the other hand, since  $E|_C \cong A \oplus K_C \otimes A^\vee$ , we have

$$h^0(C, E \otimes F|_C) = 2 + h^0(C, K_C \otimes A^{\otimes(-2)}),$$

hence  $h^0(C, E \otimes F|_C) = h^0(S, E \otimes F) + 1$ .

Q.E.D.

### 1.2. Lazarsfeld–Mukai bundles of higher rank

We study when the restriction  $E|_C$  is a simple vector bundle. Our main tool is a variant of the Bogomolov instability theorem.

**Theorem 1.3.** *Let  $S$  be a K3 surface and  $C \subset S$  a smooth curve of genus  $g \geq 4$  such that  $\text{Pic}(S) = \mathbb{Z} \cdot C$ . We fix positive integers  $r$  and  $d$  such that*

$$\rho(g, r, d) \geq 0, \quad g \geq 2r + 4 \quad \text{and} \quad d \leq \frac{3r(g - 1)}{2r + 2}.$$

*Then for any linear series  $A \in W_d^r(C)$  such that  $h^0(C, A) = r + 1$  and  $K_C \otimes A^\vee$  is globally generated, the restricted Lazarsfeld–Mukai bundle  $E|_C$  is simple.*

Note that in the special case  $\rho(g, r, d) = 0$ , the constraints from the previous statement give rise to the bound  $g > 2r + 5$ .

*Proof. Step 1.* We first establish that the natural extension (2), that is,

$$0 \longrightarrow Q_A \longrightarrow E|_C \longrightarrow K_C \otimes A^\vee \longrightarrow 0$$

is non-trivial. Assume that (2) is trivial. Then there is an injective morphism from  $K_C \otimes A^\vee$  to  $E|_C$  and hence a surjective map  $F(C) \rightarrow A$ . Then

$$G := \text{Ker}\{F(C) \rightarrow A\}$$

is a vector bundle of rank  $r + 1$  with Chern classes  $c_1(G) = (r - 1)[C]$  and

$$c_2(G) = c_2(F(C)) - c_1(F(C)) \cdot C + \text{deg}(A) = 2d + r(r - 3)(g - 1).$$

We compute the discriminant of  $G$

$$\Delta(G) = 2\text{rk}(G)c_2(G) - (\text{rk}(G) - 1)c_1^2(G) = 4d(r + 1) - 8r(g - 1) < 0,$$

hence  $G$  is unstable. Applying [HL] Theorem 7.3.4, there exists a subsheaf  $M \subset G$  with

$$\xi_{M,G}^2 \geq -\frac{\Delta(G)}{r(r + 1)^2},$$

where  $\xi_{M,G} = c_1(M)/\text{rk}(M) - c_1(G)/\text{rk}(G)$ . Setting  $c_1(M) = k \cdot [C]$  and  $s := \text{rk}(M)$ , the previous inequality becomes

$$\left(\frac{k}{s} - \frac{r - 1}{r + 1}\right)^2 (2g - 2) \geq \frac{8r(g - 1) - 4d(r + 1)}{r(r + 1)^2}.$$

Note that  $M$  destabilizes  $G$ , which coupled with the stability of  $F(C)$  yields

$$\frac{r - 1}{r + 1} \leq \frac{k}{s} < \frac{r}{r + 1},$$

implying after manipulations  $2d(r + 1) > 3(g - 1)r$ , thus contradicting the hypothesis.

*Step 2.* Assuming that  $E|_C$  is non-simple, we deduce that the extension (2) splits. We consider the exact sequence

$$H^0(S, E \otimes F) \longrightarrow H^0(C, E \otimes F|_C) \longrightarrow H^1(S, E \otimes F(-C)).$$

and it suffices to show that  $H^1(S, E \otimes F(-C)) = 0$ . Assuming this not to be the case, twisting (1) by  $E(-C)$  induces the exact sequence

$$\begin{aligned} H^0(C, A \otimes E|_C \otimes K_C^\vee) &\longrightarrow H^1(S, E \otimes F(-C)) \\ &\longrightarrow H^0(C, A) \otimes H^1(S, E(-C)). \end{aligned}$$

Since  $H^1(S, E(-C)) = 0$ , we obtain that  $H^0(C, A \otimes E|_C \otimes K_C^\vee) \neq 0$ . Furthermore,  $Q_A$  is a stable bundle and since  $\mu(Q_A \otimes A \otimes K_C^\vee) < 0$ , we find that

$$H^0(C, Q_A \otimes A \otimes K_C^\vee) = 0,$$

hence we also have the sequence induced from (2) after twisting with  $A \otimes K_C^\vee$

$$0 \longrightarrow H^0(C, E|_C \otimes K_C^\vee \otimes A) \longrightarrow H^0(C, \mathcal{O}_C) \longrightarrow H^1(C, K_C^\vee \otimes A \otimes Q_A).$$

We conclude that the coboundary map

$$H^0(C, \mathcal{O}_C) \rightarrow H^1(C, Q_A \otimes A \otimes K_C^\vee)$$

is trivial, that is,  $E|_C \cong Q_A \oplus (K_C \otimes A^\vee)$ , which completes the proof. Q.E.D.

§2. Stability of restricted Lazarsfeld–Mukai bundles

2.1. The rank 2 case

If  $C \subset S$  is an ample curve, then with one exception ( $g = 10$  and  $C$  a smooth plane sextic),  $\text{Cliff}(C)$  is computed by a pencil, see [CP] Proposition 3.3. We show that in rank 2 the semistability of the LM bundle is preserved under restriction.

**Theorem 2.1.** *Let  $S$  be a K3 surface,  $C \subset S$  an ample curve of genus  $g \geq 4$  and  $A \in W_d^1(C)$  a pencil computing  $\text{Cliff}(C)$ . If  $E_{C,A}$  is  $C$ -semistable on  $S$ , then  $E|_C$  is also semistable on  $C$ . Moreover, if  $E_{C,A}$  is  $C$ -stable on  $S$ , then  $E|_C$  is stable on  $C$ .*

*Proof.* The proof of the stability is similar, and hence we discuss the semistability part only. We write  $A = \mathcal{O}_C(D)$ , where  $D$  is an effective divisor on  $C$ . Suppose  $E|_C$  is unstable and consider an exact sequence

$$0 \rightarrow L_1 \rightarrow E|_C \rightarrow K_C \otimes L_1^\vee \rightarrow 0,$$

with  $\deg(L_1) \geq g$ . Since  $L_1 \not\subseteq A$ , the composed map  $L_1 \rightarrow E|_C \rightarrow K_C \otimes A^\vee$  must be non-zero, that is,  $L_1 = K_C(-D - D_1)$ , where  $D_1$  is an effective divisor on  $C$ . Set  $d_1 := \deg(D_1)$ . Consider the elementary modification

$$(8) \quad 0 \rightarrow V \rightarrow E \rightarrow A(D_1) \rightarrow 0$$

induced by the composition  $E \rightarrow E|_C \rightarrow A(D_1)$ . Then

$$c_1(V) = 0 \quad \text{and} \quad c_2(V) = 2d + d_1 - 2g + 2 < 0,$$

hence  $V$  is unstable with respect to any polarization and fits in an exact sequence

$$(9) \quad 0 \rightarrow M \rightarrow V \rightarrow M^\vee \otimes \mathcal{I}_{\Gamma/S} \rightarrow 0,$$

where  $\Gamma \subset S$  is a 0-dimensional subscheme and  $M$  is a divisor class that intersects positively any ample class on  $S$  and with  $M^2 > 0$ . From (8) and (9) we find that  $H^0(S, M) \cong H^0(S, V)$  and  $H^0(S, M(-C)) = 0$ . Dualizing (8), we obtain the sequence

$$0 \rightarrow F \rightarrow V^\vee \rightarrow K_C(-D - D_1) \rightarrow 0,$$

from which, using that  $V \cong V^\vee$ , we obtain

$$H^0(S, V) = H^0(C, K_C(-D - D_1)).$$

We claim that  $\text{Cliff}(A(D_1)) = \text{Cliff}(C)$ . Recall that  $h^0(S, E) = h^0(C, A) + h^1(C, A)$ , and, from the sequence (8) we write

$$h^0(S, E) \leq h^0(C, A(D_1)) + h^1(C, A(D_1)).$$

By assumption, the pencil  $A$  computes  $\text{Cliff}(C)$ , which implies

$$\begin{aligned} \text{Cliff}(C) &= g + 1 - h^0(A) - h^1(A) \\ &\geq g + 1 - h^0(A(D_1)) - h^1(A(D_1)) = \text{Cliff}(A(D_1)). \end{aligned}$$

It follows that  $\text{Cliff}(A(D_1)) = \text{Cliff}(C)$ , in particular  $K_C(-D - D_1)$  is globally generated.

Clearly,  $M \not\subseteq F$ , hence the composition

$$\varphi : M \rightarrow V \rightarrow K_C(-D - D_1)$$

is non-zero and one writes  $\text{Im}(\varphi) = K_C(-D - D_1 - D_2)$ , where  $D_2$  is an effective divisor on  $C$ . If  $D_2 \neq 0$ , then one has the sequence of inequalities

$$h^0(S, M) \leq h^0(C, K_C(-D - D_1 - D_2)) < h^0(C, K_C(-D - D_1)) = h^0(S, M),$$

a contradiction. Therefore  $M|_C = K_C(-D - D_1)$ . Viewing  $M$  as a subsheaf of  $E$ , we find  $\mu(M) = M \cdot C = \text{deg}(L_1) > \mu(E)$ , thus bringing the proof to an end. Q.E.D.

**Remark 2.2.** If  $E_{C,A}$  is stable, then it is simple and hence  $d = \lfloor \frac{g+3}{2} \rfloor$ , see [L1]. Conversely, if  $C' \subset S$  is an ample curve of genus  $g$  and gonality  $\lfloor \frac{g+3}{2} \rfloor$ , then it was shown in [LC] that the LM bundle  $E_{C,A}$  corresponding to a general curve  $C \in |\mathcal{O}_S(C')|$  and to a pencil  $A$  of degree  $\lfloor \frac{g+3}{2} \rfloor$  is  $C$ -semistable (even stable when  $g$  is odd).

**2.2. Stability of Lazarsfeld-Mukai bundles of rank four**

We show that restrictions of LM bundles of rank 4 on very general  $K3$  surfaces of genus  $g \geq 20$  are stable. Similar results were established in [V] and [FO2] for rank 2 and 3 respectively. We fix integers  $i \geq 6$  and  $\rho \geq 0$  and write

$$g := 4i - 4 + \rho \quad \text{and} \quad d := 3i + \rho,$$

so that  $\rho(g, 3, d) = \rho$ . Let  $S$  be a  $K3$  surface and  $C \subset S$  a curve of genus  $g$  such that  $\text{Pic}(S) = \mathbb{Z} \cdot C$ , and pick a globally generated linear series  $A \in W_d^3(C)$  with  $h^0(C, A) = 4$ .

*Proof of Theorem 0.1.* Our previous results show that  $E|_C$  is simple, hence indecomposable. Suppose  $E|_C$  is not stable and fix a maximal destabilizing sequence

$$0 \longrightarrow M \longrightarrow E|_C \longrightarrow N \longrightarrow 0.$$

Put  $d_N := \deg(N)$  and  $d_M := \deg(M) = 2g - 2 - d_N$ . Since  $M$  is destabilizing,

$$(10) \quad \frac{d_M}{\operatorname{rk}(M)} \geq \frac{g-1}{2}, \quad \frac{d_N}{\operatorname{rk}(N)} \leq \frac{g-1}{2}.$$

The bundle  $N$ , being a quotient of  $E$ , is globally generated. Since  $H^0(C, E|_C^\vee) = 0$ , clearly  $N \neq \mathcal{O}_C$ , therefore  $h^0(C, N) \geq 2$ . From the inequalities (10) it follows that  $\operatorname{rk}(N) > 1$ , because  $C$  has maximal gonality.

*Step 1.* We prove that  $M$  is a line bundle. Assume that, on the contrary,

$$\operatorname{rk}(M) = \operatorname{rk}(N) = 2$$

and consider the elementary modification  $G := \operatorname{Ker}\{E \rightarrow N\}$ . Its Chern classes are given as follows:

$$c_1(G) = -[C], \quad c_2(G) = d + d_N - 2(g - 1),$$

and its discriminant equals  $\Delta(G) = -64i + 110 + 8d_N - 14\rho < 0$ , because of (10). In particular, there exists a saturated subsheaf  $F \subset G$  which verifies the inequalities

$$(11) \quad \mu(G) \leq \mu(F) < \mu(E), \quad \text{and}$$

$$(12) \quad \xi_{F,G}^2 \geq -\frac{\Delta(G)}{48}.$$

Write  $c_1(F) = \alpha \cdot [C]$  and  $\operatorname{rk}(F) = \beta \leq 3$ . The above inequality (12) becomes

$$\left(\frac{\alpha}{\beta} + \frac{1}{4}\right)^2 (2g - 2) \geq -\frac{\Delta(G)}{48}.$$

We apply (11) for  $\mu(F) = \alpha(2g - 2)/\beta$  and obtain

$$-\frac{1}{4} \leq \frac{\alpha}{\beta} < \frac{1}{4},$$

hence  $\alpha = 0$ , and the inequality (12) reads in this case  $d_N \geq 5i - 10 + \rho$ . Recalling that  $d_N \leq g - 1 = 4i - 5 + \rho$ , we obtain a contradiction whenever  $i \geq 6$ .

*Step 2.* We construct an elementary modification, in order to reach a contradiction.

From (10), we have  $d_M \geq \frac{g-1}{2}$ . The composition  $M \rightarrow E|_C \rightarrow K_C \otimes A^\vee$  is not zero, for else  $M \hookrightarrow Q_A$  and since  $\mu(Q_A \otimes M^\vee) < 0$ , one contradicts the semistability of  $Q_A$ . We set  $A_1 := K_C \otimes A^\vee \otimes M^\vee$  and obtain a surjection  $F(C)|_C \rightarrow A \otimes A_1$  inducing, as before, an elementary modification

$$V := \text{Ker}\{F(C) \rightarrow A \otimes A_1\}.$$

By direct computation we show that  $\Delta(V) < 0$ . Indeed, we compute

$$c_1(V) = 2 \cdot [C], \quad c_2(V) = d + 2g - 2 - d_M, \quad \text{hence}$$

$$\Delta(V) = 8c_2(V) - 3c_1^2(V) = 8(d - d_M - g + 1) = 8(5 - d_M - i) < 0.$$

We obtain a destabilizing sheaf  $P \subset V$ , with  $\text{rk}(P) = b \leq 3$  and  $c_1(P) := a \cdot [C]$ , such that the following inequalities are both satisfied

$$(13) \quad \left(\frac{a}{b} - \frac{1}{2}\right)^2 (2g - 2) \geq -\frac{\Delta(V)}{48} \quad \text{and} \quad \mu(V) \leq \mu(P) < \mu(F(C)).$$

The second inequality gives  $\frac{1}{2} \leq \frac{a}{b} < \frac{3}{4}$ , which leaves two possibilities: either  $a = 1$  and  $b = 2$ , when via (13) one finds that  $\Delta(V) \geq 0$ , a contradiction, or else  $a = 2$  and  $b = 3$ , when inequalities (13) and (10) clash.  $\square$

### §3. Normal bundle of canonical curves of genus 7

The aim of this section is to prove Theorem 0.3 and we begin by recalling Mukai's results [M3] on canonical curves of genus 7. We choose a vector space  $U := \mathbb{C}^{10}$  and a non-degenerate quadratic form  $q : U \rightarrow \mathbb{C}$ , defining a smooth 8-dimensional quadric  $Q \subset \mathbf{P}(U) = \mathbf{P}^9$ .

The algebraic group  $\mathbf{Spin}(U)$  corresponding to the Dynkin diagram  $D_5$  admits two 16-dimensional half-spin representations  $\mathcal{S}^+$  and  $\mathcal{S}^-$ , which correspond to maximal weights  $\alpha^+$  and  $\alpha^-$  respectively. The homogeneous spaces  $V^\pm := \mathbf{Spin}(U)/P(\alpha^\pm)$  are both 10-dimensional and can be realized as the two irreducible components of the Grassmannian  $G_q(5, U)$  of projective 4-planes inside  $\mathbf{P}(U)$  which are isotropic with respect to the quadratic form  $q$ . From now on, we set

$$V := V^+ \subset \mathbf{P}(\mathcal{S}^+) = \mathbf{P}^{15}.$$

Note that  $\text{Aut}(V) = SO(10)$ . If  $\mathcal{E}$  is the restriction to  $V$  of the tautological bundle on  $G(5, 10)$ , one has an exact sequence of vector bundles

on  $V$ :

$$(14) \quad 0 \longrightarrow \mathcal{E}^\vee \longrightarrow U \otimes \mathcal{O}_V \longrightarrow \mathcal{E} \longrightarrow 0.$$

By the adjunction formula, smooth curvilinear sections of  $V$  are canonical curves of genus 7 and Mukai [M3] showed that *each* curve  $[C] \in \mathcal{M}_7$  with  $\text{Cliff}(C) = 3$  appears in this way. Precisely, there is a birational map

$$\alpha : G(7, 16) // SO(10) \dashrightarrow \overline{\mathcal{M}}_7, \quad \alpha(\Lambda) := [\Lambda \cap V],$$

where  $\Lambda \cong \mathbf{P}^6$ . Given a curve  $[C] \in \mathcal{M}_7$ , the inverse  $\alpha^{-1}([C])$  is constructed precisely via the twist of the conormal bundle on  $C$  mentioned in the introduction.

Let  $C \subset \mathbf{P}^6$  be a smooth canonical curve with  $\text{Cliff}(C) = 3$ , and set  $E := N_{C/\mathbf{P}^6}^\vee(2)$ . One has an identification  $H^0(C, E) = I_2(K_C)$  and  $E$  is a globally generated bundle. The tautological map

$$\phi_E : C \rightarrow G(5, H^0(C, E))$$

is easily shown to be injective and its image lies on  $V$ . In particular, the vector bundle  $E$  is the restricted spinorial bundle, that is,  $E = \mathcal{E}|_C$  and one has an exact sequence:

$$(15) \quad 0 \longrightarrow E^\vee \longrightarrow H^0(C, E) \otimes C \longrightarrow E \longrightarrow 0.$$

Note that  $W_4^1(C) = \emptyset$ , while  $W_5^1(C)$  is a curve. We are going to make essential use of the following fact:

**Lemma 3.1.** *Let  $C$  as above and  $A \in W_5^1(C)$ . Then there are no surjections  $E \twoheadrightarrow A$ .*

*Proof.* We proceed by contradiction. Assume that there is such a pencil  $A \in W_5^1(C)$ , then use the base point free pencil trick to write the following diagram:

$$(16) \quad \begin{array}{ccccccc} 0 & \longrightarrow & E^\vee & \longrightarrow & H^0(C, E) \otimes \mathcal{O}_C & \longrightarrow & E \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & A^\vee & \longrightarrow & H^0(C, A) \otimes \mathcal{O}_C & \longrightarrow & A \longrightarrow 0 \\ & & & & & & \downarrow \\ & & & & & & 0 \end{array}$$

In particular,  $H^0(C, E \otimes A^\vee) \neq 0$ . Via the canonical identification  $H^0(C, E) = I_2(K_C)$ , this implies that if  $L := K_C \otimes A^\vee \in W_7^2(C)$ , then the multiplication map

$$\mathrm{Sym}^2 H^0(C, L) \rightarrow H^0(C, L^{\otimes 2})$$

is not injective. This is possible only if  $L$  is not birationally very ample, in particular,  $C$  must be trigonal, which is not the case. Q.E.D.

We are now in a position to prove that the twist  $E$  of the conormal bundle of a canonical curve of genus 7 is stable.

*Proof of Theorem 0.3.* Suppose that  $0 \rightarrow F \rightarrow E \rightarrow M \rightarrow 0$  is a destabilizing sequence for the vector bundle  $E$ , that is, with  $\mu(F) \geq \mu(E) = \frac{24}{5}$ . Since  $E$  is globally generated, so is any of its quotient, in particular  $M$  too. We distinguish several possibilities, depending on the ranks that appear:

(i)  $\mathrm{rk}(F) = 4$  and  $M$  is line bundle. Then  $\mathrm{deg}(F) \geq 20$ , hence  $\mathrm{deg}(M) \leq 4$ . Since  $C$  is not tetragonal,  $h^0(C, M) \leq 1$ . Note that  $M \neq \mathcal{O}_C$ , for  $H^0(C, E^\vee) = 0$ . It follows that  $M$  is not globally generated, a contradiction.

(ii)  $\mathrm{rk}(F) = 1$  and we may assume that  $\mathrm{deg}(F) = 5$ . Suppose first that  $h^0(C, F) = 0$ , therefore  $h^0(C, K_C \otimes F^\vee) = 1$ , and hence  $K_C \otimes F^\vee$  is not globally generated. Since one has a surjection  $E^\vee(1) \twoheadrightarrow K_C \otimes F^\vee$ , we reach a contradiction by observing that  $E^\vee(1)$  is globally generated. Indeed, via Serre duality, this last statement is equivalent to the equality  $h^0(C, E(p)) = h^0(C, E) = 10$ , for every point  $p \in C$ . From the exact sequence

$$0 \rightarrow E(p) \rightarrow M_{K_C} \otimes K_C(p) \rightarrow K_C^{\otimes 3}(p) \rightarrow 0,$$

we obtain that

$$H^0(C, E(p)) = \mathrm{Ker} \left\{ H^0(C, M_{K_C} \otimes K_C(p)) \rightarrow H^0(C, K_C^{\otimes 3}(p)) \right\}.$$

The conclusion follows, since  $H^0(C, M_{K_C} \otimes K_C) = H^0(C, M_{K_C} \otimes K_C(p))$ .

Suppose now that  $h^0(C, F) \geq 1$ . The case  $h^0(C, F) \geq 2$  having been discarded in the course of proving Lemma 3, we assume that  $h^0(C, F) = 1$ , hence  $h^0(C, K_C \otimes F^\vee) = 2$ . We obtain that the multiplication map

$$\mathrm{Sym}^2 H^0(C, K_C \otimes F^\vee) \rightarrow H^0(C, K_C^{\otimes 2} \otimes F^{\otimes(-2)})$$

is not injective, which contradicts the base point free pencil trick.



(iii)  $\text{rk}(F) = 3$ , and then  $\text{deg}(F) \geq 15$ , hence  $\text{deg}(M) \leq 9$ . This time we may assume that  $F$  is stable. If  $M$  is not stable, we choose a line subbundle  $A \subset M$  of maximal degree, which we pull-back under the surjection  $E \rightarrow M$ , to obtain the exact sequence

$$0 \rightarrow G \rightarrow E \rightarrow M/A \rightarrow 0.$$

We obtain that  $\text{deg}(M/A) \leq \text{deg}(M)/2 \leq 9/2$ , that is,  $\text{deg}(M/A) \leq 4$ . In particular,  $M/A$  is not globally generated, which is again a contradiction, so we can assume that both  $F$  and  $M$  are stable vector bundles. Since  $h^0(C, M) + h^0(C, F) \geq h^0(C, E) = 10$ , the strategy is to use the fact that the Mercat statements  $(M_2)$  and  $(M_3)$  have been established for curves  $C$  of genus 7 with maximal Clifford index, that is,

$$\text{Cliff}_2(C) = \text{Cliff}_3(C) = 3,$$

see [LN3] Theorem 4.5. In particular, if both  $F$  and  $M$  contribute to their respective Clifford indices, that is,  $h^0(C, F) \geq 6$  and  $h^0(C, M) \geq 4$  respectively, then we write

$$\frac{9}{2} + 3 \leq \frac{3}{2}\gamma(F) + \gamma(M) = \frac{1}{2}(\text{deg}(F) + \text{deg}(M)) - h^0(C, F) - h^0(C, M) + 5,$$

that is,  $h^0(C, F) + h^0(C, M) \leq \frac{19}{2}$ , a contradiction.

Assume now that one of the bundles  $F$  or  $M$  does not contribute to its Clifford index. Since  $M$  is globally generated,  $h^0(C, M) \geq 2$ . We can have  $h^0(C, M) = 2$ , only when  $M = \mathcal{O}_C^{\oplus 2}$ , which is impossible, for  $\mathcal{O}_C^{\oplus 2}$  is not a direct summand of  $E$ . If  $h^0(C, M) = 3$ , then  $\text{deg}(M) \geq 7$ , and one has equality if and only if  $M = Q_L$ , where  $L \in W_7^2(C)$ . Assuming this to be the case, we choose two points  $p, q \in C$  that correspond to a node in the plane model  $\phi_L : C \rightarrow \mathbf{P}^2$ , that is,  $A := L(-p-q) \in W_5^1(C)$ . Then there is a surjection  $Q_L \twoheadrightarrow A$ , which by composition gives rise to a surjective morphism  $E \twoheadrightarrow A$ . This contradicts Lemma 3.

Thus we may assume that  $\text{deg}(M) \geq 8$ , and accordingly,  $\text{deg}(F) \leq 16$ . Then we compute

$$\gamma(F) = \mu(F) - \frac{2}{3}h^0(C, F) + 2 \leq \frac{16}{3} - \frac{14}{3} + 2 < \text{Cliff}(C),$$

which again contradicts the equality  $\text{Cliff}_3(C) = 3$ .

(iv)  $\text{rk}(F) = 2$ , and then  $\text{deg}(F) \geq 10$  and  $\text{deg}(M) \leq 14$ . We may assume this time that  $M$  is stable. If  $F$  is not stable, then it has a line subbundle  $A \hookrightarrow F$  with  $\text{deg}(A) \geq 5$ , and we are back to case (ii). Thus both  $M$  and  $F$  are stable bundles, and we proceed precisely like in case (iii).

□

It is instructive to remark that the normal bundle of a canonical curve of genus  $g < 7$  is never stable. More generally we have the following:

**Proposition 3.2.** *The normal bundle of a tetragonal canonical curve of genus  $g$  is unstable.*

*Proof.* More generally, we begin with a  $k : 1$  covering  $f : C \rightarrow \mathbf{P}^1$ , and consider the rank  $(k - 1)$ -vector bundle  $\mathcal{F}^\vee := f_*\mathcal{O}_C/\mathcal{O}_{\mathbf{P}^1}$  on the projective line. Then  $\pi : X = \mathbf{P}(\mathcal{F}) \rightarrow \mathbf{P}^1$  is a scroll of dimension  $k - 1$ , which contains the canonical curve  $C$  and which can be embedded by the tautological bundle  $\mathcal{O}_X(1)$  in  $\mathbf{P}^{g-1}$  as a variety of degree  $g - k + 1$ . Denoting by  $H, R \in \text{Pic}(X)$  the class of the hyperplane section and that of the ruling respectively, we have

$$K_X \equiv -(k - 1)H + (g - k - 1)R,$$

whereas obviously  $C \cdot H = 2g - 2$  and  $C \cdot R = k$ . We compute the degree of the normal bundle  $N_{C/X}$  and find:

$$\deg(N_{C/X}) = \deg(T_{X|C}) + \deg(K_C) = k(g + k - 1).$$

We write the usual exact sequence relating normal bundles

$$0 \longrightarrow N_{C/X} \longrightarrow N_{C/\mathbf{P}^{g-1}} \longrightarrow N_{X/\mathbf{P}^{g-1}} \otimes \mathcal{O}_C \longrightarrow 0,$$

and compare the slopes

$$\mu(N_{C/X}) = \frac{k(g + k - 1)}{k - 2} \quad \text{and} \quad \mu(N_{C/\mathbf{P}^{g-1}}) = \frac{2(g - 1)(g + 1)}{g - 2}.$$

We conclude that for  $k = 4$  and  $g \geq 6$ , the normal bundle  $N_{C/X}$  is a destabilizing subbundle of  $N_{C/\mathbf{P}^{g-1}}$ . For  $g$  at most 5, every canonical curve of genus  $g$  is a complete intersection which obviously produces a destabilizing line subbundle. Q.E.D.

## References

- [A] M. Aprodu, *Lazarsfeld-Mukai bundles and applications*, in: Commutative Algebra, Expository Papers Dedicated to David Eisenbud on the occasion of his 65th birthday (I. Peeva-ed.), Springer Verlag 2013, 1-23.

- [AF] M. Aprodu and G. Farkas, *The Green Conjecture for smooth curves lying on arbitrary K3 surfaces*, *Compositio Math.* **147** (2011), 839–851.
- [CHM] C. Ciliberto, J. Harris and R. Miranda, *On the surjectivity of the Wahl map*, *Duke Mathematical Journal* **57** (1988), 829–958.
- [CP] C. Ciliberto and G. Pareschi, *Pencils of minimal degree on curves on a K3 surface*, *Journal für die reine und angewandte Mathematik* **460** (1995), 15–36.
- [EL] L. Ein and R. Lazarsfeld, *Stability and restrictions of Picard bundles with an application to the normal bundles of elliptic curves*, in: *Complex Projective Geometry*, London Mathematical Society Lecture Notes Series 179, Cambridge University Press 1992, 149–156.
- [F] G. Farkas, *Aspects of the birational geometry of  $\overline{\mathcal{M}}_g$* , *Surveys in Differential Geometry*, Vol. 14, 2010, 57–110.
- [FO1] G. Farkas and A. Ortega, *The maximal rank conjecture and rank two Brill-Noether theory*, *Pure Applied Math. Quarterly*, **7** (2011), 1265–1296.
- [FO2] G. Farkas and A. Ortega, *Higher rank Brill-Noether on sections of K3 surfaces*, *International Journal of Mathematics* **23** (2012), 1250075.
- [GMN] I. Grzegorczyk, V. Mercat and P.E. Newstead, *Stable bundles of rank 2 with 4 sections*, *International Journal of Mathematics* **22** (2011), 1743–1762.
- [HL] D. Huybrechts and M. Lehn, *The geometry of the moduli space of sheaves*, Second Edition, Cambridge University Press 2010.
- [L1] R. Lazarsfeld, *Brill-Noether-Petri without degenerations*, *Journal of Differential Geometry*, **23** (1986), 299–307.
- [L2] R. Lazarsfeld, *A sampling of vector bundle techniques in the study of linear series*, in: *Lectures on Riemann Surfaces*, World Scientific Press, Singapore, 1989, 500–559.
- [LMN] H. Lange, V. Mercat and P.E. Newstead, *On an example of Mukai*, *Glasgow Mathematical Journal*, **54** (2012), 261–271.
- [LN1] H. Lange and P.E. Newstead, *Clifford indices for vector bundles on curves*, arXiv:0811.4680, in: *Affine Flag Manifolds and Principal Bundles* (A. Schmitt-ed.), Trends in Mathematics, 165–202, Birkhäuser 2010.
- [LN2] H. Lange and P.E. Newstead, *Further examples of stable bundles of rank 2 with 4 sections*, *Pure Applied Math. Quarterly* **7** (2011), 1517–1528.
- [LN3] H. Lange and P.E. Newstead, *Bundles of rank 3 on curves of Clifford index 3*, *Journal of Symbolic Computation* **57** (2013), 3–18.
- [LC] M. Lelli-Chiesa, *Stability of rank 3 Lazarsfeld-Mukai bundles on K3 surfaces*, *Proceedings of the London Mathematical Society* **107** (2013) 451–479.
- [Me1] V. Mercat, *Clifford’s theorem and higher rank vector bundles*, *International Journal of Mathematics* **13** (2002), 785–796.

- [M1] S. Mukai, *Biregular classification of Fano 3-folds and Fano manifolds of coindex 3*, Proceedings of the National Academy of Sciences USA, Vol. 86, 3000-3002, 1989.
- [M2] S. Mukai, *Curves and Grassmannians*, in: Algebraic Geometry and Related Topics (1992), eds. J.-H. Yang, Y. Namikawa, K. Ueno, 19-40.
- [M3] S. Mukai, *Curves and symmetric spaces I*, American Journal of Mathematics **117** (1995), 1627-1644.
- [M4] S. Mukai, *Curves and symmetric spaces II*, Annals of Mathematics **172** (2010), 1539-1558.
- [V] C. Voisin, *Sur l'application de Wahl des courbes satisfaisant la condition de Brill-Noether-Petri*, Acta Mathematica **168** (1992), 249-272.
- [W] J. Wahl, *Gaussian maps on algebraic curves*, Journal of Differential Geometry **32** (1990), 77-98.

*Institute of Mathematics "Simion Stoilow" of the Romanian Academy, Calea Griviței 21, Sector 1, RO-010702 Bucharest, Romania*  
E-mail address: [aprodu@imar.ro](mailto:aprodu@imar.ro)

*Humboldt-Universität zu Berlin, Institut für Mathematik, Unter den Linden 6 10099 Berlin, Germany*  
E-mail address: [farkas@math.hu-berlin.de](mailto:farkas@math.hu-berlin.de)

*Humboldt-Universität zu Berlin, Institut für Mathematik, Unter den Linden 6 10099 Berlin, Germany*  
E-mail address: [ortega@math.hu-berlin.de](mailto:ortega@math.hu-berlin.de)