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Compactification by GIT-stability of the moduli space of abelian varieties

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Abstract.

The moduli space $\mathcal{M}_g$ of nonsingular projective curves of genus g is compactified into the moduli $\overline{\mathcal{M}}_g$ of Deligne-Mumford stable curves of genus g. We compactify in a similar way the moduli space of abelian varieties by adding some mildly degenerating limits of abelian varieties.

A typical case is the moduli space of Hesse cubics. Any Hesse cubic is GIT-stable in the sense that its SL(3)-orbit is closed in the semistable locus, and conversely any GIT-stable planar cubic is one of Hesse cubics. Similarly in arbitrary dimension, the moduli space of abelian varieties is compactified by adding only GIT-stable limits of abelian varieties (§ 14).

Our moduli space is a projective "fine" moduli space of possibly degenerate abelian schemes *with non-classical non-commutative level structure* over $\mathbf{Z}[\zeta_N, 1/N]$ for some $N \geq 3$. The objects at the boundary are singular schemes, called PSQASes, projectively stable quasi-abelian schemes.

Contents

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§1. Introduction

The moduli of stable curves, the so-called Deligne-Mumford compactification, compactifies the moduli of nonsingular curves :

the moduli of smooth curves
= the set of all isomorphism classes of smooth curves
⊂ the set of all isomorphism classes of stable curves
= the Deligne-Mumford compactification $\overline{\mathcal{M}}_g$

The moduli of stable curves is known to be a projective scheme, while the moduli of nonsingular curves is a Zariski open subset of it.

Our problem is to do the same for moduli of smooth abelian varieties. We find certain natural limits of smooth abelian varieties similar to stable curves to compactify the moduli. In other words, we will construct a new compactification $SQ_{g,K}$, the moduli of some possibly degenerate abelian varieties with some extra structure, which contains the moduli of smooth abelian varieties with similar extra structure as a Zariski open subset. This will complete the following diagram :

the moduli of smooth AVs (= abelian varieties)
= {smooth polarized AVs + extra structure}/ isom.
⊂ {smooth polarized AVs or
singular polarized degenerate AVs + extra structure}/ isom.
= the new compactification $SQ_{g,K}$

The compactification problem of the moduli space of abelian varieties has been studied by many people :

- ◦ Satake compactification, Igusa monoidal transform of it
- ◦ Mumford toroidal compactification ([4, (1975)])
- ◦ Faltings-Chai arithmetic compactification (arithmetic version of Mumford compactification) [7, (1990)]

These are the compactifications which had been known before 1995 when the author restarted the research of compactifications. These are compactifications as spaces, not as the moduli of compact objects. In this article, we are going to construct *a natural compactification,* in fact, *projective, as the "fine/coarse" moduli space of compact geometric objects*, where

- the moduli space contains the moduli space of abelian varieties as a dense Zariski open subset,
- it is compact, which amounts to collecting enough limits,
- it is separated, which amounts to choosing the minimum possible among the above.

The following are the works closely related to the subject; first of all, the works of Mumford [20], [21] and [23] during 1966–1972, though they do not focus on compactifications directly. After 1975 there appeared Nakamura [27] and Namikawa [35], closely related to this article.

After 1999 there appeared several works on the subject: [2], [30], [1], [37] and [32]. By modifying [27], Nakamura [30] and [32] study two kinds of compactifications of the moduli space of abelian varieties (with no zero section specified and with no semi-abelian scheme action assumed). Meanwhile, Alexeev [1] and Olsson [37] study the complete moduli spaces of certain schemes with semi-abelian scheme action.

Now we shall explain how we choose our compactification $SQ_{g,K}$, which will explain why the title of this article refers to GIT-stability.

Let H be a finite Abelian group, and $V := V_H$ the unique irreducible representation of the Heisenberg group $\mathcal{G}_H$ of weight one. Let $\mathbf{P}(V)$ be the projective space of V, and $X := \mathrm{Hilb}^{\chi}_{\mathbf{P}(V)}$ the Hilbert scheme parameterizing closed subschemes of $\mathbf{P}(V)$ with Hilbert polynomials $\chi(n) = n^g|H|$. According to GIT, our problem of compactifying the moduli space is, *very roughly speaking*, reduced to studying the quotient $X_{ss}//\,\mathrm{SL}(V)$ where X_{ss} denotes the semistable locus of X with respect to $\mathrm{SL}(V)$. GIT tells us, *set-theoretically*,

$$X_{ss}//\,\mathrm{SL}(V) = \text{the set of all closed orbits in } X_{ss}. \tag{1}$$

See Section 14. This scenario has to be modified a little. In an appropriately modified scenario, the LHS of (1) is the moduli space $SQ_{g,K}$, the compactification in the title of this article, while the RHS of (1) is just the set of isomorphism classes of our degenerate abelian schemes PSQASes $(Q_0, \mathcal{L}_0)$ with $\mathcal{G}_H$-action. See Section 4, Theorem 9.8 and Theorem 14.1.3. It should be mentioned that $SQ_{g,K}$ is the fine moduli scheme for families of PSQASes over *reduced* base schemes, hence $SQ_{g,K}$ itself is also reduced.

This note is based on our lectures with the same title delivered at Kyoto university during June 11–13, 2013. It overlaps the report [31] on the same topic in many respects, though the note includes also the recent progress of the topic. In this note, we give simple proofs for the major results of [30] and [32], assuming known rather general results. We also tried to include (elementary or less elementary) proofs of the well-known related facts whose proofs are hard to find in the literature. As a whole we tried to make our presentation more accessible than [30], keeping the atmosphere of the lecture as much as possible.

In what follows throughout this article, we always consider a finite abelian group $H = \bigoplus_{i=1}^{g}(\mathbf{Z}/e_i\mathbf{Z})$, where $e_i|e_{i+1}$, and we write $N = |H| = \prod_{i=1}^{g} e_i$ and $K = K_H = H \oplus H^\vee$ ($H^\vee$: the dual of H). We call such H simply *a finite Abelian group.* We also call K *a finite symplectic Abelian group.* We also let $\mathcal{O} = \mathcal{O}_N = \mathbf{Z}[1/N, \zeta_N]$ where ζ_N is a primitive N-th root of unity.

The article is organized as follows.

Section 2 reviews the classical moduli theories of Hesse cubics with Neolithic level-3 structure or with classical level-3 structure.

Section 3 gives a new interpretation of the moduli theories in Section 2 in a non-commutative way, and then explains a new moduli theory of Hesse cubics with level-$G(3)$ structure, where $G(3)$ is a non-commutative group, the Heisenberg group. This is the model theory for all the rest. The major purpose of this article is to explain its higher dimensional analogue. See Subsec. 3.1.

In Section 4 we introduce two kinds $(P_0, \mathcal{L}_0)$ and $(Q_0, \mathcal{L}_0)$ of nice degenerate abelian schemes in arbitrary dimension to compactify the moduli space of abelian varieties. Theorem 4.6 gives an intrinsic description of those degenerate schemes $(P_0, \mathcal{L}_0)$ and $(Q_0, \mathcal{L}_0)$, where P_0 is always reduced, while Q_0 can be nonreduced.

A more direct definition of those degenerate schemes will be given in Sections 5 and 6. Especially we give a complete proof of the part $Q_\eta \simeq P_\eta \simeq G_\eta$ of Theorem 4.6. We will give two-dimensional and three-dimensional examples of PSQASes. We will also explain how a naive classical level-n structure results in a nonseparated moduli.

Section 7 reviews a rather general theory about G-action and G-linearization. We give various definitions and constructions and show their equivalence or compatibility.

In Section 8, we give a definition of level-$\mathcal{G}_H$ structure and define a quasi-projective (resp. projective) scheme $A_{g,K}$ (resp. $SQ_{g,K}$) when $e_1 \geq 3$. We show that any geometric point of $A_{g,K}$ (resp. $SQ_{g,K}$) is a nonsingular level-$\mathcal{G}_H$ PSQAS (resp. a level-$\mathcal{G}_H$ PSQAS) and vice versa.

In Section 9 we formulate the moduli functor of smooth (resp. flat) PSQASes over $\mathcal{O}_N$-schemes (resp. reduced $\mathcal{O}_N$-schemes). We will prove the representability of these functors by $A_{g,K}$ (resp. $SQ_{g,K}$) in the respective category.

In Sections 11 and 12 we see that there exists the coarse moduli algebraic space $SQ_{g,K}^{\text{toric}}$ of level-$\mathcal{G}_H$ TSQASes. This has been proved in [32] when $e_1 \geq 3$. We generalize it here to the case $e_1 \leq 2$. There is a bijective morphism from $SQ_{g,K}^{\text{toric}}$ onto $SQ_{g,K}$ if $e_1 \geq 3$. In Sections 11 and 12 many of the definitions, constructions and proofs are given in parallel to Sections 8 and 9, which we often omitted to avoid overlapping.

In Section 13 we briefly report our recent results without proofs. We define a morphism sqap from $SQ_{g,K}^{\text{toric}} \times U$ to Alexeev's complete moduli $\overline{AP}_{g,d}$ for a nonempty Zariski open subset U of $\mathbf{P}^{N-1} = \mathbf{P}(V_H)$. We see that sqap restricted to $SQ_{g,K}^{\text{toric}} \times \{u\}$ for any $u \in U$ is injective: in fact, it is almost a closed immersion. We also see that $SQ_{g,1}^{\text{toric}}$ is isomorphic to the main (reduced) component $\overline{AP}_{g,1}^{\text{main}}$ of $\overline{AP}_{g,1}$, the closure in $\overline{AP}_{g,1}$ of the moduli of abelian torsors. We emphasize that it is nontrivial to define *a well-defined morphism* sqap because singular TSQASes have a lot of continuous automorphisms.

In Section 14 we explain the set of all closed orbits and GIT stability of PSQASes. We also mention a few related topics.

We tried to give complete proofs to Sections 7-9 and to Theorem 12.1 (especially for the case $e_1 \leq 2$) in Section 12, relying in part on [30] and [32]. In the other sections we only survey mainly [30], [32] and [33].

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§2. Hesse cubics

Here we will start with a simple example.

2.1. Hesse cubics

Let k be any ring which contains $1/3$ and ζ_3, the primitive cube root of unity. A Hesse cubic curve is a curve in $\mathbf{P}_k^2$ defined by

$$C(\mu) \;:\; x_0^3 + x_1^3 + x_2^3 - 3\mu x_0 x_1 x_2 = 0 \tag{2}$$

for some $\mu \in k$, or $\mu = \infty$ (in which case we understand that $C(\infty)$ is the curve defined by $x_0x_1x_2 = 0$). We see

(i) $C(\mu)$ is nonsingular elliptic for $\mu \neq \infty, 1, \zeta_3, \zeta_3^2$,
(ii) $C(\mu)$ is a 3-gon for $\mu = \infty, 1, \zeta_3, \zeta_3^2$,
(iii) any $C(\mu)$ contains K, which is independent of μ,

$$K = \left\{[0, 1, -\zeta_3^k], [-\zeta_3^k, 0, 1], [1, -\zeta_3^k, 0]; k = 0, 1, 2\right\},$$

(iv) K is identified with the group of 3-division points by choosing $[0, 1, -1]$ as the zero, so $K \simeq (\mathbf{Z}/3\mathbf{Z})^2$ as groups,
(v) if $k = \mathbf{C}$, any Hesse cubic is the image of a complex torus $E(\omega) := \mathbf{C}/\mathbf{Z} + \mathbf{Z}\omega$ by (slightly modified) theta functions ϑ_k of level 3 (see Subsec. 2.2), and then K is the image of the 3-division points $\langle \frac{1}{3}, \frac{\omega}{3} \rangle$ of $E(\omega)$.

2.2. Theta functions

We will explain Subsec. 2.1 (v) in more detail. First let us recall standard (resp. modified) theta functions of level 3 on $E(\omega)$:

$$\theta_k(\omega, z) = \sum_{m \in \mathbf{Z}} q^{(3m+k)^2} w^{3m+k}, \quad \text{resp.}$$
$$\vartheta_k(\omega, z) = \theta_k(\omega, z + \frac{1-\omega}{2})$$

where $q = e^{2\pi i \omega/6}$, $w = e^{2\pi i z}$. They satisfy the transformation relation :

$$\theta_k(\omega, z + \frac{a + b\omega}{3}) = \zeta_3^{ak}(q^b w)^{-b} \theta_{k+b}(\omega, z),$$
$$\vartheta_k(\omega, z + \frac{a + b\omega}{3}) = \zeta_3^{ak}(q^{b-3}(-w))^{-b} \vartheta_{k+b}(\omega, z).$$

We define a mapping $\vartheta : E(\omega) \to \mathbf{P}^2$ by

$$\vartheta(\omega, z) := [\vartheta_0, \vartheta_1, \vartheta_2].$$

Let us check the second half of Subsec. 2.1 (v). For it, we rewrite

$$\vartheta_0(\omega, z) = \sum_{m \in \mathbf{Z}} q^{9m^2 - 9m} (-w)^{3m},$$
$$\vartheta_1(\omega, z) = \sum_{m \in \mathbf{Z}} q^{9m^2 - 3m - 2} (-w)^{3m+1},$$
$$\vartheta_2(\omega, z) = \sum_{m \in \mathbf{Z}} q^{9m^2 + 3m - 2} (-w)^{3m+2}.$$

Then we check $\vartheta(\omega, \frac{\ell}{3}) = [0, 1, -\zeta_3^\ell]$ and $\vartheta(\omega, \frac{\omega}{3}) = [1, -1, 0]$. First we prove $\vartheta_0(\omega, \frac{\ell}{3}) = 0$. In fact, we see

$$\begin{aligned}\vartheta_0(\omega, \frac{\ell}{3}) &= \sum_{m\in\mathbf{Z}} q^{9m^2-9m}(-1)^{3m}\\ &= \sum_{m\in\mathbf{Z}} q^{9(-m+1)^2-9(-m+1)}(-1)^{3(-m+1)}\\ &= \sum_{m\in\mathbf{Z}} q^{9m^2-9m}(-1)^{-3m+3} = -\vartheta_0(\omega, \frac{\ell}{3}),\end{aligned}$$

whence $\vartheta_0(\omega, \frac{\ell}{3}) = 0$. Moreover

$$\begin{aligned}\vartheta_1(\omega, \frac{\ell}{3}) &= \zeta_3^\ell \sum_{m\in\mathbf{Z}} q^{9m^2-3m-2}(-1)^{3m+1},\\ \vartheta_2(\omega, \frac{\ell}{3}) &= \zeta_3^{2\ell} \sum_{m\in\mathbf{Z}} q^{9m^2+3m-2}(-1)^{3m}\\ &= \zeta_3^{2\ell} \sum_{m\in\mathbf{Z}} q^{9m^2-3m-2}(-1)^{3m} = -\zeta_3^\ell\vartheta_1(\omega, \frac{\ell}{3}).\end{aligned}$$

$\vartheta(\omega, \frac{\omega}{3}) = [1, -1, 0]$ is proved similarly.

2.3. The moduli space of Hesse cubics — the Stone-age (Neolithic) level structure

With the same notation as in Subsec. 2.1, consider the moduli space $SQ_{1,3}^{\mathrm{NL}}$ of the pairs $(C(\mu), K)$ over any ring $k \ni 1/3$ and ζ_3.

Definition 2.3.1. Any pair $(C(\mu), K)$ is called *a Hesse cubic with Neolithic level-3 structure.* Let $(C(\mu), K)$ and $(C(\mu'), K)$ be two pairs of Hesse cubics with Neolithic level-3 structure. We define $(C(\mu), K) \simeq (C(\mu'), K)$ to be isomorphic if there exists an isomorphism $f : C(\mu) \to C(\mu')$ with $f_{|K} = \mathrm{id}_K$.

Claim 2.3.2. *Let $SQ_{1,3}^{\mathrm{NL}}$ be the set of isomorphism classes of $(C(\mu), K)$, and $A_{1,3}^{\mathrm{NL}}$ the subset of $SQ_{1,3}^{\mathrm{NL}}$ consisting of smooth $C(\mu)$. Then*

(i) *if $(C(\mu), K) \simeq (C(\mu'), K)$, then $\mu = \mu'$,*
(ii) *$SQ_{1,3}^{\mathrm{NL}}$ has a natural scheme structure:*

$$SQ_{1,3}^{\mathrm{NL}} \simeq \mathbf{P}_k^1 = \mathrm{Proj}\ k[\mu_0, \mu_1],$$

(iii) *this compactifies the moduli* $A_{1,3}^{\mathrm{NL}}$ *of smooth Hesse cubics:*

$$A_{1,3}^{\mathrm{NL}} \simeq \mathrm{Spec}\ k[\mu, \frac{1}{\mu^3-1}], \quad \mu = \mu_1/\mu_0,$$

where $A_{1,3}^{\mathrm{NL}}(k) = \{C(\mu); \textit{smooth}, \mu \in k\}$ *if* k *is a closed field,*

(iv) *the universal Hesse cubic over* $SQ_{1,3}^{\mathrm{NL}}$ *is given by*

$$\mu_0(x_0^3 + x_1^3 + x_2^3) - 3\mu_1 x_0 x_1 x_2 = 0. \tag{3}$$

Proof of (i). We prove (i). Suppose we are given an isomorphism

$$f : (C(\mu), K) \simeq (C(\mu'), K).$$

Since any 3 points x, y and $z \in K$ with $x + y + z = 0$ are on a line $\ell_{x,y,z}$ of $\mathbf{P}^2$, we have $\ell_{x,y,z} \cap C(\mu) = \{x, y, z\}$ and $f^*\ell_{x,y,z} = \ell_{x,y,z}$ as divisors of $C(\mu)$. Hence f is given by a 3×3 matrix A.

We shall prove that A is a scalar and $f = \mathrm{id}$. In fact, any line $\ell_{x,y}$ connecting two points $x, y \in K$ is fixed by f. Since the line $x_0 = 0$ connects $[0, 1, -1]$ and $[0, 1, -\zeta_3]$, it is fixed by f. Similarly the lines $x_1 = 0$ and $x_2 = 0$ are fixed by f, whence $f^*(x_i) = a_i x_i$ $(i = 0, 1, 2)$ for some $a_i \neq 0$. Thus A is diagonal. Since $[0, 1, -1]$ and $[-1, 0, 1]$ are fixed, we have $a_0 = a_1 = a_2$, hence A is scalar and $f = \mathrm{id}$, $\mu = \mu'$.

We do not give proofs of (ii)-(iv) here because there are complicated arguments to prove rigorously. Q.E.D.

2.4. The moduli space of smooth cubics — classical level structure

Consider the (fine) moduli space of smooth cubics over an algebraically closed field $k \ni 1/3$.

Definition 2.4.1. Let $K = (\mathbf{Z}/3\mathbf{Z})^{\oplus 2}$, e_i a standard basis of K. Let $e_K : K \times K \to \mu_3$ be a standard symplectic form of K: in other words, e_K is (multiplicatively) alternating and bilinear such that

$$e_K(e_1, e_2) = e_K(e_2, e_1)^{-1} = \zeta_3, \ e_K(e_i, e_i) = 1.$$

Let C be a smooth cubic with zero O, $C[3] = \ker(3\,\mathrm{id}_C)$ the group of 3-division points and e_C the Weil pairing of C (see [43, pp. 95–102]), that is,

$$e_C : C[3] \times C[3] \to \mu_3 \quad \text{alternating nondegenerate bilinear},$$

(see 3.3 (v)). By [20, pp. 294–295], there exists a symplectic (group) isomorphism

$$\iota : (C[3], e_C) \to (K, e_K).$$

In what follows, we identify $C(\mu)[3]$ with K by

$$O = [0, 1, -1],\ e_1 = [0, 1, -\zeta_3],\ e_2 = [1, -1, 0]. \tag{4}$$

Definition 2.4.2. The triple $(C, C[3], \iota)$ is called *a (planar) cubic with classical level-3 structure.* We define $(C, C[3], \iota) \simeq (C', C'[3], \iota')$ to be isomorphic iff there exists an isomorphism $f : C \to C'$ such that $f_{|C[3]} : C[3] \to C'[3]$ is a symplectic (group) isomorphism subject to $\iota' \cdot f = \iota$.

Claim 2.4.3. *Let $A_{1,3}^{\mathrm{CL}}$ be the set of isomorphism classes of $(C, C[3], \iota)$. Then*

(i) *any $(C, C[3], \iota)$ is isomorphic to $(C(\mu), C(\mu)[3], \iota)$ for a unique μ,*

(ii) *$(C(\mu), K, \mathrm{id}_K) \in A_{1,3}^{\mathrm{CL}}$ via (4), and*

$$\begin{aligned} A_{1,3}^{\mathrm{CL}} &= \{(C(\mu), K, \mathrm{id}_K);\ \textit{a smooth Hesse cubic}\} \\ &\simeq \operatorname{Spec} k[\mu, \frac{1}{\mu^3 - 1}], \end{aligned}$$

(iii) *we define $SQ_{1,3}^{\mathrm{CL}}$ to be the union of $A_{1,3}^{\mathrm{CL}}$ and 3-gons in Subsec. 2.1 (ii) :*

$$\begin{aligned} SQ_{1,3}^{\mathrm{CL}} :&= \{(C, C[3], \iota); C\ \textit{smooth elliptic or a 3-gon}\}/\textit{isom.} \\ &= \{(C(\mu), K, \mathrm{id}_K);\ \textit{a Hesse cubic}\} \\ &\simeq \operatorname{Proj} k[\mu_0, \mu_1], \end{aligned}$$

(v) *$A_{1,3}^{\mathrm{CL}} \simeq A_{1,3}^{\mathrm{NL}}$ and $SQ_{1,3}^{\mathrm{CL}} \simeq SQ_{1,3}^{\mathrm{NL}}$ over k.* □

Proof of (i). We prove the uniqueness of μ. Suppose that

$$f : (C(\mu), K, \mathrm{id}_K) \to (C(\mu'), K, \mathrm{id}_K)$$

is an isomorphism. Then $f \in \mathrm{GL}(3)$. Since $\mathrm{id}_K \cdot f_{|K} = \mathrm{id}_K$ by $\iota' \cdot f = \iota$, we have $f_{|K} = \mathrm{id}_K$. Hence $f = \mathrm{id} \in \mathrm{PGL}(3)$, $\mu = \mu'$ by Subsec. 2.3 (iv). See also Lemma 3.12 and Lemma 8.2.8. Q.E.D.

§3. Non-commutative level structure

3.1. For constructing a separated moduli

If we keep naively using the same definition of level structures as in Subsec. 2.4 in higher dimension, then the complete moduli will be roughly the moduli of the triples $(Z, \ker(\lambda(L)), \iota_Z)$ similar to $(C, C[3], \iota)$

$$\iota_Z : \ker(\lambda(L)) \simeq K \text{ for some } K.$$

However then we will have nonseparated moduli spaces in general. The details will be explained in Subsec. 6.8.

To construct a separated moduli, we need to find outside C an alternative for $C[3]$ embedded in C. The group $C[3]$, hence $x \in K = (\mathbf{Z}/3\mathbf{Z})^{\oplus 2}$ acts on C by translation $T_x : C \to C$. Though the action of K on C cannot be lifted to L as an action of the group K, the action of any individual element x of K can be lifted to a line bundle automorphism τ_x of L. In general τ_x and τ_y $(x, y \in K)$ do not commute so $T_x \mapsto \tau_x$ fails to be a group homomorphism. However it turns out that the non-commutative group generated by all individual liftings τ_x plays the role of an alternative for $C[3]$ embedded in C. This leads us to the notion of *a level-$G(3)$ structure*, say, *a non-commutative level structure,* where $G(3)$ is the Heisenberg group associated to K.

Remark 3.1.1. Since any elliptic curve with level-$G(3)$ structure has a section over $\mathbf{Z}[\zeta_3, 1/3]$ by [34], the level-$G(3)$ structure is a ϑ-structure of [21, II, p.78] and vice versa. A level $\mathcal{G}_H$ (or G_H-)structure is not always a ϑ-structure by [34] when $H = \mathbf{Z}/n\mathbf{Z}$ for n even in Definition 3.5.

Definition 3.2. Let k be an algebraically closed field $k \ni 1/3$. Then

(i) let C be any smooth cubic with zero O, and $L := O_C(1)$ the hyperplane bundle. Let $\lambda(L) : C \to C^\vee := \mathrm{Pic}^0(C) \simeq C$ be the map $x \to T_x^* L \otimes L^{-1}$, called the polarization morphism, where we see $\lambda(L) = 3\,\mathrm{id}_C$,

(ii) let $K := C[3] = \ker \lambda(L) \simeq (\mathbf{Z}/3\mathbf{Z})^{\oplus 2}$, and $e_K : K \times K \to \mu_3$ the Weil pairing of C. If $C = C(\mu)$ and $O = [0, 1, -1] \in C(\mu)$. Then $K = \ker(\lambda(L))$ is the same as in Subsec. 2.1 (iii).

3.3. Non-commutative interpretation of Hesse cubics

First we shall re-interpret the group $C[3]$ of 3-division points of Hesse cubics in the non-commutative way as follows.

Any translation T_x by $x \in K$ is lifted to $\gamma_x \in \mathrm{GL}(V)$, so that

$$e_K(x, y) = [\gamma_x, \gamma_y] \in \mu_3,$$

where $V = H^0(C, O_C(1)) = H^0(\mathbf{P}^2, O_{\mathbf{P}^2}(1))$. To be more precise,

(i) we define σ and τ by $\sigma(x_k) = \zeta_3^k x_k$, $\tau(x_k) = x_{k+1}$ $(k = 0, 1, 2)$, where their matrix forms are given by

$$\sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta_3 & 0 \\ 0 & 0 & \zeta_3^2 \end{pmatrix}, \quad \tau = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

(ii) σ is induced from the translation by $1/3$ because $x_k = \theta_k$ by Subsec. 2.1 (v) and

$$\theta_k(z + 1/3) = \zeta_3^k \theta_k(z),$$

(iii) τ is induced from the translation by $\omega/3$ because

$$[\theta_0, \theta_1, \theta_2](z + \omega/3) = [\theta_1, \theta_2, \theta_0](z),$$

(iv) $[\sigma, \tau] = \zeta_3$, that is, σ and τ do not commute,

$$\sigma\tau = \begin{pmatrix} 0 & 0 & 1 \\ \zeta_3 & 0 & 0 \\ 0 & \zeta_3^2 & 0 \end{pmatrix}, \quad \tau\sigma = \begin{pmatrix} 0 & 0 & \zeta_3^2 \\ 1 & 0 & 0 \\ 0 & \zeta_3 & 0 \end{pmatrix}.$$

Lemma 3.4. *Let $G(3) := \langle \sigma, \tau \rangle$ be the group generated by σ and τ. Then it is a finite group of order 27. Let $V = H^0(\mathbf{P}^2, O_{\mathbf{P}^2}(1)) = \{x_0, x_1, x_2\}$. Then V is an irreducible $G(3)$-module of weight one, where "weight one" means that $a \in \mu_3$ (center) acts by $a\,\mathrm{id}_V$.*

Proof. The first assertion is clear. See [20, Proposition 3, p. 309] or [32, Lemma 4.4] for the second assertion. Q.E.D.

The action of $G(3)$ on $H^0(C, L)$ is a special case of more general Schrödinger representations defined below.

Definition 3.5. We define $G(K) = G_H$ (resp. $\mathcal{G}(K) = \mathcal{G}_H$) to be *the Heisenberg group (finite resp. infinite)* and U_H *the Schrödinger representation of* G_H as follows:

$$\begin{aligned}
H = H(e) &:= \bigoplus_{i=1}^{g} (\mathbf{Z}/e_i\mathbf{Z}), e_i | e_{i+1}, N = |H| = \prod_{i=1}^{g} e_i,\\
K = H \oplus H^\vee&, e_{\min}(K) = e_{\min}(H) := e_1,\\
G_H &= \{(a, z, \alpha); a \in \mu_N, z \in H, \alpha \in H^\vee\},\\
\mathcal{G}_H &= \{(a, z, \alpha); a \in \mathbf{G}_m, z \in H, \alpha \in H^\vee\},\\
(a, z, \alpha) \cdot (b, w, \beta) &= (ab\beta(z), z + w, \alpha + \beta),\\
V : = V_H &= \mathcal{O}_N[H^\vee] = \bigoplus_{\mu \in H^\vee} \mathcal{O}_N\, v(\mu),\\
U_H(a, z, \alpha) v(\gamma) &= a\gamma(z) v(\alpha + \gamma).
\end{aligned}$$

Here $\mathcal{O} = \mathcal{O}_N = \mathbf{Z}[\zeta_N, 1/N]$, and $v(\mu)$ $(\mu \in H^\vee)$ is a free $\mathcal{O}_N$-basis of V_H. The group homomorphism U_H, from G_H or $\mathcal{G}_H$ to $\mathrm{End}\,(V)$, is called *Schrödinger representation.* We note

$$1 \to \mu_N \to G_H \to K \to 0 \quad \text{(exact)}$$
$$1 \to \mathbf{G}_m \to \mathcal{G}_H \to K \to 0 \quad \text{(exact)}.$$

Example 3.6. For Hesse cubics, $\mathcal{O} := \mathbf{Z}[\zeta_3, 1/3]$, $H = H^\vee = \mathbf{Z}/3\mathbf{Z}$, we identify $G(3)$ with G_H; to be precise, $G(3) = U_H(G_H)$ and

$$\sigma = U_H(1,1,0), \quad \tau = U_H(1,0,1), \quad N = 3.$$
$$V_H = \mathcal{O}[H^\vee] = \bigoplus_{k=0}^{2} \mathcal{O} \cdot v(k).$$

Let $\mathbf{P}^2 = \mathbf{P}(V_H)$. Then V_H is identified with $H^0(C, O_C(1)) = H^0(\mathbf{P}^2, O_{\mathbf{P}^2}(1))$ by the map $v(k) \mapsto x_k$ in Lemma 3.4.

Lemma 3.7. *V_H is an irreducible $\mathcal{G}_H$-$\mathcal{O}_N$-module (an irreducible G_H-$\mathcal{O}_N$-module) of weight one, unique up to equivalence. Any $\mathcal{G}_H$-$\mathcal{O}_N$-module W (resp. any G_H-$\mathcal{O}_N$-module) of finite rank is a direct sum of V_H if W is of weight one: that is, any element a in the center $\mathbf{G}_m$ (resp. μ_N) acts on W by scalar multiplication $a\,\mathrm{id}_W$.*

Proof. See Lemma 11.1.2 and [32, Lemma 4.4]. Q.E.D.

Lemma 3.8. (*Schur's lemma*) *Let R be a commutative algebra with $1/N$ and ζ_N. Let V_1 and V_2 be R-free G_H-modules of finite rank of weight one. If V_1 and V_2 are irreducible G_H-modules, and if $f : V_1 \to V_2$ and $g : V_1 \to V_2$ are $\mathcal{G}_H$-isomorphisms, then there exists a unit $c \in R^\times$ such that $f = cg$.*

Proof. See [32, Lemma 4.5]. Q.E.D.

3.9. New formulation of the moduli problem

Let k be any ring such that $k \ni \zeta_3, 1/3$ and $K = (\mathbf{Z}/3\mathbf{Z})^{\oplus 2}$. Let C be any smooth cubic, $L = O_C(1)$ the line bundle viewed as a scheme over C. By [20, p. 295] (see also [30, Lemma 7.6]) the pair (C, L) of schemes has a $G(3)$-action lifting the translation action by $C[3]$

$$\tau : G(3) \times (C, L) \to (C, L).$$

Using this $G(3)$-action, we define new level-3 structure. In a word,

- ∘ classical level-3 structure = to fix the 3-division points K
- ∘ new level-3 structure = to fix the matrix form of the action of $G(3)$ on $V \simeq H^0(C, L)$.

Definition 3.10. We define (C, ψ, τ) to be *a (planar) cubic with level-$G(3)$ structure* (or *a level-$G(3)$ cubic*) if

(i) (C, L) is a planar cubic with $L = O_C(1)$,

(ii) τ is a $G(3)$-action of weight one on the pair (C, L): that is, $\tau(a)$ acts by $(\mathrm{id}_C, a\,\mathrm{id}_L)$ for $a \in \mu_3$, the center of $G(3)$,

(iii) $\psi : C \to \mathbf{P}(V_H)$ is the inclusion, and

$$(\psi, \Psi) : (C, L) \to (\mathbf{P}(V_H), \mathbf{H})$$

is a $G(3)$-equivariant morphism by τ where $\mathbf{H}$ is the hyperplane bundle of $\mathbf{P}(V_H)$ and $\Psi : L = \psi^*\mathbf{H} \to \mathbf{H}$ the natural bundle morphism. That is,

$$(\psi, \Psi) \circ \tau(g) = S(g) \circ (\psi, \Psi) \text{ for any } g \in G(3) \tag{5}$$

with the notation in Subsec. 7.2. □

In what follows, we denote (ψ, Ψ) simply by ψ if no confusion is possible because Ψ is uniquely determined by ψ. We denote (5) by

$$\psi\tau(g) = S(g)\psi, \text{ or } \psi\tau = S\psi. \tag{6}$$

Definition 3.11. Two cubics (C, ψ, τ) and (C', ψ', τ') with level-$G(3)$ structure are defined to be isomorphic iff there exists an isomorphism

$$(f, F) : (C, L) \to (C', L')$$

such that

(i) $\psi' \cdot (f, F) = \psi$,

(ii) (f, F) is a $G(3)$-isomorphism, that is, $(f, F)\tau(g) = \tau'(g)(f, F)$ for any $g \in G(3)$.

Lemma 3.12. *Any Hesse cubic $(C(\mu), i, U_H)$ with i the inclusion of C into $\mathbf{P}(V_H)$ is a level-$G(3)$ cubic. Moreover any level-$G(3)$ cubic (C, ψ, τ) is isomorphic to a unique Hesse cubic $(C(\mu), i, U_H)$.*

Proof. Let $\mathbf{P}^2$ be $\mathbf{P}(V_H)$ and $\mathbf{H}$ the hyperplane bundle of $\mathbf{P}^2$. U_H induces an action on $H^0(\mathbf{P}^2, O_{\mathbf{P}^2}(1)) = V_H$ by Claim 7.1.5, which we denote by $H^0(U_H, O_{\mathbf{P}^2}(1))$. This is the same as the action U_H on V_H in Definition 3.5. In fact, by Subsec. 7.2 and Remark 7.3, U_H induces an action of $G(3)$ on the pair $(\mathbf{P}^2, \mathbf{H})$, which also induces an action of $G(3)$ on $H^0(\mathbf{P}^2, O_{\mathbf{P}^2}(1)) = V_H$. This is the same as U_H as is shown in Remark 7.3.

Let $O_{C(\mu)}(1) = O_{\mathbf{P}^2}(1) \otimes O_{C(\mu)}$ and $\mathbf{H}_{C(\mu)} = \mathbf{H} \times_{\mathbf{P}^2} C(\mu)$. Since $C(\mu)$ is $G(3)$-stable, $G(3)$ acts on the pair $(C(\mu), \mathbf{H}_{C(\mu)})$ by Claim 7.4.1.

Denoting the action of $G(3)$ on $\mathbf{H}_{C(\mu)}$ by the same letter U_H, we see that $(C(\mu), i, U_H)$ is a level-$G(3)$ structure.

Hence $H^0(C(\mu), O_{C(\mu)}(1))$ admits a $G(3)$-action, which we denote by $H^0(U_H, O_{C(\mu)}(1))$. Since $H^0(C(\mu), O_{C(\mu)}(1)) = H^0(\mathbf{P}^2, O_{\mathbf{P}^2}(1)) = V_H$ by restriction, we can identify $H^0(U_H, O_{C(\mu)}(1))$ with $H^0(U_H, O_{\mathbf{P}^2}(1))$ on V_H in a canonical manner. Thus we have a canonical identification

$$H^0(U_H, O_{C(\mu)}(1)) = H^0(U_H, O_{\mathbf{P}^2}(1)) = U_H.$$

By Lemma 8.2.8, any (C, ψ, τ) is isomorphic to some Hesse cubic $(C(\mu), i, U_H)$. Here we prove the uniqueness of it only. This is a new proof of Claim 2.4.3 (ii). Suppose $(C(\mu), i, U_H) \simeq (C(\mu'), i, U_H)$. Let $h : C(\mu) \to C(\mu')$ be a $G(3)$-isomorphism. Since h is linear (as is shown easily), h induces an automorphism of $(\mathbf{P}^2, O_{\mathbf{P}^2}(1))$ (also denoted h) so that we have a commutative diagram

$$\begin{array}{ccc}
H^0(\mathbf{P}^2, O_{\mathbf{P}^2}(1)) = V_H & \xrightarrow{H^0(h^*)} & H^0(\mathbf{P}^2, O_{\mathbf{P}^2}(1)) = V_H \\
\downarrow \| & & \downarrow \| \\
H^0(C(\mu'), O_{C(\mu')}(1)) & \xrightarrow{H^0(h^*)} & H^0(C(\mu), O_{C(\mu)}(1)), \\
\downarrow {\scriptstyle H^0(U_H(g), O_{C(\mu')}(1))} & & \downarrow {\scriptstyle H^0(U_H(g), O_{C(\mu)}(1))} \\
H^0(C(\mu'), O_{C(\mu')}(1)) & \xrightarrow{H^0(h^*)} & H^0(C(\mu), O_{C(\mu)}(1)),
\end{array}$$

whence

$$H^0(U_H(g), O_{C(\mu)}(1)) H^0(h^*) = H^0(h^*) H^0(U_H(g), O_{C(\mu')}(1))$$

for any $g \in G(3)$. By canonically identifying $H^0(U_H, O_{C(\mu)}(1))$ with U_H on V_H, we have

$$U_H(g) H^0(h^*) = H^0(h^*) U_H(g) \in \mathrm{End}\,(V_H)$$

for any $g \in G(3)$, where we also regard $H^0(h^*) \in \mathrm{End}\,(V_H)$. Since U_H is irreducible, $H^0(h^*)$ is a scalar by Schur's lemma. Hence $H^0(h^*) = \mathrm{id}_{V_H} \in \mathrm{PGL}(V_H)$, $h = \mathrm{id}_{\mathbf{P}(V_H)}$, $C(\mu) = C(\mu')$, $\mu = \mu'$. Q.E.D.

Remark 3.13. In the proof of Lemma 3.12, we canonically identified all the vector spaces involved to simplify the argument. This argument will be made much clearer by using $\rho(\phi, \tau)$ in Definitions 8.2.2 and 8.2.6. See Lemma 8.2.8.

Proposition 3.14. *Over* $\mathbf{Z}[\zeta_3, 1/3]$,

$$\begin{aligned} SQ_{1,3} :&= \{(C, \psi, \tau);\ a\ level\text{-}G(3)\ cubic\}/\text{isom.} \\ &= \{(C(\mu), i, U_H)\}/\text{isom.} = \{\mu \in \mathbf{P}^1\}. \end{aligned}$$

Proof. Clear from Lemma 3.12 and Lemma 8.2.8. Q.E.D.

It is this *level-*$G(3)$ *structure* that we can generalize into higher dimension so that we may obtain a separated moduli.

Remark 3.15. Suppose k is algebraically closed with $1/3$. Let $K = (\mathbf{Z}/3\mathbf{Z})^{\oplus 2}$. Let C be any cubic, and $C[3] = \ker(3\,\mathrm{id}_C)$ by choosing the zero $O \in C(k)$. Any level-$G(3)$ structure (C, ϕ, τ) gives rise to a classical level-3 structure $(C, C[3], \iota)$ as follows. First we note

$$C[3] = G(3) \cdot O.$$

Let $\pi : G(3) \to K = G(3)/[G(3), G(3)]$ be the natural homomorphism. We define $\iota : K \to C$ by

$$\iota(g \cdot O) := \pi(g).$$

Then $(C, C[3], \iota)$ is a classical level-3 structure. In fact, since $e_K(x, y) = [\gamma_x, \gamma_y]$ for a lifting γ_x of x, we have $e_K(1/3, \omega/3) = [\sigma, \tau] = \zeta_3$. Hence π defines a symplectic isomorphism $\iota : C[3] \to K$. Thus we see

$$SQ_{1,3}(k) = SQ^{\mathrm{CL}}_{1,3}(k).$$

By [34] $SQ_{1,3} \simeq SQ^{\mathrm{CL}}_{1,3}$ over $\mathbf{Z}[1/3, \zeta_3]$. See [34] for the detail.

§4. PSQAS and TSQAS

4.1. Goal

Our goal of constructing a compactification of the moduli space of abelian varieties is achieved by

(i) finding limit objects (two kinds of nice degenerate abelian schemes called PSQAS and TSQAS) (Theorems 4.5 and 4.6),
(ii) constructing the moduli $SQ_{g,K}$ as a projective scheme (Section 8),
(iii) proving that any point of $SQ_{g,K}$ is the isomorphism class of a nice degenerate abelian scheme (PSQAS) (Q_0, ϕ_0, τ_0) with level-$\mathcal{G}_H$ structure (Section 8, Theorems 8.5 and 9.8).

We recall a basic lemma from [25].

Lemma 4.2. *Let k be an algebraically closed field with $k \ni 1/N$ and H a finite Abelian group with $|H| = N$. Let (A, L) be an abelian variety over k with L an ample line bundle, $\lambda(L) : A \to A^\vee$ the polarization morphism (sending $x \mapsto T_x^* L \otimes L^{-1}$) and $\mathcal{G}(A, L)$ the group of bundle automorphisms g of L over A inducing translations of A.*

Suppose $\ker(\lambda(L)) \simeq K := H \oplus H^\vee$*. Then $\mathcal{G}(A, L) \simeq L^\times_{\ker(\lambda(L))} \simeq \mathcal{G}_H$, and any $g \in \mathcal{G}(A, L)$ induces a translation of A by some element of* $\ker(\lambda(L))$ *where $L^\times$ is the complement of the zero section in the line bundle L, and $L^\times_{\ker(\lambda(L))}$ is the pullback (restriction) of it to* $\ker(\lambda(L))$.

Proof. See [20, pp. 294–295] and [25, pp. 115-117, pp.204-211]. Q.E.D.

4.3. Limit objects

We wish to consider limits of abelian varieties.

Let R be a complete discrete valuation ring (CDVR), and $k(\eta)$ the fraction field of R and $k(0) := R/I$ the residue field. Suppose we are given an abelian scheme $(G_\eta, \mathcal{L}_\eta)$ over $k(\eta)$ and the polarization morphism

$$\lambda(\mathcal{L}_\eta) : G_\eta \to G_\eta^t := \mathrm{Pic}^0(G_\eta).$$

Let

$$K_\eta = \ker(\lambda(\mathcal{L}_\eta)), \quad \mathcal{G}(K_\eta) := \mathcal{G}(G_\eta, \mathcal{L}_\eta) \simeq (\mathcal{L}_\eta^\times)_{|K_\eta},$$

where $\mathcal{G}(G_\eta, \mathcal{L}_\eta)$ is by definition the group of bundle automorphisms of $\mathcal{L}_\eta$ over G_η which induce translations of G_η. See Lemma 4.2.

For simplicity, in what follows, we assume

(7) the field $k(0)$ contains $1/|K_\eta|$.

We apply Lemma 4.2 to $(G_\eta, \mathcal{L}_\eta)$.

Lemma 4.4. *Assume (7). Then by some base change of R if necessary, there exists a finite symplectic Abelian group K such that the diagram is commutative with exact rows:*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbf{G}_m & \longrightarrow & \mathcal{G}(K_\eta) & \longrightarrow & K_\eta & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}.} & & \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} & & \\
1 & \longrightarrow & \mathbf{G}_m & \longrightarrow & \mathcal{G}_H & \longrightarrow & H \oplus H^\vee & \longrightarrow & 0.
\end{array}$$

Theorem 4.5. (Stable reduction theorem) ([2]) *For an abelian scheme $(G_\eta, \mathcal{L}_\eta)$ and a polarization morphism $\lambda(\mathcal{L}_\eta) : G_\eta \to G_\eta^t$ over $k(\eta)$, there exist a flat projective scheme $(P, \mathcal{L}_P)$ (TSQAS) over R, by a finite base change if necessary, such that*

(1) $(P_\eta, \mathcal{L}_\eta) \simeq (G_\eta, \mathcal{L}_\eta)$,
(2) $(P, \mathcal{L}_P)$ *is normal with* $\mathcal{L}_P$ *ample, in fact,* P *is explicitly given,*
(3) P_0 *is reduced and Gorenstein with trivial dualizing sheaf.*

The following is a refined version of the above.

Theorem 4.6. (Refined stable reduction theorem) ([30, p. 703], [32, p. 98]) *For an abelian scheme* $(G_\eta, \mathcal{L}_\eta)$ *and a polarization morphism* $\lambda(\mathcal{L}_\eta) : G_\eta \to G^t_\eta$ *over* $k(\eta)$ *such that* $K_\eta \simeq K$, *there exist flat projective schemes* $(Q, \mathcal{L}_Q)$ *(PSQAS) and* $(P, \mathcal{L}_P)$ *(TSQAS) over* R, *by a finite base change if necessary, such that*

(1) $(Q_\eta, \mathcal{L}_\eta) \simeq (P_\eta, \mathcal{L}_\eta) \simeq (G_\eta, \mathcal{L}_\eta)$,
(2) $(P, \mathcal{L}_P)$ *is the normalization of* $(Q, \mathcal{L}_Q)$,
(3) P_0 *is reduced and Gorenstein with trivial dualizing sheaf,*
(4) *if* $e_{\min}(K) \geq 3$, *then* $\mathcal{L}_Q$ *is very ample,*
(5) $(Q, \mathcal{L}_Q)$ *is an étale quotient of some PSQAS* $(Q^*, \mathcal{L}_{Q^*})$ *with* $e_{\min}(\ker \lambda(\mathcal{L}_{Q^*})) \geq 3$, *hence with* $\mathcal{L}_{Q^*}$ *very ample,*
(6) $\mathcal{G}(K)$ *acts on* $(Q, \mathcal{L}_Q)$ *and* $(P, \mathcal{L}_P)$ *extending the action of* $\mathcal{G}(K_\eta)$ *on* $(G_\eta, \mathcal{L}_\eta)$.

See Definition 3.5 for $e_{\min}$. Theorem 4.6 (1) is proved in Subsec. 6.4. We call $(Q_0, \mathcal{L}_0)$ and $(P_0, \mathcal{L}_0)$ as follows:

- $(Q_0, \mathcal{L}_0)$: PSQAS — *a projectively stable quasi-abelian scheme*, which can be nonreduced,
- $(P_0, \mathcal{L}_0)$: TSQAS — *a torically stable quasi-abelian scheme* (= variety), which is always reduced.

Remark 4.7. Theorem 4.6 (2) is rather misleading. In the proof of it, we never define P to be the normalization of Q. We only construct P with P_0 reduced and $P_\eta \simeq G_\eta$. The normality of P is a consequence of the reducedness of P_0 by the following well-known Claim.

Claim 4.7.1. *Let* R *be a complete discrete valuation ring,* $S :=$ Spec R, *and* η *the generic point of* S. *Assume that* $\pi : Z \to S$ *is flat with* Z_0 *reduced and* Z_η *nonsingular. Then* Z *is normal.*

Proof. See [32, Lemma 10.3]. Q.E.D.

Remark 4.8. In dimension one, any PSQAS is a TSQAS and vice versa, which is either a smooth elliptic or an N-gon (of rational curves). Once the moduli of PSQASes (resp. TSQASes) is constructed, Theorem 4.9 will prove that *the moduli is separated*, and then Theorem 4.6 will prove that *the moduli is proper.*

Theorem 4.9. (Uniqueness [30],[32]) *In Theorem 4.6,* $(Q, \mathcal{L})$ *resp.* $(P, \mathcal{L})$ *is uniquely determined by* $(G_\eta, \mathcal{L}_\eta)$ *if* $e_{\min}(K) \geq 3$ *(resp. in any case).*

See [30, Theorem 10.4] and [32, Theorem 10.4; Claim 2, p. 124] for the detail when $e_{\min}(H) \geq 3$. See Subsec 11.10 for $e_{\min}(H) \leq 2$.

§5. PSQASes in low dimension

The purpose of this section is to show motivating examples in dimension one and two.

5.1. Hesse cubics and theta functions

Let R be a complete discrete valuation ring (CDVR), I the maximal ideal of R and q a generator (uniformizer) of I, so $I = qR$. For instance, if $R = \mathbf{Z}_3$, then we can choose $q = 3$, and if $R = k[[t]]$, k a field, then $q = t$. Let θ_k be the same as in Subsec. 2.1 (iv)

$$\theta_k(\omega, z) = \sum_{m\in\mathbf{Z}} q^{(3m+k)^2} w^{3m+k}$$

Then the power series θ_k converge I-adically.

Now we calculate the limit of $[\theta_0, \theta_1, \theta_2]$ as q tends to 0.

First we shall show a computation, which once puzzled us so much.

$$\begin{aligned}
\theta_0(q, w) &= \sum_{m\in\mathbf{Z}} q^{9m^2} w^{3m} \\
&= 1 + q^9 w^3 + q^9 w^{-3} + q^{36} w^6 + \cdots, \\
\theta_1(q, w) &= \sum_{m\in\mathbf{Z}} q^{(3m+1)^2} w^{3m+1} \\
&= qw + q^4 w^{-2} + q^{16} w^4 + \cdots, \\
\theta_2(q, w) &= \sum_{m\in\mathbf{Z}} q^{(3m+2)^2} w^{3m+2} \\
&= qw^{-1} + q^4 w^2 + q^{16} w^{-4} + q^{25} w^5 + \cdots.
\end{aligned}$$

Hence in $\mathbf{P}^2$

$$\lim_{q\to 0} [\theta_0, \theta_1, \theta_2](q, w)] = [1, 0, 0]$$

The elliptic curves converge to one point? This looks strange. The reason why we got the above is that we treated w as a constant. There is Néron model behind this strange phenomenon. We cannot explain it in detail here. Instead we show how to modify the above computation.

Let $w = q^{-1}u$ for $u \in R \setminus I$ and $\overline{u} = u \mod I$. Then we have

$$\begin{aligned}
\theta_0(q, q^{-1}u) &= \sum_{m\in\mathbf{Z}} q^{9m^2-3m}u^{3m} \\
&= 1 + q^6u^3 + q^{12}u^{-3} + q^{30}u^6 + \cdots , \\
\theta_1(q, q^{-1}u) &= \sum_{m\in\mathbf{Z}} q^{(3m+1)^2-3m-1}u^{3m+1} \\
&= u + q^6u^{-2} + q^{12}u^4 + \cdots , \\
\theta_2(q, q^{-1}u) &= \sum_{m\in\mathbf{Z}} q^{(3m+2)^2-3m-2}u^{3m+2} \\
&= q^2u^2 + q^2u^{-1} + q^{20}u^5 + q^{20}u^{-4} + \cdots .
\end{aligned}$$

Hence in $\mathbf{P}^2$

$$\lim_{q\to 0} [\theta_0, \theta_1, \theta_2](q, q^{-1}u) = [1, \overline{u}, 0]$$

Similarly

$$\begin{aligned}
&\theta_0(q, q^{-2}u) = 1 + q^3u^3 + q^{15}u^{-3} + q^{24}u^6 + \cdots , \\
&\theta_1(q, q^{-2}u) = q^{-1}u + q^{12}u^{-2} + q^8u^4 + \cdots , \\
&\theta_2(q, q^{-2}u) = u^2 + q^3u^{-1} + q^{15}u^5 + q^{24}u^{-4} + \cdots , \\
&\lim_{q\to 0} [\theta_0, \theta_1, \theta_2](q, q^{-2}u) = \lim_{q\to 0} [1, q^{-1}u, u^2] = [0, 1, 0] \quad \text{in } \mathbf{P}^2.
\end{aligned}$$

Similarly

$$\begin{aligned}
&\theta_0(q, q^{-3}u) = 1 + u^3 + q^{18}u^{-3} + q^{18}u^6 + \cdots , \\
&\theta_1(q, q^{-3}u) = q^{-2}u + q^{10}u^{-2} + q^4u^4 + \cdots , \\
&\theta_2(q, q^{-3}u) = q^{-2}u^2 + q^4u^{-1} + q^{10}u^5 + q^{28}u^{-4} + \cdots , \\
&\lim_{q\to 0} [\theta_0, \theta_1, \theta_2](q, q^{-3}u) = \lim_{q\to 0} [1, q^{-2}u, u^2] = [0, 1, \overline{u}] \quad \text{in } \mathbf{P}^2.
\end{aligned}$$

Let $w = q^{-2\lambda}u$ (a section over a finite extension of $k(\eta)$ for $\lambda \in \mathbf{Q}$) and $u \in R \setminus I$.

$$\text{(8)} \quad \lim_{q\to 0} [\theta_0, \theta_1, \theta_2](q, q^{-2\lambda}u) = \begin{cases} [1, 0, 0] & (\text{if } -1/2 < \lambda < 1/2), \\ [1, \overline{u}, 0] & (\text{if } \lambda = 1/2), \\ [0, 1, 0] & (\text{if } 1/2 < \lambda < 3/2), \\ [0, 1, \overline{u}] & (\text{if } \lambda = 3/2), \\ [0, 0, 1] & (\text{if } 3/2 < \lambda < 5/2). \\ [\overline{u}, 0, 1] & (\text{if } \lambda = 5/2), \end{cases}$$

When λ ranges in $\mathbf{R}$, the same calculation shows that the same limits repeat mod $Y = 3\mathbf{Z}$ because

$$\lim_{q\to 0} [\theta_0, \theta_1, \theta_2](q, q^{6n-a}u) = \lim_{q\to 0} [\theta_0, \theta_1, \theta_2](q, q^{-a}u).$$

Thus we see that $\lim_{\tau\to\infty} C(\mu(\tau))$ is the 3-gon $x_0x_1x_2 = 0$.

Definition 5.2. For $\lambda \in X \otimes_{\mathbf{Z}} \mathbf{R}$ fixed, let

$$F_\lambda := a^2 - 2\lambda a \quad (a \in X = \mathbf{Z}).$$

We define a Delaunay cell

$$D(\lambda) := \begin{matrix}\text{the convex closure of all } a \in X \\ \text{that attain the minimum of } F_\lambda\end{matrix}$$

By computations we see

$$D(j+\frac{1}{2}) = [j, j+1] := \{x \in \mathbf{R};\ j \le x \le j+1\},$$
$$D(\lambda) = \{j\} \quad (\text{if } j - \frac{1}{2} < \lambda < j + \frac{1}{2}),$$
$$[\bar\theta_k]_{k=0,1,2} : = \lim_{q\to 0}[\theta_k(q, q^{-2\lambda}u))]_{k=0,1,2}$$
$$\bar\theta_k = \begin{cases}\bar u^j & (\text{if } j \in D(\lambda) \cap (k+3\mathbf{Z})) \\ 0 & (\text{if } D(\lambda) \cap (k+3\mathbf{Z}) = \emptyset).\end{cases}$$

For instance $D(\frac{1}{2}) \cap (0+3\mathbf{Z}) = \{0\}$, $D(\frac{1}{2}) \cap (1+3\mathbf{Z}) = \{1\}$ and

$$\lim_{q\to 0}[\theta_k(q, q^{-1}u))] = [\bar\theta_0, \bar\theta_1, \bar\theta_2] = [\bar u^0, \bar u, 0] = [1, \bar u, 0].$$

Similarly for any $\lambda = j + (1/2)$, we have an algebraic torus as a limit

$$\{[\bar u^j, \bar u^{j+1}] \in \mathbf{P}^1; \bar u \in \mathbf{G}_m\} \simeq \mathbf{G}_m \ (= \mathbf{C}^*).$$

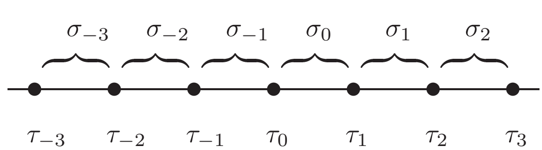

Fig. 1. Delaunay decomposition

Let $\lambda \in X \otimes \mathbf{R}$, and $\sigma = D(\lambda)$ be a Delaunay cell, and $O(\sigma)$ the stratum of $C(\infty)$ consisting of limits of $(q, q^{-2\lambda}u)$. If σ is one-dimensional, then $O(\sigma) = \mathbf{C}^*$, while $O(\sigma)$ is one point if σ is zero-dimensional. Thus we see that $C(\mu(\infty))$ is a disjoint union of $O(\sigma)$, σ being Delaunay cells mod Y, in other words, it is stratified in terms of the Delaunay decomposition mod Y.

Let $\sigma_j = [j, j+1]$ and $\tau_j = \{j\}$. Then the Delaunay decomposition (resp. the stratification of $C(\infty)$) is given in Fig. 1 (resp. Fig. 2).

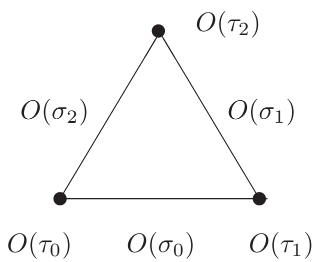

Fig. 2. A 3-gon

5.3. The complex case

To apply the computation in the last section to the moduli problem, we need to know the scheme-theoretic limit of the image of $E(\omega)$.

Now let us write

$$\theta_k(q, w) = \sum_{m \in \mathbf{Z}} q^{(3m+k)^2} w^{3m+k} = \sum_{m \in \mathbf{Z}} a(3m+k) w^{3m+k}$$

where $a(x) = q^{x^2}$ for $x \in X := \mathbf{Z}$. Let $Y = 3\mathbf{Z}$. Then θ_k is Y-invariant :

$$\theta_k = \sum_{y \in Y} a(y+k) w^{y+k}.$$

Since the curve $E(\tau)$ is embedded into $\mathbf{P}^2_{\mathbf{C}}$ by θ_k, we see

$$\begin{aligned} E(\omega) &= \operatorname{Proj} \mathbf{C}[x_k, k=0,1,2]/(x_0^3 + x_1^3 + x_2^3 - 3\mu(\omega)x_0x_1x_2) \\ &\simeq \operatorname{Proj} \mathbf{C}[\theta_k \vartheta, k=0,1,2] \qquad (9) \\ &= \operatorname{Proj} (\mathbf{C}[[a(x)w^x\vartheta, x \in X]])^{Y-\mathrm{inv}} \end{aligned}$$

where ϑ is a transcendental element of degree one, $\deg(x_k) = 1$, and $\deg(\theta_k) = 0$ and $\deg(a(x)w^x) = 0$. Recall that if $U = \mathrm{Spec}\ A$ is affine, G a finite group acting on U, then

$$U/G = \mathrm{Spec}\ A^{G\text{-inv}}.$$

So we wish to regard $E(\omega)$ as

$$E(\omega) = (\mathrm{Proj}\,(\mathbf{C}[[a(x)w^x\vartheta, x \in X]]))/Y.$$

Is this really true? Over $\mathbf{C}$, $a(x) \in \mathbf{C}^\times$, and

$$\mathbf{G}_m = \mathrm{Proj}\ \mathbf{C}[a(x)w^x\vartheta, x \in X],$$

In fact, the rhs is covered with infinitely many affine U_k

$$U_k = \mathrm{Spec}\ \mathbf{C}[a(x)w^x\vartheta/a(k)w^k\vartheta; x \in X] = \mathrm{Spec}\ \mathbf{C}[w, w^{-1}] = \mathbf{G}_m,$$

which is independent of k. Hence over $\mathbf{C}$

$$\begin{aligned} E(\omega) &\simeq \mathbf{G}_m/w \mapsto q^6 w \\ &\simeq \mathbf{G}_m/\{w \mapsto q^{2y}w; y \in 3\mathbf{Z}\} \\ &\simeq (\mathrm{Proj}\ \mathbf{C}[a(x)w^x\vartheta, x \in X])/Y. \end{aligned} \tag{10}$$

Thus we see by combining (9) and (10)

$$\begin{aligned} E(\omega) &\simeq \mathrm{Proj}\,(\mathbf{C}[[a(x)w^x\vartheta, x \in X]])^{Y-\mathrm{inv}} \\ &\simeq (\mathrm{Proj}\ \mathbf{C}[a(x)w^x\vartheta, x \in X])/Y, \end{aligned} \tag{11}$$

though we should make the convergence of infinite sum precise. In fact, this is easily justified when R is a CDVR.

5.4. The scheme-theoretic limit

We define the subring $\widetilde{R}$ of $k(\eta)[w, w^{-1}][\vartheta]$ by

$$\widetilde{R} = R[a(x)w^x\vartheta; x \in X]$$

where $a(x) = q^{x^2}$ for $x \in X$, $X = \mathbf{Z}$, and ϑ is an indeterminate of degree one, where q is the uniformizer of R. We define the action of Y on $\widetilde{R}$ by the ring homomorphism

$$S_y^*(a(x)w^x\vartheta) = a(x+y)w^{x+y}\vartheta. \tag{12}$$

where $Y = 3\mathbf{Z} \subset X$. Then what does Z look like?

$$Z = \mathrm{Proj}\ R[a(x)w^x\vartheta, x \in X]/Y.$$

Let $\mathcal{X}$ and U_n be

$$\begin{aligned}\mathcal{X} &= \text{Proj } R[a(x)w^x\vartheta, x \in X],\\ U_n &= \text{Spec } R[a(x)w^x/a(n)w^n, x \in X]\\ &= \text{Spec } R[(a(n+1)/a(n))w, (a(n-1)/a(n))w^{-1}]\\ &= \text{Spec } R[q^{2n+1}w, q^{-2n+1}w^{-1}]\\ &\simeq \text{Spec } R[x_n, y_n]/(x_ny_n - q^2),\end{aligned}$$

where U_n and U_{n+1} are glued together by

$$x_{n+1} = x_n^2y_n,\ y_{n+1} = x_n^{-1},\ x_n = q^{2n+1}w,\ y_n = q^{-2n+1}w^{-1}.$$

Fig. 3. An infinite chain

Let $\mathcal{X}_0 := \mathcal{X} \otimes_R (R/qR)$ and $V_n = \mathcal{X}_0 \cap U_n$. Then $\mathcal{X}_0$ is an infinite chain of $\mathbf{P}^1$, as in Fig. 3.

The action of the sublattice $Y = 3\mathbf{Z}$ on $\mathcal{X}_0$ is transfer by 3 components. In fact, S_{-3} sends

$$V_n \overset{S_{-3}}{\to} V_{n+3} \overset{S_{-3}}{\to} V_{n+6} \to \cdots,$$

$$(x_n, y_n) \overset{S_{-3}}{\mapsto} (x_{n+3}, y_{n+3}) = (x_n, y_n)$$

so that we have a cycle of 3 rational curves as the quotient $\mathcal{X}_0/Y$. Thus we have the same consequence as in Subsec. 5.1 by using theta functions.

5.5. The partially degenerate case in dimension two

We wish to describe any PSQAS in the partially degenerate case in dimension two. For simplicity, we shall give it directly by using theta functions. See Subsec. 6.7 for the totally degenerate case.

Case 5.5.1. First we consider the complex case. Let

$$\delta = \text{diag}(\ell, m) := \begin{pmatrix}\ell & 0\\ 0 & m\end{pmatrix}, \quad \tau = \begin{pmatrix}\tau_{11} & \tau_{12}\\ \tau_{12} & \tau_{22}\end{pmatrix}, \quad \tau_{12} = \tau_{21}.$$

Let Λ be the lattice spanned by column vectors of I_2 and $\tau\delta$, and G_η the abelian variety $\mathbf{C}^2/\Lambda$. We consider the degeneration of G_η as

$q := e^{\pi i \tau_{22}}$ tends to 0. Assume ℓ and $m \geq 3$. Following [42, Chap. VII, pp. 77–79] we define for $k = (k_1, k_2)$ $(0 \leq k_1 \leq \ell - 1, 0 \leq k_2 \leq m - 1)$,

$$\begin{aligned}\theta_k &= \sum_{n \in \mathbf{Z}^2} e^{\pi i^t(\delta n + k)\tau(\delta n + k) + 2\pi i^t(\delta n + k)z} \\ &= \sum_{n_2 \in \mathbf{Z}} q^{(mn_2 + k_2)^2} w^{mn_2 + k_2} \vartheta_{k_1}(z_1 + (mn_2 + k_2)\tau_{12}),\end{aligned}$$

where $T = \tau\delta$, $W = \delta T$ with the notation of [42], $q = e^{\pi i \tau_{22}}$ and ϑ_{k_1} is a theta function of level ℓ of one variable. Hence

$$\theta_k = \sum_{n_2 \in \mathbf{Z}} T^*_{(mn_2+k_2)\tau_{12}}(\vartheta_{k_1}) q^{(mn_2+k_2)^2} w^{mn_2+k_2}. \tag{13}$$

where (13) is a general form of algebraic theta functions in [30, Theorem 4.10 (3)].

Case 5.5.2. Now we consider the general case. In any algebraic case, we can start with the last form (13) of theta functions by [30, Theorem 4.10], where q is a uniformizing parameter of a CDVR R. In this case, $X = \mathbf{Z}$, $Y = m\mathbf{Z}$ and the Delaunay decomposition associated with this degeneration of abelian surfaces is the union of the unit intervals $[j, j+1]$ $(j \in \mathbf{Z})$ modulo Y.

Let $H = (\mathbf{Z}/\ell\mathbf{Z}) \oplus (X/Y) \simeq (\mathbf{Z}/\ell\mathbf{Z}) \oplus (\mathbf{Z}/m\mathbf{Z})$. By the theta functions θ_k we have a closed immersion of an abelian variety G_η to $\mathbf{P}(V_H)$. We compute the limit of the image of G_η as q tends to 0.

By the assumption $\ell \geq 3$, ϑ_{k_1} $(0 \leq k_1 \leq \ell - 1)$ embeds an elliptic curve into the projective space $\mathbf{P}^{\ell-1}$.

Let $w = q^{-2a-1}v$ $(a \in \mathbf{Z})$, $v \in R \setminus I$ and $I = qR$. Let $\overline{v} = v \mod I$. Then we have

$$\begin{aligned}\theta_{k_1,a}(q, u, q^{-2a-1}v) &= q^{-a^2-a} T^*_{a\tau_{12}} \vartheta_{k_1} + \cdots, \\ \theta_{k_1,a+1}(q, u, q^{-2a-1}v) &= q^{-a^2-a} T^*_{(a+1)\tau_{12}} \vartheta_{k_1} + \cdots, \\ \theta_{k_1,k_2}(q, u, q^{-2a-1}v) &\equiv 0 \bmod q^{-a^2-a+1}, \ (k_2 \neq a, a+1),\end{aligned}$$

whence

$$\begin{aligned}\lim_{q \to 0} [\theta_{k_1,k_2}(q, u, q^{-2a-1}v)]_{(k_1,k_2) \in H} &= [\theta_{k_1,a} u^a, \theta_{k_1,a+1} u^{a+1}]_{k_1} \\ &= [\underbrace{T^*_{a\tau_{12}} \vartheta_{k_1}}_{k_2 = a}, \underbrace{(T^*_{(a+1)\tau_{12}} \vartheta_{k_1}) v}_{k_2 = a+1}]_{k_1}\end{aligned}$$

with zero terms ignored. In particular, for $w = q^{-1}v$, we have

$$(14)\quad \lim_{q\to 0}[\theta_{k_1,k_2}(q,u,q^{-1}v)]_{(k_1,k_2)\in H} = [\theta_{k_1,0},\theta_{k_1,1}] = [\underbrace{\vartheta_{k_1}}_{k_2=0},\underbrace{(T^*_{\tau_{12}}\vartheta_{k_1})v}_{k_2=1}].$$

For $a = m$, we have

$$(15)\quad \lim_{q\to 0}[\theta_{k_1,k_2}(q,u,q^{-2m-1}v)]_{(k_1,k_2)\in H} = [\underbrace{T^*_{m\tau_{12}}\vartheta_{k_1}}_{k_2=0},\underbrace{(T^*_{(m+1)\tau_{12}}\vartheta_{k_1})v}_{k_2=1}]$$

Thus the limit of the abelian surface $(G_\eta, \mathcal{L}_\eta)$ as $q \to 0$ is the union of m copies of one and the same $\mathbf{P}^1$-bundle over an elliptic curve. By (14), any of the $\mathbf{P}^1$-bundle is the same compactification of the same $\mathbf{G}_m$-bundle whose extension class is given by τ_{12} through the isomorphism

$$\mathrm{Ext}(E, \mathbf{G}_m) \simeq E^\vee \simeq E \ni \tau_{12}.$$

By (14) and (15), the zero section of the first $\mathbf{P}^1$-bundle is identified with the ∞-section of the m-th $\mathbf{P}^1$-bundle by shifting by τ_{12}.

§6. PSQASes in the general case

6.1. The degeneration data of Faltings-Chai

Now we consider the general case. Let R be a complete discrete valuation ring (CDVR), $k(\eta)$ the fraction field of R, I the maximal ideal of R, q a generator (uniformizer) of I and $S = \mathrm{Spec}\ R$. Then we can construct similar degenerations of abelian varieties if we are given a lattice X, a sublattice Y of X of finite index and

$$a(x) \in k(\eta)^\times, \quad (x \in X)$$

such that the following conditions are satisfied

(i) $a(0) = 1$,
(ii) $b(x,y) := a(x+y)a(x)^{-1}a(y)^{-1}$ is a symmetric bilinear form on $X \times X$,
(iii) $B(x,y) := \mathrm{val}_q\, b(x,y)$ is positive definite,
(iv)* B is even and $\mathrm{val}_q\, a(x) = B(x,x)/2$.

We assume here a stronger condition (iv)* for simplicity.

These data do exist for any abelian scheme G_η if G_0 is a split torus. This is proved by Faltings-Chai [7].

Suppose that we are given an abelian scheme $(G_\eta, \mathcal{L}_\eta)$ and a polarization morphism

$$(16)\quad \lambda(\mathcal{L}_\eta) : G_\eta \to G^t_\eta := \mathrm{Pic}^0(G_\eta).$$

Then there exists the connected Néron model of G_η (resp. G^t_η), which we denote by G (resp. G^t). Then by finite base change if necessary we may assume G is semi-abelian, that is, an extension of an abelian scheme by an algebraic torus.

For simplicity, we assume

$$G_0 \text{ are } G^t_0 \text{ are split tori over } k(0) := R/qR. \tag{17}$$

Let

$$X = \mathrm{Hom}_{\mathrm{gp.sch.}}(G_0, \mathbf{G}_m), \quad Y = \mathrm{Hom}_{\mathrm{gp.sch.}}(G^t_0, \mathbf{G}_m). \tag{18}$$

Then both X and Y are lattices of rank g, and Y is a sublattice of X of finite index because $G_0 \to G^t_0$ is surjective. This case is called a totally degenerate case, that is, the case when $\mathrm{rank}_{\mathbf{Z}} X = \dim G_\eta$, which is what we mainly discuss here.

If G_0 is neither a torus nor an abelian variety, then the case is called a partially degenerate case. Also in the partially degenerate case we have degeneration data similar to the above $a(x)$ and $b(x, y)$, though a bit more complicated. This enables us to similarly construct a degenerating family of abelian varieties.

In what follows we consider the case where G_0 is a (split) torus $\mathbf{G}^g_{m,k(0)}$ over $k(0)$.

Lemma 6.1.1. *Let R be a CDVR, G a flat S-group scheme, and G_0 the closed fiber of G. Suppose that G_0 is a (split) torus $\mathbf{G}^g_{m,k(0)}$ over $k(0)$ for some g. Then the formal completion G_{for} of G along G_0 is isomorphic to a formal R-torus:*

$$G_{\mathrm{for}} \simeq \mathbf{G}^g_{m,R,\mathrm{for}} = \mathrm{Spf}\ R[[w^x; x \in X]]^{I\text{-}adic} \tag{19}$$

where X is a lattice of rank g.

Proof. Let $k = k(0)$. Let n be any nonnegative integer, $R_n = R/I^{n+1}$, $S_n = \mathrm{Spec}\ R_n$ and $G_n := G \times_S S_n$. By the assumption, $G_0 = \mathbf{G}^g_{m,k}$ for some g. Let $H := \mathbf{G}^g_{m,R,\mathrm{for}}$ (the formal torus over R) and $H_n = H \times_S S_n$. Hence $G_0 = H_0 = \mathbf{G}^g_{m,k}$. Let $f_0 : H_0 \to G_0$ be the identity $\mathrm{id}_{\mathbf{G}^g_{m,k}}$ of $\mathbf{G}^g_{m,k}$. Since $H_0 = \mathbf{G}^g_{m,R_0}$ is affine, the cohomology group $H^2(H_0, f_0^*\mathcal{L}ie(G_0/k))$ vanishes, where $\mathcal{L}ie(G_0/k)$ is the tangent sheaf of G_0, hence isomorphic to $O^g_{G_0}$, hence $f_0^*\mathcal{L}ie(G_0/k)) \simeq O^g_{H_0}$. By applying [6, I, Exposé III, Corollaire 2.8, p. 118] to H_1, G_1 and f_0, we see that f_0 can be uniquely lifted to an S_1-(homo)morphism $f_1 : H_1 \to G_1$ as S_1-group schemes. This lifting f_1 is an isomorphism because f_0 is an isomorphism. Similarly any isomorphism $f_n : H_n \to G_n$ as S_n-group schemes can be lifted again by [6, I, Exposé III, Corollaire 2.8, p. 118] to an S_{n+1}-isomorphism $f_{n+1} : H_{n+1} \to G_{n+1}$ as S_{n+1}-group schemes because H_n is affine, and the cohomology group $H^2(H_n, f_n^*\mathcal{L}ie(G_n/k))$ vanishes by the same argument as the $n = 0$ case. Hence $H_{\mathrm{for}} \simeq G_{\mathrm{for}}$ as S-group schemes. Q.E.D.

Lemma 6.1.2. *We have*

(1) *any line bundle on* $\mathbf{G}^g_{m,R,\mathrm{for}}$ *is trivial.*

(2) *any global section* $\theta \in \Gamma(G, \mathcal{L}^n)$ *is a formal power series of* w^x*, and we can write* θ *as*

$$\theta = \sum_{x \in X} \sigma_x(\theta) w^x \tag{20}$$

for some $\sigma_x(\theta) \in R$.

Proof. Let $R_n = R/I^{n+1}$, $A_n := R_n[w_i^{\pm 1}; i = 1, \cdots, g]$ and

$$G_n := \mathbf{G}^g_m \otimes R_n = \mathrm{Spec}\ A_n.$$

To prove the first assertion, it suffices to prove

(i) any line bundle L_0 on G_0 is trivial,

(ii) if a line bundle L on G_n is trivial on G_{n-1}, it is trivial on G_n.

Any line bundle L_0 on G_0 is linearly equivalent to $D - D'$ for some effective divisors D and D' on G_0. For proving (i) it suffices to prove that the line bundle $L' = [D]$ associated to any irreducible divisor D on G_0 is trivial. Since G_0 is affine, D is defined by a prime ideal $\mathfrak{p}$ of A_0 of height one. Since A_0 is a UFD, $\mathfrak{p}$ is generated by a single generator [19, Theorem 47, p. 141], hence it defines a trivial line bundle globally on G_0. This proves (i).

Next we prove (ii). Since G_n is an R_n-scheme, we can find an affine covering $U_j = \mathrm{Spec}\ B_j$ of G_n for some R_n-algebras B_j, and one cocycle $f_{jk} \in \Gamma(O_{U_{jk}})^\times$ (the units of $\Gamma(O_{U_{jk}})$) associated to the line bundle L on G_n such that

$$f_{ij} f_{jk} = f_{ik}. \tag{21}$$

By the assumption that L is trivial on G_{n-1} there exist $g_j \in B_j^\times$ such that $f_{ij} = g_i^{-1} g_j \mod I^n$. Let $g_{ij} = g_i g_j^{-1} f_{ij}$. Then g_{ij} is the one cocycle defining L on G_n such that $g_{ij} = 1 + a_{ij} q^n$ for some $a_{ij} \in B_{ij}$. By (21), we have $g_{ij} g_{jk} = g_{ik}$, hence

$$a_{ij} + a_{jk} = a_{ik} \quad \text{in } B_{ijk} \otimes R/I,$$

where $B_{ijk} = \Gamma(O_{U_i \cap U_j \cap U_k})$. Since $H^1(O_{G_0}) = 0$, we have $b_j \in B_i \otimes R_0$ such that $a_{ij} = -b_i + b_j$. Hence

$$g_{ij} = (1 + b_i q^n)^{-1}(1 + b_j q^n),$$

which defines the trivial line bundle on G_n. This proves (ii). Hence this completes the proof of the first assertion of Lemma 6.1.2. The second assertion of Lemma 6.1.2 follows easily from it. Q.E.D.

Theorem 6.1.3. *If G is totally degenerate, then by a suitable finite base change, there exist data $\{a(x); x \in X\}$ satisfying (i)-(iv)*. In terms of these data, we have using the expression (20)*

(v) *for any $n \geq 1$, $\Gamma(G_\eta, \mathcal{L}_\eta^n)$ is the $k(\eta)$ vector space of θ such that*

$$\sigma_{x+y}(\theta) = a(y)^n b(y, x) \sigma_x(\theta)$$

and $\sigma_x(\theta) \in k(\eta)$ for any $x \in X, y \in Y$.

The condition (v) enables us to prove the part (1) of Theorem 4.6.

6.2. Construction

So we may assume we are given the data $a(x)$ as above. Then we define $\mathcal{X}$, U_n $(n \in X)$, by

$$\begin{aligned} \mathcal{X} &= \mathrm{Proj}\ \widetilde{R}, \quad \widetilde{R} := R[a(x) w^x \vartheta; x \in X], \\ U_n &= \mathrm{Spec}\ R[a(x) w^x / a(n) w^n; x \in X] \\ &= \mathrm{Spec}\ R[(a(x)/a(n)) w^{x-n}], \end{aligned}$$

where $\widetilde{R}$ is a subring of $k(\eta)[w^x; x \in X][\vartheta]$ as in Subsec. 5.4, and $\mathcal{X}$ is a scheme locally of finite type, covered with open affine schemes U_n $(n \in X)$. Let $\mathcal{X}_{\mathrm{for}}$ be the formal completion of $\mathcal{X}$ along the special fiber.

We define $\mathcal{L}_{\mathcal{X}}$ to be the line bundle of $\mathcal{X}$ given by the homogeneous ideal of $\widetilde{R}$ generated by the degree one generator ϑ. We identify $X \times_{\mathbf{Z}} \mathbf{G}_{m,R}$ $(\simeq \mathbf{G}_{m,R}^g)$ with $\mathrm{Hom}_{\mathbf{Z}}(X, \mathbf{G}_{m,R})$. Then we have the actions S_z and T_β on $\mathcal{X}$ as follows:

$$\begin{aligned} S_z^*(a(x) w^x \vartheta) &= a(x+z) w^{x+z} \vartheta, \\ T_\beta^*(a(x) w^x \vartheta) &= \beta(x) a(x) w^x \vartheta, \text{ hence} \\ T_\beta^* S_z^*(a(x) w^x \vartheta) &= \beta(x+z) a(x+z) w^{x+z} \vartheta, \\ S_z^* T_\beta^*(a(x) w^x \vartheta) &= \beta(x) a(x+z) w^{x+z} \vartheta, \end{aligned}$$

where $z \in X$ and $\beta \in \mathrm{Hom}(X, \mathbf{G}_{m,R})$ $(\simeq \mathbf{G}_{m,R}^g)$. It follows that on $\mathcal{L}_{\mathcal{X}}$

$$S_z T_\beta = \beta(z) T_\beta S_z, \quad \text{or} \quad [S_z, T_\beta] = \beta(z)\,\mathrm{id}_{\mathcal{L}_{\mathcal{X}}}. \tag{22}$$

Let $Q_{\text{for}} := \mathcal{X}_{\text{for}}/Y := \mathcal{X}_{\text{for}}/\{S_y; y \in Y\}$:

$$\mathcal{X}_{\text{for}}/Y = (\text{Proj } R[a(x)w^x\vartheta, x \in X])_{\text{for}}/Y.$$

Then $\mathcal{L}_{\mathcal{X}}$ descends to the formal quotient Q_{for} as an ample sheaf. Hence by Grothendieck's algebraization theorem [10, III, **11**, 5.4.5] there exists a scheme $(Q, \mathcal{L})$ such that the formal completion of $(Q, \mathcal{L})_{\text{for}}$ is isomorphic to $(Q_{\text{for}}, \mathcal{L}_{\text{for}})$. This is $(Q, \mathcal{L}_Q)$ in Theorem 4.6.

Remark 6.2.1. For any connected R-scheme T, and for any T-valued points $x \in X(T) = X$ and $\beta \in \text{Hom}(X, \mathbf{G}_{m,R})(T)$, we have $\beta(x) \in \mathbf{G}_{m,R}(T) = \Gamma(O_T)^\times$. Any $\beta \in \text{Hom}(X, \mathbf{G}_{m,R})$ acts on $\mathcal{X}$ by T_β. It follows that the R-split torus $\text{Hom}(X, \mathbf{G}_{m,R})$ acts on $\mathcal{X}$ by T_β.

Definition 6.2.2. Let $H = X/Y$, $H^\vee := \text{Hom}(H, \mathbf{G}_m)$. We define $\mathcal{G}(Q, \mathcal{L}) = \mathcal{G}(P, \mathcal{L})$ to be the group generated by S_z and T_β ($z \in H = X/Y, \beta \in H^\vee$). Since $H^\vee$ is a subgroup of $\text{Hom}(X, \mathbf{G}_m)$, we infer from (22) that

$$S_z T_\beta = \beta(z) T_\beta S_z. \tag{23}$$

This is isomorphic to $\mathcal{G}_H$ in Definition 3.5 by mapping S_z (resp. T_β) to $(1, z, 0)$ (resp. $(1, 0, \beta)$). □

In what follows, we wish to prove Theorem 4.6 (1)

$$(P_\eta, \mathcal{L}_\eta) \simeq (Q_\eta, \mathcal{L}_\eta) \simeq (G_\eta, \mathcal{L}_\eta). \tag{24}$$

For doing so, we essentially need only the following.

Lemma 6.3. *With the notation in Subsec. 4.3 and Theorem 4.6, suppose $(K_\eta, e_{\text{Weil}}) \simeq (K, e_K)$ as symplectic groups. Let $Z = P$ or Q. Then there exists n_0 such that for any $n \geq n_0$ we have*

(1) $H^q(Z_0, \mathcal{L}_0^n) = H^q(Z, \mathcal{L}^n) = 0$ *for* $q \geq 1$,
(2) $\Gamma(Z_0, \mathcal{L}_0^n) = \Gamma(Z, \mathcal{L}^n) \otimes k(0)$ *is a* $k(0)$*-vector space rank* $n^g\sqrt{|K|}$,
(3) $\Gamma(P_\eta, \mathcal{L}_\eta^n) = \Gamma(P, \mathcal{L}^n) \otimes k(\eta)$,
(4) $\Gamma(P, \mathcal{L}) = \Gamma(Q, \mathcal{L})$, *which is a free* R*-module of rank* $\sqrt{|K|}$,
(5) *if* $e_{\min}(K) \geq 3$, *then* $\Gamma(Q, \mathcal{L})$ *is very ample on* Q.

Proof. This is a corollary to Serre's vanishing theorem except (4). See [30, Lemma 5.12] for (4). See [30, Lemma 6.3] for (5). Q.E.D.

6.4. Proof of $(P_\eta, \mathcal{L}_\eta) \simeq (Q_\eta, \mathcal{L}_\eta) \simeq (G_\eta, \mathcal{L}_\eta)$

By [30, Remark 3.10, p. 673] (see also [30, Remark 4.11, p. 679]), $\Gamma(P_\eta, \mathcal{L}_\eta^n)$ is a $k(\eta)$-submodule of $\Gamma(G_{\text{for}}, \mathcal{L}_{\text{for}}^n) \otimes k(\eta)$ given by

$$\left\{ \theta = \sum_{x \in X} c(x) w^x ; \begin{array}{c} c(x + ny) = b(y, x) a(y)^n c(x) \\ c(x) \in k(\eta), \text{any } x \in X, y \in Y \end{array} \right\}$$

where the I-adic convergence of θ is automatic by the condition

$$c(x+ny) = b(y,x)a(y)^n c(x).$$

This is the same as $\Gamma(G_\eta, \mathcal{L}_\eta^n)$ by Theorem 6.1.3. A $k(\eta)$-basis of $\Gamma(G_\eta, \mathcal{L}_\eta^n)$ is given for instance as $\theta_{\bar{x}}^{[n]}$ $(x \in X/nY)$

$$\theta_{\bar{x}}^{[n]} := \sum_{y\in Y} b(y,x)a(y)^n a(x) w^{x+ny} = \sum_{y\in Y} a(y)^{n-1} a(x+y) w^{x+ny}.$$

We choose $n \geq 4$ large enough so that $\mathcal{L}_\eta^n$ is very ample. Then the abelian variety G_η embedded by the linear system $\Gamma(G_\eta, \mathcal{L}_\eta^n)$ is given as the intersection of certain quadrics of $\theta_{\bar{x}}^{[n]}$ by [22, Theorem 10, p.80] (see also [40, Theorem 2.1, p. 717]). The coefficients of the defining equations are given by the Fourier coefficients of $\theta_{\bar{x}}^{[n]}$. This proves

$$(Q_\eta, \mathcal{L}_\eta) \simeq (P_\eta, \mathcal{L}_\eta) \simeq (G_\eta, \mathcal{L}_\eta).$$

where $(Q_\eta, \mathcal{L}_\eta) \simeq (P_\eta, \mathcal{L}_\eta)$ is clear. □

6.5. The Delaunay decompositions

Let X be a lattice of rank g and B a positive symmetric integral bilinear form on X associated with the degeneration data for $(\mathcal{Z}, \mathcal{L})$.

Definition 6.5.1. For a fixed $\lambda \in X \otimes_{\mathbf{Z}} \mathbf{R}$ fixed, we define a Delaunay cell σ to be the convex closure of all the integral vectors (which we call Delaunay vectors) attaining the minimum of the function

$$B(x,x) - 2B(\lambda, x) \quad (x \in X).$$

When λ ranges in $X \otimes_{\mathbf{Z}} \mathbf{R}$, we will have various Delaunay cells. Together, they constitute a locally finite polyhedral decomposition of $X \otimes_{\mathbf{Z}} \mathbf{R}$, invariant under the translation by X. We call this the Delaunay decomposition of $X \otimes_{\mathbf{Z}} \mathbf{R}$, which we denote by Del_B.

There are two types of Delaunay decomposition of $\mathbf{Z}^2 \otimes \mathbf{R} = \mathbf{R}^2$ inequivalent under the action of $\mathrm{SL}(2, \mathbf{Z})$. See Figure 4.

The Delaunay decomposition describes a PSQAS as follows.

Theorem 6.6. *Let $(Z, L) := (Q_0, \mathcal{L}_0)$ be a totally degenerate PSQAS, X the integral lattice, Y the sublattice of X of finite index and B the positive integral bilinear form on X all of which were defined in Subsec. 6.1. Let σ,τ be Delaunay cells in Del_B. Then*

(1) *for each σ there exists a subscheme $O(\sigma)$ of Z_{red}, which is a torus of dimension $\dim \sigma$ invariant under the action of the torus G_0,*

(2) *$\sigma \subset \tau$ iff $O(\sigma) \subset \overline{O(\tau)}$, where $\overline{O(\tau)}$ is the closure of $O(\tau)$ in Z,*
(3) *$\overline{O(\tau)}$ is the disjoint union of $O(\sigma)$ for all $\sigma \subset \tau$,*
(4) *$Z_{\mathrm{red}} = \bigcup_{\sigma \in \mathrm{Del}_B \bmod Y} O(\sigma)$,*
(5) *the local scheme structure of Z is completely described by B,*
(6) *L is ample, and it is very ample if $e_{\min}(X/Y) \geq 3$.*

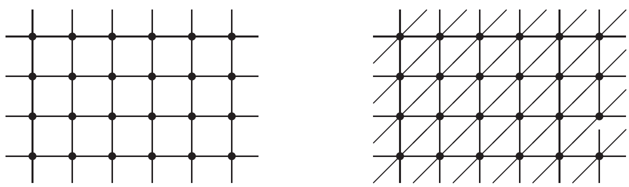

Fig. 4. Delaunay decompositions

We have similar descriptions of the partially degenerate PSQASes and of TSQASes $(P_0, \mathcal{L}_0)$ (see [2, p. 410] and [30, p. 678]).

6.7. The totally degenerate case in dimension two

We note that we learned more or less the same computation as this subsection in a letter of K. Ueno to Namikwa in 1972. We shall explain here what Figure 4 means geometrically.

We follow the construction in Subsec. 6.2. Let R be a CDVR with uniformizer q, $k(0) = R/qR$ and $X = \mathbf{Z}f_1 \oplus \mathbf{Z}f_2$ a lattice of rank two. Let ℓ and m be any positive integers, and set $Y = \mathbf{Z}\ell f_1 \oplus \mathbf{Z}mf_2$.

Case 6.7.1. Let $B(x) = x_1^2 + x_2^2$,

$$a(x) = q^{x_1^2+x_2^2} a^{2x_1x_2},\ b(x,y) = q^{2x_1y_1+2x_2y_2} a^{2x_1y_2+2y_1x_2}$$

where $a \in R^\times$, $x = x_1f_1 + x_2f_2$, $y = y_1f_1 + y_2f_2$. Then we define

$$\begin{aligned}
\mathcal{X} &= \mathrm{Proj}\ R[a(x)w^x\vartheta, x \in X], \\
U_n &= \mathrm{Spec}\ R[a(x)w^x/a(n)w^n, x \in X] \quad (n \in X) \\
&= \mathrm{Spec}\ R[(a(x)/a(n))w^{x-n}], \\
\mathcal{X}_{\mathrm{for}}/Y &= (\mathrm{Proj}\ R[a(x)w^x\vartheta, x \in X])_{\mathrm{for}}/Y.
\end{aligned}$$

Let $Q'_{\mathrm{for}} := \mathcal{X}_{\mathrm{for}}/Y$. Let $n = 0$ for simplicity. Then we have

$$\begin{aligned} U_0 &= \mathrm{Spec}\ R[a(f_1)w_1, a(f_2)w_2, a(-f_1)w_1^{-1}, a(-f_2)w_2^{-1}],\\ (U_0)_0 &= \mathrm{Spec}\ R[qw_1, qw_2, qw_1^{-1}, qw_2^{-1}] \otimes k(0)\\ &\simeq \mathrm{Spec}\ k(0)[u_1, u_2, v_1, v_2]/(u_1v_1, u_2v_2), \end{aligned}$$

where $(U_0)_0 = U_0 \otimes k(0)$. Hence $U_n \simeq U_0$ and

$$(U_n)_0 := \mathrm{Spec}\ k(0)[u_1^{(n)}, u_2^{(n)}, v_1^{(n)}, v_2^{(n)}]/(u_1^{(n)}v_1^{(n)}, u_2^{(n)}v_2^{(n)})$$

where $n = n_1f_1 + n_2f_2$, and

$$\begin{aligned} u_1^{(n)} &= q^{2n_1+1}w_1, u_2^{(n)} = q^{(2n_2+1)}w_2,\\ v_1^{(n)} &= q^{(-2n_1+1)}w_1^{-1}, v_2^{(n)} = q^{(-2n_2+1)}w_2^{-1}. \end{aligned}$$

These charts will be patched together to yield $(Q'_{\mathrm{for}})_0$.

This PSQAS $(Q'_{\mathrm{for}})_0$ is a union of ℓm copies of $\mathbf{P}^1 \times \mathbf{P}^1$, whose configuration is just the same as the Delaunay decomposition on the left hand side in Fig. 4. The first horizontal chain of ℓ rational curves is identified with the m-th horizontal chain of ℓ rational curves by shifting by multiplication by a^{2m} on each rational curve, while the first vertical chain of m rational curves is identified with the ℓ-th vertical chain of m rational curves by shifting by multiplication by $a^{2\ell}$ on each rational curve because

$$\begin{aligned} S^*_{mf_2}(w_1) &= b(f_1, mf_2)w_1 = a^{2m}w_1,\\ S^*_{\ell f_1}(w_2) &= b(\ell f_1, f_2)w_2 = a^{2\ell}w_2. \end{aligned}$$

The PSQAS $(Q'_{\mathrm{for}})_0$ is a level-$\mathcal{G}_H$ PSQAS. where $H = (\mathbf{Z}/\ell\mathbf{Z}) \oplus (\mathbf{Z}/m\mathbf{Z}) \simeq (\mathbf{Z}/e_1\mathbf{Z}) \oplus (\mathbf{Z}/e_2\mathbf{Z})$, with $e_1 = \mathrm{GCD}(\ell, m)$ and $e_2 = \ell m/e_1$.

Case 6.7.2. Let $B(x) = x_1^2 - x_1x_2 + x_2^2$,

$$a(x) = q^{x_1^2 - x_1x_2 + x_2^2},\ b(x, y) = q^{2x_1y_1 - x_1y_2 - x_2y_1 + 2x_2y_2}$$

where $x = x_1f_1 + x_2e_2$, $y = y_1f_1 + y_2e_2$. Then we define

$$\begin{aligned} \mathcal{X} &= \mathrm{Proj}\ R[a(x)w^x\vartheta, x \in X],\\ U_n &= \mathrm{Spec}\ R[a(x)w^x/a(n)w^n, x \in X] \quad (n \in X)\\ &= \mathrm{Spec}\ R[(a(x)/a(n))w^{x-n}],\\ \mathcal{X}_{\mathrm{for}}/Y &= (\mathrm{Proj}\ R[a(x)w^x\vartheta, x \in X])_{\mathrm{for}}/Y. \end{aligned}$$

Let $Q''_{\mathrm{for}} := \mathcal{X}_{\mathrm{for}}/Y$. Let $n = 0$ for simplicity. Then we have

$$\begin{aligned} U_0 &= \mathrm{Spec}\ R[qw_1, qw_1w_2, qw_2, qw_1^{-1}, qw_1^{-1}w_2^{-1}, qw_2^{-1}] \\ &\simeq \mathrm{Spec}\ k(0)[u_i; 0 \le i \le 5]/(u_{i-1}u_{i+1} - qu_i, u_iu_{i+3} - q^2) \\ (U_0)_0 &\simeq \mathrm{Spec}\ k(0)[u_i; 0 \le i \le 5]/(u_iu_j; |i - j \,(\mathrm{mod}\ 6)| \ge 2), \end{aligned}$$

where $(U_0)_0 = U_0 \otimes k(0)$.

We have a PSQAS $(Q''_{\mathrm{for}})_0$. This PSQAS $(Q''_{\mathrm{for}})_0$ is a union of ℓm copies of $\mathbf{P}^2$, whose configuration is just the same as the Delaunay decomposition on the right hand side in Fig. 4. The first horizontal chain of ℓ rational curves is identified with the m-th horizontal chain of ℓ rational curves without shifting on each rational curve, while the first vertical chain of m rational curves is identified with the ℓ-th vertical chain of m rational curves without shifting on each rational curve. The PSQAS $(Q''_{\mathrm{for}})_0$ is a level-$\mathcal{G}_H$ PSQAS for $H = (\mathbf{Z}/\ell\mathbf{Z}) \oplus (\mathbf{Z}/m\mathbf{Z})$.

Remark 6.7.3. Gunji [12] studied the defining equations of the universal abelian surface with level three structure. His universal abelian surface is the same as our universal PSQAS over the moduli space $SQ_{2,K}$ when $K = H \oplus H^\vee$, $H = (\mathbf{Z}/3\mathbf{Z})^{\oplus 2}$ and the base field is $\mathbf{C}$. He proved that the level three universal abelian surface is the intersection of 9 quadrics and 4 cubics of $\mathbf{P}^8 \times_{\mathcal{O}_3} SQ_{2,K} \times_{\mathcal{O}_3} \mathbf{C}$ [12, Theorem 8.3]. In his article Gunji determines the fibers only partially [12, pp. 95-96].

By our study [30, Theorem 11.4] (Theorem 8.5), any fiber of the universal PSQAS over $SQ_{2,K}$ is a smooth abelian surface, or a cycle of 3 rational elliptic surfaces in Subsec. 5.5, with $\ell = m = 3$, or else one of the singular surfaces in Cases 6.7.1 or 6.7.2 with $\ell = m = 3$.

Remark 6.7.4. Here we explain only a little about the local structure of $SQ_{g,K}$ for $g = 2$. It turns out that the local structure of $SQ_{g,K}$ is the same as that of a toroidal compactification, the second Voronoi compactification.

Let X be a lattice of rank two, $B(x)$ the bilinear form on X given in Case 6.7.2

$$B(x) = x_1^2 - x_1x_2 + x_2^2.$$

The Voronoi cone V_B with center B is defined to be

$$\begin{aligned} V_B &:= \{\beta : \text{positive definite bilinear form on } X \text{ with } \mathrm{Del}_\beta = \mathrm{Del}_B\} \\ &= \left\{\beta(x) := (\beta_{12} + \beta_{13})x_1^2 - 2\beta_{12}x_1x_2 + (\beta_{12} + \beta_{23})x_2^2; \beta_{ij} > 0\right\}. \end{aligned}$$

We define a chart T and a semi-universal covering $\mathcal{X}$ over T to be

$$T := T(V_B) := \mathrm{Spf}\ W(k)[[q_{ij}; i < j]],$$
$$\mathcal{X} = \mathrm{Proj}\ W(k)[[q_{ij}; i < j]][a(x)w^x\vartheta; x \in X]$$

where $W(k)$ is the Witt ring of k , $q_{ij} = q^{\beta_{ij}}$ $(1 \le i < j \le 3)$ and

$$a(x) := q^{\beta(x)} := (q_{13}^{x_1^2})(q_{23}^{x_2^2})(q_{12}^{x_1^2 - 2x_1x_2 + x_2^2}).$$

Let $\mathcal{L}_{\mathcal{X}}$ be the invertible sheaf $O_{\mathcal{X}}(1)$ on $\mathcal{X}$. We define the action of the lattice X on $\mathcal{X}$ by

$$S_z^*(a(x)w^x\vartheta) = a(x+z)w^{x+z}\vartheta.$$

Let $(\mathcal{X}_{\mathrm{for}}, \mathcal{L}_{\mathrm{for}})$ be the formal completion of $(\mathcal{X}, \mathcal{L}_{\mathcal{X}})$ along the closed subscheme $\mathcal{X}_0$ of $\mathcal{X}$ given by $q_{ij} = 0$. Let Y be a sublattice of X of finite index. We take the formal quotient of $\mathcal{X}_{\mathrm{for}}$ by Y

$$(Q_{\mathrm{for}}, \mathcal{L}_{\mathrm{for}}) := (\mathcal{X}_{\mathrm{for}}, \mathcal{L}_{\mathrm{for}})/Y,$$

where $Q_{\mathrm{for}} \otimes k(0) \simeq Q_0''$ if Y is the same as in Case 6.7.2. Moreover $(Q_{\mathrm{for}}, \mathcal{L}_{\mathrm{for}})$ is a semi-universal PSQAS over T. In other words, the deformation functor of $(Q_{\mathrm{for}}, \mathcal{L}_{\mathrm{for}}) \otimes k(0)$ is pro-represented by $W(k)[[q_{ij}; i < j]]$. Compare [27] and Subsec 9.3.

Let $\tau = (\tau_{ij})$ be a 2×2 complex symmetric matrix with positive imaginary part, and set

$$q_{12} = e^{-2\pi i \tau_{12}}, q_{13} = e^{2\pi i(\tau_{11} + \tau_{12})}, q_{23} = e^{2\pi i(\tau_{12} + \tau_{22})}.$$

These are regular parameters of $SQ_{2,K}$ at $(Q_{\mathrm{for}}, \mathcal{L}_{\mathrm{for}})$ for any K with $e_{\min}(K) \ge 3$. This is also an infinitesimally local chart of the Mumford toroidal compactification, which is in this case the so-called Voronoi compactification, or to be a little more precise, the Mumford toroidal compactification associated to the second Voronoi decomposition and some arithmetic subgroup of $\mathrm{Sp}(4, \mathbf{Z})$. See [36].

6.8. Nonseparatedness of a naive moduli

We shall explain here how a naive generalization of classical level-n structure results in a nonseparated compactification of the moduli of abelian varieties. See [27].

In three dimensional case, let X be a lattice of rank 3. We choose

$$B = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

The level-1 PSQASes $(P_0, \mathcal{L}_0)$ associated to B are parameterized by 3 nontrivial parameters [27, p. 197].

Let Del_B /X be the quotient of the Delaunay decomposition Del_B by the translation action of X. Then Del_B /X consists of three three-dimensional cells (two tetrahedra and an octahedron), eight two-dimensional cells and six one-dimensional cells and a 0-dimensional cell [27, pp. 195-196]. Each level-1 PSQAS $(P_0, \mathcal{L}_0)$ has three irreducible components, two (say, T_1, T_2) of which are $\mathbf{P}^3$ (modulo X action) and the third (say, O) of which is a rational variety distinct from $\mathbf{P}^3$. Each of the three irreducible components is a compactification of $\mathbf{G}_m^3$.

It follows that there are two different types (modulo $\mathrm{Aut}(P_0)$) of embedding of $\mathbf{G}_m^3$ into $(P_0, \mathcal{L}_0)$, that is, $\mathbf{G}_m^3 \subset T_k$ and $\mathbf{G}_m^3 \subset O$. Therefore there is a pair of R-PSQASes $(P', \mathcal{L}')$ and $(P'', \mathcal{L}'')$ such that

$$(P'_\eta, \mathcal{L}'_\eta) \simeq (P''_\eta, \mathcal{L}''_\eta), \ (\mathbf{G}_m^3 \subset P'_0) \not\simeq (\mathbf{G}_m^3 \subset P''_0).$$

This also implies that there are two inequivalent classes of classical level-n structures on the étale $(\mathbf{Z}/n\mathbf{Z})^3$-covering $(P'_0, \mathcal{L}'_0)$ of $(P_0, \mathcal{L}_0)$ as the limits of the same (isomorphic) generic fiber. This shows that a naive generalization of classical level-n structure will lead us to a nonseparated moduli.

§7. The G-action and the G-linearization

Let G be a group (scheme). The purpose of this section is to prove compatibility of various definitions about G-linearization.

7.1. The G-linearization

Definition 7.1.1. *A G-linearization on* (Z, L) is by definition the data $\{(T_g, \phi_g); g \in G\}$ satisfying the conditions

(i) T_g is an automorphism of Z, such that $T_{gh} = T_g T_h$, $T_1 = \mathrm{id}_Z$,
(ii) $\phi_g : \mathcal{L} \to T_g^*(\mathcal{L})$ is a bundle isomorphism with $\phi_1 = \mathrm{id}_L$,
(iii) $\phi_{gh} = (T_h^* \phi_g)\phi_h$ for any $g, h \in G((T)$.

We say that (Z, L) is *G-linearized* if the above conditions are true.

Remark 7.1.2. If L and L' are G-linearized, then $L \otimes L'$ is also G-linearized.

Definition 7.1.3. If (Z, L) is G-linearized, then we define a G-action τ on the pair (Z, L). Via the isomorphism $L \xrightarrow{\phi_h} T_h^*(L)$, for $x \in Z$, $\zeta \in L_x$, we define

$$\tau(h)(z, \zeta) := (T_h(z), \phi_h(z)\zeta). \tag{25}$$

Claim 7.1.4. *τ is an action of G on (Z, L).*

Proof. Via the isomorphisms

$$L \xrightarrow{\phi_h} T_h^*(L) \xrightarrow{T_h^*\phi_g} T_h^*(T_g^*(L)) = T_{gh}^*(L),$$

we see

$$\begin{aligned}\tau(g)\,(\tau(h)(z,\zeta)) &= \tau(g)\cdot(T_h(z), \phi_h(z)\zeta)\\ &= (T_g(T_h(z)), \phi_g(T_h(z))\phi_h(z)\zeta)\\ &= (T_{gh}(z), (T_h^*\phi_g\cdot\phi_h)(z)\zeta)\\ &= (T_{gh}(z), \phi_{gh}(z)\cdot\zeta) = \tau(gh)(z,\zeta).\end{aligned}$$

Hence τ is an action of G. Q.E.D.

Finally we note that if we are given an action τ of G on the pair (Z, L) of a scheme Z and a line bundle L on Z, then we have a G-linearization of L. In fact, τ is an action of G iff $T_{gh} = T_gT_h$ and $\phi_{gh} = T_h^*\phi_g \cdot \phi_h$.

Claim 7.1.5. ([20, p. 295]) *Associated to a given G-action τ on (Z, L), we define a map $\rho_{\tau,L}(g)$ of $H^0(Z, L)$ to be*

(26) $\rho_{\tau,L}(g)(\theta) := T_{g^{-1}}^*(\phi_g(\theta))$ *for any $g \in G$ and any $\theta \in H^0(Z, L)$.*

Then $\rho_{\tau,L}$ is a homomorphism.

Proof. We see

$$\begin{aligned}\rho_{\tau,L}(gh)(\theta) &= T_{h^{-1}g^{-1}}^*(\phi_{gh}\theta) = T_{g^{-1}}^*\{T_{h^{-1}}^*(T_h^*\phi_g\cdot\phi_h\theta)\}\\ &= T_{g^{-1}}^*\{T_{h^{-1}}^*(T_h^*\phi_g)\cdot(T_{h^{-1}}^*\phi_h\theta)\}\\ &= T_{g^{-1}}^*\{\phi_g\cdot(T_{h^{-1}}^*\phi_h\theta)\} = \rho_{\tau,L}(g)\rho_{\tau,L}(h)(\theta).\end{aligned}$$

Q.E.D.

7.2. The G-linearization of $O_{\mathbf{P}(V)}(1)$

Let R be any ring. Suppose we are given an action of a group G on an R-free module V of finite rank, in other words, a homomorphism $\rho : G \to \mathrm{End}\,(V)$. Let $V^\vee := \mathrm{Hom}(V, R)$ be the dual of V, $\mathbf{P}(V)$ the projective space with $V = H^0(\mathbf{P}(V), O_{\mathbf{P}(V)}(1))$, $\mathbf{H} = O_{\mathbf{P}(V)}(1)$ the hyperplane bundle of $\mathbf{P}(V)$. Then $V^\vee$ admits a natural affine R-scheme structure $\mathbf{V}^\vee$ defined by

$$\mathbf{V}^\vee = \mathrm{Spec}\ \mathrm{Sym}\,V := \mathrm{Spec}\bigoplus_{n=0}^{\infty} S^nV.$$

The action ρ of G on V induces an action of G on S^nV, hence on $\mathrm{Sym}\, V$, hence on $\mathbf{V}^\vee$, hence on the pair $(\mathbf{P}(V), \mathbf{V}^\vee - \{0\})$ of schemes. We note that $\mathbf{V}^\vee - \{0\}$ is a $\mathbf{G}_m$-bundle over $\mathbf{P}(V)$ associated with the dual of the hyperplane bundle $\mathbf{H}$ of $\mathbf{P}(V)$. Hence the action ρ of G on V induces the action on the pair $(\mathbf{P}(V), \mathbf{H})$ of schemes.

Let S be any R-scheme and $P \in \mathbf{P}(V)(S)$ any S-valued point. By choosing affine coverings $U_i := \mathrm{Spec}\ A_i$ of S if necessary, P is a collection of $P_i \in \mathbf{P}(V)(U_i)$ of (the equivalence class of) the points given by

$$\gamma_{P_i} \in \mathrm{Hom}(V, A_i)$$

such that the ideal of A_i generated by $\gamma_P(V)$ is A_i, where $\gamma_{P_i} \sim \gamma_{Q_i}$ iff $\gamma_{Q_i} = c\gamma_{P_i}$ for some $c \in A_i^\times$. Hence there are $c_{ij} \in A_{ij}^\times := \Gamma(O_{U_i \cap U_j})^\times$ such that $\gamma_{P_i} = c_{ij}\gamma_{P_j}$. In what follows, we suppose $S = U_i$ for simplicity and we identify P with γ_P.

We define an action of G on $(\mathbf{P}(V), \mathbf{V}^\vee \setminus \{0\})$ by

$$S^\vee(g)([\gamma_P], \gamma_P) := ([\gamma_P \circ \rho(g^{-1})], \gamma_P \circ \rho(g^{-1})). \tag{27}$$

Then we see,

$$\begin{aligned} S^\vee(gh)(\gamma_P) &= \gamma_P \circ \rho((gh)^{-1}) = \gamma_P \circ \rho(h^{-1})\rho(g^{-1}) \\ &= S^\vee(h)(\gamma_P)\rho(g^{-1}) = S^\vee(g)S^\vee(h)(\gamma_P). \end{aligned}$$

Thus we have an action of G on the pair $(\mathbf{P}(V), \mathbf{V}^\vee \setminus \{0\})$ by $\mathbf{G}_m$-bundle automorphisms.

Definition 7.2.1. The action $S^\vee(g)$ of $g \in G$ on $(\mathbf{P}(V), \mathbf{V}^\vee \setminus \{0\})$ induces an action on $(\mathbf{P}(V), \mathbf{H})$, which we denote by $S(g)$.

Remark 7.3. Let R be any ring, V an R-free module of finite rank, and $\rho : G \to \mathrm{End}\,(V)$ an action of G on V. Let $V^\vee := \mathrm{Hom}(V, R)$ and $\langle\ ,\ \rangle : V^\vee \times V \to R$ the dual pairing. Using this pairing we have a dual action ${}^t\rho$ of G on $V^\vee$ such that

$$\langle {}^t\rho(g)\gamma, F\rangle := \langle \gamma, \rho(g)F\rangle,$$

where $\gamma \in V^\vee$, and $F \in V$. Then ${}^t\rho(gh) = {}^t\rho(h){}^t\rho(g)$. Thus this is made into a left action of G on $\mathbf{P}(V)$ by taking $T_g(\gamma) := {}^t\rho(g^{-1})(\gamma)$. This T_g is the same as $S^\vee(g)$ in Subsec. 7.2 because

$$\begin{aligned} T_g(\gamma)(F) &= \langle {}^t\rho(g^{-1})(\gamma), F\rangle = \langle \gamma, \rho(g)^{-1}F\rangle \\ &= \gamma(\rho(g^{-1})F) = S^\vee(g)(\gamma)(F). \end{aligned}$$

Since we have the action T_g on $\mathbf{P}(V)$, Claim 7.1.5 defines a homomorphism $\rho_{T,\mathbf{H}}$ (well known as the contragredient representation of T_g). Then we have

$$\begin{aligned}(\rho_{T,\mathbf{H}}(g)F)(\gamma) :&= F(T_{g^{-1}}\gamma) = F({}^t\rho(g)\gamma)\\ &= \langle {}^t\rho(g)\gamma, F\rangle = \langle \gamma, \rho(g)F\rangle = (\rho(g)F)(\gamma),\end{aligned}$$

where $x \in V^\vee \setminus \{0\}$, $F \in V$. Hence $\rho_{T,\mathbf{H}} = \rho$.

This justifies our notation (C, i, U_H) (resp. (Z, i, U_H)) in Lemma 3.12 (resp. in Theorem 8.5) where we indicate the action on (C, L) or (Z, L) induced from U_H simply by U_H.

7.4. G-invariant closed subschemes

Let R be any ring, V an R-free module of finite rank, and G any subgroup of $\mathrm{PGL}(V)$. If Z be a G-invariant closed subscheme of $\mathbf{P}(V)$ with $L = O_Z(1)$, then the G-action of $(\mathbf{P}(V), \mathbf{H})$ keeps (Z, L) stable, hence we have an action of G on the pair (Z, L). This gives rise to a G-linearization of (Z, L).

Conversely

Claim 7.4.1. *Let (Z, L) be an R-scheme with L a G-linearized line bundle on Z, and V a G-submodule of $H^0(Z, L)$.* Suppose that V is R-free of finite rank and very ample. *Then the natural morphism (ψ, Ψ) : $(Z, L) \to (\mathbf{P}(V), \mathbf{H})$ is a G-equivariant closed immersion.*

This is a corollary to the following

Claim 7.4.2. *Let (Z, L) be an R-scheme with L a G-linearized line bundle on Z, and V a G-submodule of $H^0(Z, L)$.* Suppose that V is R-free of finite rank and base point free. *Then*

(1) *there is a G-action S on $(\mathbf{P}(V), \mathbf{H})$ in Subsec. 7.2,*
(2) *the natural morphism $(\psi, \Psi) : (Z, L) \to (\mathbf{P}(V), \mathbf{H})$ is G-equivariant.*

Proof. By Claim 7.1.5, $H^0(X, L)$ is a G-module. By the assumption V is a G-submodule of $H^0(X, L)$. Then by Subsec. 7.2 we have a G-action S on $(\mathbf{P}(V), \mathbf{H})$. With the notation in Subsec. 7.2, we define the map ψ by $\gamma_{\psi(z)}(\theta) = \theta(z)$ for $\theta \in V = H^0(Z, L)$. This defines a natural map $(\psi, \Psi) : (Z, L) \to (\mathbf{P}(V), \mathbf{H})$ because $L = \psi^*\mathbf{H}$. We prove that with respect to the G-actions τ on (Z, L) and S on $(\mathbf{P}(V), \mathbf{H})$, (ψ, Ψ) is G-equivariant. Let $(z, \zeta) \in (Z, L)$ and $P = \psi(z)$. Then we have

$$\tau(g)(z, \zeta) = (T_g(z), \phi_g(z)\zeta),\ (\psi, \Psi)(z, \zeta) = (\psi(z), \zeta). \tag{28}$$

Since $(T_g^*\phi_{g^{-1}})\phi_g = \phi_1 = \mathrm{id}_L$ by Definition 7.1.1 (iii), we see

$$\begin{aligned}\gamma_{\psi(z)} \circ \rho_L(g^{-1})(\theta) &= \gamma_{\psi(z)}(T_g^*(\phi_{g^{-1}}\theta))\\ &= (T_g^*\phi_{g^{-1}}(z)T_g^*(\theta)(z) = \phi_g^{-1}(z)T_g^*(\theta)(z)\\ &= \phi_g(z)^{-1}\theta(T_g z) = \phi_g(z)^{-1}\gamma_{\psi(T_g z)}(\theta),\end{aligned}$$

whence $[\gamma_{\psi(z)} \circ \rho_L(g^{-1})] = [\gamma_{\psi(T_g z)}] = \psi(T_g z)$. By (27), regarding ζ^{-1} as the (rational) fiber coordinate of $L^\vee$, we have

$$\begin{aligned}S^\vee(g)(\psi,\Psi)(z,\zeta^{-1}) &= ([\gamma_{\psi(z)} \circ \rho_L(g^{-1})], \gamma_{\psi(z)} \circ \rho_L(g^{-1})\zeta^{-1})\\ &= ([\gamma_{\psi(T_g z)}], \gamma_{\psi(T_g z)}\phi_g(z)^{-1}\zeta^{-1}),\end{aligned}$$

whence the fiber coordinate ζ^{-1} is transformed into $\phi_g(z)^{-1}\zeta^{-1}$ because $\psi(z)$ (resp. $\psi(T_g z)$) is a generator of the fiber of $\mathbf{H}$. Hence $S^\vee(g)$ induces the transformation $\zeta \mapsto \phi_g(z)\zeta$ on L. Thus with the notation of (28)

$$\begin{aligned}S(g)(\psi,\Psi)(z,\zeta) &= ([\gamma_{\psi(z)} \circ \rho_L(g^{-1})], \phi_g(z)\zeta) = (\psi(T_g(z)), \phi_g(z)\zeta)\\ &= (\psi,\Psi)(T_g(z), \phi_g(z)\zeta) = (\psi,\Psi)\tau(g)(z,\zeta).\end{aligned}$$

This proves that (ψ,Ψ) is G-equivariant. Q.E.D.

7.5. The G-linearization in down-to-earth terms

We quote this part from [32, p.94]. The following enables us to understand G_H-linearization in down-to-earth terms.

Claim 7.5.1. *Let $T = \mathrm{Spec}\ R$, and G a finite group. Let Z be a positive-dimensional R-flat projective scheme. L an ample G-linearized line bundle on Z. Then for any point $z \in Z$, there exists a G-invariant open affine R-subscheme U of Z such that $z \in U$ and L is trivial on U.*

Proof. See [32, Lemma 4.9]. Q.E.D.

Let $T = \mathrm{Spec}\ R$ be any affine scheme, and G a finite group. Let Z be a positive-dimensional T-flat projective scheme. Let $m : G \times_R G \to G$ be the multiplication of G, and $\sigma : G \times_R Z \to Z$ an action of G on Z. Let L be an ample G-linearized line bundle on Z. The action σ satisfies the condition:

$$\sigma(m \times \mathrm{id}_Z) = \sigma(\mathrm{id}_G \times \sigma). \tag{29}$$

Now we shall give a concrete description of the G-linearization of (Z, L) by using a nice open affine covering of Z. By Claim 7.5.1, we can choose an affine open covering $U_j := \mathrm{Spec}\,(R_j)$ $(j \in J)$ of Z such that each U_j is G-invariant and the restriction of L is trivial on each U_j.

The induced bundles $\sigma^* L$, (resp. $(\mathrm{id}_G \times \sigma)^* \sigma^*(L)$, $(m \times \mathrm{id}_Z)^* \sigma^*(L)$) are all trivial on $G \times_R U_j$ (resp. $G \times_R G \times_R U_j$ or $G \times_R G \times U_j$) with the same fiber-coordinate as L_{U_j}. Let ζ_j be a fiber-coordinate of L_{U_j}.

Now we assume that G is a constant finite group (scheme over T). Since G is affine, let $A_G := \Gamma(G, O_G)$ be the Hopf algebra of G. See [44]. Then the isomorphism $\Psi : p_2^* L \to \sigma^*(L)$ over U_j is multiplication by a unit $\psi_j(g, x) \in (A_G \otimes_R R_j)^\times$ at $(g, x) \in G \times_R U_j$. Let $A_{jk}(x)$ be the one-cocycle defining L. Then $\sigma^*(L)$ is defined by the one-cocycle $\sigma^* A_{jk}(x)$. Hence $\Psi : p_2^* L \to \sigma^*(L)$ over U_j and U_k are related by

$$\psi_j(g, x) = \frac{A_{jk}(gx)}{A_{jk}(x)} \psi_k(g, x).$$

This is the condition (ii) of Definition 7.1.1. The condition (iii) of Definition 7.1.1 is expressed as

$$\psi_j(gh, x) = \psi_j(g, hx) \psi_j(h, x).$$

§8. The moduli schemes $A_{g,K}$ and $SQ_{g,K}$

Let $H = \bigoplus_{i=1}^{g} (\mathbf{Z}/e_i\mathbf{Z})$ be a finite Abelian group with $e_i | e_{i+1}$, $e_{\min}(H) := e_1$, $K = H \oplus H^\vee$, $N = |H| = \prod_{i=1}^{g} e_i$ and $\mathcal{O}_N = \mathbf{Z}[\zeta_N, 1/N]$. The purpose of this section is to construct two schemes, projective (resp. quasi-projective) $SQ_{g,K}$ (resp. $A_{g,K}$). We will see later that $A_{g,K}$ is the fine moduli scheme of abelian varieties, which is a Zariski open subset of the projective scheme $SQ_{g,K}$. As a (geometric) point set, $SQ_{g,K}$ is the set of all GIT-stable degenerate abelian schemes (Theorem 14.1.3).

Theorem 8.1. *Let $V_H := \bigoplus_{\mu \in H^\vee} \mathcal{O}_N v(\mu)$. Let (Z, L) be a PSQAS over $k(0)$, $(Q, \mathcal{L})$ a PSQAS over a CDVR R with $\ker \lambda(\mathcal{L}) \simeq K$ such that $(Z, L) \simeq (Q, \mathcal{L}) \otimes k(0)$ and the generic fiber $(Q_\eta, \mathcal{L}_\eta)$ is an abelian variety. Let $\mathcal{V}_0 := \Gamma(Q, \mathcal{L}) \otimes k(0)$. Then*

(1) *$\dim_{k(0)} \mathcal{V}_0 = |H|$, and $\mathcal{V}_0 \simeq V_H \otimes k(0)$ as $\mathcal{G}_H$-modules,*
(2) *$\mathcal{V}_0$ is uniquely determined by (Z, L), and independent of the choice of $(Q, \mathcal{L})$,*
(3) *if $e_{\min}(H) \geq 3$, then both $\Gamma(Q, \mathcal{L})$ and $\mathcal{V}_0$ are very ample,*
(4) *if $e_{\min}(H) \geq 3$, then (Z, L) is embedded $\mathcal{G}_H$-equivariantly into $(\mathbf{P}(V_H), \mathbf{H})$ by the linear subspace $\mathcal{V}_0$ via the isomorphism $\mathcal{V}_0 \simeq V_H \otimes k(0)$ as $\mathcal{G}_H$-modules.*

Proof. By Theorem 4.6, there exists a CDVR R and a projective flat morphism $\pi : (Q, \mathcal{L}) \to \mathrm{Spec}\, R$ (resp. $\pi : (P, \mathcal{L}) \to \mathrm{Spec}\, R$) such that $(Q_0, \mathcal{L}_0) \simeq (Q, \mathcal{L}) \otimes k(0)$, and P is the normalization of Q with P_0

reduced. Then by [30, Theorems 3.9 and 4.10], for instance, here in the totally degenerate case, we have

$$\Gamma(P_0,\mathcal{L}_0)=\left\{\sum_{\bar{x}\in X/Y} c(\bar{x})\sum_{y\in Y} a(x+y)w^{x+y}\otimes k(0); c(\bar{x})\in k(0)\right\},$$

$$\Gamma(P,\mathcal{L})=\left\{\sum_{\bar{x}\in X/Y} c(\bar{x})\sum_{y\in Y} a(x+y)w^{x+y}; c(\bar{x})\in R\right\},$$

where $\bar{x}$ is the class of $x \mod Y$. Hence $\Gamma(Q,\mathcal{L})=\Gamma(P,\mathcal{L})$ because $\Gamma(Q,\mathcal{L})$ is an R-submodule of $\Gamma(P,\mathcal{L})$, and any of the generators of $\Gamma(P,\mathcal{L})$ belongs to $\Gamma(Q,\mathcal{L})$ by the construction in Subsec. 6.2. Hence

$$\mathcal{V}_0:=\Gamma(Q,\mathcal{L})\otimes k(0)=\Gamma(P,\mathcal{L})\otimes k(0)=\Gamma(P_0,\mathcal{L}_0),$$

By [32, Corollary 3.9] $(P_0,\mathcal{L}_0)$ is uniquely determined by $(Q_0,\mathcal{L}_0)$, whence $\mathcal{V}_0$ is independent of the choice of $(Q,\mathcal{L})$. This proves (2).

This $\mathcal{V}_0$ is very ample and of rank $|H|$ by Lemma 6.3 (5) if $e_{\min}(H)\geq 3$. Hence so is $\Gamma(Q,\mathcal{L})$. Since (Z,L), hence $(Q_0,\mathcal{L}_0)$, hence $(P_0,\mathcal{L}_0)$ admit a $\mathcal{G}_H$-action, $\mathcal{V}_0=\Gamma(P_0,\mathcal{L}_0)$ is a $\mathcal{G}_H$-module. Hence by Claim 7.4.1, (Z,L) is *embedded $\mathcal{G}_H$-equivariantly into* $(\mathbf{P}(V_H),\mathbf{H})$. Q.E.D.

Definition 8.1.1. Let $(Z,L)=(Q_0,\mathcal{L}_0)$ be a $k(0)$-PSQAS. We call $\mathcal{V}_0$ *a characteristic subspace* of $\Gamma(Z,L)$, and denote $\mathcal{V}_0$ by $V(Z,L)$. This $\mathcal{V}_0$ is uniquely determined by (Z,L) because $\mathcal{V}_0=\Gamma(P_0,\mathcal{L}_0)$ and $(P_0,\mathcal{L}_0)$ is uniquely determined by $(Z,L)=(Q_0,\mathcal{L}_0)$.

Remark 8.1.2. In connection with the GIT-stability of (Z,L), it is more important to know whether $V(Z,L)$ is very ample than to know whether L (that is, $\Gamma(Z,L)$) is very ample. See [30, Theorem 11.6] and Theorem 14.1.3. However [30, p. 697] conjectures $V(Z,L)=\Gamma(Z,L)$.

Definition 8.1.3. Let k be an algebraically closed field with $k\ni 1/N$ and H a finite Abelian group with $|H|=N$. Let (A,L) be an abelian variety over k. Then we define $\mathcal{G}(A,L)$ to be the bundle automorphism group which induces translations of A by $\ker(\lambda(L))$. If $\ker(\lambda(L))\simeq K:=H\oplus H^{\vee}$, then $\mathcal{G}(A,L)\simeq\mathcal{G}_H$ by Lemma 4.2.

Let $K(A,L):=\ker(\lambda(L))=\mathcal{G}(A,L)/\mathbf{G}_m$.

Remark 8.1.4. Let k be an algebraically closed field with $k\ni 1/N$ and (Z,L) any PSQAS over k. Hence there exists a PSQAS $(Q,\mathcal{L})$ over a CDVR R such that $(Z,L)\simeq(Q_0,\mathcal{L}_0)$ and the generic fiber $(Q_\eta,\mathcal{L}_\eta)$ is an abelian variety with $\ker\lambda(\mathcal{L}_\eta))\simeq K=H\oplus H^{\vee}$. Then the natural $\mathcal{G}_H$-action ($=\mathcal{G}(Q_\eta,\mathcal{L}_\eta)$) on $(Q_\eta,\mathcal{L}_\eta)$ extends to that on $(Q,\mathcal{L})$, whose restriction to $(Q_0, \mathcal{L}_0)$ is the $\mathcal{G}_H$-action on (Z, L). We denote by $\mathcal{G}(Z, L)$ the $\mathcal{G}_H$-action on (Z, L). This is determined by (Z, L) uniquely up to an automorphism of $\mathcal{G}_H$. Let $K(Z, L) := \mathcal{G}(Z, L)/\mathbf{G}_m$.

Definition 8.1.5. Let (Z, L) be a PSQAS over k. We call the action $\tau : \mathcal{G}_H \times (Z, L) \to (Z, L)$ of $\mathcal{G}_H$ *a characteristic* $\mathcal{G}_H$*-action,* or simply *characteristic*, if τ induces the natural isomorphism in Remark 8.1.4

$$\mathcal{G}_H \overset{\cong}{\to} \mathcal{G}(Z, L) \subset \mathrm{Aut}(L/Z),$$

where $\mathrm{Aut}(L/Z)$ is the bundle automorphism group of L over Z.

Remark 8.1.6. Let C be a planar cubic defined by

$$x_0^3 + \zeta_3 x_1^3 + \zeta_3^2 x_2^3 = 0.$$

This cubic C is $\mathcal{G}(3)$-invariant, hence σ and τ in Subsec. 3.3 act on C. However τ is not a translation of C. See [30, p. 712]. Therefore $\mathcal{G}(3)$ on C is not a characteristic $\mathcal{G}(3)$-action of C.

8.2. The level-$\mathcal{G}_H$ structure

Definition 8.2.1. Let k be an algebraically closed field with $k \ni 1/N$. A 6-tuple $(Z, L, V(Z, L), \phi, \mathcal{G}_H, \tau)$ or the triple (Z, ϕ, τ) over k is *a PSQAS with level-*$\mathcal{G}_H$ *structure* or *a level-*$\mathcal{G}_H$ *PSQAS* if

- (i) (Z, L) is a PSQAS $(Q_0, \mathcal{L}_0)$ over k with L very ample,
- (ii) $\tau : \mathcal{G}_H \times (Z, L) \to (Z, L)$ is a characteristic $\mathcal{G}_H$-action,
- (iii) $\phi : Z \to \mathbf{P}(V_H)$ is a $\mathcal{G}_H$-equivariant closed immersion (with respect to τ) such that $V(Z, L) = \phi^*(V_H \otimes k) \subset \Gamma(Z, L)$.

Definition 8.2.2. For a level-$\mathcal{G}_H$ PSQAS (Z, ϕ, τ) over k, let

$$\rho(\phi, \tau)(g)(v) := (\phi^*)^{-1} \rho_{\tau, L}(g) \phi^*(v) \tag{30}$$

for $v \in V_H$.

Remark 8.2.3. By Claim 7.4.2, the following condition (iv) is automatically satisfied by (Z, L) in Definition 8.2.1 :

- (iv) $(\phi, \Phi) : (Z, L) \to (\mathbf{P}(V_H), \mathbf{H})$ is a $\mathcal{G}_H$-equivariant morphism (with respect to τ) where $\mathbf{H}$ is the hyperplane bundle of $\mathbf{P}(V_H)$ and $\Phi : L = \phi^*\mathbf{H} \to \mathbf{H}$ the natural bundle morphism. That is,

$$(\phi, \Phi) \circ \tau(g) = S(\rho(\phi, \tau)g) \circ (\phi, \Phi) \text{ for any } g \in \mathcal{G}_H \tag{31}$$

with the notation of Definition 7.2.1. □

We added (iv) here for notational convenience. We denote (iii) and (iv) together by $\phi\tau = S\phi$ or $\phi\tau(g) = S(g)\phi$ for any $g \in \mathcal{G}_H$.

Definition 8.2.4. Two PSQASes (Z, ϕ, τ) and (Z', ϕ', τ') with level-$\mathcal{G}_H$ structure are defined to be *isomorphic* iff there exists a $\mathcal{G}_H$-isomorphism $f : (Z, L) \to (Z', L')$ such that $\phi' f = \phi$.

Remark 8.2.5. In Definition 8.2.4 (i), $V(Z, L) = f^* V(Z', L')$. Hence $f^* L' = L$ so that there always exists a $\mathcal{G}_H$-isomorphism of bundles $(f, F(f)) : (Z, L) \to (Z', L')$, that is,

$$(f, F(f))\tau(g) = \tau'(g)(f, F(f)) \quad \text{for any } g \in \mathcal{G}_H.$$

The line bundle L is a scheme over Z. The $\mathcal{G}_H$-isomorphism $F(f) : L \to L'$ is a $\mathcal{G}_H$-isomorphism as a (line) bundle, which induces a $\mathcal{G}_H$-isomorphism $f : Z \to Z'$. In what follows, we say this simply that $(f, F(f))$ *or* $f : (Z, L) \to (Z', L')$ *is a* $\mathcal{G}_H$*-isomorphism of bundles.*

Definition 8.2.6. (Z, ϕ, τ) is defined to be a rigid level-$\mathcal{G}_H$ PSQAS, or a PSQAS with rigid level-$\mathcal{G}_H$ structure if

(i) (Z, ϕ, τ) is a level-$\mathcal{G}_H$ PSQAS,

(ii) $\rho(\phi, \tau) = U_H$: the Schrödinger representation of $\mathcal{G}_H$.

Remark 8.2.7. A rigid object in Definition 8.2.6 is a natural generalization of a Hesse cubic. Lemma 8.2.8 shows that any PSQAS (Z, ϕ, τ) can be moved into a rigid one inside the same projective space.

Lemma 8.2.8. *Assume* $e_{\min}(K) \geq 3$*. Then for a level-*$\mathcal{G}_H$ *PSQAS* (Z, ϕ, τ) *over* k*,*

(1) *there exists a unique rigid level-*$\mathcal{G}_H$ *PSQAS* (Z, ψ, τ) *isomorphic to* (Z, ϕ, τ)*,*

(2) *there exists a unique* U_H*-invariant subscheme* (W, L) *of* $(\mathbf{P}(V_H), \mathbf{H})$ *such that* $(W, i, U_H) \simeq (Z, \psi, \tau)$*.*

Proof. By Claim 7.1.5, we have

$$\rho(\phi, \tau)(gh) = \rho(\phi, \tau)(g)\rho(\phi, \tau)(h).$$

Hence V_H is an irreducible $\mathcal{G}_H$-module of weight one through $\rho(\phi, \tau)$. By Schur's lemma, there exists $A \in \mathrm{GL}(V_H \otimes k)$ such that

$$U_H = A^{-1}\rho(\phi, \tau)A = (\phi^* A)^{-1}\rho_{\tau,L}(g)(\theta)(\phi^* A).$$

Hence it suffices to choose a closed immersion ψ by $\psi^* = \phi^* A$. Then

$$U_H = \rho(\psi, \tau) \text{ and } (Z, \phi, \tau) \simeq (Z, \psi, \tau). \tag{32}$$

The uniqueness of ψ follows from Schur's lemma (Lemma 3.8). In fact, suppose $U_H = \rho(\psi, \tau) = \rho(\phi, \tau)$. Let $\gamma := (\phi^*)^{-1}(\psi^*)$. Then

$$U_H = \rho(\phi, \tau) = \gamma\rho(\psi, \tau)\gamma^{-1} = \gamma U_H \gamma^{-1},$$

whence by Schur's lemma, γ is a nonzero scalar. Hence $\psi = \phi$.

Finally we prove the second assertion. An example of (W, i, U_H) is given by $(\psi(Z), i, U_H)$ by the first assertion. If we have another U_H-invariant PSQAS (W', j, U_H) such that $(W, i, U_H) \simeq (W', j, U_H)$, there is an isomorphism

$$f : (W, i, U_H) \to (W', j, U_H).$$

Hence $i = jf$. By the proof of the first assertion, f^* is a nonzero scalar, hence $j = i$. Hence the closed subscheme W is unique. Q.E.D.

Lemma 8.2.9. *Let k be an algebraically closed field with $k \ni 1/N$. If $e_{\min}(H) \geq 3$, then any level-$\mathcal{G}_H$ PSQAS (Z, ϕ, τ) has trivial automorphism group.*

Proof. Let f be any isomorphism $f : (Z, \phi, \tau) \to (Z, \phi, \tau)$. Hence $f\tau(g) = \tau(g)f$ for any $g \in \mathcal{G}_H$. Hence we have

$$f^*\rho_{\tau,L}(g) = \rho_{\tau,L}(g)f^* \quad \text{on } V(Z, L) \text{ for any } g \in \mathcal{G}_H.$$

Since $\rho_{\tau,L}$ is an irreducible representation of $\mathcal{G}_H$ on $V(Z, L)$, by Schur's lemma (Lemma 3.8), f^* is a scalar. Since $e_{\min}(H) \geq 3$, we have $\phi^{-1} : \phi(Z) \overset{\cong}{\to} Z$ is an isomorphism by Theorem 8.1 (5). Since f^* on $V(Z, L)$ is a nonzero scalar, $(\phi^*)^{-1} \circ f^* \circ (\phi^*)$ is a scalar isomorphism of $V_H \otimes k$, hence $\phi \circ f \circ \phi^{-1}$ is the identity of $\mathbf{P}(V_H)$, hence it is the identity of $\phi(Z)$. Hence f is the identity of Z. Q.E.D.

Lemma 8.3. *Let k be an algebraically closed field, let H be a finite Abelian group, $H^\vee$ the Cartier dual of H, $K = H \oplus H^\vee$ the symplectic Abelian group and $N = |H|$. If $k \ni 1/N$, then there exists a polarized abelian variety (A, L) over k such that the Heisenberg group $\mathcal{G}(A, L)$ of (A, L) is isomorphic to $\mathcal{G}_H \otimes k$.*

Proof. See [32, Lemma 4.2]. Q.E.D.

8.4. The Hilbert scheme $\mathrm{Hilb}^{\chi(n)}$

Let H, V_H and $\mathcal{G}_H$ be the same as in Subsec. 3.5. Let $\mathrm{Hilb}^{\chi(n)}$ be the Hilbert scheme parameterizing all the closed subscheme (Z, L) of $\mathbf{P}(V_H)$ with $\chi(Z, L^n) = n^g|H| =: \chi(n)$. Since V_H is a $\mathcal{G}_H$-module via U_H, $\mathcal{G}_H$ acts on $(\mathbf{P}(V_H), \mathbf{H})$, hence on $\mathrm{Hilb}^{\chi(n)}$. Let

$$(\mathrm{Hilb}^{\chi(n)})^{\mathcal{G}_H\text{-inv}}$$

be the fixed point set of $\mathcal{G}_H$ (the scheme-theoretic fixed points). This is a closed $\mathcal{O}_N$-subscheme of $\mathrm{Hilb}^{\chi(n)}$. Let $(Z_{\mathrm{univ}}, L_{\mathrm{univ}})$ be the pull back to $(\mathrm{Hilb}^{\chi(n)})^{\mathcal{G}_H\text{-inv}}$ of the universal subscheme of $\mathbf{P}(V_H)$ over $\mathrm{Hilb}^{\chi(n)}$. Then there is an open $\mathcal{O}_N$-subscheme U_3 of $(\mathrm{Hilb}^{\chi(n)})^{\mathcal{G}_H\text{-inv}}$ such that any geometric fiber of $(Z_{\mathrm{univ}}, L_{\mathrm{univ}})$ is an abelian variety (with zero unspecified). It is clear that $\mathcal{G}_H$ keeps U_3 stable. See [30, Subsec. 11.1].

Let $\mathrm{Aut}_{U_3}(Z_{\mathrm{univ}})$ be the relative automorphism group scheme of $(Z_{\mathrm{univ}})_{U_3}$ (see [30, Subsec. 11.1]). We define a subset U_4 of U_3 to be

$$U_4 = \left\{ s \in U_3; \begin{matrix} \text{the action of } \mathcal{G}_H \text{ on } (Z_{\mathrm{univ},s}, L_{\mathrm{univ},s}) \text{ is} \\ \text{a translation of the abelian variety } Z_{\mathrm{univ},s} \end{matrix} \right\}.$$

Since the subgroup of $\mathrm{Aut}_{U_3}(Z_{\mathrm{univ}})$ consisting of fiberwise translations is an (open and) closed subgroup $\mathbf{Z}$-scheme of $\mathrm{Aut}_{U_3}(Z_{\mathrm{univ}})$, U_4 is a closed $\mathcal{O}_N$-subscheme of U_3, which is not empty by Lemma 8.3.

We denote U_4 by $A_{g,K}$ and we define $SQ_{g,K}$ to be the closure of $A_{g,K}$ (the minimal closed $\mathcal{O}_N$-subscheme containing $A_{g,K}$)

$$SQ_{g,K} := \overline{A_{g,K}} \subset (\mathrm{Hilb}^{\chi(n)})^{G(K)\text{-inv}}. \tag{33}$$

Theorem 8.5. *Let $H = \bigoplus_{i=1}^{g}(\mathbf{Z}/e_i\mathbf{Z})$ with $e_i|e_{i+1}$ for any i and $N = \prod_{i=1}^{g} e_i$. If $e_{\min}(H) := e_1 \geq 3$, then for any algebraically closed field k with $k \ni 1/N$, we have*

$$SQ_{g,K}(k) = \left\{ (Q_0, i, U_H); \begin{matrix} Q_0 : \textit{a level-}\mathcal{G}_H \textit{ PSQAS} \\ i : Q_0 \subset \mathbf{P}(V_H) \textit{ the inclusion} \end{matrix} \right\}$$

Proof. Let x_0 be any k-point of $SQ_{g,K}$. Then for a suitable CDVR R, there exists a morphism $j : \mathrm{Spec}\ R \to SQ_{g,K}$ such that

(i) $j(0) = x_0 \in SQ_{g,K}$, and
(ii) $j(\mathrm{Spec}\ k(\eta)) \subset A_{g,K} \subset \mathrm{Hilb}^{\chi(n)}$.

In other words, there exists a projective R-flat subscheme $(Z, \mathcal{L})$ of $(\mathbf{P}(V_H), \mathbf{H})_R$ such that

(i*) $x_0 = (Z_0, \mathcal{L}_0) := (Z, \mathcal{L}) \otimes k(0) \in SQ_{g,K}$,
(ii*) $(Z_\eta, \mathcal{L}_\eta)$ is an U_H-invariant abelian variety (to more precise, invariant under the action of $U_H\mathcal{G}_H$ on $(\mathbf{P}(V_H), \mathbf{H})$) such that $\ker\lambda(\mathcal{L}_\eta) \simeq K := H \oplus H^\vee$ and the actions of $\mathcal{G}_H$ on Z_η are translations of Z_η.

where η is the generic point of S and $k(\eta)$ is the fraction field of R.

In this case, $(Z, \mathcal{L})$ is the pull back of $(Z_{\mathrm{univ}}, L_{\mathrm{univ}})$ by j. Conversely, $j : \mathrm{Spec}\ R \to SQ_{g,K}$ is induced from the subscheme $(Z, \mathcal{L})$ of $(\mathbf{P}(V_H), \mathbf{H})_R$ by the universality of $(Z_{\mathrm{univ}}, L_{\mathrm{univ}})$.

Let $i : (Z, \mathcal{L}) \to (\mathbf{P}(V_H), \mathbf{H})_R$ be the natural inclusion, and $\mathcal{V}_Z := i^*\Gamma(\mathbf{P}(V_H), \mathbf{H}) = i^*V_H \otimes R$. Clearly $\mathcal{V}_Z$ is very ample on Z. Since $j(\mathrm{Spec}\ k(\eta)) \subset A_{g,K}$, the $\mathcal{G}_H$-action on $(Z, \mathcal{L})$ induces a rigid level-$\mathcal{G}_H$

structure on $(Z_\eta, \mathcal{L}_\eta)$. That is, $(Z, \mathcal{V}_Z, \mathcal{L}, i, U_H) \otimes_R k(\eta)$ is a rigid level-$\mathcal{G}_H$ PSQAS over $k(\eta)$. In other words, $Z_\eta = i(Z_\eta)$ is also a U_H-invariant subscheme of $\mathbf{P}(V_H)$.

Meanwhile, by Theorem 4.6, by a finite base change if necessary, there exists a rigid level-$\mathcal{G}_H$ PSQAS $(Q, \mathcal{L}_Q, \phi, \tau)$ over R such that

$$(Q_\eta, \mathcal{L}_{Q,\eta}, \phi_\eta, \tau_\eta) \simeq (Z_\eta, \mathcal{L}_\eta, i_\eta, U_H).$$

By definition, $\rho(\phi, \tau) = U_H$. Hence $\phi(Q)$ is a U_H-invariant subscheme of $\mathbf{P}(V_H)_R$. Since $Z_\eta = i(Z_\eta)$ is also a U_H-invariant subscheme of $\mathbf{P}(V_H)_{k(\eta)}$, by Lemma 8.2.8 (2) (over $k(\eta)$)

$$Z_\eta = i(Z_\eta) = \phi(Q_\eta).$$

Hence their closures in $\mathbf{P}(V_H)_R$ are the same. It follows $Z = \phi(Q)$, hence $(Z_0, \mathcal{L}_0) = (\phi(Q_0), \mathcal{L}_0)$ as a subscheme of $\mathbf{P}(V_H)$. Since $\Gamma(Q, \mathcal{L}) = \phi^*\Gamma(\mathbf{P}(V_H)_R, \mathbf{H}_R)$ is very ample by Lemma 6.3 if $e_{\min}(H) \geq 3$, we have $Z_0 = \phi(Q_0) \simeq Q_0$. It follows that

$$x_0 = (Z_0, \mathcal{L}_0, i_0, U_H) \simeq (Q_0, \mathcal{L}_0, \phi_0, \tau_0),$$

which is a rigid level-$\mathcal{G}_H$ PSQAS. Q.E.D.

Corollary 8.6. *Let $|H| = N$. Under the same assumption as in Theorem 8.5, for any algebraically closed field k with $k \ni 1/N$, we have*

$$A_{g,K}(k) = \left\{ (Q_0, i, U_H); \begin{matrix} Q_0 : \textit{a level-}\mathcal{G}_H \textit{ abelian variety} \\ i : Q_0 \subset \mathbf{P}(V_H) \textit{ the inclusion} \end{matrix} \right\}.$$

§9. Moduli for PSQASes

Let $\mathcal{O} = \mathcal{O}_N$. In this section we prove

(i) $A_{g,K}$ is the fine moduli scheme for the functor of T-smooth PSQASes over $\mathcal{O}$-schemes.

(ii) $SQ_{g,K}$ is the fine moduli scheme for the functor of T-flat PSQASes over reduced $\mathcal{O}$-schemes.

9.1. T-smooth PSQASes

Let T be any $\mathcal{O}$-scheme. In this subsection we define level-$\mathcal{G}_H$ T-smooth PSQASes. Since any smooth PSQAS over a field is an abelian variety, any level-$\mathcal{G}_H$ T-smooth PSQAS is a T-smooth scheme, any of whose geometric fiber is an abelian variety. It may have no global (zero) section over T.

Definition 9.1.1. A 6-tuple $(Q, \mathcal{L}, \mathcal{V}, \phi, \mathcal{G}, \tau)$ (or a triple (Q, ϕ, τ) for brevity) is called *a T-smooth projectively stable quasi-abelian scheme* (abbr. *a T-smooth PSQAS*) of relative dimension g with level-$\mathcal{G}_H$ structure if the conditions (i)-(vi) are true:

(i) Q is a projective T-scheme with the projection $\pi : Q \to T$ surjective smooth,

(ii) $\mathcal{L}$ is a relatively very ample line bundle of Q,

(iii) $\mathcal{G}$ is a T-flat group scheme, $\tau : \mathcal{G} \times (Q, \mathcal{L}) \to (Q, \mathcal{L})$ is an action of $\mathcal{G}$ as bundle automorphisms over Q,

(iv) $\phi : Q \to \mathbf{P}(V_H)_T$ is a $\mathcal{G}$-equivariant closed T-immersion of Q,

(v) there exists $M \in \mathrm{Pic}(T)$ with trivial $\mathcal{G}$-action such that $\mathcal{L} \simeq \phi^*\mathbf{H} \otimes \pi^* M$ as $\mathcal{G}$-modules, and $\mathcal{V} = V_H \otimes_{\mathcal{O}} M$ is a locally free $\mathcal{G}$-invariant O_T-submodule [1] of $\pi_*\mathcal{L}$ of rank $|H|$ via the natural homomorphism, (see Remark 9.1.3)

(vi) for any geometric point t of T, the fiber at t $(Q_t, \mathcal{L}_t, \mathcal{V}_t, \phi_t, \mathcal{G}_t, \tau_t)$ is *a level-$\mathcal{G}_H$ smooth PSQAS* of dimension g over $k(t)$.

We call (ϕ, τ) a level-$\mathcal{G}_H$ structure on Q if no confusion is possible. We also call (Q, ϕ, τ) a level-$\mathcal{G}_H$ T-smooth PSQAS.

Remark 9.1.2. Let Q be a T-smooth TSQAS. Then $\mathrm{Aut}^0_S(Q)$ is an abelian scheme over S with zero section id_Q, hence any T-smooth TSQAS Q is an $\mathrm{Aut}^0_S(Q)$-torsor. See Theorem 13.6.5 and [33].

Remark 9.1.3. As in Definition 8.2.1 and Remark 8.2.3, ϕ in (iv) is a $\mathcal{G}$-morphism with respect to τ in the sense that

$$\phi\tau(g) = S(\rho(\phi, \tau)(g))\phi,$$

under the notation $S(\rho(\phi, \tau)(g))$ in Subsec. 7.2.

The natural homomorphism $\iota : \mathcal{V} = V_H \otimes_{\mathcal{O}} M \to \pi_*(\mathcal{L})$ is given as follows. Let $\pi_{\mathbf{P}} : \mathbf{P}(V_H)_T \to T$ be the natural projection. By the relation $\pi_{\mathbf{P}}\phi = \pi$ and the projection formula, we see

$$\pi_*(\mathcal{L}) = \pi_*(\phi^*(\mathbf{H} \otimes \pi_{\mathbf{P}}^* M)) = \pi_*(\phi^*(\mathbf{H}) \otimes \pi^* M) = (\pi_{\mathbf{P}})_*\phi_*\phi^*\mathbf{H} \otimes M,$$

while $V_H \otimes M = (\pi_{\mathbf{P}})_*(\mathbf{H}) \otimes M$. Hence ι is induced from the natural homomorphism $\mathbf{H} \to \phi_*\phi^*\mathbf{H}$. In what follows we omit ι.

Definition 9.1.4. Let (Q, ϕ, τ) be a level-$\mathcal{G}_H$ T-smooth PSQAS. Then (ϕ, τ) is called *a rigid level-$\mathcal{G}_H$ structure* if $\rho(\phi, \tau) = U_H$, where $\rho(\phi, \tau)$ is defined by

$$\rho(\phi, \tau)(g)(v) := (\phi^*)^{-1}\rho_{\tau, L}(g)\phi^*(v) \tag{34}$$

[1] $\mathcal{V} = \pi_*\mathcal{L}$ for T-smooth PSQASes.

for $v \in \mathcal{V} = \phi^* V_H \otimes_{\mathcal{O}} M$.

Definition 9.1.5. Let $\sigma_i := (Q_i, \mathcal{V}_i, \mathcal{L}_i, \phi_i, \mathcal{G}_i, \tau_i)$ be a level-$\mathcal{G}_H$ T-smooth PSQAS and $\pi_i : Q_i \to T$ the projection. Then $f : \sigma_1 \to \sigma_2$ is called *a morphism of level-$\mathcal{G}_H$ T-smooth PSQASes* if there exists $M \in \mathrm{Pic}(T)$, a T-morphism $f : Q_1 \to Q_2$ and a group scheme T-morphism $h : \mathcal{G}_1 \to \mathcal{G}_2$ such that

(i) $\phi_1 = \phi_2 \circ f$,

(ii) the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{G}_1 \times (Q_1, \mathcal{L}_1) & \xrightarrow{\tau_1} & (Q_1, \mathcal{L}_1) \\ \Big\downarrow{\scriptstyle h \times f} & & \Big\downarrow{\scriptstyle f} \\ \mathcal{G}_2 \times (Q_2, \mathcal{L}_2 \otimes_{O_T} \pi_2^*(M)) & \xrightarrow[\tau_2]{} & (Q_2, \mathcal{L}_2 \otimes_{O_T} \pi_2^*(M)). \end{array}$$

The morphism $f : \sigma_1 \to \sigma_2$ is an isomorphism if and only if $f : Q_1 \to Q_2$ is an isomorphism as schemes.

Remark 9.1.6. From Definition 9.1.5, we infer that there exists some $M \in \mathrm{Pic}(T)$ such that

(i) $\mathcal{L}_1 \simeq f^*(\mathcal{L}_2) \otimes \pi_1^*(M)$ and $\mathcal{V}_1 = \mathcal{V}_2 \otimes M$,

(ii) $(f, F(f)) : (Q_1, \mathcal{L}_1) \to (Q_2, \mathcal{L}_2 \otimes \pi_2^*(M))$ is a $\mathcal{G}_1$-morphism of bundles: that is,

$$(f, F(f)) \circ \tau_1(g) = \tau_2(g) \circ (f, F(f)), \quad g \in \mathcal{G}_1,$$

(iii) $\rho(\phi_1, \tau_1) = \rho(\phi_2, \tau_2)$. See [32, Lemma 5.5].

In particular, for any $M \in \mathrm{Pic}(T)$ with trivial $\mathcal{G}$-action,

$$(Q, \mathcal{V}, \mathcal{L}, \phi, \mathcal{G}, \tau) \simeq (Q, \mathcal{V} \otimes M, \mathcal{L} \otimes \pi^* M, \phi, \mathcal{G}, \tau).$$

Remark 9.1.7. Since any $a \in Q(T)$ (a global section of Q) acts on Q by translation, we have

$$(Q, \mathcal{V}, \mathcal{L}, \phi, \mathcal{G}, \tau) \simeq (Q, T_a^* \mathcal{V}, T_a^* \mathcal{L}, T_a^* \phi, \mathcal{G}, T_a^* \tau),$$

where $T_a^* \tau = \{T_a^* \phi_g\}$ for $\tau = \{\phi_g\}$ as $\mathcal{G}_H$-linearization. □

Lemma 9.1.8. *Assume $e_{\min}(H) \geq 3$. For a level-$\mathcal{G}_H$ T-smooth (resp. T-flat) PSQAS (Z, ϕ, τ), there exists a unique rigid level-$\mathcal{G}_H$ T-smooth (resp. T-flat) PSQAS (Z, ψ, τ) isomorphic to (Z, ϕ, τ).*

Proof. One can prove this in parallel to Lemma 8.2.8.

By Definition 9.1.1, we have a 6-tuple $(Z, L, \mathcal{V}, \phi, \mathcal{G}, \tau)$. Let $\mathcal{V} = V_H \otimes M$ for some $M \in \mathrm{Pic}(T)$. We choose an affine covering U_i of T

such that $M \otimes O_{U_i}$ is trivial. Let $Z_i := Z_T \times U_i$. Then $\phi_i := \phi_{|Z_i} : (Z_i, L_{Z_i}) \to \mathbf{P}(V_H)$ is a closed $\mathcal{G}_{U_i}$-immersion and $\rho_{\rho_i,\tau}$ is equivalent to U_H. Hence there exists $A_i \in \mathrm{GL}(V_H \otimes O_{U_i})$ such that $U_H = A_i^{-1}\rho_{\rho_i,\tau}A_i$ by Lemma 3.7. We define a closed $\mathcal{G}_{U_i}$-immersion

$$\psi_i : (Z_i, L_{Z_i}) \to (\mathbf{P}(V_H)_{U_i}, \mathbf{H}_{U_i})$$

by $\psi_i^* = \phi_i^* A_i$. Hence we have $\rho(\psi_i, \tau) = U_H$. Over $U_i \cap U_j$ we have two $\mathcal{G}_{U_i \cap U_j}$-isomorphisms

$$\psi_k^* : V_H \otimes O_{U_i \cap U_j} \simeq \mathcal{V} \otimes O_{U_i \cap U_j}, \quad (k = i, j).$$

By Lemma 3.8, there exists a unit $f_{ij} \in O_{U_i \cap U_j}^\times$ such that $\psi_i^* = f_{ij}\psi_j^*$. Hence $\psi_i = \psi_j$ over $U_i \cap U_j$ as a morphism to $\mathbf{P}(V_H)_{U_i \cap U_j}$. Thus we have a T-smooth (resp. T-flat) PSQAS (Z, ψ, τ) such that $\rho(\psi, \tau) = U_H$.

The same argument proves the Lemma for a T-flat PSQAS, though T-flat PSQASes are defined later in Subsec. 9.7. This completes the proof. Q.E.D.

Definition 9.1.9. We define a contravariant functor $\mathcal{A}_{g,K}$ from the category of $\mathcal{O}$-schemes to the category of sets by

$$\begin{aligned}\mathcal{A}_{g,K}(T) &= \text{the set of all level-}\mathcal{G}_H\ T\text{-smooth PSQASes } (Q, \phi, \tau)\\ &\quad\ \text{of relative dimension } g \text{ modulo } T\text{-isomorphism}\\ &= \text{the set of all rigid level-}\mathcal{G}_H\ T\text{-smooth PSQASes}\\ &\quad\ \text{of relative dimension } g \text{ modulo } T\text{-isomorphism}\end{aligned}$$

by Lemma 9.1.8.

9.2. Pro-representability

Let k be an algebraically closed field, and $W = W(k)$ the Witt ring of k. Let $\mathcal{C} = \mathcal{C}_W$ be the category of local Artinian W-algebra with an isomorphism $k = R/m_R$ making the following diagram commutative:

$$\begin{array}{ccc} W & \longrightarrow & R \\ \downarrow & & \downarrow \\ k & \xrightarrow{\simeq} & R/m_R. \end{array}$$

Let $\hat{\mathcal{C}}_W$ be the category of all complete local noetherian W-algebras R such that $R/m_R^n \in \mathcal{C}_W$ for every n. The morphisms in $\hat{\mathcal{C}}_W$ are local W-algebra homomorphisms. A functor $F : \mathcal{C}_W \to (Sets)$ is called *pro-representable* if there exists an $A \in \hat{\mathcal{C}}_W$ such that

$$F(R) = \mathrm{Hom}_{W\text{-hom.}}(A, R).$$

9.3. Deformation theory of abelian schemes

We briefly review [38]. Let k be an algebraically closed field. Let $\mathcal{C} = \mathcal{C}_W$. We caution that $R \in \mathcal{C}$ is not always a k-algebra.

Let A be an abelian variety over k, L_0 an ample line bundle on A, and $\lambda(L_0) : A \to A^\vee := \mathrm{Pic}^0_A$ the polarization morphism.

By Grothendieck and Mumford [38, Theorems 2.3.3, 2.4.1] the quasi-polarized moduli functor P of $(A, \lambda(L_0))$ is formally smooth if $\lambda(L_0) : A \to A^\vee$ is separable. We will explain this.

The deformation functor $M := M(A)$ of A is defined over $\mathcal{C}$ by

$$M(R) = \left\{ (X, \phi_0);\ \begin{array}{l} X \text{ is a proper } R\text{-scheme} \\ \phi_0 : X \otimes_R k \simeq A \end{array} \right\} / R\text{-isom.}$$

By Grothendieck [38, Theorem 2.2.1], M is pro-represented by

$$W(k)[[t_{i,j}; 1 \leq i, j \leq g]]$$

where $W(k)$ is the Witt ring of k.

The quasi-polarized moduli functor $P := P(A, \lambda_0)$ of $(A, \lambda(L_0))$ over $\mathcal{C}$ is defined as follows [38, pp. 240-242] :

$$P(R) = \left\{ (X, \lambda, \phi_0);\ \begin{array}{l} (X, \lambda) \text{ is an abelian } R\text{-scheme} \\ \lambda : X \to X^\vee \text{ is a homomorphism} \\ \text{such that } \lambda = \lambda(\mathcal{L}) \text{ for some } \mathcal{L} \in \mathrm{Pic}(X) \\ \phi_0 : (X, \lambda) \otimes_R k \simeq (A, \lambda_0) \end{array} \right\} / R\text{-isom.}$$

where $\lambda_0 := \lambda(L_0)$ and $X^\vee := \mathrm{Pic}^0_{X/R}$.

Thus any $(Y, \lambda, \phi_0) \in P(R)$ always has a line bundle L such that $\lambda = \lambda(L)$. This fact is used in Subsec. 9.4.

By [38, Theorem 2.3.3], $P(A, \lambda_0)$ is a pro-representable subfunctor of $M(A)$, that is, the functor $P(A, \lambda_0)$ is pro-represented by

$$\mathcal{O}_W := W(k)[[t_{i,j}; 1 \leq i, j \leq g]]/\mathfrak{a}$$

for some ideal $\mathfrak{a}$ where $\mathfrak{a}$ is generated by $\frac{1}{2}g(g-1)$ elements.

9.4. Deformations in the separably polarized case

We call $\lambda(L_0)$ (or L_0) a separable polarization if $\lambda(L_0) : A \to A^\vee$ is a separable morphism. For instance, $\lambda(L_0)$ is separable if $k \ni 1/N$ where $N = \sqrt{|\ker \lambda(L_0)|}$.

Suppose that the polarization λ_0 is separable. The ideal $\mathfrak{a}$ is generated by $t_{ij} - t_{ji}$ for any pair $i \neq j$ [38, Remark, p. 246]:

$$\mathfrak{a} = (t_{ij} - t_{ji}; 1 \leq i < j \leq g)$$

Hence $P(A, \lambda_0)$ is formally smooth of dimension $\frac{1}{2}g(g+1)$ over $W(k)$. In this case (A, λ_0) can be lifted as a formal abelian scheme $(\mathcal{X}_{\text{for}}, \lambda(\mathcal{L}_{\text{for}}))$ over $\mathcal{O}_W$, that is, there exists a system (X_n, λ_n) of polarized abelian schemes over $\mathcal{O}_{W,n} := \mathcal{O}_W/\mathfrak{m}^{n+1}$ such that

$$(X_{n+1}, \lambda_{n+1}) \otimes \mathcal{O}_n \simeq (X_n, \lambda_n),$$

where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_W$. Then by [10, III, **11**, 5.4.5], the formal scheme $\mathcal{X}$ is algebraizable, that is, there exists a polarized abelian scheme $(X, \mathcal{L})$ over Spec $\mathcal{O}_W$ such that

$$(X, \lambda(\mathcal{L})) \otimes \mathcal{O}_{W,n} \simeq (X_n, \lambda_n).$$

Let $K_{\text{su}} = \ker(\lambda(\mathcal{L}))$, $\mathcal{G}_{\text{su}} := \mathcal{G}(X, \mathcal{L}) := \mathcal{L}^{\times}_{K_{\text{su}}}$ and $\mathcal{V}_{\text{su}} := \Gamma(X, \mathcal{L})$. By [25, pp. 115-117, pp.204-211], $\mathcal{L}$ is $\mathcal{G}_{\text{su}}$-linearizable. In other words, $\mathcal{G}_{\text{su}}$ acts on $(X, \mathcal{L})$ by bundle automorphisms. Let τ_{su} be the action of $\mathcal{G}_{\text{su}}$ on $(X, \mathcal{L})$. Then $\mathcal{V}_{\text{su}}$ is an $\mathcal{O}_W$-free $\mathcal{G}_{\text{su}}$-module of rank N via $\rho_{\tau_{\text{su}},\mathcal{L}}$.

By the assumption $k \ni 1/N$, $\lambda(\mathcal{L}) : X \to X^\vee$ is separable, and K_{su} is a constant finite symplectic Abelian group of order N^2 isomorphic to $H \oplus H^\vee$ because $K_{\text{su}} \otimes_{\mathcal{O}_W} k$ is so.

If $e_{\min}(K_{\text{su}}) \geq 3$, then $\mathcal{L}$ is very ample because $\mathcal{L}_0 = L_0$ is very ample by Theorem 8.1 (Lefschetz's theorem in this case). Let $\phi_{\text{su}} : X \to \mathbf{P}(\mathcal{V}_{\text{su}}) \simeq \mathbf{P}(V_H)_{\mathcal{O}_W}$ be the embedding of X into $\mathbf{P}(\mathcal{V}_{\text{su}})$ such that $\rho(\phi_{\text{su}}, \tau_{\text{su}}) = U_H$. Thus we have a level-$\mathcal{G}_H$ $\mathcal{O}_W$-smooth PSQAS

$$(X, \mathcal{L}, \mathcal{V}_{\text{su}}, \phi_{\text{su}}, \mathcal{G}_{\text{su}}, \tau_{\text{su}}).$$

Theorem 9.5. *Let $K = H \oplus H^\vee$ and $N := |H|$. If $e_{\min}(H) \geq 3$, then the functor $\mathcal{A}_{g,K}$ of level-$\mathcal{G}_H$ smooth PSQASes over $\mathcal{O}$-schemes is represented by the quasi-projective $\mathcal{O}$-formally smooth scheme $A_{g,K}$.*

Proof. By Lemma 9.1.8, for a T-smooth PSQAS (Q, ϕ, τ) there exists a unique rigid level-$\mathcal{G}_H$ T-smooth PSQAS (Q, ψ, τ) such that (Q, ψ, τ) is T-isomorphic to (Q, ϕ, τ). Since $\mathcal{L}$ is very ample by the assumption $e_{\min}(H) \geq 3$, (Q, ψ, τ) is embedded $\mathcal{G}$-equivariantly into $(\mathbf{P}(V_H), \mathbf{H})$, whose image is contained in $A_{g,K}$, because $\rho(\psi, \tau) = U_H$. This implies that there exists a unique morphism $f : T \to A_{g,K}$ such that (Q, ψ, τ) is the pull back by f of the universal subscheme

$$(Z_{g,K} \times_{H_{g,K}} A_{g,K}, i, U_H).$$

It follows that $\mathcal{A}_{g,K}$ is represented by the quasi-projective $\mathbf{Z}[\zeta_N, 1/N]$-scheme $A_{g,K}$.

It remains to prove $A_{g,K}$ is formally smooth over $\mathbf{Z}[\zeta_N, 1/N]$. Let k be any algebraically closed field with $k \ni 1/N$, and we choose any level-$\mathcal{G}_H$ abelian variety over k

$$\sigma := (A, L_0, \Gamma(A, L_0), \phi_0, \mathcal{G}(A, L_0), \tau_0) \in \mathcal{A}_{g,K}(k).$$

By Subsec. 9.4, the quasi-polarized moduli functor $P(A, \lambda(L_0))$ is formally smooth because $\lambda(L_0) : A \to A^\vee$ is separable by $k \ni 1/N$.

We define a functor F over $\mathcal{C}$ by

$$F(R) = \{\xi := (Z, L, \mathcal{V}, \phi, \mathcal{G}, \tau) \in \mathcal{A}_{g,K}(R); \xi \otimes_R k \simeq \sigma\}$$

where we do not fix the isomorphism $\xi \otimes_R k \simeq \sigma$ in contrast with $P(A, \lambda(L_0))$. Subsec. 9.4 shows that the map $h : P(A, \lambda(L_0)) \to F$ sending $(Z, L) = (X, \mathcal{L}) \times_{\mathcal{O}_W} R$ to

$$(X, \mathcal{L}, \mathcal{V}_{\mathrm{su}}, \phi_{\mathrm{su}}, \mathcal{G}_{\mathrm{su}}, \tau_{\mathrm{su}}) \times_{\mathcal{O}_W} R$$

is surjective because $\mathcal{V}_{\mathrm{su}}$, ϕ_{su}, $\mathcal{G}_{\mathrm{su}}$ and τ_{su} are uniquely determined, . It follows from Lemma 8.2.9 that h is injective. Hence $F = P(A, \lambda(L_0))$. Hence $A_{g,K}$ is formally smooth at σ. Q.E.D.

Corollary 9.6. *$SQ_{g,K}$ is reduced.*

Proof. Since $A_{g,K}$ is $\mathcal{O}$-formally smooth, it is reduced. Since $SQ_{g,K}$ is the intersection of all closed $\mathcal{O}$-subschemes containing $A_{g,K}$, it is the intersection of all closed reduced $\mathcal{O}$-subschemes containing $A_{g,K}$ because $A_{g,K}$ is reduced. Hence $SQ_{g,K}$ is reduced. Q.E.D.

9.7. T-flat PSQASes

Definition 9.7.1. Let T be any *reduced* $\mathcal{O}$-scheme. A 5-tuple $(Q, \mathcal{L}, \mathcal{V}, \phi, \mathcal{G}, \tau)$ (or a triple (Q, ϕ, τ) for brevity) is called *a projectively stable quasi-abelian T-flat scheme* (or just *a T-flat PSQAS*) of relative dimension g with level-$\mathcal{G}_H$ structure if the conditions (ii)-(v) in Definition 9.1.1 and (i*), (vi*) are true:

- (i*) Q is a projective T-scheme with the projection $\pi : Q \to T$ surjective flat,
- (vi*) for any geometric point t of T, the fiber at t $(Q_t, \mathcal{L}_t, \phi_t, \tau_t)$ is *a PSQAS* of dimension g over $k(t)$ with level-$\mathcal{G}_H$ structure.

We also call (Q, ϕ, τ) a level-$\mathcal{G}_H$ T-PSQAS.

Definition 9.7.2. Let (Q, ϕ, τ) be a level-$\mathcal{G}_H$ T-flat PSQAS. Then (ϕ, τ) is called *a rigid level-$\mathcal{G}_H$ structure* if $\rho(\phi, \tau) = U_H$.

Definition 9.7.3. Let $(Q_i, \mathcal{V}_i, \mathcal{L}_i, \phi_i, \mathcal{G}_i, \tau_i)$ be level-$\mathcal{G}_H$ T-PSQASes and $\pi_i : Q_i \to T$ a flat morphism (structure morphism) with T *reduced.* Then $f : Q_1 \to Q_2$ is called *a morphism of level-$\mathcal{G}_H$ T-PSQASes* if the conditions in Definition 9.1.5 are true.

Definition 9.7.4. The category Sch_{red} of reduced schemes is a subcategory of the category Sch of schemes with

$$\begin{aligned} \mathrm{Obj}(Sch_{\mathrm{red}}) &= \text{reduced schemes},\\ \mathrm{Mor}(Sch_{\mathrm{red}}) &= \text{morphisms in the category of schemes}. \end{aligned}$$

Definition 9.7.5. We define a contravariant functor $\mathcal{SQ}_{g,K}$ from the category Sch_{red} of *reduced* $\mathcal{O}$-schemes to the category of sets by

$$\begin{aligned} \mathcal{SQ}_{g,K}(T) &= \text{the set of all level-}\mathcal{G}_H\ T\text{-flat PSQASes } (Q, \phi, \tau)\\ &\qquad \text{of relative dimension } g \text{ modulo } T\text{-isomorphism}\\ &= \text{the set of all rigid level-}\mathcal{G}_H\ T\text{-flat PSQASes}\\ &\qquad \text{of relative dimension } g \text{ modulo } T\text{-isomorphism} \end{aligned}$$

by Lemma 9.1.8.

Theorem 9.8. *Suppose* $e_{\min}(K) \geq 3$. *Let* $N := \sqrt{|K|}$. *The functor* $\mathcal{SQ}_{g,K}$ *of level-*$G(K)$ *PSQASes* (Q, ϕ, τ) *over* reduced *schemes is represented by the projective reduced* $\mathcal{O}_N$*-scheme* $SQ_{g,K}$.

Proof. This is proved in parallel to Theorem 9.5. Properness of $SQ_{g,K}$ follows from Theorem 4.6. See [30, Theorem 10.4] for a more precise statement. Since $SQ_{g,K}$ is a proper subscheme of the projective scheme $\mathrm{Hilb}^{\chi(n)}$ in Subsec. 8.4, it is projective. Q.E.D.

§10. The functor of TSQASes

10.1. TSQASes over k

We introduced two kinds of nice classes of degenerate abelian schemes, PSQASes and TSQASes in Theorem 4.6.

It is TSQASes that we discuss in this section. They are nonsingular abelian varieties, or reduced even if singular, and therefore easier to handle than PSQASes. However the very-ampleness criterion (Theorem 4.6 (4)) fails for $(P, \mathcal{L}_P)$, and because of this defect, we cannot expect the existence of the fine moduli scheme for TSQASes.

Let H be any finite Abelian group, $N = |H|$, k an algebraically closed field with $k \ni 1/N$, $K = H \oplus H^\vee$ and $\mathcal{O} = \mathcal{O}_N$.

Remark 10.1.1. Let (Z, L) be any TSQAS over k. Hence there exist an Abelian group H and a flat family $(P, \mathcal{L})$ over a CDVR R with $k = R/m$ given in Theorem 4.6 such that $(Z, L) \simeq (P_0, \mathcal{L}_0)$, P_0 is reduced, and the generic fiber $(P_\eta, \mathcal{L}_\eta)$ is an abelian variety with $\ker \lambda(\mathcal{L}_\eta)) \simeq K_H = H \oplus H^\vee$. Hence we have an action of $\mathcal{G}_H$ on (Z, L). See Remark 8.1.4. We denote by $\mathcal{G}(Z, L)$ the $\mathcal{G}_H$-action on (Z, L). This is determined by (Z, L) uniquely up to an automorphism of $\mathcal{G}_H$. In the totally degenerate case, the action of $\mathcal{G}(Z, L)$ is explicitly written as S_x and T_a $(x \in X/Y, a \in X \times_{\mathbf{Z}} \mathbf{G}_m)$. See Definition 6.2.2.

Definition 10.1.2. Let (Z, L) be a TSQAS over k. We call $\tau : \mathcal{G}_H \times (Z, L) \to (Z, L)$ *a characteristic* $\mathcal{G}_H$*-action,* or simply *characteristic*, if this action of $\mathcal{G}_H$ induces the natural isomorphism in Remark 10.1.1

$$\mathcal{G}_H \overset{\cong}{\to} \mathcal{G}(Z, L) \subset \mathrm{Aut}(L/Z).$$

Definition 10.1.3. Let (Z, L) be a TSQAS over k. We define $(Z, L, \phi^*, \mathcal{G}_H, \tau)$ (denoted often (Z, ϕ^*, τ) or (Z, L, ϕ^*, τ)) to be a level-$\mathcal{G}_H$ TSQAS if if the conditions (i)-(iii) are true:

(i) (Z, L) is a PSQAS $(P_0, \mathcal{L}_0)$ over k with L ample,
(ii) $\tau : \mathcal{G}_H \times (Z, L) \to (Z, L)$ is a characteristic $\mathcal{G}_H$-action,
(iii) $\phi^* : V_H \otimes k \to H^0(Z, L)$ is a $\mathcal{G}_H$-isomorphism.

Definition 10.1.4. We define level-$\mathcal{G}_H$ k-TSQASes $(Z_1, L_1, \phi_1^*, \tau_1)$ and $(Z_2, L_2, \phi_2^*, \tau_2)$ to be *isomorphic* if there exists a $\mathcal{G}_H$-isomorphism $f : (Z_1, L_1) \to (Z_2, L_2)$ such that $f^*\phi_1^* = c\phi_2^*$ for some nonzero $c \in k$.

10.2. T-smooth TSQASes

Let T be any $\mathcal{O}$-scheme. In this subsection we define level-$\mathcal{G}_H$ T-smooth TSQASes. The level-$\mathcal{G}_H$ T-smooth TSQASes are essentially the same as level-$\mathcal{G}_H$ T-smooth PSQASes in Subsec. 9.1. The only difference from Subsec. 9.1 is that we define them without any restriction on $e_{\min}(H)$. Since any smooth TSQAS over a field is an abelian variety, any level-$\mathcal{G}_H$ T-smooth TSQAS is a level-$\mathcal{G}_H$ abelian scheme over T possibly with no zero section over T.

Definition 10.2.1. A 5-tuple $(P, \mathcal{L}, \phi^*, \mathcal{G}, \tau)$ (or a triple (P, ϕ^*, τ) for brevity) is called *a* T*-smooth PSQAS* of relative dimension g with level-$\mathcal{G}_H$ structure if the conditions (i)-(v) are true:

(i) P is a projective T-scheme with the projection $\pi : P \to T$ surjective smooth,
(ii) $\mathcal{L}$ is a relatively ample line bundle of P,
(iii) $\mathcal{G}$ is a T-flat group scheme, $\tau : \mathcal{G} \times (P, \mathcal{L}) \to (P, \mathcal{L})$ is an action of $\mathcal{G}$ on $(P, \mathcal{L})$ as bundle automorphism,

(iv) there exists a $\mathcal{G}$-isomorphism $\phi^* : V_H \otimes_{\mathcal{O}} M \overset{\cong}{\to} \pi_* \mathcal{L}$ for some $M \in \mathrm{Pic}(T)$ with trivial $\mathcal{G}$-action,

(v) for any geometric point t of T, the fiber at t $(P_t, \mathcal{L}_t, \phi_t^*, \mathcal{G}_t, \tau_t)$ is *a level-$\mathcal{G}_H$ smooth TSQAS* of dimension g over $k(t)$.

We call (ϕ^*, τ) a level-$\mathcal{G}_H$ structure on P if no confusion is possible. We also call (P, ϕ^*, τ) a level-$\mathcal{G}_H$ T-smooth TSQAS.

Definition 10.2.2. Let (P, ϕ^*, τ) be a level-$\mathcal{G}_H$ T-smooth TSQAS. Then (ϕ^*, τ) is called *a rigid level-$\mathcal{G}_H$ structure* if $\rho(\phi^*, \tau) = U_H$, where $\rho(\phi^*, \tau)$ is defined by

$$\rho(\phi^*, \tau)(g)(\theta) := (\phi^*)^{-1} \rho_{\tau, L}(g)(\theta)\phi^* \tag{35}$$

for $\theta \in \mathcal{V} := \phi^* V_H \otimes_{\mathcal{O}} M$. If ϕ^* defines a morphism $\phi : Z \to \mathbf{P}(V_H)_T$, then $\rho(\phi^*, \tau) = \rho(\phi, \tau)$ with the notation in Definition 9.1.4.

Definition 10.2.3. Let $(P_k, \mathcal{L}_k, \phi_k^*, \mathcal{G}_k, \tau_k)$ be a level-$\mathcal{G}_H$ T-smooth TSQAS and $\pi_k : P_k \to T$ the projection (structure morphism). Then $f : P_1 \to P_2$ is called *a morphism of level-$\mathcal{G}_H$ T-smooth TSQASes* if there exists $M \in \mathrm{Pic}(T)$, a T-morphism $f : P_1 \to P_2$ and a group scheme T-morphism $h : \mathcal{G}_1 \to \mathcal{G}_2$ such that

(i**) $f^* \phi_2^* = c\phi_1^*$ for some unit $c \in H^0(O_T)^\times$,

(ii**) the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{G}_1 \times (P_1, \mathcal{L}_1) & \xrightarrow{\tau_1} & (P_1, \mathcal{L}_1) \\ \Big\downarrow{\scriptstyle h\times f} & & \Big\downarrow{\scriptstyle f} \\ \mathcal{G}_2 \times (P_2, \mathcal{L}_2 \otimes_{O_T} \pi_2^*(M)) & \xrightarrow[\tau_2]{} & (P_2, \mathcal{L}_2 \otimes_{O_T} \pi_2^*(M)). \end{array}$$

The same is true as in Remark 9.1.6 by replacing $\rho(\phi_k, \tau_k)$ by $\rho(\phi_k^*, \tau_k)$.

Lemma 10.2.4. *For a level-$\mathcal{G}_H$ T-smooth (resp. T-flat) TSQAS (Z, ϕ^*, τ), there exists a unique rigid level-$\mathcal{G}_H$ T-smooth (resp. T-flat) TSQAS (Z, ψ^*, τ) such that*

(1) *(Z, ψ^*, τ) is isomorphic to (Z, ϕ^*, τ),*

(2) *ψ^* is the $\mathcal{G}_H$-isomorphism with $\rho(\psi^*, \tau) = U_H$, unique up to nonzero constant multiple.*

Proof. One can prove this in parallel to Lemma 8.2.8. Q.E.D.

Definition 10.2.5. We define a contravariant functor $\mathcal{A}_{g,K}$ (the functor of level-$\mathcal{G}_H$ smooth TSQASes) from the category of $\mathcal{O}$-schemes to the category of sets by

$$\begin{aligned}\mathcal{A}^{\text{toric}}_{g,K}(T) &= \text{the set of all level-}\mathcal{G}_H\ T\text{-smooth TSQASes } (Q, \phi^*, \tau)\\ &\quad \text{of relative dimension } g \text{ modulo } T\text{-isomorphism}\\ &= \text{the set of all rigid level-}\mathcal{G}_H\ T\text{-smooth TSQASes}\\ &\quad \text{of relative dimension } g \text{ modulo } T\text{-isomorphism}\end{aligned}$$

by Lemma 10.2.4.

10.3. T-flat TSQASes

Definition 10.3.1. Let T be any *reduced* $\mathcal{O}$-scheme. A 5-tuple $(P, \mathcal{L}, \phi^*, \mathcal{G}, \tau)$ (or a triple (P, ϕ^*, τ) for brevity) is called *a T-flat TSQAS* of relative dimension g with level-$\mathcal{G}_H$ structure if the conditions (ii)-(iv) in Definition 10.2.1 and (i*), (v*) are true:

(i*) P is a T-scheme with the projection $\pi : P \to T$ surjective flat,

(v*) for any geometric point t of T, the fiber at t $(P_t, \mathcal{L}_t, \phi^*_t, \tau_t)$ is *a TSQAS* of dimension g over $k(t)$ with level-$\mathcal{G}_H$ structure.

We also call (P, ϕ^*, τ) a level-$\mathcal{G}_H$ T-TSQAS.

Definition 10.3.2. Let (P, ϕ^*, τ) be a level-$\mathcal{G}_H$ T-flat TSQAS. Then (ϕ^*, τ) is called *a rigid level-$\mathcal{G}_H$ structure* if $\rho(\phi^*, \tau) = U_H$.

Definition 10.3.3. Let $(P_i, \mathcal{L}_i, \phi^*_i, \mathcal{G}_i, \tau_i)$ be level-$\mathcal{G}_H$ T-TSQASes and $\pi_i : P_i \to T$ a flat morphism (structure morphism) with T *reduced*. Then $f : P_1 \to P_2$ is called *an isomorphism of level-$\mathcal{G}_H$ T-TSQASes* if the conditions in Definition 10.2.3 are true.

Definition 10.3.4. The category $\mathit{Space}_{\text{red}}$ of reduced algebraic spaces is a subcategory of the category Space of algebraic spaces with

$$\begin{aligned}\text{Obj}(\mathit{Space}_{\text{red}}) &= \text{reduced algebraic spaces},\\ \text{Mor}(\mathit{Space}_{\text{red}}) &= \text{morphisms in the category of algebraic spaces.}\end{aligned}$$

Definition 10.3.5. We define a contravariant functor $\mathcal{SQ}^{\text{toric}}_{g,K}$ from the category $\mathit{Space}_{\text{red}}$ of *reduced* algebraic $\mathcal{O}$-spaces to the category of sets by

$$\begin{aligned}\mathcal{SQ}^{\text{toric}}_{g,K}(T) &= \text{the set of all level-}\mathcal{G}_H\ T\text{-flat TSQASes } (P, \phi^*, \tau)\\ &\quad \text{of relative dimension } g \text{ modulo } T\text{-isomorphism}\\ &= \text{the set of all rigid level-}\mathcal{G}_H\ T\text{-flat TSQASes}\\ &\quad \text{of relative dimension } g \text{ modulo } T\text{-isomorphism}\end{aligned}$$

by Lemma 10.2.4.

§11. The moduli spaces $A_{g,K}^{\mathrm{toric}}$ and $SQ_{g,K}^{\mathrm{toric}}$

Let H be a finite Abelian group, $K = K_H := H \oplus H^\vee$ and $N = |H|$, and let $\mathcal{O} = \mathcal{O}_N$. In this section we recall from [32, § 9] how to construct the algebraic space $SQ_{g,K}^{\mathrm{toric}}$ parameterizing level-$\mathcal{G}_H$ TSQASes.

The construction in Subsec. 11.2–11.6 is carried out *without any change regardless of the value of* $e_{\min}(H)$. We do not assume $e_{\min}(H) \geq 3$ unless otherwise mentioned.

We summarize this section in Summary 11.11 at the end.

11.1. Preliminaries

Let k be any algebraically closed field with $k \ni 1/N$. In this subsection we list some basic properties of a level-$\mathcal{G}_H$ TSQAS $(P_0, \mathcal{L}_0)$ over k that we use in what follows.

Lemma 11.1.1. *Let k be any algebraically closed field with $k \ni 1/N$. Let $(P_0, \mathcal{L}_0, \phi_0^*, \mathcal{G}(P_0, \mathcal{L}_0), \tau_0)$ be a level-$\mathcal{G}_H$ TSQAS over k, and therefore a closed fiber of the TSQAS $(P, \mathcal{L})$ over a CDVR R with the generic fiber P_η an abelian variety. Then*

(1) *P_0 is nonsingular if and only if it is an abelian variety,*
(2) *P_0 is reduced,*
(3) *$\mathcal{L}_0$ is ample, and $n\mathcal{L}_0$ is very ample for $n \geq 2g+1$,*
(4) *$H^q(P_0, n\mathcal{L}_0) = 0$ for any $q > 0, n > 0$,*
(5) *$\chi(P_0, n\mathcal{L}_0) = n^g|H|$ for any $n > 0$,*
(6) *the action $\mathcal{G}(P_0, \mathcal{L}_0)$ of $\mathcal{G}_H$ on $(P_0, \mathcal{L}_0)$ is characteristic, that is, it is induced from $\mathcal{G}(P_\eta, \mathcal{L}_\eta)$, where any of the latter induces a translation of an abelian variety P_η.*

Proof. (1) follows from Theorem 4.6. For (2)–(5), see [2] or [32, Theorem 2.11, p. 79]. (6) is proved (and defined) in a manner similar to Remark 8.1.4 and Definition 10.1.2. Q.E.D.

Lemma 11.1.2. *Let n be any positive integer, and $d = Nn+1$. We define $U_{d,H}$ on the $\mathcal{O}$-module V_H in Definition 3.5 by*

$$U_{d,H}(a, z, \alpha)v(\beta) = a^d\beta(z)^d v(\alpha + \beta). \tag{36}$$

We denote V_H by $V_{d,H}$ if $\mathcal{G}_H$ acts on V_H via $U_{d,H}$. Then

(1) *$V_{d,H}$ is an irreducible $\mathcal{G}_H$-module of weight d,*
(2) *let W be any $\mathcal{O}$-free $\mathcal{G}_H$-module of finite rank. If $\mathcal{G}_H$ acts on W with weight d: that is, the center $\mathbf{G}_m$ of $\mathcal{G}_H$ acts on W by $a^d\,\mathrm{id}_W$, then W is equivalent to $W_0 \otimes_{\mathcal{O}} V_{d,H}$ as $\mathcal{G}_H$-module, where W_0 is an $\mathcal{O}$-module with trivial $\mathcal{G}_H$-action.*

Proof. We denote the action of $g \in \mathcal{G}_H$ on W by $U(g)$, and we write $U(g) = U(a, z, \alpha)$ for $g = (a, z, \alpha) \in \mathcal{G}_H$. Let $W(\chi) = \{w \in W; U(h)w = \chi(h)w$ for any $h \in H\}$. By [32, p. 89], we have

$$W = \bigoplus_{\chi \in H^\vee} W(\chi), \quad W(\chi) = U(1, 0, \chi)W(0). \tag{37}$$

Therefore $W(0) \neq 0$ if $W \neq 0$.

For any $w \in W(0)$, we define $v(\chi, w) = U(1, 0, \chi)w$ for $\chi \in H^\vee$.

By imitating [32, p. 89], we infer

$$\begin{aligned} U(1, z, 0) \cdot v(\chi, w) &= U(\chi(z)(1, 0, \chi))w = \chi(z)^d v(\chi, w), \\ U(1, 0, \alpha) \cdot v(\chi, w) &= U(1, 0, \chi + \alpha) \cdot w = v(\chi + \alpha, w), \end{aligned}$$

whence

$$\begin{aligned} U(a, z, \alpha) \cdot v(\chi, w) &= U(a(1, 0, \alpha)(1, z, 0)(1, 0, \chi))w \\ &= U(a\chi(z)(1, 0, \chi + \alpha)(1, z, 0))w \\ &= a^d \chi(z)^d v(\chi + \alpha, w). \end{aligned} \tag{38}$$

We define a homomorphism $F : W(0) \otimes V_{d,H} \to W$ by

$$F(w \otimes v(\chi)) = v(\chi, w) \tag{39}$$

where $w \in W(0)$ and $v(\chi) \in V_{d,H}$. Here $W(0)$ in the left hand side of (39) is regarded as a trivial $\mathcal{G}_H$-module, while $W(0)$ in the right hand side of (39) is an $\mathcal{O}$-submodule of W. Then by (36) and (38), F is a $\mathcal{G}_H$-homomorphism:

$$F(w \otimes U_{d,H}(g)(v(\chi))) = U(g)v(\chi, w).$$

In view of (37), W is spanned by $v(\chi, w)$ for $w \in W(0)$ and $\chi \in H^\vee$. Hence F is surjective. By (37), W and $W(0) \otimes V_{d,H}$ are $\mathcal{O}$-modules of the same rank. Hence F is an isomorphism. Q.E.D.

11.2. $\mathrm{Hilb}^P(X/T)$

Let (X, L) be a polarized $\mathcal{O}$-scheme with L very ample and $P(n)$ an arbitrary polynomial. Let $\mathrm{Hilb}^P(X)$ be the Hilbert scheme parameterizing all closed subschemes Z of X with $\chi(Z, nL_Z) = P(n)$. As is well known $\mathrm{Hilb}^P(X)$ is a projective $\mathcal{O}$-scheme.

Let T be a projective scheme, (X, L) a flat projective T-scheme with L an ample line bundle of X, and $\pi : X \to T$ the projection. Then for an arbitrary polynomial $P(n)$, let $\mathrm{Hilb}^P(X/T)$ be the scheme parameterizing all closed subschemes Z of X with $\chi(Z, nL_Z) = P(n)$ such that Z is contained in fibers of π. Then $\mathrm{Hilb}^P(X/T)$ is a closed $\mathcal{O}$-subscheme of $\mathrm{Hilb}^P(X) \times_{\mathcal{O}} T$. See [3, Chap. 9].

11.3. The scheme $H_1 \times H_2$

Choose and fix a coprime pair of natural integers d_1 and d_2 such that $d_1 > d_2 \geq 2g+1$ and $d_\nu \equiv 1 \mod N$. This pair does exist because it is enough to choose prime numbers d_1 and d_2 large enough such that $d_\nu \equiv 1 \mod N$ and $d_1 > d_2$. We choose integers q_ν such that $q_1 d_1 + q_2 d_2 = 1$.

We consider a $\mathcal{G}_H$-module

$$W_\nu(K) := W_\nu \otimes V_{d_\nu,H} \simeq V_{d_\nu,H}^{\oplus N_\nu}$$

where $N_\nu = d_\nu^g$ and W_ν is a free $\mathcal{O}$-module of rank N_ν with trivial $\mathcal{G}_H$-action. Let σ_ν be the natural action of $\mathcal{G}_H$ on $W_\nu(K)$. In what follows we always consider σ_ν.

Let H_ν ($\nu = 1, 2$) be the Hilbert scheme parameterizing all closed polarized subschemes (Z_ν, L_ν) of $\mathbf{P}(W_\nu(K))$ such that

(a) Z_ν is $\mathcal{G}_H$-stable,

(b) $\chi(Z_\nu, nL_\nu) = n^g d_\nu^g |H|$, where $L_\nu = \mathbf{H}(W_\nu(K)) \otimes O_{Z_\nu}$ is the hyperplane bundle of Z_ν .

Since (a) and (b) are closed conditions, H_ν is a closed (hence projective) subscheme of $\mathrm{Hilb}^{\chi_\nu}(\mathbf{P}(W_\nu(K))$ where $\chi_\nu(n) = n^g d_\nu^g |H|$.

Let $\mathcal{O} = \mathcal{O}_N$. Let X_ν be the universal subscheme of $\mathbf{P}(W_\nu(K))$ over H_ν. Let $X = X_1 \times_{\mathcal{O}} X_2$ and $H_3 = H_1 \times_{\mathcal{O}} H_2$. Let $p_\nu : X_1 \times_{\mathcal{O}} X_2 \to X_\nu$ be the ν-th projection, $\pi : X \to H_3$ the natural projection. Hence X is a subscheme of $\mathbf{P}(W_1(K)) \times_{\mathcal{O}} \mathbf{P}(W_2(K)) \times_{\mathcal{O}} H_3$, flat over $H_3 = H_1 \times_{\mathcal{O}} H_2$.

We note that $\mathbf{H}(W_\nu(K))$ has a $\mathcal{G}_H$-linearization $\{\psi_g^{(\nu)}\}$, which we fix once for all. Since $\mathcal{G}_H$ transforms any closed $\mathcal{G}_H$-stable subscheme Z of $\mathbf{P}(W_\nu(K))$ onto itself, it follows that $\mathcal{G}_H$ acts on H_ν trivially. Hence, $\mathcal{G}_H$ transforms any fiber X_u of $\pi : X \to H_3$ onto X_u itself.

11.4. The scheme U_1

The aim of this and the subsequent subsections is to construct a new compactification of the moduli space of abelian varieties as the quotient of a certain $\mathcal{O}$-subscheme of $\mathrm{Hilb}^P(X/H_3)$ by $\mathrm{PGL}(W_1) \times \mathrm{PGL}(W_2)$.

Let B be the pullback to X of a very ample line bundle on H_3. Let $M_\nu = p_\nu^*(\mathbf{H}(W_i(K))) \otimes O_X$ and

$$M = d_2 M_1 + d_1 M_2 + B. \tag{40}$$

Then M is a very ample line bundle on X. Since M_ν is $\mathcal{G}_H$-linearized and B is trivially $\mathcal{G}_H$-linearized, M is $\mathcal{G}_H$-linearized.

Let $P(n) = (2nd_1d_2)^g|H|$. Let $\mathrm{Hilb}^P(X/H_3)$ be the Hilbert scheme parameterizing all closed subschemes Z of X contained in the fibers of $\pi : X \to H_3$ with $\chi(Z, nM_Z) = P(n)$, and Z^P be the universal subscheme of X over it. We denote $\mathrm{Hilb}^P(X/H_3)$ by H^P for brevity. Now using the double polarization trick of Viehweg, we define U_1 to be the subset of H^P consisting of all subschemes (Z, M_Z) of (X, M) with the properties

(i) Z is $\mathcal{G}_H$-stable,
(ii) $d_2 L_1 = d_1 L_2$, where $L_i = M_i \otimes O_Z$.

By Lemma 8.3, U_1 is a nonempty closed $\mathcal{O}$-subscheme of H^P. See [32, Subsec. 9.3].

11.5. The scheme U_2

Let U_2 be the open subscheme of U_1 consisting of all subschemes (Z, M_Z) of (X, M) such that besides (i)-(ii) the following are satisfied:

(iii) $p_{\nu|Z}$ is an isomorphism ($\nu = 1, 2$),
(iv) Z is reduced with $h^0(Z, O_Z) = 1$,
(v) $d_\nu L$ is very ample on Z, where $L = (q_1 M_1 + q_2 M_2) \otimes O_Z$,
(vi) $\chi(Z, nL) = n^g |H|$ for $n > 0$,
(vii) $H^q(Z, nL) = 0$ for $q > 0$ and $n > 0$,
(viii) $H^0(p_\nu^*) : W_\nu(K) \otimes k(u) \to \Gamma(Z, d_\nu L)$ is surjective (hence an isomorphism by (vi) and (vii)) for $\nu = 1, 2$.

Let $(Z, M_Z) \in \mathrm{Hilb}^P$. By (ii) and (v), we have $L = q_1 L_1 + q_2 L_2$ for $L_i = M_i \otimes O_Z$. Since $d_1 q_1 + d_2 q_2 = 1$, we have $L_\nu = d_\nu L$ by (ii). (iii) is an open condition by [3, Chap. 9, Lemma 7.5]. It is clear that (iv)-(viii) are open conditions. It follows that U_2 is a nonempty open $\mathcal{O}$-subscheme of U_1. See [32, Subsec. 9.5].

11.6. The schemes $U^\dagger_{g,K}$ and U_3

See [32, Subsec. 9.7]. First we note that if $(Z, L) \in U_2$, then $L = q_1 L_1 + q_2 L_2$. On each L_ν we have a $\mathcal{G}_H$-action on (Z, L_ν) induced from the $\mathcal{G}_H$-action ($= \mathcal{G}_H$-linearization) on Z^P induced from those $\mathcal{G}_H$-actions on $\mathbf{P}(W_\nu(K))$. By Remark 7.1.2, we have a $\mathcal{G}_H$-linearization on (Z, L). In what follows, we mean this $\mathcal{G}_H$-action on Z or (Z, L) by the (characteristic) $\mathcal{G}_H$-action on (Z, L) when $(Z, L) \in U_2$.

The locus $U_{g,K}$ of abelian varieties (with the zero not necessarily chosen) is an open subscheme of U_2. In fact, $U_{g,K}$ is the largest open $\mathcal{O}$-subscheme among all the open $\mathcal{O}$-subschemes H' of U_2 such that

(α) the projection $\pi_{H'} : Z^P \times_{H^P} H' \to H'$ is smooth over H',
(β) at least one geometric fiber of $\pi_{H'}$ is an abelian variety for each irreducible component of H'.

In general, the subset H'' of U_2 over which the projection $\pi_{H''} : Z^P \times_{H^P} H'' \to H''$ is smooth is an open $\mathcal{O}$-subscheme of U_2. By [26, Theorem 6.14], any geometric fiber of $\pi_{U_{g,K}}$ is a polarized abelian variety. See also [30, p. 705] and [32, p. 116].

Next we define $U^{\dagger}_{g,K}$ to be the subset of $U_{g,K}$ parameterizing all subschemes $(A, L) \in U_{g,K}$ such that

(ix) the K-action on A induced from the $\mathcal{G}_H$-action on (A, L) is effective and contained in $\mathrm{Aut}^0(A)$.

We see that $U^{\dagger}_{g,K}$ is a nonempty open $\mathcal{O}$-subscheme of $U_{g,K}$.

Finally we define U_3 to be the closure of $U^{\dagger}_{g,K}$ in U_2. It is the smallest closed $\mathcal{O}$-subscheme of U_2 containing $U^{\dagger}_{g,K}$.

We denote the pull back to $U^{\dagger}_{g,K}$ (resp. U_3) of the universal subscheme of X over $H^P = \mathrm{Hilb}^P(X/H_3)$ by

$$(A_{\mathrm{univ}}, L_{\mathrm{univ}}) \quad \text{resp.} \quad (Z_{\mathrm{univ}}, L_{\mathrm{univ}}). \tag{41}$$

□

Theorem 11.7. *Let R be a CDVR, $S :=$ Spec R, and η the generic point of S. Let h be a morphism from S into U_3. Let $(Z, \mathcal{L})$ be the pullback by h of the universal subscheme $(Z_{\mathrm{univ}}, L_{\mathrm{univ}})$ (41) such that $(Z_\eta, \mathcal{L}_\eta)$ is a polarized abelian variety. Then after a finite base change if necessary, $(Z, \mathcal{L})$ is isomorphic to $(P, \mathcal{L}_P)$ in Theorem 4.6. In particular, $(Z_0, \mathcal{L}_0)$ is a TSQAS over $k(0)$.*

Proof. The outline of the proof of Theorem is as follows. The generic fiber $(Z_\eta, \mathcal{L}_\eta)$ of $(Z, \mathcal{L})$ is an abelian variety. By Theorem 4.6 there exists an R^*-TSQAS $(P, \mathcal{L}_P)$ after a suitable base change Spec R^* of Spec R. So we have two flat families $(Z, \mathcal{L})_{R^*}$ and $(P, \mathcal{L}_P)$ over R^*, which we can now compare. For each of $(Z, \mathcal{L})$ and $(P, \mathcal{L}_P)$, we can find a natural level-$\mathcal{G}_H$ structure extending a level-$\mathcal{G}_H$ structure of $(Z_\eta, \mathcal{L}_\eta)$ $(= (P_\eta, \mathcal{L}_{P,\eta}))$. Then we can prove they are isomorphic. See [32, Theorem 10.4] for the details when $e_{\min}(H) \geq 3$. The case $e_{\min}(H) \leq 2$ is proved by reducing to the case $e_{\min}(H) \geq 3$ by Claims in Subsec. 11.10. See Claim 11.10.3. Q.E.D.

Theorem 11.8. *Let $G = \mathrm{PGL}(W_1) \times \mathrm{PGL}(W_2)$ and k an algebraically closed field with $k \ni 1/N$. Then*

(1) $U_3(k) = \left\{ (Z, L) \in U_2(k);\ \begin{array}{c}\text{a level-}\mathcal{G}_H \text{ TSQAS with}\\ \text{characteristic } \mathcal{G}_H \text{ action}\end{array} \right\}$

(2) *let $(Z, L) \in U_3(k)$ and $(Z', L') \in U_3(k)$ where $L = M \otimes O_Z$ and $L' = M \otimes O_{Z'}$ with the notation of Subsec 11.4 Eq.(40). Then the following are equivalent:*

  (a) *(Z, L) is $\mathcal{G}_H$-isomorphic to (Z', L') with respect to their characteristic $\mathcal{G}_H$-action in the sense of Remark 8.2.5,*

(b) *(Z, L) and (Z', L') have the same G-orbit.*

Proof. (1) is a corollary of Theorem 11.7. By the first assertion, any $(Z, L) \in U_3(k)$ has a natural characteristic $\mathcal{G}_H$-action. Thus (2) makes sense. See [32, Lemma 11.1] for a proof of (2). Q.E.D.

Theorem 11.9. *Let $G = \mathrm{PGL}(W_1) \times \mathrm{PGL}(W_2)$. Then*

(1) *$U^{\dagger}_{g,K}$ and U_3 are G-invariant,*

(2) *the action of G on $U^{\dagger}_{g,K}$ is proper and free (resp. proper with finite stabilizer) if $e_{\min}(H) \geq 3$ (resp. if $e_{\min}(H) \leq 2$),*

(3) *the action of G on U_3 is proper with finite stabilizer.*

(4) *the uniform geometric and uniform categorical quotient of U_3 (resp. $U^{\dagger}_{g,K}$) by G exists as a separated algebraic $\mathcal{O}$-space, which we denote by $SQ^{*\,\mathrm{toric}}_{g,K}$ (resp. $A^{\mathrm{toric}}_{g,K}$).*

See [32, Sec. 10-11] for Theorems 11.7 -11.9 when $e_{\min}(H) \geq 3$.

11.10. The case $e_{\min}(H) \leq 2$

Theorems 11.7-11.9 for $e_{\min}(H) \leq 2$ are proved in the same manner as in the case $e_{\min}(H) \geq 3$ by using the following Claims.

Claim 11.10.1. *Let k be an algebraically closed field with $k \ni 1/N$, $K = H \oplus H^{\vee}$ and $N = |H|$. Let (P, L) be a TSQAS over k with L $\mathcal{G}_H$-linearized and $\mathcal{G}(P, L) \simeq \mathcal{G}_H$, and n any positive integer (≥ 3) prime to both N and the characteristic of k. Then there exists a TSQAS $(P^{\dagger}, L^{\dagger})$ over k with the pull back $L^{\dagger}$ of L $\mathcal{G}_{H^{\dagger}}$-linearized which is an étale Galois covering of (P, L) with Galois group $H^{\dagger}/H \simeq (\mathbf{Z}/n\mathbf{Z})^g$, where H (resp. $H^{\dagger}$) is a maximal isotropic subgroup of $K := K(P, L) = H \oplus H^{\vee}$ (resp. of $K^{\dagger} := K(P^{\dagger}, L^{\dagger}) = H^{\dagger} \oplus (H^{\dagger})^{\vee} = K \oplus (\mathbf{Z}/n\mathbf{Z})^{2g}$).*

Proof. We denote the given TSQAS (P, L) by $(P_0, \mathcal{L}_0)$. Let R be a CDVR, $(P, \mathcal{L})$ an R-flat family such that

(i) the generic fiber $(P_{\eta}, \mathcal{L}_{\eta})$ is a level-$\mathcal{G}_H$ abelian variety,

(ii) the closed fiber $(P_0, \mathcal{L}_0)$ of $(P, \mathcal{L})$ is the given TSQAS with torus part T_0 and abelian part $(A_0, \mathcal{M}_0)$.

Since P_0 is a $k(0)$-TSQAS with $T_0 = \mathrm{Hom}(X, \mathbf{G}_m)$ for some lattice X of rank g'', there exists a sublattice Y of X such that $K(P_0, \mathcal{L}_0) = K(A_0, \mathcal{M}_0) \oplus (X/Y) \oplus (X/Y)^{\vee}$. See [30, 5.14] and Definition 6.2.2. Therefore it is enough to construct an étale $H^{\dagger}/H \simeq (\mathbf{Z}/n\mathbf{Z})^g$-covering $(A_0^{\dagger}, M_0^{\dagger})$ of (A_0, M_0) as above.

Hence we may assume P_0 is an abelian variety. In what follows we denote $(P_0, \mathcal{L}_0)$ by (A, L). Let $A[m] = \ker(m\,\mathrm{id}_A)$ for any positive integer m. By the assumption, $A[n^2] \simeq (\mathbf{Z}/n^2\mathbf{Z})^{2g}$ and $N^2 = |K(A, L)|$.

Let L' be the pull back of L by $n\,\mathrm{id}_A$. Then by [25, p. 56, Corollary 3; p. 71 (iv)] there exists $M \in \mathrm{Pic}^0(A)$ such that $L' = L^{n^2} \otimes M$. For a line bundle F on A, we denote by ϕ_F the homomorphism $A \to A^\vee$ defined by $x \mapsto T_x^* F \otimes F^{-1}$. Then by [25, p. 57, Corollary 4] $\phi_{L'} = \phi_{L^{n^2}} = n^2\phi_L$. Since $T_x^* M = M$, we have

$$K(A, L') := \ker(\phi_{L'}) = \ker n^2\phi_L = K(A, L^{n^2}) \supset A[n^2].$$

Since n is prime to N, we have $A[n^2] \cap K(A, L) = \{0\}$, hence

$$K(A, L') = K(A, L^{n^2}) = K(A, L) \oplus A[n^2].$$

For a maximal isotropic subgroup $G^\dagger$ ($\simeq (\mathbf{Z}/n^2\mathbf{Z})^g$) of $A[n^2]$, we define $\Delta^\dagger := (n\mathbf{Z}/n^2\mathbf{Z})^g$. It is the unique subgroup of $G^\dagger$ isomorphic to $(\mathbf{Z}/n\mathbf{Z})^g$. We set $A^\dagger := A/\Delta^\dagger$, and $\pi : A \to A^\dagger$ the projection. Now we have a diagram with $\varpi\pi = n\,\mathrm{id}_A$:

$$A \overset{\pi}{\to} A^\dagger = A/\Delta^\dagger \overset{\varpi}{\to} A/A[n] \simeq A.$$

As a subgroup of $K(A, L')$, we have

$$A[n^2] = \{0\} \oplus G^\dagger \oplus (G^\dagger)^\vee,$$
$$A[n] = \{0\} \oplus \Delta^\dagger \oplus (G^\dagger/\Delta^\dagger)^\vee,$$

where in particular $A[n]$ is a totally isotropic subgroup of $A[n^2]$.

Let $L^\dagger := \varpi^*(L)$. Then $L' = \pi^*(L^\dagger)$. Let $(\Delta^\dagger)^\perp$ be the orthogonal complement of $\Delta^\dagger$ in $K(A, L')$. Then by [20, p. 291]

$$K^\dagger := K(A^\dagger, L^\dagger) \simeq (\Delta^\dagger)^\perp/\Delta^\dagger,$$

where we see $(\Delta^\dagger)^\perp = K(A, L) \oplus G^\dagger \oplus (G^\dagger/\Delta^\dagger)^\vee$, where $(G^\dagger/\Delta^\dagger)^\vee \simeq (n\mathbf{Z}/n^2\mathbf{Z})^g$. Let H be a maximal isotropic subgroup of $K(A, L)$. Let $H^\dagger := H \oplus \{0\} \oplus (G^\dagger/\Delta^\dagger)^\vee \subset K^\dagger$. Then $H^\dagger$ is a maximal isotropic subgroup of $K^\dagger$ with $(H^\dagger)^\vee = H^\vee \oplus (G^\dagger/\Delta^\dagger) \oplus \{0\}$. It follows

$$K^\dagger \simeq K(A, L) \oplus (G^\dagger/\Delta^\dagger) \oplus (G^\dagger/\Delta^\dagger)^\vee \simeq H^\dagger \oplus (H^\dagger)^\vee. \tag{42}$$

Hence the covering $\varpi : A^\dagger \to A$ is étale with Galois group

$$A[n]/\Delta^\dagger \simeq (G^\dagger/\Delta^\dagger)^\vee \simeq H^\dagger/H \simeq (\mathbf{Z}/n\mathbf{Z})^g,$$

and $\mathcal{L}^\dagger$ is $\mathcal{G}_{H^\dagger}$-linearized by (42). This proves Claim 11.10.1. Q.E.D.

Claim 11.10.2. (See also [32, Lemma 6.7]) Let R be a complete discrete valuation ring, $k(\eta)$ the fraction field of R and $S := \operatorname{Spec} R$. Let (Z_i, ϕ_i^*, τ_i) $(i = 1, 2)$ be rigid-$\mathcal{G}_H$ S-TSQASes whose generic fibers are abelian varieties. If (Z_i, ϕ_i^*, τ_i) are $k(\eta)$-isomorphic, then they are S-isomorphic.

Claim 11.10.2 follows from the following Claim 11.10.3.

Claim 11.10.3. *With the same notation as above, let $(P, \mathcal{L})$ be an S-TSQAS with generic fiber $(P_\eta, \mathcal{L}_\eta)$ an abelian variety. Then $(P, \mathcal{L})$ is the normalization of a modified Mumford family with generic fiber $(P_\eta, \mathcal{L}_\eta)$ by a finite base change if necessary.*

Proof. Let n be a positive integer ≥ 3 prime to the characteristic of $k(0)$ and $|H|$. In view of Claim 11.10.1, by a finite base change $S^\dagger$ of S and then by taking the pull back of $(P, \mathcal{L})$ to $S^\dagger$, we have an étale $H^\dagger/H \simeq (\mathbf{Z}/n\mathbf{Z})^g$-covering $(P_0^\dagger, \mathcal{L}_0^\dagger)$ of $(P_0, \mathcal{L}_0)$ such that $K(P_0^\dagger, \mathcal{L}_0^\dagger) = H^\dagger \oplus (H^\dagger)^\vee$. From now, we denote $S^\dagger$ by S, and $(P, \mathcal{L}) \times_S S^\dagger$ by $(P, \mathcal{L})$.

Let P_{for} be the formal completion of P along P_0. By [11, Corollaire 8.4], there is a category equivalence between étale coverings of P_0 and étale coverings of P_{for}. Hence there exists a formal scheme $(P_{\text{for}}^\dagger, \mathcal{L}_{\text{for}}^\dagger)$ which is an étale $(\mathbf{Z}/n\mathbf{Z})^g$-covering of $(P_{\text{for}}, \mathcal{L}_{\text{for}})$. Then there exists a projective S-scheme $(P^\dagger, \mathcal{L}^\dagger)$ algebraizing $(P_{\text{for}}^\dagger, \mathcal{L}_{\text{for}}^\dagger)$ which is an étale $(\mathbf{Z}/n\mathbf{Z})^g$-covering of $(P, \mathcal{L})$ with $\mathcal{L}^\dagger$ the pull back of $\mathcal{L}$. It follows that the generic fiber $(P_\eta^\dagger, \mathcal{L}_\eta^\dagger)$ is a polarized abelian variety, and $(P_0^\dagger, \mathcal{L}_0^\dagger)$ is a reduced $k(0)$-TSQAS and $P^\dagger$ is normal by Claim 4.7.1.

Since $n \geq 3$, by [32, 10.4] $(P^\dagger, \mathcal{L}^\dagger)$ is the normalization of a modified Mumford family with generic fiber $(P_\eta^\dagger, \mathcal{L}_\eta^\dagger)$. By [11, Corollaire 8.4] $(P, \mathcal{L})$ is the quotient of $(P^\dagger, \mathcal{L}^\dagger)$ by $(\mathbf{Z}/n\mathbf{Z})^g$, because $(P_0, \mathcal{L}_0)$ is the quotient of $(P_0^\dagger, \mathcal{L}_0^\dagger)$ by $(\mathbf{Z}/n\mathbf{Z})^g$. Hence $(P, \mathcal{L})$ is the normalization of a modified Mumford family with generic fiber $(P_\eta, \mathcal{L}_\eta)$. This proves the Claim. Q.E.D.

Summary 11.11. Let k be an algebraically closed field with $k \ni 1/N$. Let $H^P := \operatorname{Hilb}^P(X/H_3)$ be as in Subsec. 11.4. We define the schemes U_k, $U_{g,K}$ and $U_{g,K}^\dagger$ as follows:

$$U_1 = \{(Z, L_1, L_2) \in H^P; \text{(i)-(ii) are true}\},$$

$$U_2 = \{(Z, L) \in U_1; \text{(iii)-(viii) are true}\},$$

$$U_{g,K}(k) = \{(Z, L) \in U_2(k); (Z, L) \text{ is an abelian variety over } k\},$$

$$U_{g,K}^\dagger(k) = \{(Z, L) \in U_{g,K}(k); \text{(ix) is true}\},$$

$$U_3 = \text{the closure of } U_{g,K}^\dagger \text{ in } U_2.$$

Then

(1) U_1 is a closed $\mathcal{O}$-subscheme of H^P, while U_2, $U_{g,K}$ and $U^{\dagger}_{g,K}$ are nonempty $\mathcal{O}$-subschemes of U_1 such that $U^{\dagger}_{g,K} \subset U_{g,K} \subset U_2$, and

$$U^{\dagger}_{g,K}(k) = \left\{(A, L) \in U_2(k); \begin{matrix}\text{an abelian variety over } k \text{ with} \\ \text{characteristic } \mathcal{G}_H\text{-action}\end{matrix}\right\}$$

$$U_3(k) = \left\{(Z, L) \in U_2(k); \begin{matrix}\text{a level-}\mathcal{G}_H \text{ TSQAS over } k \text{ with} \\ \text{characteristic } \mathcal{G}_H \text{ action}\end{matrix}\right\},$$

(2) $(Z', L') \in U_3(k)$, $(Z, L) \in U_3(k)$ are $\mathcal{G}_H$-isomorphic iff they are in the same G-orbit, where $G = \mathrm{PGL}(W_1) \times \mathrm{PGL}(W_2)$,

(3) there exists a nice quotient $A^{\mathrm{toric}}_{g,K}$ of $U^{\dagger}_{g,K}$ by G,

(4) there exists a nice quotient $SQ^{*\,\mathrm{toric}}_{g,K}$ of U_3 by G,

(5) let $SQ^{\mathrm{toric}}_{g,K} := (SQ^{*\,\mathrm{toric}}_{g,K})_{\mathrm{red}}$.

See [32, Corollaries 10.5, 10.6] for $U_3(k)$.

§12. Moduli for TSQASes

Let $\mathcal{O} = \mathcal{O}_N$. In this section we prove

(i) $A^{\mathrm{toric}}_{g,K}$ is the coarse moduli algebraic $\mathcal{O}$-space for the functor of level-$\mathcal{G}_H$ smooth TSQASes over algebraic $\mathcal{O}$-spaces for any $e_{\min}(K)$,

(ii) $A^{\mathrm{toric}}_{g,K} \simeq A_{g,K}$ if $e_{\min}(K) \geq 3$, which is the fine moduli scheme.

We also see

(iii) $SQ^{\mathrm{toric}}_{g,K}$ is the coarse moduli algebraic $\mathcal{O}$-space for the functor of level-$\mathcal{G}_H$ flat TSQASes over reduced algebraic $\mathcal{O}$-spaces,

(iv) if $e_{\min}(K) \geq 3$, there exists a natural morphism $\mathrm{sq} : SQ^{\mathrm{toric}}_{g,K} \to SQ_{g,K}$, which is surjective and bijective on $SQ^{\mathrm{toric}}_{g,K}$, and the identity on $A_{g,K}$, hence $SQ^{\mathrm{toric}}_{g,K}$ is a projective $\mathcal{O}$-scheme.

Theorem 12.1. *Let $K = H \oplus H^{\vee}$ and $N := |H|$.*

(1) *If $e_{\min}(H) \geq 3$, then $A^{\mathrm{toric}}_{g,K} \simeq A_{g,K}$ and $\mathcal{A}^{\mathrm{toric}}_{g,K}$ is represented by the quasi-projective formally smooth $\mathcal{O}$-scheme $A_{g,K}$,*

(2) *if $e_{\min}(H) \leq 2$, then $\mathcal{A}^{\mathrm{toric}}_{g,K}$ has a normal coarse moduli algebraic $\mathcal{O}$-space $A^{\mathrm{toric}}_{g,K}$.*

Proof. We can prove this almost in parallel to Theorem 9.5.

Let $\mathcal{O} = \mathcal{O}_N$. Let d_ν, W_ν and $W_\nu(K) = W_\nu \otimes_{\mathcal{O}} V_{d_\nu,H}$ be the same as in Subsec. 11.3. Similarly let (X_ν, L_ν), H_ν, (X, L) and $H_3 = H_1 \times_{\mathcal{O}} H_2$ be the same as in Subsections 11.4–11.5.

Step 1. Let T be any $\mathcal{O}$-scheme, and $(P, \mathcal{L}, \phi^*, \mathcal{G}, \tau)$ any level-$\mathcal{G}_H$ T-smooth TSQAS with $\pi : P \to T$ the projection. Then we define a natural morphism $\bar{\eta} : T \to A_{g,K}^{\text{toric}}$ as follows.

The sheaf $\pi_*(d_\nu \mathcal{L})$ is a vector bundle of rank $d_\nu^g N$ over T. Let U_i be an affine covering of T which trivializes both $\pi_*(d_\nu \mathcal{L})$. Then

$$\Gamma(U_i, \pi_*(d_\nu \mathcal{L})) = \Gamma(P_{U_i}, d_\nu \mathcal{L}) \simeq (\mathcal{W}_\nu)_{U_i} \otimes_{\mathcal{O}} V_{d_\nu, H}$$

for some locally O_T-free module $\mathcal{W}_\nu$ of rank d_ν^g with trivial $\mathcal{G}$-action.

Since $d_\nu \mathcal{L}_t$ is very ample, we can choose closed $\mathcal{G}$-immersions

$$(\phi_\nu)_{U_i} : P_{U_i} \to \mathbf{P}(W_\nu(K))_{U_i}$$

by the linear system associated to $\pi_*(d_\nu \mathcal{L})_{U_i}$ such that

$$\rho((\phi_\nu)^*_{U_i}, \tau_{U_i}) = \mathrm{id}_{W_\nu} \otimes U_{d_\nu, H} \tag{43}$$

We caution that $(\phi_\nu)_{U_i}$ is not unique, there is freedom of isomorphisms by $\mathrm{GL}(W_\nu, O_{U_i})$.

By (43) the image of $(\phi_\nu)_{U_i}$ is $\mathcal{G}$-invariant, so the image of $(\phi_\nu)_t$ is $\mathcal{G}_H$-invariant for any $t \in T$, Since $\mathcal{L} = q_1 d_1 \mathcal{L} + q_2 d_2 \mathcal{L}$, $\mathcal{L}_{U_i}$ is $\mathcal{G}_{U_i}$-linearized. Hence $(P_{U_i}, \mathcal{L}_{U_i})$ has a $\mathcal{G}_{U_i}$-action, that is, fiberwise $(P_t, \mathcal{L}_t)$ has a $\mathcal{G}_H$-action. By the definition of level-$\mathcal{G}_H$ TSQASes, this $\mathcal{G}_H$-action on $(P_t, \mathcal{L}_t)$ is characteristic. Hence the image of $(\phi_\nu)_{U_i}$ is contained in $U_{g,K}^\dagger$ by Theorem 11.8 or Summary 11.11. It follows that $(P_{U_i}, \mathcal{L}_{U_i})$ is the pull back by a morphism $U_i \to U_{g,K}^\dagger$ of the universal subscheme (X, H_3) in Subsec 11.3.

On $U_i \cap U_j$, $\Gamma(U_i, \pi_*(d_\nu \mathcal{L}))$ and $\Gamma(U_j, \pi_*(d_\nu \mathcal{L}))$ are identified by $\mathrm{GL}(W_\nu \otimes \Gamma(O_{U_i \cap U_j}))$. Thus we have a morphism

$$j : T \to U_{g,K}^\dagger / \mathrm{PGL}(W_1) \times \mathrm{PGL}(W_2) = A_{g,K}^{\text{toric}},$$

where $G = \mathrm{PGL}(W_1) \times \mathrm{PGL}(W_2)$. This induces a morphism of functors

$$f : \mathcal{A}_{g,K}^{\text{toric}} \to h_W, \quad W := A_{g,K}^{\text{toric}}. \tag{44}$$

The argument so far is true regardless of the value of $e_{\min}(H)$.

Step 2. Now we assume $e_{\min}(H) \geq 3$.

Step 2-1. Any level-$\mathcal{G}_H$ T-smooth TSQAS is a level-$\mathcal{G}_H$ T-smooth PSQAS with $\mathcal{V} = \pi_*(\mathcal{L})$, and vice versa. Hence the functors are the same : $\mathcal{A}_{g,K}^{\text{toric}} = \mathcal{A}_{g,K}$.

Step 2-2. Now we assume $e_{\min}(H) \geq 3$. There is the universal subscheme over $U_{g,K}^\dagger$ (41)

$$(A_{\text{univ}}, \mathcal{V}_{\text{univ}}, L_{\text{univ}}, \phi_{\text{univ}}, \mathcal{G}_{\text{univ}}, \tau_{\text{univ}})$$

where $\mathcal{G}_{\mathrm{univ}} = \mathcal{G}_H \times U^{\dagger}_{g,K}$, $\tau_{\mathrm{univ}} = U_H$ (acting on $\mathbf{P}(V_H)_{U^{\dagger}_{g,K}}$), $\mathcal{V}_{\mathrm{univ}} = V_H \otimes O_{U^{\dagger}_{g,K}}$ and we choose a closed immersion $\phi_{\mathrm{univ}} : A_{\mathrm{univ}} \to \mathbf{P}(V_H)_{U^{\dagger}_{g,K}}$, such that $\rho(\phi_{\mathrm{univ}}, \tau_{\mathrm{univ}}) = U_H$. This is a rigid level-$\mathcal{G}_H$ $U^{\dagger}_{g,K}$-smooth PSQAS. Hence we have a morphism $\eta^{\dagger} : U^{\dagger}_{g,K} \to A_{g,K}$ because $A_{g,K}$ is the fine moduli scheme of $\mathcal{A}_{g,K}$ by Theorem 9.5. Since the morphism $\eta^{\dagger}$ is $G = \mathrm{PGL}(W_1) \times \mathrm{PGL}(W_2)$-invariant, we have a morphism

$$\bar{\eta} : A^{\mathrm{toric}}_{g,K} \to A_{g,K}.$$

Step 2-3. Conversely since $A_{g,K}$ is the fine moduli scheme for $\mathcal{A}_{g,K}$, there exists the universal level-$\mathcal{G}_H$ PSQAS

$$\pi_A : (Z_A, \mathcal{V}_A, L_A, \phi_A, \mathcal{G}_A, \tau_A) \to A_{g,K}.$$

Then we apply Step 1 to the universal level-$\mathcal{G}_H$ PSQAS over $A_{g,K}$. We have a morphism from $A_{g,K}$ to $A^{\mathrm{toric}}_{g,K}$, which is evidently the inverse of $\bar{\eta}$. This proves that $\bar{\eta}$ is an isomorphism. This proves the first assertion of Theorem 12.1 by Theorem 9.5. See [32, Lemma 11.5].

Step 3. We consider next the case $e_{\min}(H) \leq 2$. By Step 1 (44), we have a morphism of functors $f : \mathcal{A}^{\mathrm{toric}}_{g,K} \to h_W$ where $W := A^{\mathrm{toric}}_{g,K}$. To prove that $A^{\mathrm{toric}}_{g,K}$ is a coarse moduli algebraic $\mathcal{O}$-space for $\mathcal{A}^{\mathrm{toric}}_{g,K}$, it remains to prove

(a) $f(\mathrm{Spec}\ k) : \mathcal{A}^{\mathrm{toric}}_{g,K}(\mathrm{Spec}\ k) \to A^{\mathrm{toric}}_{g,K}(\mathrm{Spec}\ k)$ is bijective for any algebraically closed field k over $\mathcal{O}$,

(b) For any algebraic $\mathcal{O}$-space V, and any morphism $g : \mathcal{A}^{\mathrm{toric}}_{g,K} \to h_V$, there is a unique morphism $\chi : h_W \to h_V$ such that $g = \chi \circ f$,

where $W = A^{\mathrm{toric}}_{g,K}$, h_V is the functor defined by $h_V(T) = \mathrm{Hom}(T, V)$.

The assertion (b) is proved similarly to Step 1 and Step 2-2.

The assertion (a) follows from Theorem 11.8. In fact, let

$$\sigma_j := (Z_j, L_j, \phi^*_j, \mathcal{G}_H, \tau_j)$$

be a level $\mathcal{G}_H$ smooth k-TSQAS. Since $A^{\mathrm{toric}}_{g,K}$ is the orbit space of $U_{g,K}$ by $G := \mathrm{PGL}(W_1) \times \mathrm{PGL}(W_2)$, (Z_1, L_1) and (Z_2, L_2) determine the same point of $A^{\mathrm{toric}}_{g,K}$ iff (Z_1, L_1) and (Z_2, L_2) have the same G-orbit. By Theorem 11.8, (Z_1, L_1) and (Z_2, L_2) have the same G-orbit iff (Z_1, L_1) and (Z_2, L_2) are $\mathcal{G}_H$-isomorphic with respect to their characteristic $\mathcal{G}_H$-action in the sense of Remark 8.2.5. Thus it suffices to prove that $\sigma_1 \simeq \sigma_2$ iff (Z_1, L_1) and (Z_2, L_2) are $\mathcal{G}_H$-isomorphic.

If $\sigma_1 \simeq \sigma_2$, then by definition $(Z_1, L_1) \simeq (Z_2, L_2)$.

Conversely assume $(Z_1, L_1) \simeq (Z_2, L_2)$ $\mathcal{G}_H$-isomorphic with respect to their characteristic $\mathcal{G}_H$-action. Let $f : (Z_1, L_1) \to (Z_2, L_2)$ be the $\mathcal{G}_H$-isomorphism. Hence $(f^*)^{-1}\rho_{\tau_1,L_1}(g)f^* = \rho_{\tau_2,L_2}(g)$. Meanwhile we can choose a $\mathcal{G}_H$-isomorphism $\phi_j^* : V_H \otimes k \to \Gamma(Z_j, L_j)$ such that $\rho(\phi_j^*, \tau_j) = U_H$. Let $h := (\phi_1^*)^{-1}f^*\phi_2^*$. Then we see $U_H h = hU_H$. Since U_H is an irreducible representation of $\mathcal{G}_H$, h is a nonzero scalar. Hence $f^*\phi_2^* = c\phi_1^*$ for some unit c. It follows from Definition 10.2.3 that $\sigma_1 \simeq \sigma_2$. This proves (a). Thus $A_{g,K}^{\text{toric}}$ is a coarse moduli algebraic $\mathcal{O}$-space for $\mathcal{A}_{g,K}^{\text{toric}}$.

Step 4. Finally we prove that $A_{g,K}^{\text{toric}}$ is reduced for $e_{\min}(H) \le 2$. We use the same notation as in the proof of Theorem 9.5. Let k be any algebraically closed field with $k \ni 1/N$, (A, L_0) be an abelian variety over k with L_0 $\mathcal{G}_H$-linearized, and τ_0 be the $\mathcal{G}_H$-action associated to the $\mathcal{G}_H$-linearization of L_0. Let $\sigma_0 := (A, L_0, \phi_0^*, \mathcal{G}_H, \tau_0)$ be a rigid level-$\mathcal{G}_H$ k-smooth TSQAS.

Let $\mathcal{C} = \mathcal{C}_W$ be the category of local Artinian W-algebra with $k = R/m_R$. We define a subfunctor $F := F_{\sigma_0}$ of $\mathcal{A}_{g,K}^{\text{toric}}$ by

$$F(R) = \left\{\sigma := (Z, L, \phi^*, (\mathcal{G}_H)_R, \tau) \in \mathcal{A}_{g,K}^{\text{toric}}(R); \sigma \otimes k \simeq \sigma_0\right\}$$

where $R \in \mathcal{C}$ and the isomorphism $\sigma \otimes k \simeq \sigma_0$ is not fixed in F.

Let $(X, \mathcal{L})$, $K_{\text{su}} = \ker(\lambda(\mathcal{L}))$, $\mathcal{G}_{\text{su}} := \mathcal{G}(X, \mathcal{L}) := \mathcal{L}_{K_{\text{su}}}^{\times}$, $\mathcal{V}_{\text{su}} := \Gamma(X, \mathcal{L})$ and the action τ_{su} of $\mathcal{G}_{\text{su}}$ on $(X, \mathcal{L})$ be the same as in Subsec 9.4. Since $\lambda(\mathcal{L}) : X \to X^\vee$ is separable, K_{su} is isomorphic to $(H \oplus H^\vee)_{\mathcal{O}_W}$, hence $\mathcal{G}_{\text{su}} \simeq (\mathcal{G}_H)_{\mathcal{O}_W}$. If $e_{\min}(K_{\text{su}}) \ge 3$, we choose the unique closed $\mathcal{G}_H$-immersion ϕ_{su} of X into $\mathbf{P}(\mathcal{V}_{\text{su}}) \simeq \mathbf{P}(V_H)_{\mathcal{O}_W}$ such that $\rho(\phi_{\text{su}}, \tau_{\text{su}}) = U_H$. If $e_{\min}(K_{\text{su}}) \le 2$, then we choose the unique $\mathcal{G}_H$-isomorphism $\phi_{\text{su}}^* : (V_H)_{\mathcal{O}_W} \to \Gamma(X, \mathcal{L})$ such that $\rho(\phi_{\text{su}}^*, \tau_{\text{su}}) = U_H$. In any case we have a level-$\mathcal{G}_H$ smooth TSQAS over $\mathcal{O}_W$

$$(X, \mathcal{L}, \mathcal{V}_{\text{su}}, \phi_{\text{su}}^*, \mathcal{G}_{\text{su}}, \tau_{\text{su}}).$$

Now we shall define a morphism of functors $h : P(A, \lambda(L_0)) \to F$ over $\mathcal{C} = \mathcal{C}_W$. Let $R \in \mathcal{C}$. By Subsec 9.4, for $(Z, \lambda(L)) \in P(A, \lambda(L_0))(R)$, $R \in \mathcal{C}$, we have a unique morphism

$$\rho \in \text{Hom}(\text{Spec } R, \text{Spf } \mathcal{O}_W) = \text{Hom}_{\hat{\mathcal{C}}}(\mathcal{O}_W, R)$$

such that $(Z, \lambda(L)) = \rho^*(X, \lambda(\mathcal{L}))$. Then we define

$$h(Z, \lambda(L)) = \rho^*(X, \mathcal{L}, \mathcal{V}_{\text{su}}, \phi_{\text{su}}^*, \mathcal{G}_{\text{su}}, \tau_{\text{su}}) \in F(R).$$

One can check that this is well-defined.

Subsec. 9.4 shows that $h(R) : P(A, \lambda(L_0))(R) \to F(R)$ is surjective for any $R \in \mathcal{C}$. In general, h is not injective. Let

$$G_0 := \mathrm{Aut}(\sigma_0) = \{f \in \mathrm{Aut}(A); f(0) = 0,\ f^*\sigma_0 \simeq \sigma_0\},$$

where 0 is the zero of A. Since $f^*L_0 \simeq L_0$ for any $f \in G_0$, we have $f^*(3L_0) \simeq 3L_0$. Since $3L_0$ is very ample, G_0 is an algebraic k-group. G_0 has trivial connected part because $f(0) = 0$ for any $f \in G_0$. Hence G_0 is a finite group scheme, acting nontrivially on $P(A, \lambda(L_0))$. Then

$$\begin{aligned} F(R) &= P(A, \lambda(L_0))(R)/G_0 \\ &= \mathrm{Hom}(\mathcal{O}_W/\mathfrak{a}, R)/G_0 \\ &= \mathrm{Hom}((\mathcal{O}_W/\mathfrak{a})^{G_0-\mathrm{inv}}, R) \end{aligned}$$

whence F is pro-represented by $(\mathcal{O}_W/\mathfrak{a})^{G_0-\mathrm{inv}}$, which is normal. This proves that the formal completion of any local ring of $A^{\mathrm{toric}}_{g,K}$ is normal. Hence it satisifies (R_1) and (S_2) by Serre's criterion. See Remark 12.1.1. This implies that any local ring of $A^{\mathrm{toric}}_{g,K}$ satisfies (R_1) and (S_2). Hence $A^{\mathrm{toric}}_{g,K}$ is normal. Q.E.D.

Remark 12.1.1. Let A be a noetherian local ring. Then A is normal if and only if (R_1) and (S_2) are true for A, where

(1) (S_2) is true if and only if $\mathrm{depth}(A_p) \geq \inf(2, \mathrm{ht}(p))$ for all $p \in \mathrm{Spec}\,(A)$,

(2) (R_1) is true if and only if A is codimension one regular.

See [19, Theorem 39] and [10, IV_2, 5.8.5 and 5.8.6].

Theorem 12.2. ([32]) *Let $N = |H|$ and $SQ^{\mathrm{toric}}_{g,K} = (SQ^{*\,\mathrm{toric}}_{g,K})_{\mathrm{red}}$. For any $K = H \oplus H^\vee$, the functor $\mathcal{SQ}^{\mathrm{toric}}_{g,K}$ of level-$\mathcal{G}_H$ TSQASes (P, ϕ^*, τ) over reduced algebraic $\mathcal{O}$-spaces is coarsely represented by a proper (hence separated) reduced algebraic $\mathcal{O}$-space $SQ^{\mathrm{toric}}_{g,K}$.*

Proof. We imitate the proof of Theorem 12.1. Let

$$(P \xrightarrow{\pi} T, L, \phi^*, \mathcal{G}, \tau)$$

be a level-$\mathcal{G}_H$ T-flat TSQAS with T reduced. Then by Step 1 of Theorem 12.1, we have a morphism

$$j : T \to U_3/G = SQ^{*\,\mathrm{toric}}_{g,K},$$

where $G = \mathrm{PGL}(W_1) \times \mathrm{PGL}(W_2)$. Hence we have a morphism

$$j_{\mathrm{red}} : T_{\mathrm{red}} = T \to (SQ^{*\,\mathrm{toric}}_{g,K})_{\mathrm{red}} =: SQ^{\mathrm{toric}}_{g,K}.$$

This induces a morphism of functors

$$f : \mathcal{SQ}^{\text{toric}}_{g,K} \to h_W, \quad W = SQ^{\text{toric}}_{g,K}. \tag{45}$$

As in Theorem 12.1 Step 3, it remains to prove

(a) $f(\text{Spec } k) : \mathcal{SQ}^{\text{toric}}_{g,K}(\text{Spec } k) \to SQ^{\text{toric}}_{g,K}(\text{Spec } k)$ is bijective for any algebraically closed field k over $\mathcal{O}$,

(b) For any algebraic $\mathcal{O}$-space V, and any morphism $g : \mathcal{SQ}^{\text{toric}}_{g,K} \to h_V$, there is a unique morphism $\chi : h_W \to h_V$ such that $g = \chi \circ f$,

where h_V is the functor defined by $h_V(T) = \text{Hom}(T, V)$. For a reduced space T, $h_V(T) = h_{V_{\text{red}}}(T)$, that is, $h_V = h_{V_{\text{red}}}$ over $\mathit{Space}_{\text{red}}$. Hence we may assume V is reduced.

We shall prove (b). Let $g : \mathcal{SQ}^{\text{toric}}_{g,K} \to h_V$ be any morphism for a reduced algebraic $\mathcal{O}$-space V. The universal subscheme $(Z_{\text{univ}}, L_{\text{univ}})$ has a natural $\mathcal{G}_H$-action which is characteristic for any fiber $(Z_{\text{univ},u}, L_{\text{univ},u})$ $(u \in U_3)$. We choose $\phi^*_{\text{univ}} = \text{id}_{V_H \otimes O_{U_3}}$. Thus we have a rigid level-$\mathcal{G}_H$ U_3-flat TSQAS $(Z_{\text{univ}}, L_{\text{univ}}, \phi^*_{\text{univ}}, \mathcal{G}_H, \tau_{\text{univ}})$ over U_3. Hence by $g : \mathcal{SQ}^{\text{toric}}_{g,K} \to h_V$ we have a morphism $\widetilde{\chi} : U_3 \to V$, which turns out to be G-invariant. Hence we have a morphism $\bar{\chi} : SQ^{*\,\text{toric}}_{g,K} \to V$, hence $\chi := \bar{\chi}_{\text{red}} : SQ^{\text{toric}}_{g,K} \to V_{\text{red}} = V$. It is clear that $g = \chi \circ f$.

By the same argument as in the proof of Theorem 12.1 Step 3 (a), we see $\mathcal{SQ}^{\text{toric}}_{g,K}(\text{Spec } k) = SQ^{*\,\text{toric}}_{g,K}(k) = SQ^{\text{toric}}_{g,K}(k)$. This proves (a). This completes the proof. Q.E.D.

Theorem 12.3. ([32]) *Suppose $e_{\min}(K) \geq 3$. Then*

(1) *both $SQ_{g,K}$ and $SQ^{\text{toric}}_{g,K}$ are compactifications of $A_{g,K}$,*

(2) *there exists a bijective $\mathcal{O}$-morphism*

$$\text{sq} : SQ^{\text{toric}}_{g,K} \to SQ_{g,K}$$

extending the identity of $A_{g,K}$,

(3) *their normalizations are isomorphic :*

$$(SQ^{\text{toric}}_{g,K})^{\text{norm}} \simeq (SQ_{g,K})^{\text{norm}}.$$

Corollary 12.4. *$SQ^{\text{toric}}_{g,K}$ is a projective scheme if $e_{\min}(K) \geq 3$.*

Proof. Since $SQ^{\text{toric}}_{g,K}$ is finite over $SQ_{g,K}$ and $SQ_{g,K}$ is a scheme, $SQ^{\text{toric}}_{g,K}$ is a scheme by [18, Theorem 4.1, p. 169], hence it is a projective scheme because $SQ_{g,K}$ is projective by (33). Q.E.D.

§13. Morphisms to Alexeev's complete moduli spaces

In this section

(i) we briefly review Alexeev [1],

(ii) then report that

  (a) any T-flat TSQAS has a canonical semi-abelian action,

  (b) $SQ_{g,1}^{\mathrm{toric}} \simeq \overline{AP}_{g,1}^{\mathrm{main}}$.

Definition 13.1. [1] Let k be an algebraically closed field. A g-dimensional semiabelic k-pair of degree d is a quadruple $(G, P, \mathcal{L}, \Theta)$ such that

(i) P is a connected seminormal *complete* k-variety, and any irreducible component of P is g-dimensional,

(ii) G is a semi-abelian k-scheme acting on P,

(iii) there are only finitely many G-orbits,

(iv) the stabilizer subgroup of every point of P is connected, reduced and lies in the torus part of G,

(v) $\mathcal{L}$ is an ample line bundle on P with $h^0(P, \mathcal{L}) = d$,

(vi) Θ is an effective Cartier divisor of P with $\mathcal{L} = O_P(\Theta)$ which does not contain any G-orbits.

Recall that a variety Z is said to be *seminormal* if any bijective morphism $f : W \to Z$ with W reduced is an isomorphism.

Definition 13.2. Let T be a scheme. A g-dimensional semiabelic T-pair of degree d is a quadruple $(G, P \overset{\pi}{\to} T, \mathcal{L}, \Theta)$ such that

(i) G is a semi-abelian group T-scheme of relative dimension g,

(ii) P is a proper flat T-scheme, on which G acts,

(iii) $\mathcal{L}$ is a π-ample line bundle on P with $\pi_*(\mathcal{L})$ locally free,

(iv) any geometric fiber $(G_t, P_t, \mathcal{L}_t, \Theta_t)$ $(t \in T)$ is a stable semiabelic pair of degree d.

Definition 13.3. We define two functors: for any scheme T

$$\overline{\mathcal{AP}}_{g,d}(T) = \left\{(G, P \overset{\pi}{\to} T, D);\ \text{semi-abelic } T\text{-pair of degree } d\right\} /T\text{-isom.},$$

$$\mathcal{AP}_{g,d}(T) = \left\{\begin{matrix}(G, A \overset{\pi}{\to} T, D);\ \text{semi-abelic } T\text{-pair of degree } d \\ G \text{ is an abelian } T\text{-scheme}\end{matrix}\right\} /T\text{-isom.}.$$

Theorem 13.4. (Alexeev [1, 5.10.1])

(1) *The component $\overline{\mathcal{AP}}_{g,d}$ of the moduli stack of semiabelic pairs containing the moduli stack $\mathcal{AP}_{g,d}$ of abelian pairs as well as pairs of the same numerical type is a proper Artin stack with finite stabilizer,*

(2) *It has a proper coarse moduli algebraic space $\overline{AP}_{g,d}$ over* $\mathbf{Z}$.

13.5. The components of $\overline{AP}_{g,d}$

In order to compare $\overline{AP}_{g,d}$ with $SQ^{\mathrm{toric}}_{g,K}$ we consider the pullback of $\overline{AP}_{g,d}$ to $\mathcal{O}_d$, which we denote $\overline{AP}_{g,d}$ by abuse of notation. Let $\overline{AP}^{\mathrm{main}}_{g,d}$ be the closure of $AP_{g,d}$ in $\overline{AP}_{g,d}$. $\overline{AP}^{\mathrm{main}}_{g,d} \neq \overline{AP}_{g,d}$ in general.

We define some algebraic subspaces of $\overline{AP}_{g,d}$ as follows:

$$\begin{aligned} AP_{g,d} &= \{(A,D) \in \overline{AP}_{g,d}; A : \text{ nonsingular}\},\\ AP_{g,K} &= \{(A,D) \in AP_{g,d}; \ker(\lambda(D)) \simeq K\},\\ \overline{AP}_{g,K} &= \text{the closure of } AP_{g,K} \text{ in } \overline{AP}_{g,d},\\ \overline{AP}^{\mathrm{main}}_{g,d} &= \text{the closure of } AP_{g,d} \text{ in } \overline{AP}_{g,d}. \end{aligned}$$

Then we see

(i) $AP_{g,d}$ is the union of $AP_{g,K}$ with $\sqrt{|K|} = d$,
(ii) $\overline{AP}^{\mathrm{main}}_{g,d}$ is a proper separated algebraic subspace of $\overline{AP}_{g,d}$,
(iii) $\dim AP_{g,d} = \dim \overline{AP}_{g,K} = \dim \overline{AP}^{\mathrm{main}}_{g,d} = g(g+1)/2 + d - 1$.

13.6. The semi-abelian group action on a T-TSQAS

The purpose of this subsection to construct a semiabelian group action on any T-flat TSQAS. See [33].

Lemma 13.6.1. *Let $(P_0, \mathcal{L}_0)$ be a totally degenerate TSQAS over k. Let X be a lattice of rank g associated to P_0, Del_B the Delaunay decomposition of $X_{\mathbf{R}}$ also associated to P_0, and $\mathrm{Del}^{(d)}_B$ the set of all d-dimensional Delaunay cells in Del_B. Let $\tau \in \mathrm{Del}^{(g-1)}_B$ and $\sigma_i \in \mathrm{Del}^{(g)}_B$ $(i = 1, 2)$ be Delaunay cells such that $\tau = \sigma_1 \cap \sigma_2$. Let $Z(\sigma_i) = \overline{O(\sigma_i)}$ be the irreducible component of P_0 corresponding to σ_i. Then P_0 is, along $O(\tau)$, isomorphic to the subscheme of $O(\tau) \times \mathbf{A}^2_k$ given by*

$$\mathrm{Spec}\ \Gamma(O_{O(\tau)})[\zeta_1, \zeta_2]/(\zeta_1\zeta_2),$$

where $\mathbf{A}^2_k = \mathrm{Spec}\ k[\zeta_1, \zeta_2]$: the two-dimensional affine space over k. Here $Z(\sigma_i)$ is given by $\zeta_i = 0$, and P_0 is, along $O(\tau)$, the union of $Z(\sigma_1)$ and $Z(\sigma_2)$, while $O(\tau)$ $(\simeq \mathbf{G}^{g-1}_{m,k})$ is given by $\zeta_1 = \zeta_2 = 0$, which is a Cartier divisor of each $Z(\sigma_i)$.

Remark 13.6.2. Instead of proving Lemma 13.6.1 here, we revisit Case 6.7.1 to illustrate the situation. In this case, $P_0 = Q_0$, and we recall the open affine subset $U_0(0)$ of P_0:

$$\begin{aligned} (U_0)_0 &= \mathrm{Spec}\ R[qw_1, qw_2, qw_1^{-1}, qw_2^{-1}] \otimes k(0)\\ &\simeq \mathrm{Spec}\ k(0)[u_1, u_2, v_1, v_2]/(u_1v_1, u_2v_2), \end{aligned}$$

where $(U_0)_0 = U_0 \otimes k(0)$.

Let $\tau = [0,1] \times \{0\} \in \mathrm{Del}_B^{(1)}$. Then there are exactly two Delaunay cells $\sigma = \sigma_i$ $(i = 1, 2)$ such that $\tau \subset \sigma$ and $\sigma \in \mathrm{Del}_B^{(2)}$, where

$$\sigma_1 = [0,1] \times [0,1], \quad \sigma_2 = [0,1] \times [-1,0].$$

We see

$$O(\tau) \simeq \mathrm{Spec}\ k(0)[u_1^{\pm 1}, u_2, v_1, v_2]/(u_2, v_1, v_2) \simeq \mathrm{Spec}\ k(0)[u_1^{\pm}].$$

Let $(U_0)_0(\tau)$ be the subset of $(U_0)_0$ where u_1 is invertible. Then we have

$$\begin{aligned}(U_0)_0(\tau) &= \mathrm{Spec}\ k(0)[u_1^{\pm}, u_2, v_2]/(u_2 v_2),\\ Z(\sigma_1) &= \mathrm{Spec}\ k(0)[u_1^{\pm 1}, u_2, v_2]/(u_2),\\ Z(\sigma_2) &= \mathrm{Spec}\ k(0)[u_1^{\pm 1}, u_2, v_2]/(v_2).\end{aligned}$$

This is what is meant by "along $O(\tau)$" in Lemma 13.6.1.

Definition 13.6.3. Let P_0 be a (not necessarily totally degenerate) $k(0)$-TSQAS of dimension g. Let $\mathrm{Sing}\,(P_0)$ be the singular locus of P_0. Let $\Omega^1_{P_0}$ be the sheaf of germs of regular one-forms over P_0, and $\Theta_{P_0} := \mathcal{H}om_{O_{P_0}}(\Omega^1_{P_0}, O_{P_0}) = \mathcal{D}er(O_{P_0})$. Then we define $\widetilde{\Omega}_{P_0}$ to be the sheaf of germs of rational one-forms ϕ over P_0 such that

(i) ϕ is regular outside $\mathrm{Sing}\,(P_0)$, and it has log poles at a generic point of every $(g-1)$-dimensional irreducible component of $\mathrm{Sing}\,(P_0)$ (we say ϕ has log poles on P_0),

(ii) the sum of the residues of ϕ along every $(g-1)$-dimensional irreducible component of $\mathrm{Sing}\,(P_0)$ is equal to zero.

These conditions make sense in view of Lemma 13.6.1.

Lemma 13.6.4. *Let P_0 be a (not necessarily totally degenerate) $k(0)$-TSQAS of dimension g. We define $\Theta^{\dagger}_{P_0}$ and $\Omega^{\dagger}_{P_0}$ by.*

$$\Theta^{\dagger}_{P_0} := \mathcal{H}om_{O_{P_0}}(\widetilde{\Omega}_{P_0}, O_{P_0}), \quad \Omega^{\dagger}_{P_0} := \mathcal{H}om_{O_{P_0}}(\Theta^{\dagger}_{P_0}, O_{P_0}).$$

Then we have $\Theta^{\dagger}_{P_0} \simeq O_{P_0}^{\oplus g}$, $\Omega^{\dagger}_{P_0} \simeq O_{P_0}^{\oplus g}$.

We note that by [39, p. 112], the tangent space of the automorphism group $\mathrm{Aut}(P_0)$ is given by $H^0(P_0, \Theta_{P_0})$.

Theorem 13.6.5. *Let T be a reduced scheme, $(P \xrightarrow{\pi} T, \mathcal{L})$ a T-TSQAS. Let $\widetilde{\Omega}_{P/T}$ be the sheaf as in Definition 13.6.3, $\Theta^{\dagger}_{P/T}$ the O_P-dual of $\widetilde{\Omega}_{P/T}$ and $\Omega^{\dagger}_{P/T}$ the O_P-dual of $\Theta^{\dagger}_{P/T}$. We define $\mathrm{Aut}^{\dagger}_T(P)$ to* *be the maximal closed subgroup T-scheme of* $\mathrm{Aut}_T(P)$ *which keep* $\Omega^{\dagger}_{P/T}$ *stable, and* $\mathrm{Aut}^{\dagger 0}_T(P)$ *the fiberwise identity component of* $\mathrm{Aut}^{\dagger}_T(P)$*, that is, the minimal open subgroup T-scheme of* $\mathrm{Aut}^{\dagger}_T(P)$*. Then*

(1) $\mathrm{Aut}^{\dagger}_T(P)$ *is flat over T, and the fiber* $(\mathrm{Aut}^{\dagger}_T(P))_t$ *has the tangent space* $H^0(P_t, \Theta^{\dagger}_{P_t})$ *for any geometric point t of T,*

(2) $\mathrm{Aut}^{\dagger 0}_T(P)$ *is a semi-abelian group scheme over T, flat over T.*

Theorem 13.7. ([33]) *Let* $N = \sqrt{|K|}$*. We define a map* sqap *by*

$$SQ^{\mathrm{toric}}_{g,K} \ni (P, \mathcal{L}, \phi^*, \tau) \times [v] \mapsto (\mathrm{Aut}^{\dagger 0}(P), P, \mathcal{L}, \mathrm{Div}\,\phi^*(v)) \in \overline{AP}_{g,K},$$

where $v \in V_H$, $\mathrm{Div}\,\phi^*(v)$ *is a Cartier divisor of P defined by* $\phi^*(v)$*. Then there exists a nonempty Zariski open subset U of* $\mathbf{P}(V_H)$ *such that*

(1) sqap *is a well-defined finite Galois morphism from* $SQ^{\mathrm{toric}}_{g,K} \times U$ *but it is not surjective,*

(2) *for any* $u \in U$*,*

  (a) $\mathrm{sqap} : SQ^{\mathrm{toric}}_{g,K} \times \{u\} \to \overline{AP}_{g,K}$ *is proper injective,*

  (b) $\mathrm{sqap} : A^{\mathrm{toric}}_{g,K} \times \{u\} \to AP_{g,K}$ *is an injective immersion.*

Details will appear in [33].

Corollary 13.8. $SQ^{\mathrm{toric}}_{g,1} \simeq \overline{AP}^{\mathrm{main}}_{g,1}$.

Remark 13.8.1. Assume Theorem 13.6.5. Then Corollary 13.8 is proved as follows. The scheme $U^{\dagger}_{g,1}$ is reduced, as is shown in the same manner as in Theorem 12.1, hence the closure U_3 of $U^{\dagger}_{g,1}$ is also reduced. Over U_3 we have a universal family

$$(Z_{\mathrm{univ}}, L_{\mathrm{univ}})_{U_3} := (Z_{\mathrm{univ}}, L_{\mathrm{univ}}) \times_{H^P} U_3.$$

Since U_3 is reduced and any fiber of $(Z_{\mathrm{univ}}, L_{\mathrm{univ}})_{U_3}$ is a TSQAS by Theorem 11.9, we can apply Theorem 13.6.5.

Since $A_{g,1} \simeq AP_{g,1}$ by $d = 1$, it is reduced by Theorem 12.1. Hence the closure $\overline{AP}^{\mathrm{main}}_{g,1}$ of $AP_{g,1}$ in $\overline{AP}_{g,1}$ is reduced because it is the intersection of all closed algebraic subspaces of $\overline{AP}_{g,1}$ containing $AP_{g,1} = (AP_{g,1})_{\mathrm{red}}$, hence it is the intersection of all closed reduced algebraic subspaces of $\overline{AP}_{g,1}$ containing $(AP_{g,1})_{\mathrm{red}}$.

It follows from Theorem 12.1 that we have a G-morphism from U_3 to $\overline{AP}^{\mathrm{main}}_{g,1}$ where $G = \mathrm{PGL}(W_1) \times \mathrm{PGL}(W_2)$. By the universality of the categorical quotient, we have a morphism $\mathrm{sqap} : SQ^{\mathrm{toric}}_{g,1} \to \overline{AP}^{\mathrm{main}}_{g,1}$, which is an isomorphism over $A_{g,1}$. Since $SQ^{\mathrm{toric}}_{g,1}$ is proper, sqap is surjective. The forgetful map

$$\overline{AP}_{g,1}^{\text{main}} \ni (G, P, \mathcal{L}, \Theta) \mapsto (P, \mathcal{L}) \in SQ_{g,1}^{\text{toric}}$$

is the left inverse of sqap. This proves $SQ_{g,1}^{\text{toric}} \simeq \overline{AP}_{g,1}^{\text{main}}$ because both $SQ_{g,1}^{\text{toric}}$ and $\overline{AP}_{g,1}^{\text{main}}$ are reduced.

§14. Related topics

14.1. Stability

Let us look at the following example. Let $X = \text{Spec}\ \mathbf{C}[x, y]$ and $\mathbf{G}_m = \text{Spec}\ \mathbf{C}[s, s^{-1}]$. Then $\mathbf{G}_m$ acts on X by $(x, y) \mapsto (sx, s^{-1}y)$. Let $(a, b) \in X$ and let $O(a, b)$ be the $\mathbf{G}_m$-orbit of (a, b). The (categorical) quotient of X by $\mathbf{G}_m$ is given by

$$X//\mathbf{G}_m = \text{Spec}\ \mathbf{C}[t], \quad (t = xy).$$

Any closed $\mathbf{G}_m$-orbit is either $O(a, 1)$ $(a \neq 0)$ or $O(0, 0)$. Hence by mapping $t = a$ (resp. $t = 0$) to the orbit $O(a, 1)$ (resp. $O(0, 0)$), the quotient $X//\mathbf{G}_m$ is identified with the set of closed orbits. This is a very common phenomenon. The same is true in general.

Theorem 14.1.1. (Seshadri-Mumford) *Let $X = \text{Proj}\ B$ be a projective scheme over a closed field k, and G a reductive algebraic k-group acting linearly on B (hence on X). Then there exists an open subscheme X_{ss} of X consisting of all semistable points in X, and a quotient Y of X_{ss} by G, that is, $Y = \text{Proj}\,(R)$, where R is the graded subring of B of all G-invariants. To be more precise, there exist a G-invariant morphism π from X_{ss} onto Y such that*

(1) *For any k-scheme Z on which G acts, and for any G-equivariant morphism $\phi : Z \to X$ there exists a unique morphism $\bar{\phi} : Z \to Y$ such that $\bar{\phi} = \pi\phi$,*

(2) *For given points a and b of X_{ss}*

$$\pi(a) = \pi(b) \text{ if and only if } \overline{O(a)} \cap \overline{O(b)} \neq \emptyset$$

where the closure is taken in X_{ss},

(3) *$Y(k)$ is regarded as the set of G-orbits closed in X_{ss}.*

See [26, p.38, p.40] and [41, p. 269].

A reductive group in Theorem 14.1.1 is by definition an algebraic group whose maximal solvable normal subgroup is an algebraic torus; for example $\text{SL}(n)$ and $\mathbf{G}_m$ are reductive.

The following is well known.

Theorem 14.1.2. ([9], [24]) *For a connected curve C of genus greater than one with dualizing sheaf ω_C, the following are equivalent:*

(1) *C is a stable curve, (moduli-stable)*
(2) *the n-th Hilbert point of C embedded by $|\omega_C^m|$ ($m \geq 10$) is GIT-stable for n large,*
(3) *the Chow point of C embedded by $|\omega_C^m|$ ($m \geq 10$) is GIT-stable.*

Proof. The proof goes as (2) $\Longrightarrow$ (1) $\Longrightarrow$ (3) $\Longrightarrow$ (2).
We explain only who proved these and where.
By [9, Chap. 2], let $\pi : Z_{U_C} \to U_C$ be the universal curve such that
(i) $X_h := \pi^{-1}(h)$ ($h \in U_C$) is a connected curve of genus g and degree $d = n(2g-2)$ embedded by the linear system $\omega_{X_h}^n$ into $\mathbf{P}^N$ ($N = d - g$),
(ii) the m_0-th Hilbert point $H_{m_0}(X_h)$ of X_h is $\mathrm{SL}(N{+}1)$-semistable,

where m_0 is a fixed positive integer large enough.

Then by [9, Theorem 1.0.1, p. 26], X_h is a semistable curve, that is, a reduced connected curve with nodal singularities only, any of whose nonsingular rational irreducible components meets the other irreducible components of X_h at two or more points. For any semistable curve X, ω_X is ample if and only if X is a stable curve. Hence (2) implies (1).

By [24, Theorem 5.1], if C is a stable curve, $\Phi_n(C)$, the image of C by the linear system ω_C^n, is Chow-stable. Thus (1) implies (3). (3) implies (2) by [8] and [26, Prop. 2.18, p. 65]. See [26, p. 215]. Q.E.D.

We have an analogous theorem for PSQASes.

Theorem 14.1.3. *Let $K = H \oplus H^\vee$, $N = |H|$, $N = |H|$, and k an algebraically closed field with $k \ni 1/N$.*

Suppose $e_{\min}(H) \geq 3$, and (Z, L) is a closed subscheme of $\mathbf{P}(V)$. Suppose moreover that (Z, L) is smoothable into an abelian variety whose Heisenberg group is isomorphic to $\mathcal{G}_H$. Then the following are equivalent:

(1) *(Z, L) is a level-$\mathcal{G}_H$ PSQAS, (moduli-stable)*
(2) *any Hilbert point of (Z, L) of large degree is GIT-stable,*
(3) *(Z, L) is stable under (a conjugate of) $\mathcal{G}_H$.*

See [30, Theorem 11.6] and [31, Theorems 10.3, 10.4].

Remark 14.1.4. In Table 1 we mean by GIT-stable that the cubic has a closed PGL(3)-orbit in the semistable locus. See [31] for details.

By Table 1, a planar cubic is GIT-stable if and only if it is either a smooth elliptic curve or a 3-gon. This is a special case of Theorem 14.1.3.

Table 1. Stability of cubics

curves (sing.)	stability	stab. gr.
smooth elliptic	GIT-stable	finite
3 lines, no triple point	GIT-stable	2 dim
a line+a conic, not tangent	semistable, not GIT-stable	1 dim
irreducible, a node	semistable, not GIT-stable	$\mathbf{Z}/2\mathbf{Z}$
3 lines, a triple point	not semistable	1 dim
a line+a conic, tangent	not semistable	1 dim
irreducible, a cusp	not semistable	1 dim

14.2. Arithmetic moduli

Katz and Mazur [15] constructed an integral model $X(n)$ of the moduli scheme of elliptic curves with level n-structure. Level structure is generalized as A-generators of the group of n-division points for $A = (\mathbf{Z}/n\mathbf{Z})^{\oplus 2}$. For any $n \geq 3$, $X(n)$ is a regular $\mathbf{Z}$-flat scheme such that $X(n) \otimes \mathbf{Z}[1/n, \zeta_n] \simeq SQ_{1,A}$. If $n = 3$, $X(3) \otimes \mathbf{F}_3$ is a union of four copies of $\mathbf{P}^1$, intersecting at the unique supersingular elliptic curve over $\mathbf{F}_9$.

This $X(n)$ is the model that we wish to generalize to the higher dimensional case, using our PSQASes or TSQASes. This will be discussed somewhere else.

14.3. The other compactifications

It is still unknown whether $AP^{\text{main}}_{g,1}$ (or $SQ^{\text{toric}}_{g,1}$) is normal or not. Therefore it is not yet known whether $AP^{\text{main}}_{g,1}$ (or $SQ^{\text{toric}}_{g,1}$) is the Voronoi compactification, one of the toroidal compactifications associated to the second Voronoi cone decomposition. There will exist a flat family of PSQASes or TSQASes over the Voronoi compactification. This will define, by the universality of the target, a morphism from the Voronoi compactification to $\overline{AP}^{\text{main}}_{g,1}$ (or $SQ^{\text{toric}}_{g,1}$) or $SQ^{\text{toric}}_{g,K}$ for some K once we check the family is algebraic. The author conjectures that $SQ^{\text{toric}}_{g,K}$ is normal, hence isomorphic to the Voronoi(-type) compactification.

Added in proof:

Definition 3.11 has to be replaced by the following

Definition 3.11. Two cubics (C, ψ, τ) and (C', ψ', τ') with level-$G(3)$ structure are defined to be isomorphic iff there exist isomorphisms

$$(f, F) : (C, L) \to (C', L'),$$
$$(h, H) : (\mathbf{P}(V), \mathbf{H}) \to (\mathbf{P}(V), \mathbf{H})$$

such that

(i) $\psi' \cdot (f, F) = (h, H)\psi$,

(ii) (f, F) is a $G(3)$-isomorphism, that is, $(f, F)\tau(g) = \tau'(g)(f, F)$ for any $g \in G(3)$.

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